# KONTSEVICH DEFORMATION QUANTIZATION AND FLAT CONNECTIONS

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ABSTRACT. In [13], the second author used the Kontsevich deformation quantization technique to define a natural connection $\omega_n$ on the compactified configuration spaces $\overline{C}_{n,0}$ of $n$ points on the upper half-plane. Connections $\omega_n$ take values in the Lie algebra of derivations of the free Lie algebra with $n$ generators. In this paper, we show that $\omega_n$ is flat.

The configuration space $\overline{C}_{n,0}$ contains a boundary stratum at infinity which coincides with the (compactified) configuration space of $n$ points on the complex plane. When restricted to this stratum, $\omega_n$ gives rise to a flat connection $\omega_n^\infty$. We show that the parallel transport $\Phi$ defined by the connection $\omega_3^\infty$ between configuration $1(23)$ and $(12)3$ verifies axioms of an associator.

We conjecture that $\omega_n^\infty$ takes values in the Lie algebra $\mathfrak{t}_n$ of infinitesimal braids. If correct, this conjecture implies that $\Phi \in \exp(\mathfrak{t}_3)$ is a Drinfeld's associator. Furthermore, we prove $\Phi \neq \Phi_{KZ}$ showing that $\Phi$ is a new explicit solution of the associator axioms.

## 1. INTRODUCTION

The Kontsevich proof of the formality conjecture and the construction of the star product on $\mathbb{R}^d$ equipped with a given Poisson structure make use of integrals of certain differential forms over compactified configuration spaces $\overline{C}_{n,m}$ of points on the upper half-plane. Here $n$ points are free to move in the upper half-plane, $m$ points are bound to the real axis, and we quotient by the diagonal action of the group $z \mapsto az + b$ with $a \in \mathbb{R}_+, b \in \mathbb{R}$.

In this paper, we use the same ingredients to study a certain connection $\omega_n$ on $\overline{C}_{n,0}$ with values in the Lie algebra of derivations of the free Lie algebra with $n$ generators. This connection was introduced by the second author in [13]. One of our results is flatness of $\omega_n$.

The compactified configuration space $\overline{C}_{n,0}$ contains a boundary stratum "at infinity" which coincides with the configuration space of $n$ points on the complex plane (quotient by the diagonal action $z \mapsto az + b$ with $a \in \mathbb{R}_+, b \in \mathbb{C}$). Over this boundary stratum, the connection $\omega_n$ restricts to the connection $\omega_n^\infty$ with values in the Lie algebra $\mathfrak{krv}_n$ defined in [3]. We conjecture that in fact $\omega_n$ takes values in the Lie algebra $\mathfrak{t}_n \subset \mathfrak{krv}_n$ defined by the infinitesimal braid relations.

Let $\Phi$ be the parallel transport defined by the connection $\omega_3^\infty$ for the straight path between configurations $1(23)$ and $(12)3$ of 3 points on the complex plane. We show that $\Phi$ verifies axioms of an associator with values in the group $KRV_3 = \exp(\mathfrak{krv}_3)$. If the conjecture of the previous paragraph holds true, then $\Phi \in \exp(\mathfrak{t}_3)$, and it becomes a Drinfeld's associator. The key ingredient in the proof of the pentagon axiom if the flatness property of $\omega_n^\infty$. The construction of $\Phi$ is parallel to the construction of the Knizhnik-Zamolodchikov associator $\Phi_{KZ}$ in [6] with $\omega_3^\infty$

replacing the Knizhnik-Zamolodchikov connection. Furthermore, one can show that $\Phi$ is even, and hence $\Phi \neq \Phi_{KZ}$.

While this paper was in preparation, we learnt of the work [11] proving our conjecture stated above.

The plan of the paper is as follows. In Section 2, we review some standard facts about the Kontsevich déformation quantization technique and free Lie algebras. In Section 3, we prove flatness of the connection $\omega_n$. Section 4 contains the proof of associator axioms for the element $\Phi$.

## 2. Deformation quantization and free Lie algebras

Many sources are now available on the Kontsevich formula for quantization of Poisson brackets (see *e.g.* [5]). For convenience of the reader, we briefly recall the main ingredients of [10] for $\mathbb{R}^d$ and the construction [13] of the connection $\omega_n$.

### 2.1. Free Lie algebras and their derivations.

2.1.1. *Free Lie algebras and derivations.* Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathfrak{lie}_n = \mathfrak{lie}(x_1, \ldots, x_n)$ be the degree completion of the graded free Lie algebra over $\mathbb{K}$ withgenerators $x_1, \ldots, x_n$ of degree one. We shall denote by $\mathfrak{der}_n$ the Lie algebra of derivations of $\mathfrak{lie}_n$. An element $u \in \mathfrak{der}_n$ is completely determined by its values on generators, $u(x_1), \ldots, u(x_n) \in \mathfrak{lie}_n$. The Lie algebra $\mathfrak{der}_n$ carries a grading induced by the one of $\mathfrak{lie}_n$.

**Definition 1.** *A derivation $u \in \mathfrak{der}_n$ is called* tangential *if there exist $a_i \in \mathfrak{lie}_n, i = 1, \ldots, n$ such that $u(x_i) = [x_i, a_i]$.*

Tangential derivations form a Lie subalgebra $\mathfrak{tder}_n \subset \mathfrak{der}_n$. Elements of $\mathfrak{tder}_n$ are in one-to-one correspondence with $n$-tuples of elements of $\mathfrak{lie}_n$, $(a_1, \ldots, a_n)$, which verify the condition that $a_k$ has no linear term in $x_k$ for all $k$. By abuse of notations, we shall often write $u = (a_1, \ldots, a_n)$. For two elements of $\mathfrak{tder}_n$, $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$, we have $[u, v]_{\mathfrak{tder}} = (c_1, \ldots, c_n)$ with

$$(1) \qquad\qquad c_k = u(b_k) - u(a_k) + [a_k, b_k]_{\mathfrak{lie}}.$$

**Definition 2.** *A derivation $u = (a_1, \ldots, a_n) \in \mathfrak{tder}_n$ is called* special *if $u(x) = \sum_i [x_i, a_i] = 0$ for $x = \sum_{i=1}^{n} x_i$.*

We shall denote the space of special derivations by $\mathfrak{sder}_n$. It is obvious that $\mathfrak{sder}_n \subset \mathfrak{tder}_n$ is a Lie subalgebra. Both $\mathfrak{tder}_n$ and $\mathfrak{sder}_n$ integrate to prouniportent groups denoted by $TAut_n$ and $SAut_n$, respectively. In more detail, $TAut_n$ consists of automorphisms of $\mathfrak{lie}_n$ such that $x_i \mapsto \mathrm{Ad}_{g_i} x_i = g_i x_i g_i^{-1}$, where $g_i \in \exp(\mathfrak{lie}_n)$. Similarly, elements of $SAut_n$ are tangential automorphisms of $\mathfrak{lie}_n$ with an extra property $x = \sum_{i=1}^{n} x_i \mapsto x$.

