THE LOGARITHMIC FORMALITY QUASI-ISOMORPHISM

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Abstract. In this paper we give a detailed proof of a famous statement of M. Kontsevich in [11, Subsection 4.1], where it is claimed that replacing the standard angular function by a logarithmic one in the formulæ for the integral weights in the L_∞-quasi-isomorphism $\mathcal{U}$ constructed in [12, Section 6] yields a different L_∞-quasi-isomorphism $\mathcal{U}^{\log}$.

In particular, we prove the convergence of the integral weights appearing in $\mathcal{U}^{\log}$ and the L_∞-relations for $\mathcal{U}^{\log}$, which follow from a variant of Stokes' Theorem on compact, oriented manifolds with corners for differential forms with poles of order 1 along the boundary. Finally, we prove that $\mathcal{U}^{\log}$ satisfies the globalization requirements.

1. Introduction

In [12, Section 6], M. Kontsevich has explicitly constructed an L_∞-quasi-isomorphism $\mathcal{U}$ between the differential graded (shortly, from now on, dg) Lie algebras $T_{\mathrm{poly}}(A)$ and $D_{\mathrm{poly}}(A)$, where $T_{\mathrm{poly}}(X)$, resp. $D_{\mathrm{poly}}(X)$, denotes the graded vector space of totally skew-symmetric multi-derivations of $A = \mathbb{K}[x_1,\dots,x_d]$, for a field $\mathbb{K} \supseteq \mathbb{C}$, resp. multi-differential operators on A. The quasi-isomorphism property translates here into the fact that $\mathcal{U}$ extends the well-known Hochschild–Kostant–Rosenberg quasi-isomorphism of dg vector spaces.

Among the many corollaries of the existence of $\mathcal{U}$ are *i*) the fact that every Poisson manifold $(X, \hbar\pi)$, $\hbar$ being a formal parameter, admits a deformation quantization (see [12, Sections 1, 7], [6, 7]), *ii*) the "complex-geometric counterpart" of Duflo's Theorem [9] sketched by M. Kontsevich in [12, Subsection 8.4] and proved in detail in [4], and *iii*) a proof of the famous Kashiwara–Vergne Conjecture, see [2, 16].

The nicest feature of $\mathcal{U}$ is its universal and explicit formula: in other words, the construction of $\mathcal{U}$ does not depend on the dimension d of A, and revolves around the notion of certain admissible directed graphs with two types of vertices. With such a graph Γ, M. Kontsevich associates a multi-differential operator on $T_{\mathrm{poly}}(A)$ with values in $D_{\mathrm{poly}}(A)$ and an integral weight ϖ_Γ *via*

$$\varpi_\Gamma = \int_{C^+_{n,m}} \omega_\Gamma, \qquad \omega_\Gamma = \prod_{e \in E(\Gamma)} \omega_e, \qquad \omega(z_1, z_2) = \frac{1}{2\pi} d\arg\left(\frac{z_1 - z_2}{\overline{z}_1 - z_2}\right).$$

Here, $E(\Gamma)$ denotes the set of edges of the graph Γ, which is assumed to be implicitly endowed with a total order; $C^+_{n,m}$ is the configuration space of n points in the complex upper half-plane $\mathbb{H}^+$ and m ordered points on $\mathbb{R}$ *modulo* rescalings and real translations. Finally, every directed edge e of Γ yields a natural projection π_e from $C^+_{n,m}$ onto $C^+_{2,0}$ or $C^+_{1,1}$: ω_e is the pull-back of ω with respect to π_e. *A priori*, it is not clear that the integral weights ϖ_Γ converge. Furthermore, the L_∞-relations translate into an infinite family of quadratic relations among the integral weights. The convergence of the integral weights ϖ_Γ has been proved by M. Kontsevich by constructing well-suited compactifications $\overline{C}^+_{n,m}$ *à la Fulton–MacPherson*: these are compact, oriented manifolds with corners, and ω extends smoothly to $\overline{C}^+_{2,0}$, as well as the projections π_e, whence also ω_Γ. Furthermore, since ω_Γ is closed, Stokes' Theorem for compact, oriented manifolds with corners applies here, and it translates into the quadratic relations yielding the L_∞-relations.

One year later, M. Kontsevich has proposed in [11, Subsection 4.1, F)] to replace ω by $\omega^{\log}$ in the formulæ for $\mathcal{U}$, where

$$\omega^{\log}(z_1, z_2) = \frac{1}{2\pi i} d\log\left(\frac{z_1 - z_2}{\overline{z}_1 - z_2}\right),$$

and has claimed without proof that one obtains in this way a different L_∞-quasi-isomorphism $\mathcal{U}^{\log}$ from $T_{\mathrm{poly}}(A)$ to $D_{\mathrm{poly}}(A)$.

This claim is not as simple as it would appear at first sight. First of all, the presence of the logarithm in $\omega^{\log}$ readily shows that, unlike $\omega^{\log}$ does not extend to $\overline{C}^+_{2,0}$, because it has a pole of order 1 along the boundary stratum of codimension 1 corresponding to the collapse of its two arguments to a single point in $\mathbb{H}^+$: hence, *a priori*, $\omega^{\log}_\Gamma$ is a form with possibly poles along the boundary, hence the corresponding integral weight $\varpi^{\log}_\Gamma$ may diverge. Furthermore, provided the integral weights $\varpi^{\log}_\Gamma$ converge, one needs a variant of Stokes' Theorem on compact, oriented manifolds with corners for differential forms admitting poles along the boundary in order to prove the L_∞-relations for $\mathcal{U}^{\log}$.

Here, we address in detail both questions. More precisely, we first prove that, for admissible graphs Γ whose number of edges equals the dimension of the corresponding configuration space, $\omega_\Gamma^{\log}$ extends indeed to a complex-valued, real analytic form of top degree on the compactified configuration space: this solves the convergence problem. Further, we prove that, if the number of edges of Γ equals the dimension of the corresponding configuration space minus 1, $\omega_\Gamma^{\log}$ is a closed form with poles of order 1 along the boundary: we may then apply the corresponding variant of Stokes' Theorem on compact, oriented manifolds with corners for differential forms with poles of order 1 along the boundary to derive the desired quadratic identities yielding the L_∞-relations. Finally, we prove that the L_∞-quasi-isomorphism $\mathcal{U}^{\log}$ satisfies the globalization requirements, see [12, Section 7], [8, Section 4] and [17, Sections 6, 8, 9].

Remark 1.1. In [15], we consider a family of singular propagators ω^t over $\mathbb{R}$ which interpolates between the standard argument propagator and the logarithmic propagator. The main results of the present paper generalize to ω^t (*i.e.* convergence, L_∞-property, globalization): still, this paper has been finished a long time before [15] has been initiated, and many ideas leading to the more general results in [15] took inspiration from the techniques presented in detail here.

Acknowledgments. We warmly thanks S. Merkulov for many useful discussions, for the careful reading of the first drafts of the present note, for many useful suggestions, for his interest in the topic and for providing the interesting reference [10]. The third author thanks heartily G. Felder for the interest he showed toward the subject of the paper and for many stimulating and useful discussions.

2. Notation and conventions

We consider here the ground field $\mathbb{K} = \mathbb{C}$.

We will always use the abbreviation "dg" for "differential graded". More precisely, a dg vector space V is a pair (V, d_V) consisting of a $\mathbb{Z}$-graded vector space V over $\mathbb{K}$ and an endomorphism d_V of V of degree 1, which squares to 0, in other words, a dg vector space over $\mathbb{K}$ is a complex of vector spaces over $\mathbb{K}$.

By $[\bullet]$ we denote the degree-shifting functor on the tensor category `grVect` of $\mathbb{Z}$-graded vector spaces over $\mathbb{K}$; the degree-shifting functor $[\bullet]$ acts accordingly on dg vector spaces as well. The tensor structure on `grVect` is specified by the graded tensor product over $\mathbb{K}$ (unadorned tensor products are always meant to be over the ground field $\mathbb{K}$, unless otherwise explicitly stated); obviously, the tensor product of two dg vector spaces is again a dg vector space, the differential being specified by the graded Leibniz rule with respect to the tensor product.

The fact that `grVect` admits a tensor structure implies that with an object V of `grVect` we may associate its (graded) symmetric algebra $\mathrm{S}(V)$, its exterior algebra $\wedge(V)$ and its tensor algebra $\mathrm{T}(V)$: their definitions mimic the corresponding definitions for a vector space concentrated in degree 0 with obvious due modifications inherited from the grading. We also observe that $\mathrm{S}(V)$ and $\mathrm{T}(V)$ admit structures of (graded) coassociative coalgebras and of (graded) associative algebras; the corresponding structures on $\mathrm{S}(V)$ are even cocommutative and commutative respectively. By $\mathrm{S}^+(V)$ we denote the cocommutative, coassociative coalgebra without counit, *i.e.* $\mathrm{S}^+(V) = \bigoplus_{n\geq 1} \mathrm{S}^n(V)$ (is is customary to set $\mathrm{T}^0(V) = \mathrm{S}^0(V) = \mathbb{K}$).

When dealing with objects and morphisms of the category `grVect`, we tacitly assume the validity of Koszul's sign rule with respect to the relevant grading.

2.1. L_∞-algebras and -morphisms.

We need a brief *memento* of L_∞-algebras and related morphisms.

An L_∞-algebra structure on an object $\mathfrak{g}$ of `grVect` consists of a coderivation Q of degree 1, which additionally squares to 0, on the cocommutative, cofree coassociative coalgebra without counit $\mathrm{S}^+(\mathfrak{g}[1])$.

The fact that $\mathrm{S}^+(\mathfrak{g}[1])$ is a cofree, coassociative, cocommutative coalgebra implies that Q is uniquely specified by its Taylor components $Q_n : \mathrm{S}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$, $n \geq 1$, *via* the assignment

$$Q(x_I) = \sum_{\substack{J\subseteq I\\ |J|\geq 1}} \varepsilon(J, I) Q_{|J|}(x_J) x_{I\smallsetminus J},$$

where x_I is any monomial of degree $|I| \geq 1$ in $\mathrm{S}^+(\mathfrak{g}[1])$, for some set of indices I, and for a subset J of I of cardinality bigger or equal than 1, $\varepsilon(J, I)$ is the sign specified by the rule $x_I = \varepsilon(J, I) x_J x_{I\smallsetminus J}$.

Remark 2.1. There is an isomorphism of vector spaces over $\mathbb{K}$ between $\mathrm{S}^n(\mathfrak{g}[1])$ and $\wedge^n(\mathfrak{g})[n]$ (the *décalage* isomorphism), through which the Taylor component Q_n of Q may be also regarded as a morphism of degree $2 - n$ from $\wedge^n(\mathfrak{g})$ to $\mathfrak{g}$.

The condition that Q squares to 0 is equivalent to an infinite family of quadratic identities between the Taylor components of Q, *i.e.*

$$\sum_{\substack{J\subseteq I\\ |J|\leq 1}} \varepsilon(J, I) Q_{|I|-|J|+1}(Q_{|J|}(x_J) x_{I\smallsetminus J}) = 0,$$

for any choice of I of cardinality at least 1.

In particular, Q_1 defines a structure of dg vector space on $\mathfrak{g}$; the second quadratic identity implies that Q_1 satisfies the graded Leibniz rule with respect to the graded skew-symmetric map from $\mathfrak{g}\otimes\mathfrak{g}$ to $\mathfrak{g}$ of degree 0 specified by Q_2 (*i.e.* Q_1 and Q_2 are compatible). Finally, the third quadratic identity implies that Q_2 satisfies the graded Jacobi identity up to an explicit homotopy Q_3 with respect to Q_1. In other words, the cohomology of an L_∞-algebra $\mathfrak{g}$ with respect to Q_1 has a structure of graded Lie algebra. On the other hand, an L_∞-algebra structure on $\mathfrak{g}$ with non-trivial Taylor coefficients Q_n for $n=1,2$ is equivalent to a structure of dg Lie algebra on $\mathfrak{g}$.

Finally, given two L_∞-algebras $(\mathfrak{g}_i,Q_i)$, an L_∞-morphism F from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is a morphism of degree 0 of cocommutative, cofree, coassociative coalgebras without counits from $\mathrm{S}^+(\mathfrak{g}_1[1])$ to $\mathrm{S}^+(\mathfrak{g}_2[1])$, which additionally intertwines the corresponding codifferentials.

