ON TRIVIALITY OF THE KASHIWARA-VERGNE PROBLEM FOR QUADRATIC LIE ALGEBRAS

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ABSTRACT. We show that the Kashiwara-Vergne (KV) problem for quadratic Lie algebras (that is, Lie algebras admitting an invariant scalar product) reduces to the problem of representing the Campbell-Hausdorff series in the form $\ln(e^x e^y) = x + y + [x, a(x,y)] + [y, b(x,y)]$, where $a(x,y)$ and $b(x,y)$ are Lie series in x and y. This observation explains the existence of explicit rational solutions of the quadratic KV problem [17, 2] whereas constructing an explicit rational solution of the full KV problem would probably require the knowledge of a rational Drinfeld associator. It also gives, in the case of quadratic Lie algebras, a direct proof of the Duflo theorem (implied by the KV problem).

Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathfrak{lie}_n$ be the degree completion of the free Lie algebra with n generators. The Campbell-Hausdorff series $\mathrm{ch}(x, y) = \ln(e^x e^y)$ is an element of $\mathfrak{lie}_2$. Similarly, $\mathrm{ch}(x_1, \ldots, x_n) = \ln(e^{x_1} \ldots e^{x_n})$ is an element of $\mathfrak{lie}_n$. For $t \in \mathbb{K}^*$, let $\mathrm{ch}_t(x_1, \ldots, x_n) = t^{-1}\mathrm{ch}(tx_1, \ldots, tx_n)$. Note that $\mathrm{ch}_t(x_1, \ldots, x_n)$ is analytic in t.

The universal enveloping algebra of $\mathfrak{lie}_n$, $U(\mathfrak{lie}_n) = \mathrm{Ass}_n$ is the degree completion of the free associative algebra with n generators. For an element $\alpha \in \mathrm{Ass}_n$ there is a unique decomposition $\alpha = \alpha_0 + \sum_{i=1}^n (\partial_i \alpha) x_i$ with $\alpha_0 \in \mathbb{K}$ and $\partial_i \alpha \in \mathrm{Ass}_n$. Extend the adjoint action of $\mathfrak{lie}_n$ to an algebra homomorphism $\mathrm{ad} : \mathrm{Ass}_n \to \mathrm{End}(\mathfrak{lie}_n)$. It is easy to see that for $\alpha \in \mathfrak{lie}_n$ one has

$$\frac{d}{ds} \alpha(x_1, \ldots, x_i + sz, \ldots, x_n)|_{s=0} = \mathrm{ad}(\partial_i \alpha)z.$$

We denote by $C_n \subset \mathrm{Ass}_n$ the subspace spanned by commutators ($ab - ba \in C_n$ for all $a, b \in \mathrm{Ass}_n$). Let τ be the unique anti-involution on Ass_n defined by the property $\tau(\alpha) = -\alpha$ for all $\alpha \in \mathfrak{lie}_n$. Denote by $A_n \subset \mathrm{Ass}_n$ the eigenspace of τ corresponding to the eigenvalue (-1), $\alpha \in A_n$ if $\tau(\alpha) = -\alpha$. Define $\mathfrak{tr}_n = \mathrm{Ass}_n / C_n$ the graded vector space of cyclic words in n letters, and denote by $\mathrm{tr} : \mathrm{Ass}_n \to \mathfrak{tr}_n$ the natural projection. In particular, we have $\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\beta\alpha)$ for all $\alpha, \beta \in \mathrm{Ass}_n$. Similarly, define $\mathfrak{tr}_n^{quad} = \mathrm{Ass}_n / \langle A_n, C_n \rangle$ and $\mathrm{tr}^{quad} : \mathrm{Ass}_n \to \mathfrak{tr}_n^{quad}$ the corresponding projection. Here $\langle A_n, C_n \rangle$ is the subspace of Ass_n spanned by A_n and C_n. The definition implies $\mathrm{tr}^{quad}(\alpha) = \mathrm{tr}^{quad}(\tau(\alpha))$ for all $\alpha \in \mathrm{Ass}_n$ and $\mathrm{tr}^{quad}(\alpha\beta) = \mathrm{tr}^{quad}(\beta\alpha)$ for all $\alpha, \beta \in \mathrm{Ass}_n$. Note that for all $\alpha \in \mathfrak{lie}_n$ and all k odd we have $\mathrm{tr}^{quad}(\alpha^k) = \mathrm{tr}^{quad}(\tau(\alpha^k)) = (-1)^k \mathrm{tr}^{quad}(\alpha^k) = 0$.