The family of Lie algebras $\mathfrak{tder}_n$ is equipped with simplicial Lie homomorphisms $\mathfrak{tder}_n \to \mathfrak{tder}_{n+1}$. For instance, for $u = (a,b) \in \mathfrak{tder}_2$ we define

$$
\begin{aligned}
u^{1,2} &= (a(x,y), b(x,y), 0)\,, \\
u^{2,3} &= (0, a(y,z), b(y,z))\,, \\
u^{12,3} &= (a(x+y,z), a(x+y,z), b(x+y,z)),
\end{aligned}
$$

and similarly for other simplicial maps. These Lie homomorphisms integrate to group homomorphisms of $TAut_n$ and $SAut_n$.

2.1.2. *Cyclic words.* Let $Ass_n^+ = \prod_{k=1}^{\infty} Ass^k(x_1, \ldots, x_n)$ be the graded free associative algebra (without unit) with generators $x_1, \ldots, x_n$. Every element $a \in Ass_n^+$ admits a unique decomposition of the form $a = \sum_{i=1}^{n}(\partial_i a)x_i$, where $a_i \in Ass_n$ ($Ass_n$ is a free associative algebra with unit).

We define the graded vector space $\mathrm{cy}_n$ as a quotient

$$\mathrm{cy}_n = Ass_n^+ / \langle (ab - ba); a, b \in Ass_n \rangle.$$

Here $\langle (ab-ba); a, b \in Ass_n \rangle$ is the subspace of $Ass_n^+$ spanned by commutators. The multiplication map of $Ass_n^+$ does not descend to $\mathrm{cy}_n$ which only has a structure of a graded vector space. We shall denote by $\mathrm{tr} : Ass_n^+ \to \mathrm{cy}_n$ the natural projection. By definition, we have $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in Ass_n$ imitating the defining property of trace. In general, graded components of $\mathrm{cy}_n$ are spanned by words of a given length modulo cyclic permutations.

**Example** 1. *The space $\mathrm{cy}_1$ is isomorphic to the space of formal power series in one variable without constant term, $\mathrm{cy}_1 \cong xk[[x]]$. This isomorphism is given by the following formula,*

$$f(x) = \sum_{k=1}^{\infty} f_k x^k \mapsto \sum_{k=1}^{\infty} f_k \mathrm{tr}(x^k).$$

2.1.3. *Divergence.* Let $u = (a_1, \ldots, a_n) \in \mathfrak{tder}_n$. We define the divergence as

$$\mathrm{div}(u) = \sum_{i=1}^{n} \mathrm{tr}(x_i (\partial_i a_i)).$$

It is a 1-cocycle of $\mathfrak{tder}_n$ with values in $\mathrm{cy}_n$ (see Proposition 3.6 in [3]).

We define $\mathfrak{krv}_n \subset \mathfrak{sder}_n \subset \mathfrak{tder}_n$ as the Lie algebra of special derivation with vanishing divergence. Hence, $u = (a_1, \ldots, a_n) \in \mathfrak{krv}_n$ is a solution of two equations: $\sum_{i=1}^{n}[x_i, a_i] = 0$ and $\sum_{i=1}^{n} \mathrm{tr}(x_i(\partial_i a_i)) = 0$. We shall denote by $KRV_n = \exp(\mathfrak{krv}_n)$ the corresponding prounipotent group.

## 2.2. Kontsevich construction.

2.2.1. *Configurations spaces.* We denote by $C_{n,m}$ the configuration space of $n$ distincts points in the upper half plane and $m$ points on the real line modulo the diagonal action of the group $z \mapsto az + b$ ($a \in \mathbb{R}_+, b \in \mathbb{R}$). In [10], Kontsevich constructed compactifications of spaces $C_{n,m}$ denoted by $\overline{C}_{n,m}$. These are manifolds with corners of dimension $2n-2+m$. We denote by $\overline{C}_{n,m}^+$ the connected component of $\overline{C}_{n,m}$ with real points in the standard order (*id.* $\overline{1} < \overline{2} < \cdots < \overline{m}$).

The compactified configuration space $\overline{C}_{2,0}$ (the "Kontsevich eye") is shown on Fig. 1. The upper and lower eyelids correspond to one of the points ($z_1$ or $z_2$) on the real line, left and right corners of the eye are configurations with $z_1, z_2 \in \mathbb{R}$

and $z_1 > z_2$ or $z_1 < z_2$. The boundary of the iris takes into account configurations where $z_1$ and $z_2$ collapse inside the complex plane. The angle along the iris keeps track of the angle at which $z_1$ approaches $z_2$.
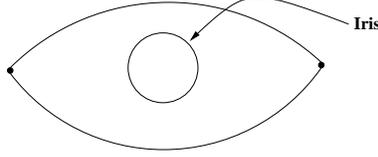


FIGURE 1. Variety $\overline{C}_{2,0}$.

2.2.2. *Graphs.* A graph $\Gamma$ is a collection of vertices $V_\Gamma$ and oriented edges $E_\Gamma$. Vertices are ordered, and the edges are ordered in a way compatible with the order of the vertices. We denote by $G_{n,2}$ the set of graphs with $n+2$ vertices and $2n$ edges verifying the following properties:

  i - There are $n$ vertices of the first type $1, 2, \cdots, n$ and 2 vertices of the second type $\overline{1}, \overline{2}$
  ii - Edges start from vertices of the first type, 2 edges per vertex.
  iii - Source and target of an edge are distinct.
  iv - There are no multiple edges (same source and target).

We are interested in the case of *linear graphs*. That is, vertices of the first type admit at most one incoming edge. Such graphs are superpositions of Lie type graphs (graphs with one root as on Fig. 2) and wheel type graphs (graph with one oriented loop, as on Fig. 3).
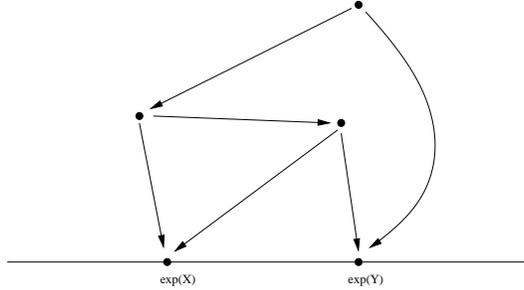


FIGURE 2. Lie type graph with symbol $\Gamma(x, y) = [[x, [x, y]], y]$.