The fact that F is a morphisms of coalgebras from $\mathrm{S}^+(\mathfrak{g}_1[1])$ to $\mathrm{S}^+(\mathfrak{g}_1[1])$ and the cofreeness of such coalgebras implies that F is uniquely determined by its Taylor components $F_n:\mathrm{S}^n(\mathfrak{g}_1[1])\to\mathfrak{g}_2[1]$, for $n\geq 1$, namely

$$F(x_I)=\sum_{\substack{p\geq 1,\ J_1\sqcup\cdots\sqcup J_p=I\\ |J_i|\geq 1,\ i=1,\dots,p}}\varepsilon(J_1,\dots,J_p,I)F_{|J_1|}(x_{J_1})\cdots F_{|J_p|}(x_{J_p}),$$

with the same notation as above.

Remark 2.2. Using the *décalage* isomorphism, the Taylor component F_n may be also regarded as a map of degree $1-n$ from $\wedge^n(\mathfrak{g}_1)$ to $\mathfrak{g}_2$.

The fact that F intertwines Q_i, $i=1,2$, is equivalent to an infinite family of (non-commutative) polynomial identities with respect to the Taylor components of F and Q_i, $i=1,2$; in other words,

$$\sum_{\substack{J\subseteq I\\ |J|\geq 1}}\varepsilon(J,I)F_{|I|-|J|+1}(Q_{1,|J|}(x_J)x_{I\smallsetminus J})=\sum_{\substack{p\geq 1,\ J_1\sqcup\cdots\sqcup J_p=I\\ |J_i|\geq 1,\ i=1,\dots,p}}\varepsilon(J_1,\dots,J_p,I)Q_{2,p}(F_{|J_1|}(x_{J_1})\cdots F_{|J_p|}(x_{J_p})).$$

The first polynomial identity is equivalent to the fact that F_1 defines a morphism of dg vector spaces between $(\mathfrak{g}_i,Q_{i,1})$, $i=1,2$; further, the second polynomial identity implies that F_1 intertwines $Q_{i,2}$, $i=1,2$, up to the explicit homotopy F_2 with respect to $Q_{i,1}$, $i=1,2$. Both statements imply that F_1 descends to a morphism of graded Lie algebras on the corresponding cohomologies, which, together with previous arguments, motivates the following definition.

Definition 2.3. An L_∞-morphism F between L_∞-algebras $(\mathfrak{g}_i,Q_i)$, $i=1,2$, is an L_∞-quasi-isomorphism, if its first Taylor component F_1 is a quasi-isomorphism on the corresponding cohomologies.

3. Compactified Configuration spaces *à la* Kontsevich

In the present Section we discuss the open configuration spaces C_A and $C^+_{A,B}$ introduced in [12, Section 5] and the corresponding compactifications, for A a finite set and B a finite (totally) ordered set. We will discuss the structure of compact manifold with corners on the compactified configuration spaces: in particular, for later computational purposes, we will focus our attention on local coordinates around any boundary stratum of any codimension.

Some constructions presented here owe to [3, Part IV] and [13, 14], where local coordinates for compactified configuration spaces are presented with a lot of details. Motivated by these presentations, we write down our own account on the subject: we use the equivalent language of nested subsets for the combinatorial description of boundary strata instead of the language of trees. More importantly, although the construction of the compactification of the open configuration spaces needs the coordinates in standard position, as was pointed out by Kontsevich in [12], the construction of local coordinates near boundary strata of any codimension actually need smooth or real analytic sections of the principal bundles associated with open configuration spaces. It turns out that, for explicit computations, the section of coordinates in standard position is not the best available: this is where we differ mostly from the treatments in the above references, in that we use complex coordinates for certain “problematic” boundary strata.

3.1. Manifolds with corners.

From now on, by a manifold with corners is always meant, unless otherwise stated, a real analytic manifold with corners.

A (compact) oriented manifold with corners X of dimension d is a (compact) differentiable manifold in the usual sense, whose local charts are diffeomorphic to $U_{p,q}=\mathbb{R}_+^p\times\mathbb{R}^q$, $p+q=d$, and $\mathbb{R}_+=\{x\geq 0\}$, for some $p\geq 1$ and $p+q=d$. Thus, locally, a manifold with corners looks like $U_{p,q}$, for $p\geq 1$ and $p+q=d$.

The real analyticity of X enters into play when characterizing transition function: a self-map $\phi=(\phi_1,\dots,\phi_d)$ of $\mathbb{R}^d$ is an analytic isomorphism of $U_{p,q}$, if ϕ is an analytic diffeomorphism of $\mathbb{R}^d$ which restricts to an analytic diffeomorphism of $U_{p,q}$ which satisfies the properties

$$\phi_i=x_i\psi_i,\ \ \psi_i=1+x_i\widetilde{\psi}_i,\ \ i=1,\dots,p,$$

for $\widetilde{\psi}_i$ real analytic. Observe that there is a natural action of $\mathfrak{S}_p \times \mathfrak{S}_q$ on $U_{p,q}$, and we may compose analytic isomorphisms with this action: the result is also called an analytic isomorphism of $U_{p,q}$. A manifold with corners X as before is locally modeled on $U_{p,q}$, in such a way that transition functions are analytic isomorphisms in the above sense.

The set $U_{p,q}$ admits a boundary stratification, *i.e.* $U_{p,q} \supseteq \partial U_{p,q} \supseteq \partial^2 U_{p,q} \supseteq \cdots$ into boundary strata of codimension 1, 2, *etc.*. It is easy to verify that analytic isomorphisms preserve the boundary stratification of $U_{p,q}$.

Therefore, a manifold with corners X of dimension d admits a boundary stratification $X \supseteq \partial X \supseteq \partial^2 X \supseteq \cdots$. Typically, we will consider an orientable manifold with corners X: the orientation of X induces in a natural way orientations on all boundary strata thereof.

3.1.1. *Differential forms on manifold with corners and regularization morphism.* Let us begin by considering the local model $U_{p,q}$ for a manifold with corners. By $\mathcal{O}(U_{p,q})$ we denote the algebra of complex-valued, real analytic functions on $U_{p,q}$: an easy computation shows that analytic isomorphisms act on $\mathcal{O}(U_{p,q})$.

More generally, we consider the graded algebra

$$\Omega^\bullet_{\mathrm{p,log}}(U_{p,q}) = \mathcal{O}(U_{p,q})\left[\log(x_1),\dots,\log(x_p),\frac{dx_1}{x_1},\dots,\frac{dx_p}{x_p},dx_1,\dots,dx_{p+q}\right]$$

of complex-valued, real analytic differential forms on $U_{p,q}$ with poles and logarithmic singularities along $\partial U_{p,q}$. We assign degree 0 and 1 to $\log(x_i)$ and dx_i, dx_i/x_i respectively. $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$, endowed with the de Rham differential, obviously becomes a complex.

A computation in local coordinates by means of analytic isomorphisms, see [1, Lemma 1.2], implies the complex $(\Omega^\bullet_{\mathrm{p,log}}(X), d)$ of forms on X with poles and logarithmic singularities along ∂X can be defined for a general manifold with corners X.

Similarly, one can define the graded subspaces $\Omega^\bullet(X)$, $\Omega^\bullet_{\log}(X)$, $\Omega^\bullet_1(X)$ of complex-valued, real analytic forms, of forms with only logarithmic singularities, of forms with poles of order 1 and of forms with poles of order 1 along ∂X respectively.

We further need the graded subspace $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ of $\big(\Omega^\bullet_{\mathrm{p,log}}(U_{p,q}), d\big)$, whose elements are differential forms with poles and logarithmic singularities along $\partial U_{p,q}$ which look as follows:

$$\omega = \sum_{i=1}^p \frac{dx_i}{x_i}\omega_i + \eta,\ \omega_i \in \mathcal{O}(U_{p,q})[x_1\log(x_1),\dots,\log(x_i),\dots,x_p\log(x_p),dx_1,\dots,dx_d],\ i=1,\dots,p,$$
$$\eta \in \mathcal{O}(U_{p,q})[x_1\log(x_1),\dots,x_p\log(x_p),\log(x_1)^{q_1}dx_1,\dots,\log(x_p)^{q_p}dx_p,dx_{p+1},\dots,dx_p],\ q_i \geq 0,\ i=1,\dots,p.$$

In other words, a general element ω of $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ has the following properties: *i)* ω has only poles of order 1 of the form dx_i/x_i, *ii)* the coefficient ω_i corresponding to the pole dx_i/x_i has only logarithmic singularities, and, for $j \neq i$, whenever $\log(x_j)$ or any non-negative power thereof appears in ω_i, it is always accompanied by x_j, and *iii)* η has only logarithmic singularities and whenever $\log(x_i)$ or any non-negative power thereof appears, it is always accompanied either by x_i or a non-negative power thereof or by dx_i, $i=1,\dots,p$. In [1, Lemma 1.4] it has been proved that $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ is acted on by analytic isomorphisms, hence it makes sense to define $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(X)$ for a general manifold with corners X.

We will be mostly interested in the graded subspaces $\Omega^\bullet_{\log}(X)$, $\Omega^\bullet_1(X)$ and $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(X)$ of $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$. (Observe that, with the exception of $\Omega^\bullet_1(X)$, none of them is a subcomplex.)

The regularization of an element ω of $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ along a stratum $D = \{x_1 = \cdots = x_l = 0\}$, $1 \leq l \leq p$ of codimension l of $U_{p,q}$ is defined as

$$\mathrm{Reg}_D(\omega) = \omega\left(x_i = \log(x_i) = dx_i = \frac{dx_i}{x_i} = 0,\ i=1,\dots,l\right).$$

From [1, Lemma 1.6s], it is known that the map Reg_D is a morphism of complexes from $\big(\Omega^\bullet_{\mathrm{p,log}}(U_{p,q}), d\big)$ to $\big(\Omega^\bullet_{\mathrm{p,log}}(D), d\big)$, and moreover it commutes with the action of analytic isomorphisms: hence, for a manifold with corners X and a boundary stratum $\partial_D X$ thereof, there is a well-defined morphism of complexes $\mathrm{Reg}_D : \big(\Omega^\bullet_{\mathrm{p,log}}(X), d\big) \to \big(\Omega^\bullet_{\mathrm{p,log}}(\partial_D X), d\big)$, which is called the regularization morphism along $\partial_D X$.

It is quite clear that the regularization morphism restricts to a graded morphism of graded vector spaces on $\Omega^\bullet_{\log}(X)$, $\Omega^\bullet_1(X)$ and $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(X)$ of $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$.

3.1.2. *Variants of Stokes' Theorem for manifold with corners.* We now quote here the variant of Stokes' Theorem which will be needed throughout the whole paper; for its complete proof, we refer to [1, Subsubsection 1.1.2].

The following Theorem will be used to prove the L_∞-property of the family of stable formality morphisms we will introduce later on.

Theorem 3.1. *Let X be a compact, oriented manifold with corners of degree $d \geq 2$. Further, consider an element ω of $\Omega_1^{d-1}(X)$, which satisfies the two additional properties:*

i) its exterior derivative $d\omega$ is a complex-valued, real analytic form of top degree on X, and

ii) the regularization $\mathrm{Reg}_{\partial X}(\omega)$ along the boundary strata ∂X of codimension 1 of X is a complex-valued, real analytic form on ∂X.

Then, the integral over X of $d\omega$ and the integral over ∂X of $\mathrm{Reg}_{\partial X}(\omega)$ exist and the following identity holds true:

$$\int_X d\omega = \int_{\partial X} \mathrm{Reg}_{\partial X}(\omega). \tag{1}$$

3.2. **Open configuration spaces.** We denote by $\mathbb{H}^+$ the complex upper half-plane. We further denote by $[n]$ the set $\{1,\dots,n\}$; $[n]$ is naturally ordered.

The configuration space C_A is defined as

$$C_A = \mathrm{Conf}_A/G_3 = \left\{p \in \mathbb{C}^A \,|\, p(a) \neq p(a') \text{ if } a \neq a'\right\}/G_3,$$

where G_3 is the semi direct product $\mathbb{R}^+ \ltimes \mathbb{C}$, which acts diagonally on $\mathbb{C}^A$ *via*

$$(\lambda,\mu)p = \lambda p + \mu,\ \ \lambda \in \mathbb{R}^+,\ \ \mu \in \mathbb{C}.$$

It is clear that G_3 is a real Lie group of dimension 3, whose action on Conf_A is free precisely when $2|A| - 3 \geq 0$: then, C_A is a smooth real manifold of dimension $2|A| - 3$.