Let $\mathfrak{tder}_n$ be the Lie algebra of derivations of $\mathfrak{lie}_n$ with an extra property that for $u \in \mathfrak{tder}_n$ there exist $a_1, \ldots, a_n \in \mathfrak{lie}_n$ such that $u(x_i) = [x_i, a_i]$. Elements of $\mathfrak{tder}_n$ are called tangential derivations. If we assume that a_i does not contain a linear term proportional to x_i, the correspondence between tangential derivations and n-tuples $(a_1, \ldots, a_n)$ is one-to-one. We define simplicial maps $\mathfrak{tder}_n \to \mathfrak{tder}_{n+1}$.

For instance, for $u = (A, B) \in \mathfrak{tder}_2$ we introduce

$$
\begin{aligned}
u^{1,2} &= (A(x,y), B(x,y), 0), \\
u^{2,3} &= (0, A(y,z), B(y,z)), \\
u^{12,3} &= (A(\mathrm{ch}(x,y), z), A(\mathrm{ch}(x,y), z), B(\mathrm{ch}(x,y), z)), \\
u^{1,23} &= (A(x, \mathrm{ch}(y,z)), B(x, \mathrm{ch}(y,z)), B(x, \mathrm{ch}(y,z))).
\end{aligned}
$$

Recall (Proposition 3.6, [5]) that $\mathrm{div}(u) = \sum_{i=1}^{n} \mathrm{tr}(x_i(\partial_i a_i))$ is a 1-cocycle on $\mathfrak{tder}_n$ with values in $\mathfrak{tr}_n$. Similarly, we define $\mathrm{div}^{quad}(u) = \sum_{i=1}^{n} \mathrm{tr}^{quad}(x_i(\partial_i a_i))$. It is a 1-cocyle on $\mathfrak{tder}_n$ with values in $\mathfrak{tr}_n^{quad}$. It is easy to see that the divergence transforms in a natural way under simplicial maps. For example, for $u \in \mathfrak{tder}_2$ we define $g(x,y) = \mathrm{div}(u) \in \mathfrak{tr}_2$ and we have

$$
\begin{aligned}
\mathrm{div}(u^{1,2}) &= g(x,y), & \mathrm{div}(u^{2,3}) &= g(y,z), \\
\mathrm{div}(u^{12,3}) &= g(\mathrm{ch}(x,y), z), & \mathrm{div}(u^{1,23}) &= g(x, \mathrm{ch}(y,z)).
\end{aligned}
$$

For example, to prove the third equation we use that

$$
\mathrm{tr}(x(\partial_x a(\mathrm{ch}(x,y), z)) + y(\partial_y a(\mathrm{ch}(x,y), z))) = \mathrm{tr}(\mathrm{ch}(x,y)(\partial_1 a)(\mathrm{ch}(x,y), z)).
$$

The same transformation properties under simplicial maps hold for div^{quad}.

Let TAut_n be the subgroup of automorphisms of $\mathfrak{lie}_n$ with an extra property that for each $g \in \mathrm{TAut}_n$ there exist $b_1, \ldots, b_n \in \mathfrak{lie}_n$ such that $g(x_i) = \exp(\mathrm{ad}_{b_i})x_i$. The group TAut_n is isomorphic to $\mathfrak{tder}_n$ with group multiplication defined by the Campbell-Hausdorff series. Simplicial maps lift to group homomorphisms $\mathrm{TAut}_n \to \mathrm{TAut}_{n+1}$.