2.2.3. *The angle map and Kontsevich weights.* Let $p$ and $q$ be two points on the upper half plane. Consider the hyperbolic angle map on $C_{2,0}$:

$$(2) \qquad \phi_h(p, q) = \arg\left(\frac{q - p}{q - \overline{p}}\right) \in \mathbb{T}^1.$$

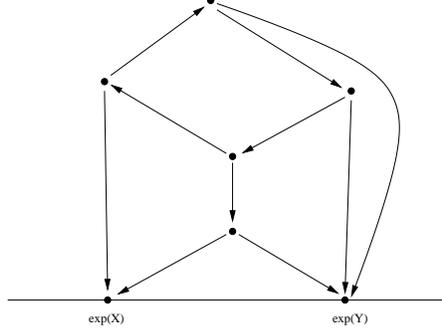This function admits a continuous extension to the compactification $\overline{C}_{2,0}$.

FIGURE 3. Wheel type graph with symbol $\Gamma(x, y) = \mathrm{tr}(y^2[x, y]x)$.

Consider a graph $\Gamma \in G_{n,2}$, and draw it in the upper half plane with vertices of the second type on the real line. By restriction, each edge $e$ defines an angle map $\phi_e$ on $\overline{C}_{n,2}^+$. The ordered product

$$(3) \qquad \Omega_\Gamma = \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e$$

is a regular $2n$-form on $\overline{C}_{n,2}^+$ (which is a $2n$-dim compact space).

**Definition 3.** *The Kontsevich weight of $\Gamma$ is given by the following formula,*

$$(4) \qquad w_\Gamma = \frac{1}{(2\pi)^{2n}} \int_{\overline{C}_{n,2}^+} \Omega_\Gamma.$$

2.3. **Campbell-Hausdorff and Duflo formulas.** Lie type graphs in $G_{n,2}$ are binary rooted trees. Hence, to each $\Gamma \in G_{n,2}$ one can associate a Lie word $\Gamma(x, y) \in \mathfrak{lie}_2$ of degree $2n$ in variables $x, y$ (see Fig. 2). Similarly, if $\Gamma$ is a wheel type graph, it corresponds to an element $G(x, y) \in \mathrm{cy}_2$ (see Fig. 3).

Recall the definition of the Duflo density function

$$\mathrm{duf}(x, y) = \frac{1}{2} \left( j(x) + j(y) - j(\mathrm{ch}(x, y)) \right) \in \mathrm{cy}_2,$$

where $\mathrm{ch}(x, y) = \log(e^x e^y)$ is the Campbell-Hausdorff series and

$$j(x) = \sum_{n \geq 2} \frac{b_n}{n \cdot n!} \mathrm{tr}(x^n)$$

with $b_n$ the Bernoulli numbers. The following Theorem relates functions $\mathrm{ch}(x, y)$ and $\mathrm{duf}(x, y)$ to the Kontsevich graphical calculus.

**Theorem 1** ([9], [4]). *The following identities hold true:*

$$(5) \qquad \mathrm{ch}(x, y) = x + y + \sum_{n \geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type } (n,2)}} w_\Gamma \Gamma(x, y),$$

$$(6) \qquad \mathrm{duf}(x, y) = \sum_{n \geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{wheel type } (n,2)}} \frac{w_\Gamma}{m_\Gamma} \Gamma(x, y),$$

*where $m_\Gamma$ is the order of the symmetry group of the graph $\Gamma$.*

Here *geometric* means that graphs are not labeled. Note that the definition of both $\Gamma(x, y)$ and $w_\Gamma$ requires an order on the set of edges, but the product $w_\Gamma \Gamma(x, y)$ is independent of this order. Even though $\mathrm{ch}(x, y)$ and $\mathrm{duf}(x, y)$ are defined over rationals, some of the coefficients $w_\Gamma$ are very probably irrational (see example of [7]).

2.4. $\xi$-**deformation.** In [13], one studies the following deformation for the Campbell-Hausdorff formula. Let $\xi \in \overline{C}_{2,0}$, $\Gamma \in G_{n,2}$, and let $\pi$ be the natural projection from $\overline{C}_{n+2,0}$ onto $\overline{C}_{2,0}$. We define the coefficients $w_\Gamma(\xi)$ for $\xi \in \overline{C}_{2,0}$ as

$$w_\Gamma(\xi) = \frac{1}{(2\pi)^{2n}} \int_{\pi^{-1}(\xi)} \Omega_\Gamma.$$

Functions $w_\Gamma(\xi)$ are smooth over $C_{2,0}$, and they are continuous over the compactification $\overline{C}_{2,0}$. The $\xi$-deformation of the Campbell-Hausdorff series $\mathrm{ch}_\xi(x, y)$ is defined as

$$(7) \qquad \mathrm{ch}_\xi(x, y) = x + y + \sum_{n \geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type } (n,2)}} w_\Gamma(\xi) \Gamma(x, y).$$

In a similar fashion, we introduce a deformation of the Duflo function,

$$\mathrm{duf}_\xi(x, y) = \sum_{n \geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{wheel type } (n,2)}} \frac{w_\Gamma(\xi)}{m_\Gamma} \Gamma(x, y).$$

For $\xi = (0, 1)$ (the right corner of the eye on Fig. 1), the expression (7) is given by the standard Campbell-Hausdorff series, and for $\xi$ in the position $\alpha$ on the iris, the Kontsevich Vanishing Lemma implies $\mathrm{ch}_\alpha(x, y) = x + y$. By the results of [12], for $\xi = (0, 1)$ the Duflo function $\mathrm{duf}_\xi(x, y)$ coincides with the standard Duflo function, and for $\xi$ on the iris one has $\mathrm{duf}_\alpha(x, y) = 0$.