For a finite set A and a finite (totally) ordered set B, we define the open configuration space $C^+_{A,B}$ as

$$C^+_{A,B} = \mathrm{Conf}^+_{A,B}/G_2 = \left\{(p,q) \in (\mathbb{H}^+)^A \times \mathbb{R}^B \,|\, p(a) \neq p(a') \text{ if } a \neq a',\, q(b) < q(b') \text{ if } b < b'\right\}/G_2,$$

where G_2 is the semi-direct product $\mathbb{R}^+ \ltimes \mathbb{R}$, which acts diagonally on $(\mathbb{H}^+)^A \times \mathbb{R}^B$ *via*

$$(\lambda,\mu)(p,q) = (\lambda p + \mu, \lambda q + \mu),\ \ \lambda \in \mathbb{R}^+,\ \ \mu \in \mathbb{R}.$$

The action of the 2-dimensional Lie group G_2 on such $|A| + |B|$-tuples is free, precisely when $2|A| + |B| - 2 \geq 0$: in this case, $C^+_{A,B}$ is a smooth real manifold of dimension $2|A| + |B| - 2$.

Finally, for $A = [n]$ and $B = [m]$, we use the simpler notation C_n and $C^+_{n,m}$.

3.2.1. *Examples of configuration spaces and sections.* We discuss some examples of configuration spaces as introduced before. In particular, we introduce some *ad hoc* local sections of the corresponding principal bundles, through which we define local coordinates for the corresponding configuration spaces.

i) We consider C_n, for $n \geq 2$. We then define a global section of C_n *via*

$$C_n \ni [(z_1,\dots,z_n)] \mapsto \left(0, \frac{z_2 - z_1}{|z_2 - z_1|}, \frac{z_3 - z_1}{|z_2 - z_1|}, \dots, \frac{z_n - z_1}{|z_2 - z_1|}\right) \in \mathrm{Conf}_n.$$

By forgetting the first point in the n-tuple on the right-hand side of the previous assignment, we may identify the image of C_n with respect to this section with

$$\left\{(e^{i\varphi}, u_1, \dots, u_{n-2}) \in S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C} \smallsetminus \{0\}) :\ w_i \neq e^{i\varphi}\right\},$$

and finally we get the useful identification

$$C_n \cong S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C} \smallsetminus \{0,1\})$$

via the diffeomorphism

$$C_n \ni (e^{i\varphi}, u_1, \dots, u_{n-2}) \mapsto (e^{i\varphi}, e^{-i\varphi}u_1, \dots, e^{-i\varphi}u_{n-2}) \in S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C} \smallsetminus \{0,1\}).$$

ii) We consider $C^+_{0,m}$, for $m \geq 2$. We may define a global section of $C^+_{0,m}$ *via*

$$C^+_{0,m} \ni [(x_1,\dots,x_m)] \mapsto \left(\frac{x_2 - x_1}{x_m - x_1}, \dots, \frac{x_{m-1} - x_1}{x_m - x_1}\right) \in \triangle_{m-2},\ i = 2,\dots,m-1,$$

where $\triangle_{m-2}$ is the open unit simplex in $\mathbb{R}^{m-2}$.

iii) We consider $C^+_{1,m}$, for $m \geq 0$. We may define a global section of $C^+_{1,m}$ *via*

$$C^+_{1,m} \ni [(z_1, x_1, \dots, x_m)] \mapsto \left(\frac{x_1 - \mathrm{Re}(z_1)}{\mathrm{Im}(z_1)}, \dots, \frac{x_m - \mathrm{Re}(z_1)}{\mathrm{Im}(z_1)}\right) \in \mathbb{R}^m,$$

where the points in the m-tuple on the right-hand side are naturally ordered. Further, the open subset $\{(x_1,\dots,x_m) \in \mathbb{R}^m :\ -\infty < x_1 < \cdots < x_m < \infty\}$ is diffeomorphic to the open unit simplex $\triangle_m$ in $\mathbb{R}^m$ by simultaneously applying the $\arg(i - \bullet)/\pi$ the m-tuple on the right-hand side of the previous assignment, $\arg(\bullet)$ denoting the argument function from $\mathbb{H}^+$ to $(0,\pi)$, such that $\arg(i) = \pi/2$.

iv) More generally, for $n, m \geq 1$, we consider $C^+_{n,m}$, and we have *e.g.* either the global section

$$C^+_{n,m} \ni [(z_1,\dots,z_n,x_1,\dots,x_m)] \mapsto \left(\frac{z_2-\mathrm{Re}(z_1)}{\mathrm{Im}(z_1)},\dots,\frac{z_n-\mathrm{Re}(z_1)}{\mathrm{Im}(z_1)},\frac{x_1-\mathrm{Re}(z_1)}{\mathrm{Im}(z_1)},\dots,\frac{x_m-\mathrm{Re}(z_1)}{\mathrm{Im}(z_1)}\right) \in (\mathbb{H}^+)^{n-1}\times\mathbb{R}^m,$$

or

$$C^+_{n,m} \ni [(z_1,\dots,z_n,x_1,\dots,x_m)] \mapsto \left(\frac{z_1-x_1}{|z_1-x_1|},\dots,\frac{z_n-x_1}{|z_1-x_1|},0,\dots,\frac{x_m-x_1}{|z_1-x_1|}\right) \in S^1\times(\mathbb{H}^+)^{n-2}\times\mathbb{R}^m.$$

The configuration space $C_{2,0}$ is particularly important and deserves an own name, the Eye; we will revisit the Eye in detail later on.

v) The open configuration space C_A admits a standard global section, which is precisely the global section that mapping an equivalence class $[(\underline{z}_A)] = [(z_{a_1},\dots,z_{a_{|A|}})]$ to its unique representative satisfying the two equations $\sum_{a\in A} z_a = 0$, $\sum_{a\in A}|z_a|^2 = 1$ (such a representative is said to be in standard position). As a consequence, C_A is a bounded open subset of $\mathbb{C}^A$.

We denote a point of $C^+_{A,B}$ by $[(\underline{z}_{A\sqcup B})] = [(z_{a_1},\dots,z_{b_1},\dots)]$, where $b_1 < \dots < b_{|B|}$. There is an obvious embedding $\mathrm{Conf}^+_{A,B} \hookrightarrow \mathrm{Conf}_{A\sqcup\overline{A}\sqcup B}$, given explicitly by $(\underline{z}_{A\sqcup B}) \mapsto (\underline{z}_A, \overline{\underline{z}}_A, \underline{z}_B)$, where $\underline{z}_A$ denotes the $|A|$-tuple of points labeled by A, while $\overline{z}$ denotes the complex conjugate of z in $\mathbb{C}$. As G_2 is an obvious subgroup of G_3, the previous embedding descends to a map from $C^+_{A,B}$ to $C_{A\sqcup\overline{A}\sqcup B}$; as either the tuple $\underline{z}_B$ is in $\mathbb{R}^B$ (if $B\neq\varnothing$) or as, if $B=\varnothing$, $[(\underline{z}'_{A\sqcup\overline{A}})] = [(\underline{z}_{A\sqcup\overline{A}})]$ in $C_{A\sqcup\overline{A}}$, then $\lambda\underline{z}_A+\mu = \underline{z}'_A$ and $\lambda\overline{\underline{z}}_A+\mu = \overline{\underline{z}}'_A+\mu$, for λ in $\mathbb{R}^+$ and μ in $\mathbb{C}$, it follows that the above map descends to an embedding $C^+_{A,B}$ into $C_{A\sqcup\overline{A}\sqcup B}$.

In particular, there is also a standard section for $C^+_{A,B}$: it is the section mapping any equivalence class $[(\underline{z}_{A\sqcup B})]$ to its unique representative satisfying the two conditions $\sum_{a\in A}\mathrm{Re}(z_a)+\sum_{b\in B} z_B = 0$, $\sum_{a\in A}|z_a|^2 + \sum_{b\in B} z_b^2 = 1$. Thus, $C^+_{A,B}$ is a bounded open subset of $(\mathbb{H}^+)^A\times\mathbb{R}^B$.

The examples *i*)-*v*) show that Conf_A and $\mathrm{Conf}^+_{A,B}$ admit structures of trivial principal bundles over C_A and $C^+_{A,B}$, due to the existence of global sections: such sections will be used throughout to construct (global) coordinates systems on C_A and $C^+_{A,B}$. It is not difficult to verify that two different such coordinate systems are related to each other by real analytic transformations.

It is pretty clear that both $\mathrm{Conf}^+_{A,B}$ and Conf_A are oriented open manifolds as well as G_2 and G_3. Therefore, the corresponding quotients $C^+_{A,B}$ and C_A inherit the structure of oriented open manifolds by any local section of $\mathrm{Conf}^+_{A,B}$ and Conf_A to be orientation-preserving.

In other words, by means of a local section, we have $\mathrm{Conf}^+_{A,B}\cong C^+_{A,B}\times G_2$ and $\mathrm{Conf}_A\cong C_A\times G_3$, thus a canonical orientation of $\mathrm{Conf}^+_{A,B}$ and Conf_A orients $C^+_{A,B}$ and C_A. We will return to orientations later on.

3.3. **Compactified configuration spaces.** The key point in the whole upcoming construction lies in the combinatorics of the boundary strata of well-suited compactifications of the previously introduced open configuration spaces: the need for a compactification also arises to ensure that certain *a priori* ill-defined integrals over the open configuration spaces truly converge, while the combinatorics of the boundary strata is related to certain important quadratic identities between the aforementioned integrals.

3.3.1. *The compactified configuration spaces* $\overline{C}_A$ *and* $\overline{C}^+_{A,B}$. We briefly recall the explicit construction of the compactified configuration spaces $\overline{C}_A$ and $\overline{C}^+_{A,B}$, associated respectively with C_A and $C^+_{A,B}$.

We denote by $\mathbb{S}^A_0(1)$, for a finite set A as above, the subset of $\mathbb{C}^A$ cut out by the equations $\sum_{a\in A} z_a = 0$ and $\sum_{a\in A}|z_a|^2 = 1$, see also Subsubsection 3.1.1, *v*): it is obviously a real smooth, compact submanifold of $\mathbb{C}^A$ of dimension $2|A|-3$. Observe that $\mathbb{S}^A_0(1)$ contains all diagonals in $\mathbb{C}^A$, except the smallest diagonal subset, where all components of a tuple in $\mathbb{C}^A$ are equal.

We first consider C_A: for any subset $\widetilde{A}\subset A$ of cardinality $2\leq|\widetilde{A}|\leq|A|-1$, there is a natural projection from C_A onto $C_{\widetilde{A}}$. Then, the compactified configuration space $\overline{C}_A$ is defined as the closure of the natural embedding

$$C_A \hookrightarrow \prod_{\substack{\widetilde{A}\subseteq A\\ 2\leq|\widetilde{A}|\leq|A|}} \mathbb{S}^{\widetilde{A}}_0(1),$$

where we have used the standard section of Subsubsection 3.1.1, *v*), to construct embeddings $C_{\widetilde{A}}\hookrightarrow\mathbb{S}^{\widetilde{A}}_0(1)$.

The compactification $\overline{C}^+_{A,B}$ is defined in a similar way, recalling the embedding of open configuration spaces $C^+_{A,B}\hookrightarrow C_{A\sqcup\overline{A}\sqcup B}$: then, $\overline{C}^+_{A,B}$ is defined as the closure of the natural embedding

$$C^+_{A,B}\hookrightarrow C_{A\sqcup\overline{A}\sqcup B}\hookrightarrow \prod_{\substack{C\subseteq A\sqcup\overline{A}\sqcup B\\ 2\leq|C|\leq 2|A|+|B|}} \mathbb{S}^C_0(1),$$

where the rightmost product is over subsets C of $A \sqcup \overline{A} \sqcup B$ coming from subsets of $A \sqcup B$ in the natural way.