The Kashiwara-Vergne (KV) problem [12] can be stated in the following way:

Kashiwara-Vergne problem: Find a pair of Lie series in two variables $A, B \in \mathfrak{lie}_2$ such that

$$
(1) \qquad (1 - \exp(-\mathrm{ad}_x))A(x,y) + (\exp(\mathrm{ad}_y) - 1)B(x,y) = x + y - \mathrm{ch}(y,x),
$$

and

$$
(2) \qquad \mathrm{tr}(x(\partial_x A) + y(\partial_y B)) = \frac{1}{2} \mathrm{tr}(f(x) + f(y) - f(\mathrm{ch}(x,y))),
$$

where $f(x) = x/(e^x - 1) - 1 + x/2 = \sum_{k=2}^{\infty} B_k x^k / k!$ is the generating series of Bernoulli numbers.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$. Then, the positive solution of the KV problem implies the Duflo theorem [11] for $\mathfrak{g}$ (an isomorphism $Z(U\mathfrak{g}) \cong (S\mathfrak{g})^{\mathfrak{g}}$ between the center of the universal enveloping algebra and the ring of invariant polynomials) and the cohomology isomorphism $H(\mathfrak{g}, U\mathfrak{g}) \cong H(\mathfrak{g}, S\mathfrak{g})$ [16, 14]. For $\mathbb{K} = \mathbb{R}$, one also obtains the extension of the Duflo isomorphism to germs of invariant distributions [6], [7].

Let $\mathfrak{g}$ be a finite dimensional quadratic Lie algebra over $\mathbb{K}$. That is, $\mathfrak{g}$ carries an invariant non-degenerate symmetric bilinear form (*e.g.* $\mathfrak{g}$ is semi-simple, but not necessarily, see [13]). Then, the Duflo theorem (both algebraic and analytic versions) as well as the cohomology isomorphism $H(\mathfrak{g}, U\mathfrak{g}) \cong H(\mathfrak{g}, S\mathfrak{g})$ follow from a weaker version of the KV problem:

Quadratic Kashiwara-Vergne problem: Find a pair of Lie series in two variables $A, B \in \mathfrak{lie}_2$ which verify equation (1) and

$$
(3) \qquad \mathrm{tr}^{quad}(x(\partial_x A) + y(\partial_y B)) = \frac{1}{2} \mathrm{tr}^{quad}(f(x) + f(y) - f(\mathrm{ch}(x,y))),
$$

with $f(x) = x/(e^x - 1) - 1 + x/2$.

Remark 1. This reduction of the second KV equation from $\mathfrak{tr}_2$ to $\mathfrak{tr}_2^{quad}$ is related to the following property of traces in the adjoint representation of a quadratic Lie algebra $\mathfrak{g}$. Let $\tau_\mathfrak{g}$ be the unique involution of $U\mathfrak{g}$ such that $\tau_\mathfrak{g}(\alpha) = -\alpha$ for all $\alpha \in \mathfrak{g}$. Then, for all $\alpha \in U\mathfrak{g}$ we have $\mathrm{tr}_\mathfrak{g} \, \mathrm{ad}(\tau(\alpha)) = \mathrm{tr}_\mathfrak{g} \, \mathrm{ad}(\alpha)$. At the level of free Lie algebras, this property leads to replacing tr by tr^{quad}.

It is obvious that equation (1) admits many solutions. Indeed, let $a(x, y)$ and $b(x, y)$ be Lie series given by the following formulas

$$a(x, y) = \frac{1 - \exp(-\mathrm{ad}_x)}{\mathrm{ad}_x} A(x, y) \quad , \quad b(x, y) = \frac{\exp(\mathrm{ad}_y) - 1}{\mathrm{ad}_y} B(x, y).$$

Then, equation (1) takes the form

$$[x, a(x, y)] + [y, b(x, y)] = x + y - \mathrm{ch}(y, x).$$

The right hand side is given by a series in Lie monomials of degree greater or equal to two. Since each Lie monomial starts either with a Lie bracket with x or with a Lie bracket with y, we obtain a solution of equation (1) for each explicit presentation of the Campbell-Hausdorff formula in terms of Lie monomials (*e.g.* using the Dynkin formula) . Furthermore, one can classify solutions of the homogeneous equation $[x, a] + [y, b] = 0$ using Lemma in Section 6, [10] (see below), or using the technique of [9].

The full KV problem admits a solution using the Kontsevich deformation quantization technique [3], and there is another solution [5] using the Drinfeld's theory of associators [10]. At the same time, it is known that the quadratic KV problem is much easier. In particular, it admits explicit rational solutions [17], [2] (whereas in the general case, it is plausible that an explicit rational solution of the KV problem amounts to finding an explicit rational associator). There are also two elementary proofs of the Duflo theorem for quadratic Lie algebras: one using Clifford calculus [1], and one using the Kontsevich integral in knot theory [8].