2.5. **Connection $\omega_2$.** In [13], one defines a connection on $C_{2,0}$ with values in $\mathfrak{tder}_2$,

$$\omega_2 = \big( F_\xi(x, y), G_\xi(x, y) \big).$$

Here $F_\xi$ and $G_\xi$ are 1-forms on $C_{2,0}$ taking values in $\mathfrak{lie}_2$. They satisfy the following two (Kashiwara-Vergne type) equations (see Theorem 1 and Theorem 2 in [13])

$$(8) \qquad \mathrm{d}\,\mathrm{ch}_\xi(x, y) = \omega_2(\mathrm{ch}_\xi(x, y))$$

$$(9) \qquad \mathrm{d}\,\mathrm{duf}_\xi(x, y) = \omega_2(\mathrm{duf}_\xi(x, y)) + \mathrm{div}(\omega_2),$$

where d is the de Rham differential on $C_{2,0}$, and $\omega_2$ acts on $\mathrm{ch}_\xi(x, y)$ and $\mathrm{duf}_\xi(x, y)$ as a derivation of $\mathfrak{lie}_2$.

Let us briefly recall the construction of $\omega_2$. We will denote by $A, B \in G_{n,2}$ connected graphs of Lie type, and we define an extended graph $\vec{1}A$ (resp. $B\vec{1}$) as a graph with an additional edge starting at $\overline{1}$ (resp. $\overline{2}$) and ending at the root of $A$ (resp. $B$), see Fig. 4. Note that $n = 0$ is allowed, but since the source and the target of an edge must be distinct, $\Gamma$ will have a single edge starting at $\overline{1}$ and ending at $\overline{2}$, or starting at $\overline{2}$ and ending at $\overline{1}$.

Draw the extended graph in the upper half plane (with vertices of the second type corresponding to $\xi \in \overline{C}_{2,0}$). Then,

$$\Omega_{\restriction A} = \bigwedge_{e \in E_{\restriction A}} \mathrm{d}\phi_e$$

is a $2n + 1$-form on $\overline{C}_{n+2,0}$ (which is a $2n + 2$-dim compact space). The push forward along the natural projection $\pi : \overline{C}_{n+2,0} \to \overline{C}_{2,0}$ yields a 1-form on $\overline{C}_{2,0}$, $\omega_{\restriction A} = \pi_*(\Omega_{\restriction A})$.
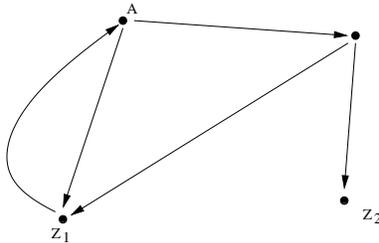


FIGURE 4. Extended graph

The connection $\omega_2 = (F_\xi(x,y), G_\xi(x,y))$ is defined by the following formula[1],

(10)
$$
\begin{cases}
F_\xi(x,y) &= \displaystyle\sum_{n \geq 0} \sum_{\substack{A \text{ graph of} \\ \text{Lie type } (n,2)}} \omega_{\restriction A} A(x,y) \\
G_\xi(x,y) &= \displaystyle\sum_{n \geq 0} \sum_{\substack{B, \text{graph of} \\ \text{Lie type } (n,2)}} \omega_{B \restriction} B(x,y)
\end{cases}
$$

2.6. **Definition of $\omega_n$.** We now extend this construction to an arbitrary number of vertices of the second type.

Consider $\Gamma \in G_{p,n}$ a connected graph of Lie type with vertices of the second type labeled $\overline{1}, \ldots, \overline{n}$. Define the extended graph $\Gamma^{(i)}$ by adding an edge from the vertex $\overline{i}$ to the root of $\Gamma$. Consider the natural projection $\pi : \overline{C}_{p+n,0} \to \overline{C}_{n,0}$ and take the push forward 1-form $\omega_{\Gamma^{(i)}} = \pi_*(\Omega_{\Gamma^{(i)}})$. We define 1-forms with values in $\mathfrak{lie}_n$

$$F_i = \sum_{p \geq 0} \sum_{\substack{\Gamma \text{ graph of} \\ \text{Lie type } (p,n)}} \omega_{\Gamma^{(i)}} \Gamma(x_1, \ldots, x_n),$$

and $\omega_n = (F_1, \ldots, F_n)$ yields a connection with values in $\mathfrak{tder}_n$.

The connection $\omega_n$ is smooth over $C_{n,0}$. Over the compactification $\overline{C}_{n,0}$, it belongs to the class $L^1$ when restricted to piece-wise differentiable curves. Hence, along such curves all iterated integrals converge, and there is a unique solution of the initial value problem $\mathrm{d}g = -g\omega$ with $g(z_0) = 1$ for the base point $z_0$ (e.g. by using the Grönwall's Lemma). Therefore, parallel transports are well defined.

The same applies to restrictions of $\omega_n$ to boundary strata of $\overline{C}_{n,0}$ of dimension at least one. For instance, in the case of $\overline{C}_{2,0}$ one can consider a path along the eyelid, or a generic path from the corner of $\overline{C}_{2,0}$ to the iris.

We will need restrictions of $\omega_n$ to various boundary strata of co-dimension one of $\overline{C}_{n,0}$. First of all, there is a stratum "at infinity" equal to the configuration space

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[1] For $n = 0$, $A = y$ and $B = x$.

$C_n$ of $n$ points on the complex plane (modulo the diagonal action of the group $z \mapsto az + b$ for $a \in \mathbb{R}_+, b \in \mathbb{C}$). We denote the corresponding connection $\omega_n^\infty$. It is given by the same formula as $\omega_n$ with the configuration space $\overline{C}_{n,0}$ replaced by $\overline{C}_n$, and the hyperbolic angle replaced with the Euclidean angle.

Next, for $q$ points collapsing inside the upper half plane, we have a stratum of the form $C_q \times C_{n-q+1,0}$. We denote the natural projections by $\pi_1$ and $\pi_2$, and obtain an expression for the connection

$$\omega_n|_{C_q \times C_{n-q+1,0}} = \pi_1^*(\omega_q^\infty)^{1,2,\ldots,q} + \pi_2^* \omega_{n-q+1}^{12\ldots q, q+1,\ldots,n}.$$

A similar property holds for the connection $\omega_n^\infty$ for the stratum $C_q \times C_{n-q+1}$ corresponding to $q$ points collapsing together,

$$\omega_n^\infty|_{C_q \times C_{n-q+1}} = \pi_1^*(\omega_q^\infty)^{1,2,\ldots,q} + \pi_2^*(\omega_{n-q+1}^\infty)^{12\ldots q, q+1,\ldots,n}.$$

In the case when $q$ points are collapsing to the point on the real axis, we obtain the stratum $C_{q,0} \times C_{n-q,1}$, and for the connection we get

$$\omega_n|_{C_{q,0} \times C_{n-q,1}} = \pi_1^* \omega_q^{1,2,\ldots,q} + \pi_2^* \omega_{n-q+1}^{12\ldots q,q+1,\ldots,n}|_{C_{n-q,1}}.$$

Note that the restriction of the connection form $\omega_n$ to the boundary stratum $C_{n-1,1}$ corresponds to configurations with the point $z_1$ on the real axis, and it has the following property: its first component (as an element of $\mathfrak{tder}_n$) vanishes since the 1-form $d\phi_e$ vanishes when the source of the edge $e$ is bound to the real axis.