3.3.2. *The boundary stratification of $\overline{C}_A$ and $\overline{C}^+_{A,B}$: combinatorics.* For the main computations, we need an explicit description of the boundary stratification of the compactified configuration spaces $\overline{C}^+_{A,B}$ and $\overline{C}_A$. First, we describe the boundary stratification of $\overline{C}_A$ and $\overline{C}^+_{A,B}$ from a combinatorial point of view, and deduce from the combinatorics well-suited local coordinates, from which the structure of smooth manifolds with corners will become apparent.

We consider first $\overline{C}_A$. From a combinatorial point of view, a general boundary stratum of codimension $1 \le p \le |A|-2$ is in one-to-one correspondence with a family $\{A_1, \dots, A_p\}$ of subsets of A of cardinality $2 \le |A_i| \le |A|-1$, $i = 1, \dots, p$, and such that either $A_i \cap A_j = \varnothing$ or $A_i \subset A_j$ or $A_i \supset A_j$: such a family $\{A_1, \dots, A_p\}$ is called nested. For a given nested family $\{A_1, \dots, A_p\}$ as above, we define the star of A_i (denoted $\mathrm{star}(A_i)$), for $i = 1, \dots, p$, as the subfamily of $\{A_1, \dots, A_p\}$ of subsets of A_i which are maximal with respect to the partial order $\subset$; we set $A_0 = A$, and accordingly we may define its star $\mathrm{star}(A_0)$. By its very definition, $\mathrm{star}(A_0)$ is never empty, while $\mathrm{star}(A_i)$ may be empty, for $i = 1, \dots, p$. Furthermore, $i \neq j$, $\mathrm{star}(A_i) \cap \mathrm{star}(A_j) = \varnothing$: namely, if $A_i \cap A_j = \varnothing$, the claim is obvious, while, if *e.g.* $A_i \subset A_j$, and $\mathrm{star}(A_i) \cap \mathrm{star}(A_j) \neq \varnothing$, there would exist A_k such that $A_k \subset A_i$ and $A_k \subset A_j$, and A_k is a maximal subset of both A_i and A_j with respect to the partial order $\subset$, which is in contradiction with $A_k \subset A_i \subset A_j$.

Accordingly, the stratum $\partial_{A_1,\dots,A_p}\overline{C}_A$ of codimension p is isomorphic to the product of compactified configuration spaces

$$\partial_{A_1,\dots,A_p}\overline{C}_A \cong \prod_{i=0}^{p} \overline{C}_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}, \tag{2}$$

where $A_i \smallsetminus \mathrm{star}(A_i)$ is a short-hand notation for the complement of the elements of $\mathrm{star}(A_i)$ inside A_i, and $\{\bullet\}^{\mathrm{star}(A_i)}$ denotes a set of cardinality $|\mathrm{star}(A_i)|$.

We then consider $\overline{C}^+_{A,B}$. Combinatorially, a general boundary stratum of codimension $1 \le p \le |A| + |B|$ of $\overline{C}^+_{A,B}$ is in one-to-one correspondence with a family $\{C_1, \dots, C_p\}$ of p subsets C_i either of A or of $A \sqcup B$ with the following properties:

i) if $C_i = A_i$, then $2 \le |C_i| \le |A|$;
ii) if $C_i = A_i \sqcup B_i$, B_i consists of consecutive elements of B, either A_i or B_i may be empty; if A_i is empty, $2 \le |B_i| \le |B|$, if B_i is empty, $1 \le |A_i| \le |A|$;
iii) the family is nested in the following sense: either $C_i \cap C_j = \varnothing$, or $C_i \subset C_j$ or $C_j \subset C_i$, where the partial order $\subset$ is such that $C_i = A_i \subset C_j = A_j \sqcup B_j$ means $A_i \subset A_j$ in the usual sense, but $C_i = A_i \not\supset C_j = A_j \sqcup B_j$, even if $A_i \supset A_j$ in the usual sense.

Again, we set $C_0 = A \sqcup B$. Observe that $\subset$ as previously defined yields a partial order on nested families of the form $\{C_1, \dots, C_p\}$: thus, the definition of $\mathrm{star}(C_i)$, $i = 0, \dots, p$, still makes sense with due modifications. In particular, observe that, if $C_i = A_i \sqcup B_i$, then $\mathrm{star}(C_i)$ may contain elements $C_j = A_j$ or $C_j = A_j \sqcup B_j$, while, if $C_i = A_i$, $\mathrm{star}(C_i)$ contains only elements of the form $C_j = A_j$, by definition of the new partial order $\subset$. Other than that, the stars of $\{C_0, C_1, \dots, C_p\}$ exhaust $\{C_1, \dots, C_p\}$ and stars of distinct elements C_i, C_j of $\{C_1, \dots, C_p\}$ are disjoint. We introduce a further piece of notation: if $C_i = A_i \sqcup B_i$, by $\mathrm{star}_A(C_i)$, resp. $\mathrm{star}_{A,B}(C_i)$, we denote the subsets $C_j = A_j$, resp. $C_j = A_j \sqcup B_j$, in $\mathrm{star}(C_i)$. Obviously, $\mathrm{star}(C_i) = \mathrm{star}_A(C_i) \sqcup \mathrm{star}_{A,B}(C_i)$.

Accordingly, the boundary stratum $\partial_{C_1,\dots,C_p}\overline{C}^+_{A,B}$ is isomorphic to a product of compactified configuration spaces

$$\partial_{C_1,\dots,C_p}\overline{C}^+_{A,B} \cong \prod_{i=0}^{p} \overline{C}^+_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}}, \tag{3}$$

where now

$$\overline{C}^+_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}} = \begin{cases} \overline{C}_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}, & C_i = A_i, \\ \overline{C}^+_{(A_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}_A(C_i)|}, (B_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}_{A,B}(C_i)|}}, & C_i = A_i \sqcup B_i, \end{cases} \tag{4}$$

where $A_i \smallsetminus \mathrm{star}(C_i)$, resp. $(B_i \smallsetminus \mathrm{star}(C_i))$, is a short-hand notation for the complement of all A_j in A_i, resp. of B_j in B_i, for $C_j = A_j$ or $C_j = A_j \sqcup B_j$ in $\mathrm{star}(C_i)$.

3.3.3. *The boundary stratification of $\overline{C}_A$ and $\overline{C}^+_{A,B}$: local coordinates.* Now that we have devised a meaningful combinatorics for the boundary stratification of $\overline{C}_A$ and $\overline{C}^+_{A,B}$, we may construct a corresponding structure of smooth manifold with corners on both compactified configuration spaces, and thus give a precise meaning to (2) and (3).

Let us begin by discussing local coordinates near a boundary stratum of $\overline{C}_A$ of codimension p of the form $\partial_{A_1,\dots,A_p}\overline{C}_A$, where $\{A_1, \dots, A_p\}$ is a nested family of subsets of A. With any subset A_i, we should associate a compactified configuration space $\overline{C}_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$: we first introduce the notation

$$A_i = \left\{a_1^{A_i}, \dots, a_{|A_i|}^{A_i}\right\},\ A_i \smallsetminus \mathrm{star}(A_i) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|} = \left\{a_{i_k}^{A_i} \in A_i \smallsetminus \mathrm{star}(A_i);\ a_{A_j},\ A_j \in \mathrm{star}(A_i)\right\}.$$

We consider now a general element a of A, which may be exhausted by a chain of nested elements of $\{A_1, \dots, A_p\}$ as follows: by construction, either a in $A_0 \smallsetminus \mathrm{star}(A_0)$ or a in $A_{i_1}^a$, for $A_{i_1}^a$ in $\mathrm{star}(A_0)$. In the second case, a is either in $A_{i_1}^a \smallsetminus \mathrm{star}(A_{i_1}^a)$ or in $A_{i_2}^a$, for $A_{i_2}^a$ in $\mathrm{star}(A_{i_1}^a)$. Once again, in the second case, we may proceed as before, until a is in $A_{i_{q_a}}^a \smallsetminus \mathrm{star}(A_{i_{q_a}}^a)$, for some $A_{i_{q_a}}^a$. The corresponding chain of nested elements is denoted by $\{A_{i_1}^a, \dots, A_{i_{q_a}}^a\}$, where $A_{i_1}^a \supset \cdots \supset A_{i_{q_a}}^a$, where we adopt the convention that, if $q_a = 0$, then $A_{i_0} = A_0 = A$. Observe that, by its very construction, a uniquely determines its exhausting chain $\{A_{i_1}^a, \dots, A_{i_{q_a}}^a\}$.

We consider the open configuration space $C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$, for which we choose a smooth section as in Subsubsection 3.1.1: the corresponding set of local coordinates is denoted, according to the previously introduced notation, by

$$\{z_{i_k}^{A_i};\ z_{A_j},\ A_j \in \mathrm{star}(A_i)\} \in C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}.$$

With any A_i, we additionally associate a parameter ρ_{A_i}, which ranges in an interval $[0, \varepsilon_i)$, for ε_i sufficiently small; we finally set $\rho_{A_0} = \rho_A = 1$.

Then, a set of local coordinates for $\overline{C}_A$ near the boundary stratum $\partial_{A_1,\dots,A_p} \overline{C}_A$ is specified by the Formula

$$\prod_{i=1}^{p} [0, \varepsilon_i) \times \prod_{i=0}^{p} C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}} \ni \prod_{i=0}^{p} \left(\rho_{A_i}; \underline{z}_{i_k}^{A_i}; \underline{z}_{A_j}\right) \mapsto \left\{A \ni a \mapsto z_a = \sum_{j=1}^{q_a} \left(\prod_{k=1}^{j-1} \rho_{A_{i_k}^a}\right) z_{A_{i_j}^a}\right\} \in \overline{C}_A. \tag{5}$$

In Formula (5), we have used the short-hand notation $\underline{z}_{i_k}^{A_i}$ for the corresponding tuple of local coordinates, and similarly for $\underline{z}_{A_j}$; when $j = q_a$ in the sum on the rightmost expression in Formula (5), then $z_{A_{i_{q_a}}^a} = z_{i_a}$, for the unique index i_a such that in $A_{i_{q_a}}^a$, $a = a_{i_a}$. According to the convention that, if $q_a = 0$, then $A_{i_0} = A$, we set $z_a = z_{i_a}^{A_0}$ for the unique index i_a such that in $A \smallsetminus \mathrm{star}(A)$, $a = a_{i_a}$. Finally, we have used the convention that a product with a negative number of factors is 1.

Remark 3.2. There is an important *caveat* to be made at this point: for ease of later computations and of notation, the $2|A| - 3$-tuple on the right-hand side of Formula (5) is meant to be an equivalence class of the corresponding tuple in $\mathbb{C}^A$ with respect to the action of G_3. Hence, if we look for coordinates *stricto sensu*, we have to additionally choose a section of Conf_A. See Subsection 4.1 for an example.

It will be convenient for later computations to choose local sections for $C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$ as in Subsubsection 3.1.1, i): this means that in $(\underline{z}_{i_k}^{A_i}; \underline{z}_{A_j})$, one of the coordinate is set to be 0, another is is set to be 1 and all of them are multiplied by an element of S^1. With this in mind, the expression on the right-hand side of Formula (5) can be re-written as

$$A \ni a \mapsto z_a = \sum_{j=1}^{q_a} \left(\prod_{k=1}^{j-1} \rho_{A_{i_k}^a}\right) z_{A_{i_j}^a} = \sum_{j=1}^{q_a} \left(\prod_{k=1}^{j-1} w_{A_{i_k}^a}\right) z_{A_{i_j}^a},$$

where $w_{A_{i_k}^a} = \rho_{A_{i_k}^a} e^{i\varphi_{A_{i_k}^a}}$, where the angle coordinate $\varphi_{A_{i_k}^a}$ is associated with $A_{i_k}^a$ by means of a section of $C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$ from Subsubsection 3.1.1, i) when $A_i = A_{i_k}^a$, and the parameter $\rho_{A_{i_k}^a}$ is viewed as the other corresponding polar coordinate with respect to $\varphi_{A_{i_k}^a}$.

The discussion of local coordinates near a boundary stratum of $\overline{C}_{A,B}^{+}$ of codimension p of the form $\partial_{C_1,\dots,C_p} \overline{C}_{A,B}^{+}$, where $\{C_1, \dots, C_p\}$ is a nested family of subsets of $A \sqcup B$, is similar to the previous one. We point out the relevant modifications.