These simplifications in the quadratic case are explained by the following theorem:

Theorem 1. *Every solution of equation* (1) *verifies equation* (3).

This observation shows that the quadratic KV problem reduces to equation (1). In particular, this explains why rational solutions of the quadratic KV problem are easy to obtain: every rational factorization of the Campbell-Hausorff series of the form $\mathrm{ch}(y, x) = x + y - [x, a(x, y)] - [y, b(x, y)]$ gives rise to such a solution. It also explains why easy proofs of the Duflo theorem for quadratic Lie algebras are available: in this case, the Duflo theorem follows from the fact that the Campbell-Hausdorff series factorizes according to equation (1).

To prepare the proof of Theorem 1, we recall Lemma in Section 6, [10]. Let $\mathfrak{tr}_n^2$ be the linear span of expressions of the form $\mathrm{tr}(ab)$ for $a, b \in \mathfrak{lie}_n$. The lemma states that there is a one-to-one correspondence between elements $p \in \mathfrak{tr}_n^2$ and n-tuples $a_1, \ldots, a_n \in \mathfrak{lie}_n$ satisfying $\sum_{i=1}^{n}[x_i, a_i] = 0$. This correspondence is given by formula $\frac{d}{ds} p(x_1, \ldots, x_i + sz, \ldots, x_n)|_{s=0} = \mathrm{tr}(za_i)$.

Proposition 2. *Let $a_1, \ldots, a_n \in \mathfrak{lie}_n$ such that $\sum_{i=1}^n [x_i, a_i] = 0$, and let $u \in \mathfrak{tder}_n$ be the tangential derivation defined by the n-tuple $(a_1, \ldots, a_n)$. Then,*

$$\mathrm{div}^{quad}(u) = \sum_{i=1}^n \mathrm{tr}^{quad}(x_i(\partial_i a_i)) = 0$$

Proof. Let $p \in \mathfrak{tr}_2^2$ be an element generating the derivation u. Consider

$$
\begin{aligned}
\tfrac{\partial^2}{\partial s \partial t} p(x_1, \ldots, x_i + sz_1 + tz_2, \ldots x_n)|_{s=t=0} &= \tfrac{\partial}{\partial t} \mathrm{tr}(z_1 a_i(x_1, \ldots, x_i + tz_2, \ldots, x_n))|_{t=0} \\
&= \mathrm{tr}(z_1(\mathrm{ad}(\partial_i a)z_2)) \\
&= \mathrm{tr}((\mathrm{ad}(\tau(\partial_i a))z_1)z_2).
\end{aligned}
$$

Since the left hand side is symmetric in z_1, z_2, we conclude that $\partial_i a_i = \tau(\partial_i a_i)$. Then,

$$
\begin{aligned}
\mathrm{div}^{quad}(u) &= \sum_{i=1}^n \mathrm{tr}^{quad}(x_i(\partial_i a_i)) \\
&= \sum_{i=1}^n \mathrm{tr}^{quad}(\tau(x_i(\partial_i a_i))) \\
&= -\sum_{i=1}^n \mathrm{tr}^{quad}((\partial_i a_i)x_i) \\
&= -\mathrm{div}^{quad}(u),
\end{aligned}
$$

and $\mathrm{div}^{quad}(u) = 0$, as required. $\qquad\square$

Remark 2. Here we presented an algebraic proof of Proposition 2 suggested to us by Michele Vergne. One can also give a proof using graphical calculus as in Section 5.1, [15].

The next proposition summarizes known properties of equation (1).

Proposition 3. *The following three statements are equivalent:*

- *$A, B \in \mathfrak{lie}_2$ is a solution of equation (1).*
- *For all $t \in \mathbb{K}^*$, the tangential derivation $u_t \in \mathfrak{tder}_2$ defined by formula $u_t(x) = t^{-1}[x, A(tx, ty)], u_t(y) = t^{-1}[y, B(tx, ty)]$ verifies equation*

$$u_t(\mathrm{ch}_t(x, y)) = \frac{d}{dt}\mathrm{ch}_t(x, y).$$

- *The solution $F_t \in \mathrm{TAut}_2$ of the differential equation*

$$F_t^{-1}\frac{dF_t}{dt} = u_t$$

with initial condition $F_0 = 1$ verifies equation $F_t(\mathrm{ch}_t(x, y)) = x + y$.