## 3. Zero curvature equation and applications

One of our main results is flatness of the connection $\omega_n$.

### 3.1. The zero curvature equation.

**Theorem 2.** *The connection $\omega_n$ is flat. That is, the following 2-form on $C_{n,0}$ vanishes,*

$$(11) \qquad d\omega_n + \frac{1}{2}[\omega_n, \omega_n] = 0$$

*Proof :*  The argument is based on the Stokes formula, and we give details in the case of $\omega_2$. The case of arbitrary $n$ is treated in a similar fashion.

Let $C_\xi$ be a small circle around $\xi \in C_{2,0}$, $\Delta_\xi$ be the corresponding disk, and consider $\pi^{-1}(\Delta_\xi)$. Since the forms $\Omega_{\vdash A}, \Omega_{B \dashv}$ are closed, we have

$$\int_{\pi^{-1}(\Delta_\xi)} d\left(\sum_A \Omega_{\vdash A} A(x,y), \sum_B \Omega_{B \dashv} B(x,y)\right) = 0.$$

By applying Stokes formula and the definition of the connection $\omega_2$, one obtains

$$(12) \qquad 0 = \int_{C_\xi} \omega_2 + \int_{\bigcup_{z \in \Delta_\xi} \partial(\pi^{-1}(z))} \left(\sum_A \Omega_{\vdash A} A(x,y), \sum_B \Omega_{B \dashv} B(x,y)\right).$$

By using again the Stokes's formula (on the disk $\Delta_\xi$), one can rewrite the first term in the form $\int_{\Delta_\xi} d_\xi \omega_2$.

For the second term, one obtains contributions from the boundary strata of co-dimension one. The usual arguments in Kontsevich theory rule out strata where more than two points collapse (by the Kontevich Vanishing Lemma), and strata corresponding to collapse of internal edges (by Jacobi identity). The remaining

strata correspond to collapsing of a vertex of the first type and a vertex of the second type. Figures below illustrate different cases of such boundary strata (for the first component):

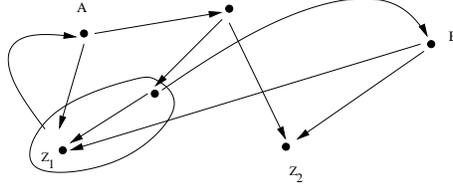  - Fig. 5 computes terms of the type $[x, \omega_{\upharpoonright B} B(x,y)] \cdot \partial_x \left( \omega_{\upharpoonright A} A(x,y) \right)$.



FIGURE 5.

  - Fig. 6 represents terms of the type $[y, \omega_{B\upharpoonright} B(x,y)] \cdot \partial_y \left( \omega_{\upharpoonright A} A(x,y) \right)$.
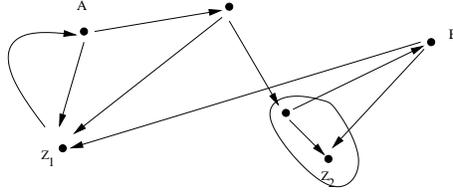


FIGURE 6.

  - Fig. 7 computes terms of the type $[\omega_{\upharpoonright A} A(x,y), \omega_{\upharpoonright B} B(x,y)]$. This term appears only once. It corresponds to the componentwise bracket in $\mathfrak{lie}_2 \times \mathfrak{lie}_2$

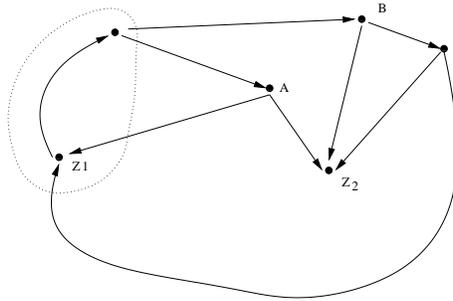$$\frac{1}{2}[\omega_2(x,y), \omega_2(x,y)]_{\mathfrak{lie}_2}.$$



FIGURE 7.

These are exactly the three terms of the bracket $\frac{1}{2}[\omega_2, \omega_2]_{\mathfrak{tder}}$ (see equation (1)). By equation (12), one gets

$$\int_{\Delta_\xi} \left( \mathrm{d}\omega_2 + \frac{1}{2}[\omega_2, \omega_2] \right) = 0.$$

Since the curvature $\mathrm{d}\omega_2 + \frac{1}{2}[\omega_2, \omega_2]$ is a continuous function of $\xi$, we conclude that it vanishes on $C_{2,0}$. ∎

3.2. **Parallel transport and symmetries.** In this Section we discuss various properties of the connection $\omega_2$, including the induced holonomies and their symmetries.

3.2.1. *Parallel transport.* Since $\omega_2$ is flat, the equation

$$\mathrm{d}\,g = -g\omega_2,$$

has a local solution on $C_{2,0}$ with values in $TAut_2 = \exp(\mathfrak{tder}_2)$. By abuse of notations we write $(u, v) \in TAut_2$ for an element acting on generators by $x \mapsto \mathrm{Ad}_u x = uxu^{-1}, y \mapsto \mathrm{Ad}_v y = vyv^{-1}$.

Take the initial data $g_\alpha = 1$ for $\alpha$ on the iris of $\overline{C}_{2,0}$ (see Fig. 8), and consider a path from $\alpha$ to $\xi$. The value at $\xi$ for the parallel transport is well-defined since the connection is integrable, and by the flatness property it only depends of the homotopy class of the path. Integrating the equation (8), we obtain $g_\xi(\mathrm{ch}_\xi(x, y)) = \mathrm{ch}_\alpha(x, y) = x + y$.
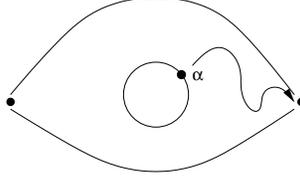
Recall [1] that for $\xi = (0, 1)$ this parallel transport $F$ defines a solution of the Kashiwara-Vergne conjecture [8]. We conclude that this solution is independent of the choice of a path in the trivial homotopy class (the straight line joining $\alpha = 0$ and $\xi = (0, 1)$).