With any subset C_i, we should associate a compactified configuration space *via* the rule (4). We borrow previous notation as in

$$(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \bullet^{|\mathrm{star}(C_i)|} = \left\{c_{i_k}^{C_i} \in C_i \smallsetminus \mathrm{star}(C_i);\ c_{C_j},\ C_j \in \mathrm{star}(C_i)\right\},$$

where points $c_{i_k}^{C_i}$, whenever they belong to B, are ordered according to the fact that B_i consists of consecutive points, and, for C_j in $\mathrm{star}_{A,B}(C_i)$, c_{C_j}, its position with respect to the order on B is specified by the position B_j, if $C_j = A_j \sqcup B_j$, for $B_j \neq \varnothing$, otherwise, if $B_j = \varnothing$, it will be necessary to specify its position: in such a situation, we tacitly consider its position among the other elements.

The construction of the unique exhausting chain of a or b may be borrowed *verbatim* from previous considerations.

We choose (local) sections for $C^{+}_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}}$, whose corresponding set of local coordinates is

$$\left\{z_{i_k}^{C_i};\ z_{C_j},\ C_j \in \mathrm{star}(C_i)\right\} \in C^{+}_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}},$$

where, if $C_i = A_i$, the corresponding coordinates are in $\mathbb{C}$, while, for $C_i = A_i \sqcup B_i$, the corresponding coordinates are in $\mathbb{H}^+$, resp. $\mathbb{R}$, if they are associated with an index coming from A, resp. B. Again, we additionally associate with any C_i a parameter ρ_{C_i} in $[0, \varepsilon_{C_i})$, for ε_{C_i} sufficiently small.

Then, Formula (5) can be copied almost *verbatim* by keeping track of the previously discussed modifications:

$$\prod_{i=1}^{p}[0,\varepsilon_i) \times \prod_{i=0}^{p} C^+_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}} \ni \prod_{i=0}^{p}\left(\rho_{C_i}; \underline{z}^{C_i}_{i_k}; \underline{z}_{C_j}\right) \mapsto \left\{ C \ni c \mapsto z_c = \sum_{j=1}^{q_c}\left(\prod_{k=1}^{j-1} \rho_{C^c_{i_k}}\right) z_{C^c_{i_j}} \right\} \in \overline{C}^+_{A,B}. \tag{6}$$

Of course, the $2|A|+|B|$-tuple on the right-hand side of Formula (6) must be regarded as the corresponding equivalence class in $C^+_{A,B}$ with respect to the action of G_2.

For later computations, it is useful to choose local sections for $C^+_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}}$, for $C_i = A_i$, as in Subsubsection 3.1.1, i), as well: this means that in $(\underline{z}^{C_i}_{k_i}; \underline{z}_{C_j})$, one of the coordinate is 0, another one is 1 and all coordinates are multiplied by an element of S^1 and we set, whenever it makes sense, $w_{C^c_{i_k}} = \rho_{C^c_{i_k}} e^{i\varphi_{C^c_{i_k}}}$, where the angle coordinate $\varphi_{C^c_{i_k}}$ is associated with $C^c_{i_k}$ by means of the aforementioned local section of $C_{(C_i \smallsetminus \mathrm{star}(C_i)) \sqcup \{\bullet\}^{|\mathrm{star}(C_i)|}}$ when $C_i = C^c_{i_k}$.

The only choices involved in the constructions of the local coordinates (5) and (6) lies in the choice of sections of open configuration spaces: thus, coordinates changes are related to changes of sections. It is possible to choose real analytic global sections, and such choices imply in turn that the corresponding coordinate changes are as in Subsubsection 3.2.1.

4. The logarithmic propagator

We are now going to introduce and discuss in some detail the logarithmic propagator, first introduced in [11, Subsection 4.1, F)]. For this purpose, we need a detailed discussion of the compactified configuration space $\overline{C}^+_{2,0}$, the Eye.

4.1. **The Eye and its boundary stratification.**

We now describe in detail the compactified configuration space $\overline{C}^+_{2,0}$.

By the discussion in Subsection 3.2, $\overline{C}^+_{2,0}$ has a boundary stratification with three boundary strata of codimension 1 and two boundary strata of codimension 2, which are associated respectively with the nested families of subsets $A_1 = \{1,2\}$, $A_1 = \{1\} \sqcup \varnothing$, $A_1 = \{2\} \sqcup \varnothing$, $\{\{1\} \sqcup \varnothing, \{2\} \sqcup \varnothing\}$ and $\{\{2\} \sqcup \varnothing, \{1\} \sqcup \varnothing\}$. Observe that the order in the last two boundary strata reflects the relative positions of the two subsets. Geometrically, the corresponding boundary strata are given by

$$\begin{aligned}
\partial_{\{1,2\}} \overline{C}^+_{2,0} &\cong \overline{C}^+_{1,0} \times \overline{C}_2,\\
\partial_{\{1\}\sqcup\varnothing} \overline{C}^+_{2,0} &\cong \overline{C}^+_{1,1} \times \overline{C}^+_{1,0},\\
\partial_{\{2\}\sqcup\varnothing} \overline{C}^+_{2,0} &\cong \overline{C}^+_{1,1} \times \overline{C}^+_{1,0},\\
\partial_{\{\{1\}\sqcup\varnothing,\{2\}\sqcup\varnothing\}} \overline{C}^+_{2,0} &\cong \overline{C}^+_{0,2} \times \overline{C}^+_{1,0} \times \overline{C}^+_{1,0},\\
\partial_{\{\{2\}\sqcup\varnothing,\{1\}\sqcup\varnothing\}} \overline{C}^+_{2,0} &\cong \overline{C}^+_{0,2} \times \overline{C}^+_{1,0} \times \overline{C}^+_{1,0}.
\end{aligned}$$

To visualize $\overline{C}^+_{2,0}$ and its boundary stratification, we need local coordinates near the given boundary strata. For this purpose, we have to choose sections of all open configuration spaces involved.

Therefore, we choose *e.g.* the section of $C^+_{2,0}$, which fixes the first point to i, hence yielding $C^+_{2,0} \cong \mathbb{H}^+ \smallsetminus \{i\}$, the one of $C^+_{1,1}$ fixing the vertex in $\mathbb{H}^+$ to i, yielding $C^+_{1,1} \cong \mathbb{R}$; the one of $C^+_{1,0}$ which identifies it with $\{i\}$, the one of $C^+_{0,2}$ which identifies it with the pair of points $\{0,1\}$ and the one which identifies C_2 with S^1.

Then, according to Subsubsection 3.3.3, Formulæ (5) and (6), local coordinates near the boundary strata of $\overline{C}^+_{2,0}$ are given by

$$\begin{aligned}
&[0,\varepsilon) \times \overline{C}^+_{1,0} \times \overline{C}_2 \ni (\rho, i, \varphi) \mapsto i + \rho e^{i\varphi} \in \mathbb{H}^+ \smallsetminus \{i\},\\
&[0,\varepsilon) \times \overline{C}^+_{1,1} \times \overline{C}^+_{1,0} \ni (\rho, x, i) \mapsto x + \rho i \in \mathbb{H}^+ \smallsetminus \{i\},\\
&[0,\varepsilon) \times \overline{C}^+_{1,1} \times \overline{C}^+_{1,0} \ni (\rho, x, i) \mapsto \frac{i - x}{\rho} \in \mathbb{H}^+ \smallsetminus \{i\},\\
&[0,\varepsilon)^2 \times \overline{C}^+_{0,2} \times \overline{C}^+_{1,0} \times \overline{C}^+_{1,0} \ni (\rho_1, \rho_2, 0, 1, i, i) \mapsto \frac{1 + \rho_2 i}{\rho_1} \in \mathbb{H}^+ \smallsetminus \{i\},\\
&[0,\varepsilon)^2 \times \overline{C}^+_{0,2} \times \overline{C}^+_{1,0} \times \overline{C}^+_{1,0} \ni (\rho_1, \rho_2, 0, 1, i, i) \mapsto \frac{-1 + \rho_1 i}{\rho_2} \in \mathbb{H}^+ \smallsetminus \{i\}.
\end{aligned}$$

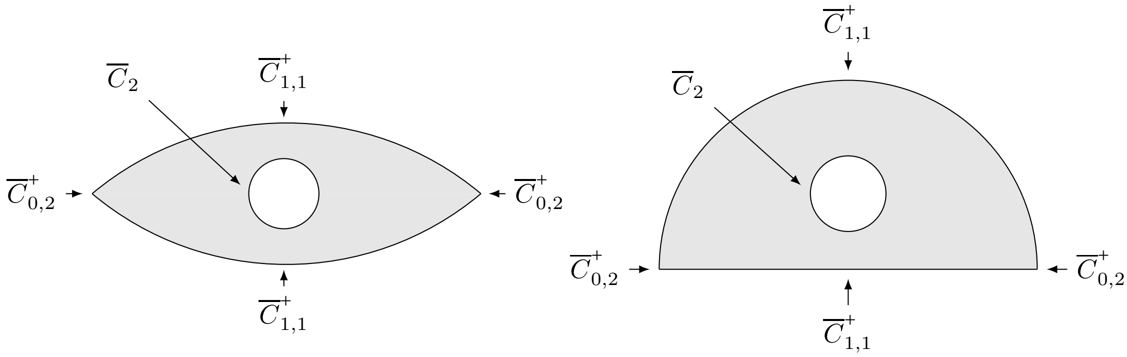

FIGURE 1. The Eye.

We first observe that the previous formulæ are deduced immediately from Formulæ (5) and (6) by using the previous sections of open configuration spaces. The only choice, which may look problematic according to the definition of smooth manifold with corners from Subsubsection 3.2.1, is the identification $C^+_{1,1} \cong \mathbb{R}$. Furthermore, the last two local charts are redundant, *i.e.* we may cover $\overline{C}^+_{2,0}$ with the first three coordinate charts. In order to illustrate this last fact and to simultaneously prove that $\overline{C}^+_{2,0}$ admits an atlas, whose coordinate changes are analytic isomorphisms as in Subsubsection 3.2.1, we consider *e.g.* a point of the form $x + \rho i$ in $\mathbb{H}^+ \smallsetminus \{i\}$, for x sufficiently big and ρ sufficiently small: then, there is an analytic diffeomorphism $(\rho, x) \mapsto (\rho x, x)$ for such a region, and setting $\rho_1 = 1/x$, $\rho_2 = \rho$, then $x + \rho i$ can be mapped homeomorphically to the class $[(\rho_1 i, 1 + \rho_2 i)]$, and now both ρ_i, $i = 1, 2$, are sufficiently small. On the other hand, we consider $(i - x)/\rho$, for $-x$ sufficiently big and ρ sufficiently small as in the previous situation, and we consider the same analytic diffeomorphism $(\rho, x) \mapsto (\rho x, x)$: setting once again $\rho_1 = -1/x$ and $\rho_2 = \rho$, $(i - x)/\rho$ is mapped homeomorphically to the class $[(\rho_2 i, 1 + \rho_1 i)]$. Introducing the coordinates (ρ_1, ρ_2) as in the fourth chart, we see that the second and the third chart overlap on the fourth chart, where they are identified by means of the analytic isomorphism $(\rho_1, \rho_2) \mapsto (\rho_2, \rho_1)$ as in Subsubsection 3.2.1. Similar computations prove that the second and the third chart overlap on the fifth chart, and the corresponding coordinates are identified by means of the same analytic isomorphism as before.

These computations justify the graphical representation of $\overline{C}^+_{2,0}$ in Figure 1.

4.2. **The logarithmic propagator.** We define on $\mathrm{Conf}^+_{2,0}$ the smooth, multi-valued function

$$\mathrm{Conf}^+_{2,0} \ni (z_1, z_2) \mapsto \frac{1}{2\pi i} \log\left(\frac{z_1 - z_2}{\overline{z}_1 - z_2}\right) \in \mathbb{C},$$

for some choice of a complex logarithm $\log(\bullet)$. Observe that, up to addition of constant factors, it can be re-written as the sum of two functions: the first, $\log(z_1 - z_2)$ is a multivalued function on $\mathrm{Conf}^+_{2,0}$, while the second, $\log(\overline{z}_1 - z_2)$, for a suitable choice of a complex logarithm, is a well-defined, smooth function on $\mathrm{Conf}^+_{2,0}$.