Proof. For equivalence of the first and second statements, see Lemma 3.2, [12]. Equivalence of the second and third statements is obvious, see also Theorem 5.2, [5]. $\qquad\square$

Note that given $F \in \mathrm{TAut}_2$ verifying $F(\mathrm{ch}(x, y)) = x + y$ (as in Proposition 3), one can construct $\mathcal{F} \in \mathrm{TAut}_n$ by formula $\mathcal{F} = F^{1,2}F^{12,3} \ldots F^{1\ldots(n-1),n}$ such that

$$
\begin{aligned}
\mathcal{F}(\mathrm{ch}(x_1, \ldots, x_n)) &= F^{1,2}F^{12,3} \ldots F^{1\ldots(n-1),n}(x_1 + \cdots + x_n) \\
&= F^{1,2}F^{12,3} \ldots F^{1\ldots(n-2),n-1}(\mathrm{ch}(x_1, \ldots, x_{n-1}) + x_n) \\
&= \ldots \\
&= F^{1,2}(\mathrm{ch}(x_1, x_2) + x_3 + \cdots + x_n) \\
&= x_1 + \cdots + x_n.
\end{aligned}
$$

Proposition 4. *Let $u \in \mathfrak{tder}_n$ such that $u(\mathrm{ch}(x_1, \ldots, x_n)) = 0$. Then, $\mathrm{div}^{quad}(u) = 0$.*

Proof. Define $v = \mathrm{Ad}_{\mathcal{F}} u \in \mathfrak{tder}_n$. We have

$$v(x_1 + \cdots + x_n) = \mathcal{F}(u(\mathcal{F}^{-1}(x_1 + \cdots + x_n))) = \mathcal{F}(u(\mathrm{ch}(x_1, \ldots, x_n))) = 0.$$

Then,

$$\mathrm{div}^{quad}(u) = \mathrm{div}^{quad}(\mathrm{Ad}_{\mathcal{F}^{-1}}(v)) = \mathcal{F}^{-1} \cdot \mathrm{div}^{quad}(v) = 0,$$

where we used Proposition 2 and the cocycle property of div^{quad}. $\qquad\square$

Proposition 5. *Let $A, B \in \mathfrak{lie}_2$ be a solution of equation* (1)*, and $u \in \mathfrak{tder}_2$ be the corresponding tangential derivation. Then,*

(4)
$$U = u^{1,2} + u^{12,3} - u^{1,23} - u^{2,3}$$

verifies $U(\mathrm{ch}(x, y, z)) = 0$.

Proof. Consider $\mathrm{ch}_t(x, y, z) = \mathrm{ch}_t(\mathrm{ch}_t(x, y), z)$. We have,

$$
\begin{aligned}
\tfrac{d}{dt}\,\mathrm{ch}_t(x, y, z)|_{t=1} &= (\tfrac{d}{dp}\,\mathrm{ch}_p(\mathrm{ch}_q(x, y), z) + \tfrac{d}{dq}\,\mathrm{ch}_p(\mathrm{ch}_q(x, y), z))_{p=q=1} \\
&= u^{12,3}(\mathrm{ch}(x, y, z)) + u^{1,2}(\mathrm{ch}(x, y, z)).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\tfrac{d}{dt}\,\mathrm{ch}_t(x, y, z)|_{t=1} &= (\tfrac{d}{dp}\,\mathrm{ch}_p(x, \mathrm{ch}_q(y, z)) + \tfrac{d}{dq}\,\mathrm{ch}_p(x, \mathrm{ch}_q(y, z)))_{p=q=1} \\
&= u^{1,23}(\mathrm{ch}(x, y, z)) + u^{2,3}(\mathrm{ch}(x, y, z)).
\end{aligned}
$$

By combining these two equations we arrive at $U(\mathrm{ch}(x, y, z)) = 0$. $\qquad\square$

Proposition 6. *Let $A, B \in \mathfrak{lie}_2$ be a solution of equation* (1)*, $u \in \mathfrak{tder}_2$ be the corresponding tangential derivation, and $g = \mathrm{div}^{quad}(u) \in \mathrm{tr}_2^{quad}$. Then, there is $h \in \mathrm{tr}_1^{quad}$ such that $g(x, y) = h(x) + h(y) - h(\mathrm{ch}(x, y))$.*

The main part of the proof is borrowed from [5], Theorem 2.1. We reproduce it here for convenience of the reader.