3.2.2. *Holonomy.* Solutions of equation $\mathrm{d}\,g = -g\omega_2$ are not globally defined on $\overline{C}_{2,0}$ because of the holonomy around the iris.

**Lemma 1.** *The restriction of $\omega_2$ to the iris is equal to $\omega_\theta = \frac{\mathrm{d}\,\theta}{2\pi}(y, x)$. The holonomy around the Iris $H_{2\pi}$ is given by the inner automorphism $(\exp(x + y), \exp(x + y)) \in TAut_2$.*

*Proof :* When the point $\xi \in \overline{C}_{2,0}$ reaches the iris, the Kontsevich angle map degenerates to the Euclidian angle map on the complex plan, and the connection $\omega_2$ is replaced by $\omega_2^\infty$

$$\omega_2^\infty = \Big( \sum w_{\ulcorner A}^\infty A(x, y), \sum w_{B\urcorner}^\infty B(x, y) \Big).$$

Since the Euclidian angle is rotation invariant, so is the 1-form $w_{\ulcorner A}^\infty$. Hence, it is sufficient to compute $\int_{\mathbb{T}^1} w_{\ulcorner A}^\infty$. By Kontsevich Vanishing Lemma (see [10] § 6.6), integrals of 3 and more angle 1-forms vanish. Therefore, for $A$ a nontrivial graph

one gets $\int_{\mathbb{T}^1} w^\infty_{\rhd A} = 0$ which implies $w^\infty_{\rhd A} = 0$. As a result, we obtain the connection $\omega_\theta$ by adding two trivial graph contributions,

$$\omega_\theta = \frac{\mathrm{d}\,\theta}{2\pi}(y,0) + \frac{\mathrm{d}\,\theta}{2\pi}(0,x) = \frac{\mathrm{d}\,\theta}{2\pi}(y,x).$$

Let's integrate the equation $\mathrm{d}_\theta g = -g\omega_\theta$ over the boundary of the Iris. Note that $t = (y,x)$ is actually an inner derivation since $t(x) = [x,y] = [x, x+y]$ and $t(y) = [y,x] = [y, x+y]$. We conclude that the parallel transport around the iris is given by $H_\theta = \exp(\theta t/2\pi) = (\exp(\theta(x+y)/2\pi), \exp(\theta(x+y)/2\pi))$. In particular, for $\theta = 2\pi$ we obtain $H_{2\pi} = (\exp(x+y), \exp(x+y))$, as required. ∎

3.2.3. *Symmetries of the connection.* Consider the following involutions on $\overline{C}_{2,0}$,

$$\sigma_1 : (z_1, z_2) \mapsto (z_2, z_1) \quad \text{and} \quad \sigma_2 : (z_1, z_2) \mapsto (-\bar{z}_1, -\bar{z}_2).$$

Identifying $\overline{C}_{2,0}$ with the Kontsevich eye (see Fig. 1), $\sigma_1$ is the reflection with respect to the center of the eye, and $\sigma_2$ is the reflection with respect to the vertical axis (see Fig. 1).

We shall denote by $\tau_1$ and $\tau_2$ the following involutions of $\mathfrak{tder}_2$,

$$\begin{aligned}
\tau_1 &: & (F(x,y), G(x,y)) &\mapsto (G(y,x), F(y,x)), \\
\tau_2 &: & (F(x,y), G(x,y)) &\mapsto (F(-x,-y), G(-x,-y)).
\end{aligned}$$

They lift to involutions of $TAut_2$.

**Proposition 1.** *The connection $\omega_2$ verifies*

$$\sigma_1^*(\omega_2) = \tau_1(\omega_2) \quad , \quad \sigma_2^*(\omega_2) = \tau_2(\omega_2)$$

*Proof :* The involution $\sigma_1$ simply exchanges the colors $x$ and $y$ of all graphs which induces the involution $\tau_1$ on $\mathfrak{tder}_2$.

The involution $\sigma_2$ flips the sign of the one form $d\phi_e$ for each edge (since the reflection changes sign of the Euclidean angle), and changes the orientation of each integration over a complex variable. Hence, for a graph with $n$ internal vertices we collect $-1$ to the power $(2n+1) + n \equiv n+1 \ (mod\ 2)$. Corresponding rooted trees have exactly $n+1$ leaves. Hence, one should change a sign of each leaf which results in applying the involution $\tau_2$. ∎

Let $F \in TAut_2$ be the parallel transport of the equation $\mathrm{d}g = -g\omega_2$ for the straight path between the the position 0 on the iris to the right corner of the eye $\overline{C}_{2,0}$. Since the path is invariant under the composition $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$, the parallel transport is invariant under $\tau = \tau_1 \tau_2 = \tau_2 \tau_1$, $\tau(F) = F$.

In order to discuss the involution $\tau_1$ and $\tau_2$ separately, we need the following Lemma.

**Lemma 2.** *The parallel transport along the lower eyelid in the counter-clockwise direction is equal to $R = (\exp(y), 1) \in TAut_2$.*

*Proof :* The connection restricted to the lower eyelid has a trivial second component because edges starting from the real line give rise to a vanishing 1-form. Write the corresponding parallel transport as $R = (g(x,y), 1) \in TAut_2$. Integrating the equation $\mathrm{d}\,\mathrm{ch}_\xi(x,y) = \omega_2(\mathrm{ch}_\xi(x,y))$ along the lower eyelid, we obtain

$$R(\mathrm{ch}(x,y)) = \mathrm{ch}(\mathrm{Ad}_{g(x,y)}x, y) = \mathrm{ch}(y,x),$$

and this equation implies $g(x,y) = \exp(y)$, as required.                    ∎

Note that the path along the upper eyelid (oriented in the counter-clockwise direction) can be obtained by applying the involution $\sigma_1$ to the lower eyelid. Hence, the corresponding parallel transport is given by $\tau_1(R) = R^{2,1}$.