Anyway, it is clear that the above function descends to a smooth, multi-valued function on $C^+_{2,0}$, whence its exterior derivative yields a smooth, well-defined closed 1-form $\omega_{\log}$ on $C^+_{2,0}$: the 1-form $\omega_{\log}$ is called the logarithmic propagator. The logarithmic propagator is better written down explicitly as

$$\omega_{\log}(z_1, z_2) = \frac{1}{2\pi i}\left(\frac{dz_1 - dz_2}{z_1 - z_2} - \frac{d\overline{z}_1 - dz_2}{\overline{z}_1 - z_2}\right)$$

First of all, it is easy to verify that $\omega_{\log}$ on $C^+_{2,0}$ extends to a smooth 1-form on the two boundary strata $\partial_{\{1\}\sqcup\varnothing}\overline{C}^+_{2,0}$ and $\partial_{\{2\}\sqcup\varnothing}\overline{C}^+_{2,0}$ of $\overline{C}^+_{2,0}$. Namely, if the first, resp. second, argument in $C^+_{2,0}$ approaches $\mathbb{R}$, $\omega_{\log}$ restricts to 0, resp. to the smooth, exact 1-form

$$\omega_{\log}(z_1, z_2) = \frac{1}{\pi} d\arg(z_1 - z_2),$$

if we choose *e.g.* the principal branch of the complex logarithm, for which the argument function $\arg(\bullet)$ on $\mathbb{H}^+$ takes its values in $(0, \pi)$ and $\arg(i) = \pi/2$. This can be also verified by an easy computation using the previously introduced local coordinates on the Eye near the two aforementioned boundary strata.

The propagator $\omega_{\log}$, unlike the standard propagator from [12], does not extend to a complex-valued, real analytic 1-form on $\overline{C}^+_{2,0}$. Namely, let us consider the previous local coordinates of $\overline{C}^+_{2,0}$ near the stratum $\partial_{\{1,2\}}\overline{C}^+_{2,0}$, whence

$$\omega_{\log}(i, i+\rho e^{i\varphi}) = \frac{1}{2\pi i}\left(\frac{dw}{w} - \frac{dw}{w+2i}\right),\ w = \rho e^{i\varphi} \in B_\varepsilon(0) \smallsetminus \{0\},$$

for ε sufficiently small. Observe that, by considering the corresponding coordinate chart associated with the section fixing the second argument to i, the logarithmic propagator takes the form

$$\omega_{\log}(i+\rho e^{i\varphi}, i) = \frac{1}{2\pi i}\left(\frac{dw}{w} - \frac{d\overline{w}}{\overline{w}-2i}\right),\ w = \rho e^{i\varphi} \in B_\varepsilon(0) \smallsetminus \{0\},$$

for ε sufficiently small. In particular, the restriction of $\omega_{\log}$ to a small neighborhood of the boundary stratum $\partial_{\{1,2\}}\overline{C}^+_{2,0}$ of $\overline{C}^+_{2,0}$ is a complex-valued, smooth 1-form on a small punctured disk around 0 with a simple pole of order 1 at the origin.

Proposition 4.1. *The complex-valued, real analytic* 1*-form* $\omega_{\log}$ *on* $C^+_{2,0}$ *extends to a real analytic exact* 1*-form on the boundary strata* $\partial_{\{1\}\sqcup\varnothing}\overline{C}^+_{2,0}$ *and* $\partial_{\{2\}\sqcup\varnothing}\overline{C}^+_{2,0}$ *of* $\overline{C}^+_{2,0}$*, while, identifying a small neighborhood of the boundary stratum* $\partial_{\{1,2\}}\overline{C}^+_{2,0}$ *in* $\overline{C}^+_{2,0}$ *with a small punctured disk around the origin,* $\omega_{\log}$ *restricts to a closed, complex-valued real analytic* 1*-form with a simple pole of order* 1 *at the origin.*

5. The L_∞-quasi-isomorphism with the logarithmic propagator

In the present Section, we illustrate the explicit construction of the L_∞-quasi-isomorphism $\mathcal{U}$ from [12] using the logarithmic propagator: roughly speaking, the construction is very easy, namely, we just take *verbatim* from [12] the formulæ for the L_∞-quasi-isomorphism and replace everywhere the standard propagator with the logarithmic one. However, due to Proposition 4.1, the replacement is not "smooth": the fact that $\omega_{\log}$, unlike the standard propagator, does not extend smoothly to the whole compactified configuration space $\overline{C}^+_{2,0}$ may cause some convergence problems in the integral weights. Additionally, the L_∞-property for $\mathcal{U}$ follows from Stokes's Theorem in [12]: the fact that $\omega_{\log}$ has a simple pole along the boundary stratum $\partial_{\{1,2\}}\overline{C}^+_{2,0}$ implies that Stokes' Theorem is not immediately available and requires some care.

5.1. Polar and complex coordinates.

The motivation for the present subsection comes from Subsubsections 3.2.1, i), and 3.3.3: namely, observe that the parameter ρ_{C_i}, for $C_i = A_i$, $|A_i| \geq 2$, together with the angle variable φ_{C_i}, can be considered as a pair of polar coordinates, for any C_i as before.

Using polar coordinates (ρ,φ) in $(0,\infty)\times S^1$, we may identify $\mathbb{C}\smallsetminus\{0\}$ with $(0,\infty)\times S^1$ *via* $w = \rho e^{i\varphi}$. Polar coordinates on $\mathbb{R}^+\times S^1$ define the real blow-up $\mathrm{Bl}(\mathbb{R}^2,\{0\})$ of $\mathbb{C}$ at the origin: more precisely, we first identify S^1 with the real projective line $\mathbb{RP}^1$, and define $\mathrm{Bl}(\mathbb{R}^2,\{0\})$ as the total space of the tautological vector bundle of rank 2 over $\mathbb{RP}^1$ with the natural projection onto $\mathbb{R}^2 = \mathbb{C}$, *i.e.*

$$\widehat{\mathbb{C}} = \mathrm{Bl}(\mathbb{R}^2,\{0\}) = \left\{([x_0 : x_1],(y_0,y_1)) \in \mathbb{RP}^1\times\mathbb{R}^2 :\ (y_0,y_1)\in[x_0:x_1]\right\},$$

where $[x_0 : x_1]$ denote homogeneous coordinates of $\mathbb{RP}^1$.

The natural projection $\pi : \widehat{\mathbb{C}} \to \mathbb{C}$ restricts to a diffeomorphism between the complement of the trivial section of $\widehat{\mathbb{C}}$ and $\mathbb{C}\smallsetminus\{0\}$, and the inverse image of $\{0\}$ with respect to π identifies with $\mathbb{RP}^1 = S^1$. Polar coordinates are well-defined on $\widehat{\mathbb{C}}$: observe that $\{\rho = 0\}$ identifies precisely with S^1.

With respect to the complex coordinate w on $\mathbb{C}\smallsetminus\{0\}$, the smooth 1-forms dw/w, $d\overline{w}/\overline{w}$ do obviously not extend to $\mathbb{C}$ due to their singularity at $w = 0$. Similarly neither do the smooth 2-forms $dwd\overline{w}/w$, $dwd\overline{w}/\overline{w}$ and $dwd\overline{w}/|w|$. However, the previous three 2-forms extend to smooth 2-forms of type $(1,1)$ to $\widehat{\mathbb{C}}$, because

$$\frac{dwd\overline{w}}{w} = -2ie^{-i\varphi}d\rho d\varphi, \qquad \frac{dwd\overline{w}}{\overline{w}} = -2ie^{i\varphi}d\rho d\varphi, \qquad \frac{dwd\overline{w}}{|w|} = -2id\rho d\varphi.$$

5.2. Admissible graphs and integral weights.

The integral weights appearing in the explicit formulæ for $\mathcal{U}$ are described in terms of admissible graphs, borrowing the denomination from [12, Subsection 6.2] (up to combinatorial factors, which are taken care of in a different way, see Subsection 6.3, where the notation is inspired by the arguments in [5, Appendix A]).

We consider two finite sets A, B, where B is endowed with a total order. A general admissible graph Γ of type (A,B) is a directed graph with vertex set $A\sqcup B$, such that

i) no directed edge departs from a vertex in B, and

ii) Γ admits neither multiple edges nor short loops (*i.e.* between any two vertices of Γ there is at most one directed edge, and no edge may depart and end at the same vertex).

It is customary to call a vertex in A, resp. B, vertex of the first, resp. second type. We will sometimes use the notation $V(\Gamma) = A \sqcup B$ and $E(\Gamma)$ for the set of edges of Γ; the set of admissible graphs of type (A, B) is denoted by $\mathcal{G}_{A,B}$.

Observe now that any directed edge e of Γ in $\mathcal{G}_{A,B}$ determines a unique natural projection π_e from $C^+_{A,B}$ onto $C^+_{2,0}$ or $C^+_{1,1}$ (if both endpoints or the starting point of e lie in $\mathbb{H}^+$).

Definition 5.1. The (logarithmic) integral weight associated with Γ in $\mathcal{G}_{A,B}$, which we denote by $\varpi^{\log}_\Gamma$, is defined *via*

$$\varpi^{\log}_\Gamma = \int_{C^+_{A,B}} \omega^{\log}_\Gamma, \qquad \omega^{\log}_\Gamma = \prod_{e \in E(\Gamma)} \omega^{\log}_e, \qquad \omega^{\log}_e = \pi^*_e(\omega_{\log}). \tag{7}$$

First of all, observe that, if $|E(\Gamma)| \neq 2|A| + |B| - 2$, then $\varpi^{\log}_\Gamma$ is automatically trivial. Second, as $\omega_{\log}$ is a 1-form, the product in (7) needs an ordering to be well-defined, which translates into an ordering of $E(\Gamma)$: we may choose any ordering here. As we will see later on, the ordering is uninfluent, because any sign indeterminacy in the above product of forms will be compensated by a similar sign indeterminacy in the corresponding multidifferential operator.

The key-point right now lies in the fact that the integral (7) is not obviously convergent, despite having at our disposal a nice compactified configuration space $\overline{C}^+_{A,B}$: because of Proposition 4.1, the integrand $\omega^{\log}_\Gamma$ on $C^+_{A,B}$ does not obviously extend to the compactification.

Proposition 5.2. *For a general admissible graph Γ of type (A, B) with $|E(\Gamma)| = 2|A| + |B| - 2$, $\omega^{\log}_\Gamma$ belongs to $\Omega^{2n+m-2}(\overline{C}^+_{A,B})$.*

Proof. We need to prove that $\omega^{\log}_\Gamma$, for Γ as above, extends to a complex-valued, real analytic form of top degree near a general boundary stratum $\partial_{C_1,\dots,C_p}\overline{C}^+_{A,B}$ of codimension p: for this purpose, we use the local coordinates (5) and (6), where we have picked sections for open configuration spaces C_A as in Subsubsection 3.1.1, i).

As already remarked in Subsection 4.2, the logarithmic propagator can be written as a sum of two terms: the first one, coming from a multi-valued function on $C^+_{2,0}$, is a holomorphic logarithmic differential which is singular when its two arguments approach in $\mathbb{H}^+$, while the second one does is complex-valued, real analytic on $\overline{C}^+_{2,0}$ by direct inspection.

Using local coordinates near the given boundary stratum as in (5) and (6), the multiplicative property of the complex logarithm and the computations in Subsection 4.2, it is not difficult to verify that, near the boundary stratum $\partial_{C_1,\dots,C_p}\overline{C}^+_{A,B}$, factors in the product form $\omega^{\log}_\Gamma$ may be written as

$$\frac{1}{2\pi i}\frac{dw_{C_i}}{w_{C_i}} + \dots = \frac{1}{2\pi i}\frac{d\rho_{C_i}}{\rho_{C_i}} + \frac{d\varphi_{C_i}}{2\pi} + \dots,$$

for any subset $C_i = A_i$, $|A_i| \geq 2$, and where $\dots$ denotes complex-valued, real analytic summands.

Being ω^t_Γ of top degree on $C^+_{A,B}$, near the given boundary stratum every summand of it must be proportional to a form which contains all differentials: in particular, it must contain the 2-form $dw_{C_i} d\overline{w}_{C_i}$, for every $C_i = A_i$, $|A_i| \geq 2$.