Proof. Propositions 4 and 5 imply that for $U \in \mathfrak{tder}_3$ defined by equation (4) we have $\mathrm{div}^{quad}(U) = 0$. That is,

(5)
$$
\begin{aligned}
0 = \mathrm{div}^{quad}(U) &= \mathrm{div}^{quad}(u^{1,2}) + \mathrm{div}^{quad}(u^{12,3}) - \mathrm{div}^{quad}(u^{1,23}) - \mathrm{div}^{quad}(u^{2,3}) \\
&= g(x, y) + g(\mathrm{ch}(x, y), z) - g(x, \mathrm{ch}(y, z)) - g(y, z).
\end{aligned}
$$

Consider an auxiliary equation with the Campbell-Hausdorff series $\mathrm{ch}(x, y)$ replaced by $x + y$. In more detail, we look for all $g(x, y) \in \mathrm{tr}_2^{quad}$ which verify

(6)
$$g(x, y) + g(x + y, z) - g(x, y + z) - g(y, z) = 0.$$

In order to solve this equation, we first put $x \mapsto sx, y \mapsto x, z \mapsto z$ to get

$$g(sx, x) + g((1 + s)x, z) - g(sx, x + z) - g(x, z) = 0.$$

Similarly, by putting $x \mapsto x, y \mapsto z, z \mapsto sz$ we obtain

$$g(x, z) + g(x + z, sz) - g(x, (1 + s)z) - g(z, sz) = 0.$$

Since equation (6) preserves the degree, we can assume without loss of generality that g is homogeneous of degree n. Then, we have

$$
\begin{aligned}
ng(x, z) &= \tfrac{d}{ds}\,(g((1 + s)x, z) + g(x, (1 + s)z))|_{s=0} \\
&= \tfrac{d}{ds}\,(g(sx, x + z) + g(x + z, sz) - g(sx, x) - g(z, sz))|_{s=0}.
\end{aligned}
$$

That is, we get $g(x,z) = \text{tr}^{quad}((\alpha x + \beta z)(x+z)^{n-1} - \alpha x^n - \beta z^n)$. This expression vanishes for n odd because $\text{tr}^{quad}(x^n) = \text{tr}^{quad}(\tau(x^n)) = (-1)^n \text{tr}^{quad}(x^n)$ (and by a similar argument for other terms). For n even, we compute

$$g(x,y) + g(x+y,z) - g(x,y+z) - g(y,z)$$
$$= (\beta - \alpha)\text{tr}^{quad} y((x+y)^{n-1} + (y+z)^{n-1} - (x+y+z)^{n-1} - y^{n-1}).$$

In particular, the coefficient in front of $\text{tr}^{quad}(y^{n-2}(xz+zx))$ is equal to $(\beta - \alpha)(n-2)$ which implies $\beta = \alpha$ for $n \neq 2$. That is, for $n \neq 2$ we get

$$g(x,z) = \alpha \text{tr}^{quad}((x+z)^n - x^n - z^n).$$

Furthermore, for $n = 2$ we obtain

$$g(x,z) = (\alpha + \beta)\text{tr}^{quad}(xz) = \frac{\alpha + \beta}{2}\,\text{tr}^{quad}((x+z)^2 - x^2 - z^2).$$

In summary, all solutions of equation (6) are of the form $g(x,z) = h(x) + h(z) - h(x+z)$ for $h \in \text{tr}_1^{quad}$.