**Proposition 2.** *The element $F \in TAut_2$ verifies the following identities,*

$$F = e^{t/2}\tau_1(F)\tau_1(R^{-1}) = e^{-t/2}\tau_1(F)R.$$

*Proof :*   These equations express the flatness condition for two contractible paths shown on Fig.9.                    ∎

These equations can be re-interpreted as the property of the parallel transport under the involution $\tau_1$,

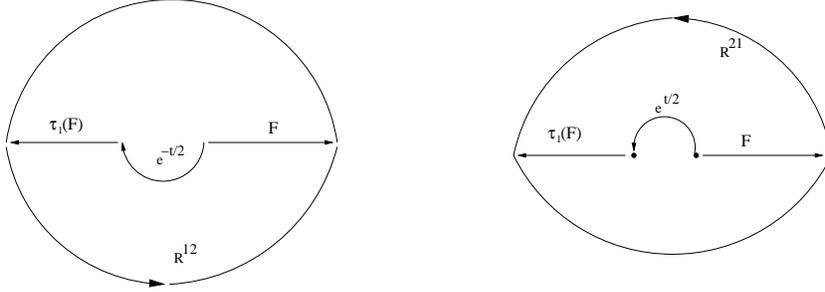$$F^{2,1} = \tau_1(F) = e^{-t/2}FR^{2,1} = e^{t/2}FR^{-1}.$$



FIGURE 9.  The $R$ matrix

## 4. CONNECTION $\omega_3$ AND ASSOCIATORS

4.1. **Connection $\omega_3^\infty$.** An important element of our construction is the connection $\omega_3^\infty$ which is built using the Kontsevich technique applied to the complex plane equipped with the Euclidean angle form.

**Example 2.** *Consider $\omega_3^\infty$ for 3 points situated on the real line at the positions $0, s, 1$ (see Fig. 10). The Euclidean angle form is $\mathrm{d}\theta$ with $\tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$. One of the simplest trees is shown on Fig. 10. The corresponding 3-form is given by the following expression,*

$$(13) \quad \frac{1}{1+(\frac{y}{x})^2}\mathrm{d}\left(\frac{y}{x}\right) \wedge \frac{1}{1+(\frac{y}{x-s})^2}\mathrm{d}\left(\frac{y}{(x-s)}\right) \wedge \frac{1}{1+(\frac{y}{x-1})^2}\mathrm{d}\left(\frac{y}{(x-1)}\right) =$$

$$-\frac{y^2}{(x^2+y^2)((x-s)^2+y^2)((x-1)^2+y^2)}dx \wedge dy \wedge ds.$$

*By [2] §I.1, the orientation is given by $-dx \wedge dy \wedge ds$, and one gets*

$$\omega_{\Gamma(1)}^\infty = -\frac{1}{8\pi^2}\left(\frac{\log(1-s)}{s} + \frac{\log(s)}{(1-s)}\right)\mathrm{d}s.$$

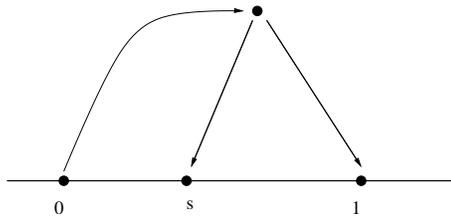*This 1-form is integrable (semi-algebraic), and one has $\int_0^1 \omega_{\Gamma(1)}^\infty = \frac{1}{24}$.*

FIGURE 10.

**Remark 1.** *Let $\alpha : z \mapsto \bar{z}$ be a the complex conjugation, and let $\kappa$ be an involution of $\mathfrak{tder}_3$ defined by formula*

$$(a(x,y,z), b(x,y,z), c(x,y,z)) \mapsto (a(-x,-y,-z), b(-x,-y,-z), c(-x,-y,-z)).$$

*Then, $\alpha^* \omega_3^\infty = \kappa(\omega_3^\infty)$. The proof is similar to the one of Proposition 1.*

**Proposition 3.** *Connection $\omega_3^\infty$ is flat and takes values in $\mathfrak{krv}_3$.*

*Proof :* The flatness condition $d\omega_3^\infty + \frac{1}{2}[\omega_3^\infty, \omega_3^\infty] = 0$ is obtained by replacing the hyperbolic angle form on the upper half-plane by the Euclidean angle form on the complex plane in the proof of Theorem 2.

In the same way, equations (8) and (9) are valid for the connection $\omega_n^\infty$. By Kontsevich Vanishing Lemma (Lemma 6.6 [10]), $w_\Gamma^\infty(\xi) = 0$ for Kontsevich coefficient associated to nontrivial tree graphs. Hence, $\mathrm{ch}_\xi^\infty(x,y,z) = x+y+z$. Furthermore, Shoikhet's result [12] implies $\mathrm{duf}_\xi^\infty(x,y,z) = 0$. Therefore, differential equations (8) and (9) yield

$$\omega_3^\infty(x+y+z) = 0 \quad , \quad \mathrm{div}(\omega_3^\infty) = 0.$$

That is, $\omega_3^\infty$ takes values in $\mathfrak{krv}_3$ as required. ∎

4.2. **Associator.** We will use the following notation. Recall that $T = (u,v) \in TAut_2$ is an automorphism of $\mathfrak{lie}_2$ acting by

$$(x,y) \mapsto (\mathrm{Ad}_u x, \mathrm{Ad}_v y).$$

We denote $T^{1,2} = (u(x,y), v(x,y), 1) \in TAut_3$, $T^{12,3} = (u(x+y,z), u(x+y,z), v(x+y,z)) \in TAut_3$ etc. For $F \in TAut_2$ the parallel transport from the iris to the right corner of the eye $\overline{C}_{2,0}$, we define

$$\Phi = F^{1,23} F^{23} (F^{12,3} F^{12})^{-1} \in TAut_3.$$

This element is the main topic of study in this Section.

**Proposition 4.** *The element $\Phi$ coincides with the parallel transport for the equation $\mathrm{d}g = -g\omega_3^\infty$ between positions $1(23)$ to $(12)3$.*

*Proof :* Consider the following path in the configuration space $\overline{C}_{3,0}$:

First, place $z_1, z_2, z_3$ on the stratum at infinity and move them along the horizontal line (the real axis of the complex plane at infinity) from the position $1(23)$ ($z_2$ and $z_3$ collapsed) to the position $(12)3$ ($z_1$ and $z_2$ collapsed). The connection at infinity is $\omega_3^\infty$, and we denote the corresponding parallel transport by $\Phi^\infty$.

Next, make $z_3$ descend from the stratum at infinity to plus infinity of the real axis of the upper half-plane (this corresponds to moving to the right corner of the

eye for the points (12) and 3). On this stratum, the connection is $\omega_2^{12,3}$, and the parallel transport is given by $F^{12,3}$.