By the previous computations, the possible singular terms are holomorphic logarithmic differentials dw_{C_i}/w_{C_i}; every such term must be paired with an anti-holomorphic differential $d\overline{w}_{C_i}$ because of degree reasons. It follows directly from the shape of the logarithmic propagator that the corresponding anti-holomorphic differential $d\overline{w}_{C_i}$ comes from a complex-valued, real analytic term. Hence, the two factors produce the 2-form $dw_{C_i} d\overline{w}_{C_i}/w_{C_i}$, which extends to $\widehat{\mathbb{C}}$ by the computations in Subsection 5.1.

Equivalently, we may use polar coordinates instead of complex coordinates. Observe first that the angle coordinate φ_{C_i} corresponding to ρ_{C_i} may appear either in the factor

$$\frac{1}{2\pi i}\frac{d\rho_{C_i}}{\rho_{C_i}} + \frac{d\varphi_{C_i}}{2\pi} + \dots$$

or in a complex-valued, real analytic term as $\rho_{C_i} d\varphi_{C_i}$: this follows immediately from the fact that the denominator in the logarithmic propagator, as already remarked, yields complex-valued, real analytic forms. In every summand there can be at most one term like $d\rho_{C_i}/\rho_{C_i}$: degree reasons further imply that $d\rho_{C_i}/\rho_{C_i}$ is either paired with a factor $\rho_{C_i} d\varphi_{C_i}$, which compensates the singularity, or with a factor $d\varphi_{C_i}$.

Let us consider the second case and assume we have a summand which is a product of 1-form with a factor $d\rho_{C_i}/\rho_{C_i}$ and $d\varphi_{C_i}$: the previous computations imply that $d\varphi_{C_i}$ must come from another copy of the 1-form

$$\frac{1}{2\pi i}\frac{d\rho_{C_i}}{\rho_{C_i}} + \frac{\mathrm{d}\varphi_{C_i}}{2\pi}.$$

Thus, there is exactly another summand which looks exactly as the previous one, but where the positions of $d\rho_{C_i}/\rho_{C_i}$ and $d\varphi_{C_i}$ are swapped: since 1-forms anti-commute with each other, the two terms sum up to 0.

Hence, whenever $d\rho_{C_i}/\rho_{C_i}$ appears in a non-trivial term, it is always accompanied by $\rho_{C_i}d\varphi_{C_i}$.

Finally observe that the previous arguments do not apply to the situation, where Γ admits a univalent vertex, *i.e.* a vertex joined by exactly one directed edge: the corresponding integral weight $\varpi_\Gamma^{\log}$ vanishes because of dimensional reasons (we thank S. Merkulov for this observation). □

It thus makes sense to consider the integral weight (7) for any admissible graph Γ in $\mathcal{G}_{A,B}$: if $|E(\Gamma)| = 2|A| + |B| - 2$, Proposition 5.2 guarantees that the integral exists, otherwise we set $\varpi_\Gamma^{\log} = 0$ for dimensional reasons.

5.3. **Singular forms associated with admissible graphs.** One of the main difficulties in [12] lies in proving that the explicit formulæ for the pre-L_∞-quasi-isomorphism $\mathcal{U}$ in terms of admissible graphs define indeed an L_∞-morphism in the sense of Subsection 2.1: as already remarked, this reduces to proving an infinite series of quadratic relations among the integral weights, which is in turn a consequence of the combinatorial structure of the boundary stratification of compactified configuration spaces of the form $\overline{C}_{A,B}^+$ and of the properties of the standard propagator in [12].

The main technical tool in proving the quadratic relations between the integral weights in [12] is Stokes' Theorem for a smooth, compact manifold with corners: observe that, in view of Proposition 4.1, such a Stokes' theorem is not immediately available for our purposes.

In fact, it may be that a differential form $\omega_\Gamma^{\log}$, for Γ in $\mathcal{G}_{A,B}$ and such that $|E(\Gamma)| = 2|A| + |B| - 3$, presents some singularities along certain boundary strata: Proposition 4.1 shows that it is indeed the case for the differential form $\omega_{\log} = \omega_\Gamma^{\log}$, where Γ is the unique admissible graph in $\mathcal{G}_{2,0}$ with one arrow connecting its two vertices of the first type. Observe that $\omega_\Gamma^{\log}$ extends to a complex-valued, real analytic 1-form on all boundary strata of $\overline{C}_{2,0}^+$, except for the stratum, where the two points collapse together in $\mathbb{H}^+$: there, it has a pole of order 1.

This fact finds its natural generalization in the following proposition.

Proposition 5.3. *For a general admissible graph Γ of type (A,B) with $|E(\Gamma)| = 2|A| + |B| - 3$, $\omega_\Gamma^{\log}$ belongs to $\Omega_1^{2n+m-3}(\overline{C}_{A,B}^+)$.*

Proof. Let us consider a general boundary stratum of $\overline{C}_{A,B}^+$ of the form $\partial_{C_1,\dots,C_p}\overline{C}_{A,B}^+$ of codimension p as in the proof of Proposition 5.2.

We may repeat the very same computations therein to prove that the only possible singularities arise precisely from subsets C_i of the nested family $\{C_1,\dots,C_p\}$ of the form $C_i = A_i$, where $|A_i| \geq 2$: more precisely, the possible singularities arise from factors of the shape

$$\frac{1}{2\pi i}\frac{dw_{C_i}}{w_{C_i}} + \cdots = \frac{1}{2\pi i}\frac{d\rho_{C_i}}{\rho_{C_i}} + \frac{d\varphi_{C_i}}{2\pi} + \dots,$$

for $C_i = A_i$, $|A_i| \geq 2$.

In the present situation the degree of $\omega_\Gamma^{\log}$ is $2|A| + |B| - 3$, *i.e.* it differs by 1 from the dimension of $\overline{C}_{A,B}^+$. The proof of Proposition 5.2 implies that a summand in $\omega_\Gamma^{\log}$ may contain a product of singular 1-forms of the type dw_{C_i}/w_{C_i} and, for each index i associated with $C_i = A_i$, $|A_i| \geq 2$, there is precisely one such singular form. The corresponding form has degree equal to the dimension of $C_{A,B}^+$ minus 1, hence the argument in the final part of the proof of Proposition 5.2 implies that there may be at most one index i, such that the corresponding singular 1-form dw_{C_i}/w_{C_i} is not paired with a regular 1-form proportional to $d\overline{w}_{C_i}$ so as to produce the 2-form $dw_{C_i}d\overline{w}_{C_i}/w_{C_i}$.

If we use polar coordinates instead of complex ones, the previous arguments imply that in every summand of $\omega_\Gamma^{\log}$ there may be at most one index i associated with a subset $C_i = A_i$, $|A_i| \geq 2$, for which the summand contains a singular factor $d\rho_{C_i}/\rho_{C_i}$.

The consequence is that near a boundary stratum as above, $\omega_\Gamma^{\log}$ may have summands containing at most one singular term dw_{C_i}/w_{C_i} for $C_i = A_i$, $|A_i| \geq 2$. The remaining terms are complex-valued, real analytic by means of previous computations.

Let us now write $\omega_\Gamma^{\log}$ near the chosen boundary stratum as

$$\omega_\Gamma^{\log} = \sum_{\substack{C_i = A_i\\ |A_i| \geq 2}} \frac{dw_{C_i}}{w_{C_i}}\omega_{C_i,\Gamma} + \eta_\Gamma,$$

where $\omega_{C_i,\Gamma}$, η_Γ are complex-valued, real analytic.

For a $C_i = A_i$, $|A_i| \geq 2$, we further consider the form

$$\mathrm{Reg}_{\{\rho_{C_i}=0\}}(\omega_\Gamma^{\log}) = \left(\sum_{\substack{j\neq i,\ C_j = A_j \\ |A_j|\geq 2}} \frac{dw_{C_j}}{w_{C_j}} \omega_{C_j,\Gamma} + \eta_\Gamma \right)\Bigg|_{\rho_{C_i}=0}.$$

Observe that $\{\rho_{C_i} = 0\}$ in $\widehat{\mathbb{C}}$ corresponds to $\{w_{C_i} = \overline{w}_{C_i} = 0\}$ in $\mathbb{C}$. We claim that this form is complex-valued, real analytic on the given stratum. We may use here a slight variant of the arguments in the proof of Proposition 5.2: since we consider the regularization at $\{\rho_{C_i} = 0\}$ of $\omega_\Gamma^{\log}$ and the degree of the corresponding form equals the dimension of the stratum, whenever the singular differential dw_{C_j}/w_{C_j} appears, the corresponding differential $d\overline{w}_{C_j}$, which comes from complex-valued, real analytic terms, must also appear.

Equivalently, with respect to polar coordinates associated with any subset $C_i = A_i$, $|A_i| \geq 2$, the logarithmic differential $d\rho_{C_j}/\rho_{C_j}$ in a summand of $\mathrm{Reg}_{\{\rho_{C_i}=0\}}(\omega_\Gamma^{\log})$ is always paired with the corresponding angle differential $d\varphi_{C_j}$ because of degree reasons. If $d\varphi_{C_j}$ is paired with ρ_{C_j}, it compensates the singular term. However, it is possible that a summand contains a factor $d\rho_{C_j}/\rho_{C_j}$ paired only with $d\varphi_{C_j}$: then again, by the same arguments at the end of the proof of Proposition 5.2, there is exactly another summand which is equal to the previous one up to the exchange of the positions of $d\rho_{C_j}$ and $d\varphi_{C_j}$. This follows from the fact that $\omega_\Gamma^{\log}$ is a product form and from the multiplicative property of the complex logarithm. It follows that both terms cancel each other.

Therefore, it follows that $\mathrm{Reg}_{\{\rho_{C_i}=0\}}(\omega_\Gamma^{\log})$ is complex-valued, real analytic, for any $C_i = A_i$, $|A_i| \geq 2$.

□

6. The L_∞-property and Stokes' Theorem

We now come to the core of this note, *i.e.* the L_∞-property for the local L_∞-quasi-isomorphism $\mathcal{U}$, which is constructed using the same strategy as in [12] but replacing the standard propagator by the logarithmic one.

Before entering into the technical details, it is better to recall the explicit formulæ for $\mathcal{U}$: in fact, from now on, we will write $\mathcal{U}^{\log}$ for the pre-L_∞-morphism constructed following the prescriptions of [12, Subsection 6.2] and [5, Appendix A] but replacing the standard propagator by the logarithmic one.

6.1. **The relevant dg Lie algebras.** We consider $X = \mathbb{R}^d$, for $d \geq 1$.

Let us introduce the two relevant dg Lie algebras entering into play here; we refer to [12, Subsubsections 3.4.2, 4.6.1] for more details. The dg Lie algebra $T_{\mathrm{poly}}(X)$, $D_{\mathrm{poly}}(X)$, of polyvector fields, resp. polidifferential operators, on X is defined as

$$T_{\mathrm{poly}}(X) = \bigoplus_{p\geq -1} T^p_{\mathrm{poly}}(X) = \bigoplus_{p\geq -1} \Gamma(X, \wedge^{p+1}(TX)),\ \text{resp. } T_{\mathrm{poly}}(X) = \bigoplus_{p\geq -1} D^p_{\mathrm{poly}}(X),$$

where $D^p_{\mathrm{poly}}(X)$ is the subspace of Hochschild cochains of degree $p+1$ on $A = C^\infty(X)$ consisting of $\mathbb{R}$-linear maps from $A^{\otimes(p+1)}$ to A, which are differential operators in each entry.

Observe that $T^{-1}_{\mathrm{poly}}(X) = D^{-1}_{\mathrm{poly}}(X) = A$, $T^0_{\mathrm{poly}}(X) = \mathfrak{X}(X)$ (smooth vector fields on X) and $D^0_{\mathrm{poly}}(X) = D(X)$ (global differential operators on A).

Now, $T^0_{\mathrm{poly}}(X)$ is a Lie algebroid over $T^{-1}_{\mathrm{poly}}(X)$: we may therefore extend the standard Lie bracket on $T^0_{\mathrm{poly}}(X)$ on $T_{\mathrm{poly}}(X)$ by means of the graded Leibniz rule with respect to the standard wedge product $\cup$ on $T_{\mathrm{poly}}(X)$. In this way, $(T_{\mathrm{poly}}(X), 0, [\bullet,\bullet])$ becomes a dg Lie algebra with trivial differential. The graded Lie bracket on $T_{\mathrm{poly}}(X)$ is called the Schouten–Nijenhuis bracket.