Getting back to equation (5), let $g = \sum_{k \geq n} g_k$ be a solution, where g_k are homogeneous components of degree k. By taking the degree n part of equation (5) we recover equation (6) for g_n. Hence, there is $h_n \in \text{tr}_1^{quad}$ such that $g_n(x,y) = h_n(x) + h_n(y) - h_n(x+y)$. Consider $\tilde{g}(x,y) = g(x,y) - (h_n(x) + h_n(y) - h_n(\text{ch}(x,y)))$. It is easy to see that $\tilde{g}$ still verifies equation (5), and that it starts in degree $n+1$. Proceeding by induction, we show that all solutions of equation (5) are of the form $g(x,y) = h(x) + h(y) - h(\text{ch}(x,y))$ for some $h \in \text{tr}_1^{quad}$, as required. $\square$

We conclude the proof of the main result of this paper with the following Proposition.

Proposition 7. *Let $A, B \in \mathfrak{lie}_2$ be a solution of* (1), *$u \in \mathfrak{tder}_2$ be the corresponding tangential derivation, and assume* $\text{div}^{quad}(u) = h(x) + h(y) - h(\text{ch}(x,y))$ *for some $h \in \text{tr}_1^{quad}$. Then, $h(x) = \text{tr}^{quad} f(x)$ for $f = x/(e^x - 1) - 1 + x/2$.*

In the proof, we follow the ideas [4] (see Remark 4.3). The main part of the argument is the same as in the proof of Proposition 6.1 in [5]. We reproduce it for convenience of the reader.

Proof. Write $A(x,y) = ax + \alpha(\text{ad}_x)y + \ldots, B(x,y) = bx + \beta(\text{ad}_x)y + \ldots$, where $a, b \in \mathbb{K}, \alpha, \beta \in \mathbb{K}[[x]]$, and $\ldots$ stand for terms containing at least two y's. Replace $y \mapsto sy$ in equation (1), and compute the first and second derivatives in s at $s = 0$. The first derivative yields

$$y - \frac{\text{ad}_x}{e^{\text{ad}_x} - 1}\,y = (1 - e^{-\text{ad}_x})\alpha(\text{ad}_x)y - b[x,y],$$

and we obtain

$$\alpha(t) = b\,\frac{t}{1 - e^{-t}} - \frac{t}{(e^t - 1)(1 - e^{-t})} + \frac{1}{1 - e^{-t}}.$$

Note that elements of degree two in y of $\mathfrak{lie}_2$ are in bijection with skew-symmetric formal power series in two variables,

$$a(u,v) = \sum_{i,j=0}^{\infty} a_{i,j} u^i v^j \mapsto \sum_{i,j=0}^{\infty} a_{i,j}[\text{ad}_x^i\, y, \text{ad}_x^j\, y]$$

The second derivative of (1) gives the following equality in formal power series,

$$\frac{1}{2}\frac{(u+v)(e^u-e^v)-(u-v)(e^{u+v}-1)}{(e^{u+v}-1)(e^u-1)(e^v-1)}=(1-e^{-(u+v)})a_2(u,v)+\frac{b}{2}(u-v)+(\beta(v)-\beta(u)),$$

where the left hand side corresponds to the second derivative of the Campbell-Hausdorff series $-\mathrm{ch}(sy,x)$, and $a_2(u,v)$ represents the second derivative of $A(x,sy)$. By putting $v=-u$ in the last equation we obtain,

$$\beta_{odd}(t)=\frac{b}{2}\,t-\frac{1}{2}\frac{t}{(e^t-1)(1-e^{-t})}+\frac{1}{4}\frac{e^t+1}{e^t-1}\,.$$

Here $\beta_{odd}(t)=(\beta(t)-\beta(-t))/2$.

Finally, consider equation

$$\mathrm{tr}^{quad}(x(\partial_x A)+y(\partial_y B))=\frac{1}{2}\,\mathrm{tr}^{quad}(f(x)+f(y)-f(\mathrm{ch}(x,y))),$$

and compute the contribution linear in y (that is, of the form $\mathrm{tr}^{quad}(a(x)y)$) on the left hand side and on the right hand side. Note that $\mathrm{tr}^{quad}(x^n)$ and $\mathrm{tr}^{quad}(x^{n-1}y)$ vanish for n odd. Hence, it is sufficient to look at even degrees. In this way, we obtain

$$\beta_{odd}(t)-\alpha_{odd}(t)=-\frac{1}{2}\frac{df}{dt}$$

which implies

$$f(t)=\frac{t}{e^t-1}-1+\frac{t}{2},$$

as required.

$\square$

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