Continue with descending both $z_1$ and $z_2$ to the real axis. The connection on this stratum is $\omega_2^{1,2}$, and the parallel transport gives $F^{1,2}$.

Then, move $z_2$ to the vicinity of $z_3$ along the real axis of the upper half-plane. The parallel transport is trivial since the connection vanishes along the real axis.

Finally, lift $z_2$ and $z_3$ from the real axis and make them collapse on each other (parallel transport $(F^{2,3})^{-1}$), and lift $z_1$ from the real axis and make it collapse with $z_2 = z_3$ (parallel transport $(F^{1,23})^{-1}$).

Thus, we made a loop and returned to the position $1(23)$ at infinity. This loop is contractible, and the total parallel transport is trivial by the flatness property of the connection. Hence,

$$\Phi^\infty F^{12,3} F^{1,2} (F^{2,3})^{-1} (F^{1,23})^{-1} = 1,$$

and we obtain

$$\Phi^\infty = F^{1,23} F^{2,3} (F^{12,3} F^{1,2})^{-1} = \Phi,$$

as required.                                                                          ■

We have $\Phi = F^{1,23} F^{23} (F^{12,3} F^{12})^{-1}$, and the first term of $\Phi$ is given by

$$\Phi(x,y,z) = 1 - \frac{1}{24}([y,z], -[x,z], [y,z]) + \ldots = 1 + \frac{1}{24}[t^{1,2}, t^{2,3}] + \ldots,$$

with $t^{1,2} = (y,x,0)$ and $t^{2,3} = (0,z,y)$. Here we used the fact that $\int_0^1 \omega_{\Gamma^{(1)}}^\infty = \frac{1}{24}$ (see Example considered above), and the minus sign is coming from the orientation of the boundary stratum $C_3 \subset \partial C_{3,0}$. The main properties of the element $\Phi$ are summarized in the following Theorem.

**Theorem 3.** *The element $\Phi$ satisfies associator axioms.*

*Proof :*   The axioms to verify are as follows: $\Phi$ is a group like element and it is a solution of the following equations,

$$\Phi^{3,2,1} \Phi^{1,2,3} = 1 \quad (i)$$

$$\Phi^{1,2,34} \Phi^{12,3,4} = \Phi^{2,3,4} \Phi^{1,23,4} \Phi^{1,2,3} \quad (ii)$$

$$\exp\left(\pm \frac{1}{2} t_{12}\right) \Phi^{3,1,2} \exp\left(\pm \frac{1}{2} t_{13}\right) \Phi^{2,3,1} \exp\left(\pm \frac{1}{2} t_{23}\right) \Phi^{1,2,3} =$$

$$\exp\left(\pm \frac{1}{2}(t_{12} + t_{13} + t_{23})\right) \quad (iii)$$

(i) - $(\Phi^{1,2,3})^{-1}$ is the parallel transport between positions $1(23)$ to $(12)3$. Let $\beta$ be the reflection with respect to the vertical axis. Then, similar to Proposition 1, we obtain $\beta^* \omega_3^\infty = (\omega_3^\infty)^{3,2,1}$. Hence, we get $\Phi^{3,2,1} = (\Phi^{1,2,3})^{-1}$, as required.

(iii) - Consider the following path:

$$1(23) \mapsto (12)3 \mapsto (21)3 \mapsto 2(13) \mapsto 2(31) \mapsto (23)1 \mapsto 1(23).$$

Here the last step is by moving the collapsed pair (23) around the point 1 along the iris of the corresponding $\overline{C}_{2,0}$ stratum. By the flatness property, the total parallel transport is trivial. This gives exactly the pair of hexagonal equations *(iii)* by using the equation (i) and the fact that $c = t_{12} + t_{13} + t_{23}$ is central in $\mathfrak{soer}_3$ (see

Proposition 3.4, [3]). The plus or minus sign in equation *(iii)* depends on the choice of the (clockwise or anti-clockwise) semi-circle for each exchange of two points (*e.g.* 1 moves above 2 or below 2 in the move $(12)3 \mapsto (21)3$).

(ii) - Consider four points $z_1 = 0, z_2 = s, z_3 = t, z_4 = 1$ on the horizontal line (the real axis of the complex plane) representing a point of the configuration space of the complex plane placed at infinity of $\overline{C}_{4,0}$. The path

$$((12)3)4 \mapsto (1(23))4 \mapsto 1((23)4) \mapsto 1(2(34))$$

is contractible. Hence, the parallel transport defined by the flat connection $\omega_4^\infty$ is trivial. It is easy to see that it reproduces the pentagon equation *(ii)* (see Fig. 11).
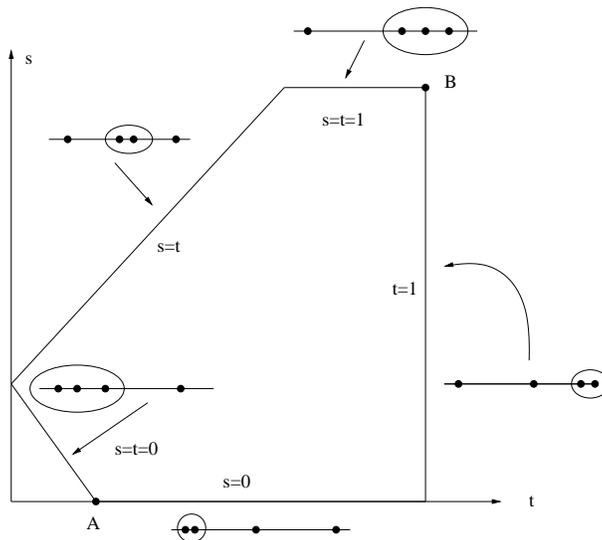


FIGURE 11. The compactification space of 4 positions on a line

∎

Note that $\Phi$ is an element of the group $KRV_3$ which contains the subgroup $T_3 = \exp(\mathfrak{t}_3)$. If $\Phi$ is actually an element of $T_3$, it becomes a Drinfeld associator. Since $\Phi$ is even ($\kappa(\Phi) = \Phi$, by Remark 1), it does not coincide the Knizhnik-Zamolodchikov associator (the only known Drinfeld associator defined by an explicit formula).

**Conjecture 1.** *The element $\Phi$ is a Drinfeld associator. That is, $\Phi \in T_3 \subset KRV_3$.*

In a recent work [11], Severa and Willwacher prove this conjecture affirming that the element $\Phi$ is indeed a new Drinfeld associator admitting a presentation as a parallel transport of the flat connection $\omega_3^\infty$ defined by explicit formulas.

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