On the other hand, there are obvious inclusions

$$D_{\mathrm{poly}}(X)^p \subseteq \mathrm{Hom}(A^{p+1}, A),\ p \geq -1,$$

thus we may regard $D_{\mathrm{poly}}(X)$ as a graded subspace of the (shifted) Hochschild cochain complex of A,

$$C^\bullet(A,A)[1] = \bigoplus_{p\geq -1} \mathrm{Hom}(A^{p+1}, A).$$

It is well-known that the Hochschild cochain complex $C^\bullet(A,A)[1]$ is endowed with a lot of interesting algebraic structures, one of the most relevant being the B_∞-structure: for the present applications, it suffices to know that the B_∞-structure on $C^\bullet(A,A)[1]$ gives rise to a graded Lie bracket $[\bullet,\bullet]$, the Gerstenhaber bracket, and the associative algebra structure on A yields, by means of the Gerstenhaber bracket, and the Hochschild differential d_{H}. The B_∞-structure restricts to the graded subspace $D_{\mathrm{poly}}(X)$, which comes endowed with a dg Lie algebra structure $(D_{\mathrm{poly}}(X), d_{\mathrm{H}}, [\bullet,\bullet])$. We observe that the restriction of the Gerstenhaber bracket on $D^0_{\mathrm{poly}}(X)$ equals the standard commutator between differential operators; the Hochschild differential is explicitly given by $d_{\mathrm{H}} = [m,\bullet]$, where m

is the standard product on A, which may be regarded as an element of $D_{\mathrm{poly}}(X)^1$, which additionally satisfies $[m,m]=0$.

In general, given a dg Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}}, [\bullet,\bullet])$, an element γ of degree 1 satisfying

$$d_{\mathfrak{g}}\gamma + \frac{1}{2}[\gamma,\gamma] = 0$$

is called a Maurer–Cartan element, for short, MC element, of $\mathfrak{g}$. It is not difficult to prove that $d_\gamma = d_{\mathfrak{g}} + [\gamma,\bullet]$ defines a differential on $\mathfrak{g}$, compatible with the Lie bracket: the corresponding dg Lie algebra $(\mathfrak{g}, d_\gamma, [\bullet,\bullet])$ is the twist of $\mathfrak{g}$ by the MC element γ.

Remark 6.1. The notion of MC elements makes sense also for L_∞-algebras under some assumptions on convergence of infinite sums in $\mathfrak{g}$, and accordingly there is a notion of twist for L_∞-algebras and for morphisms between L_∞-algebras: we will review it in detail later on, when dealing with globalization issues.

For example, if we consider the dg Lie algebra $T_{\mathrm{poly}}(X)$ with trivial differential, a MC element of $T_{\mathrm{poly}}(X)$ is a Poisson bivector π (*i.e.* a bivector field on X which satisfies $[\pi,\pi]=0$). On the other hand, the standard product m on A is a MC element for $D_{\mathrm{poly}}(X)$ by a direct computation using the Gerstenhaber bracket.

6.2. **The Hochschild–Kostant–Rosenberg Theorem.** We consider the dg Lie algebras $T_{\mathrm{poly}}(X)$ and $D_{\mathrm{poly}}(X)$ endowed with the previously discussed structures. In particular, we observe that the cohomology of $T_{\mathrm{poly}}(X)$ equals itself: the Hochschild–Kostant–Rosenberg Theorem (shortly, from now on, HKR Theorem) puts into relationship the cohomology of the dg Lie algebra $D_{\mathrm{poly}}(X)$ with $T_{\mathrm{poly}}(X)$.

Theorem 6.2. *The dg vector spaces $(T_{\mathrm{poly}}(X),0)$ and $(D_{\mathrm{poly}}(X),d_{\mathrm{H}})$ are quasi-isomorphic by means of the explicit natural morphism of graded A-modules (the HKR quasi-isomorphism)*

$$\begin{aligned} &\mathrm{HKR}:(T_{\mathrm{poly}}(X),0)\to(D_{\mathrm{poly}}(X),d_{\mathrm{H}}),\\ &\partial_{i_1}\cdots\partial_{i_p}\mapsto\left\{a_1\otimes\cdots\otimes a_p\mapsto\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}(-1)^\sigma\partial_{\sigma(i_1)}(a_1)\cdots\partial_{\sigma(i_p)}(a_p)\right\}, \end{aligned} \tag{8}$$

where we have chosen a system of global coordinates $\{x_i\}$ on X (thus, $\{\partial_i\}$ is a system of graded generators over A for $T_{\mathrm{poly}}(X)$) and we have omitted wedge products for the sake of simplicity.

Proof. A complete proof is given in [12, Subsubsubsection 4.6.1.1]. □

It is well-known that the quasi-isomorphism (8) is not compatible with Lie brackets: still, the main result of [12] shows that the induced map on cohomology is indeed compatible with the corresponding Lie brackets.

6.3. **The logarithmic pre-L_∞-morphism.** Recalling Subsection 2.1, we may regard $T_{\mathrm{poly}}(X)$ and $D_{\mathrm{poly}}(X)$ as L_∞-algebras. We now construct a pre-L_∞-morphism from $T_{\mathrm{poly}}(X)$ to $D_{\mathrm{poly}}(X)$, *i.e.* setting $\mathfrak{g}_1 = T_{\mathrm{poly}}(X)$ and $\mathfrak{g}_2 = D_{\mathrm{poly}}(X)$, we define a morphism $\mathcal{U}^{\log}$ of coalgebras from $\mathrm{S}^+(\mathfrak{g}_1[1])$ to $\mathrm{S}^+(\mathfrak{g}_2[1])$ with the help of the logarithmic propagator from Subsection 4.2. This is equivalent to specifying the corresponding Taylor components

$$\mathcal{U}_n^{\log}:\wedge^n(\mathfrak{g}_1)\to\mathfrak{g}_2[1-n],\ n\geq1,$$

which we now construct explicitly following the prescriptions in [12, Subsections 6.1-6.3] with due modifications.

Let us identify $T_{\mathrm{poly}}(X)$ with the graded A-module $(A[\theta_1,\dots,\theta_d])[1]$, where $\{\theta_i\}$ denotes a set of graded variables of degree 1, which commute with A and anticommute with each other (one may think of θ_i as ∂_i with shifted degree). We further consider the graded linear endomorphism τ of $T_{\mathrm{poly}}(X)^{\otimes 2}$ defined *via*

$$\tau=\partial_{\theta_i}\otimes\partial_{x_i},$$

where of course summation over repeated indices is understood. Observe that τ is well-defined (*i.e.* it does not depend on the choice of linear coordinates on X) and has degree -1.

Furthermore, we define a closed, complex-valued, real analytic 1-form on $C_{2,0}^+$ with values in the graded algebra of graded endomorphisms of $T_{\mathrm{poly}}(X)$ *via* $\omega_\tau^{\log}=\omega_{\log}\otimes\tau$: its total degree (*i.e.* form degree and endomorphism degree) equal 0. Proposition 4.1 implies that $\omega_\tau^{\log}$ extends to a complex-valued, real analytic, exact form on all boundary strata of $\overline{C}_{2,0}^+$, except for the boundary stratum $\partial_{\{1,2\}}\overline{C}_{2,0}^+$, along which it has a simple pole of order 1.

Let us now fix two non-negative integers (n,m), with which we associate the set $\mathcal{G}_{n,m}$ of admissible graphs of type (n,m): recalling the beginning of Subsection 5.2, we use the short-hand notation $\mathcal{G}_{n,m}=\mathcal{G}_{[n],[m]}$, where $[n]=\{1,\dots,n\}$ and $[m]$ is endowed with the natural ordering.

We consider an element Γ of $\mathcal{G}_{n,m}$, and we assume that $|E(\Gamma)|=2n+m-2$. We associate with Γ a natural morphism

$$\mathcal{U}_\Gamma^{\log}:\wedge^n(\mathfrak{g}_1)\to\mathfrak{g}_2[1-n]$$

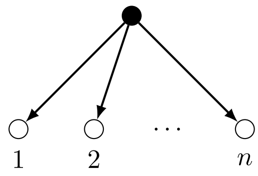

FIGURE 2. The only admissible graph in $\mathcal{G}_{1,n}$; filled circles denote vertices of the first type, while circles denote vertices of the second type.

by setting

$$\left(\mathcal{U}_\Gamma^{\log}(\gamma_1,\dots,\gamma_n)\right)(a_1\otimes\cdots\otimes a_m)=\mu_{m+n}\left(\int_{C_{n,m}^+}\omega_{\tau,\Gamma}^{\log}(\gamma_1\otimes\cdots\otimes\gamma_n\otimes a_1\otimes\cdots\otimes a_m)\right),$$

where

$$\omega_{\tau,\Gamma}^{\log}=\prod_{e\in E(\Gamma)}\omega_{\tau,e}^{\log},\qquad \omega_{\tau,e}^{\log}=\pi_e^*(\omega_{\log})\otimes\tau_e,$$

τ_e being the graded endomorphism of $T_{\mathrm{poly}}(X)^{\otimes(m+n)}$ which acts as τ on the two factors of $T_{\mathrm{poly}}(X)$ corresponding to the initial and final point of the edge e, and μ_{m+n} denotes the multiplication map from $T_{\mathrm{poly}}(X)^{m+n}$ to $T_{\mathrm{poly}}(X)$, followed by the natural projection from $T_{\mathrm{poly}}(X)$ onto A. The integration operation acts obviously on the form-part of $\omega_{\tau,\Gamma}^{\log}$.

The operator-valued form $\omega_{\tau,\Gamma}^{\log}$ does not depend on the ordering on $E(\Gamma)$, because each factor has total degree 0.

More importantly, we observe that, in view of Proposition 5.2, if Γ has the property that $|E(\Gamma)|=2n+m-2$, the operator-valued form $\omega_{\tau,\Gamma}^{\log}$ is a form of top-degree on $C_{n,m}^+$, hence extends to an operator-valued, real analytic form of top degree on $\overline{C}_{n,m}^+$: this implies that the operator $\mathcal{U}_\Gamma^{\log}$ is defined by a convergent integral, hence exists.

Finally, we set

$$\mathcal{U}_n^{\log}=\sum_{m\geq 0}\sum_{\Gamma\in\mathcal{G}_{n,m}}\mathcal{U}_\Gamma^{\log},\ n\geq 1. \tag{9}$$

By the previous arguments, Formula (9) yields a well-defined pre-L_∞-morphism $\mathcal{U}^{\log}$ from $T_{\mathrm{poly}}(X)$ to $D_{\mathrm{poly}}(X)$.

6.4. **The HKR quasi-isomorphism revisited.** We consider the first Taylor component $\mathcal{U}_1^{\log}$ of the pre-L_∞-morphism (9): by its very definition,

$$\mathcal{U}_1^{\log}=\sum_{m\geq 0}\sum_{\Gamma\in\mathcal{G}_{1,m}}\mathcal{U}_\Gamma^{\log},$$

and has degree 0.

For $m=0$, the only graph in $\mathcal{G}_{1,0}$ has a single vertex of the first type and no edge: with it, we associate the trivial operator. Let us consider $m\geq 1$: then, there is only one admissible graph Γ in $\mathcal{G}_{1,m}$, which is depicted in Figure 2.

We immediately see that $\mathcal{U}_\Gamma^{\log}$ is A-linear on $T_{\mathrm{poly}}(X)$, hence it suffices to evaluate it on a polyvector of the form $\partial_{i_1}\cdots\partial_{i_m}=\theta_{i_1}\cdots\theta_{i_m}$, for $1\leq i_1<\cdots<i_m\leq m$.

It suffices now to observe from Proposition 4.1 that each operator-valued 1-form $\omega_{\tau,e_i}^{\log}$, $i=1,\dots,m$, extends to $\overline{C}_{1,m}^+$; moreover, the form part of $\omega_{\tau,e_i}^{\log}$ equals the form part of ω_{τ,e_i}, where the logarithmic propagator is replaced by the standard one introduced in [12, Subsection 6.2].

Therefore, the very same computations in [12, Subsubsection 6.4.3] apply to the explicit computation of $\mathcal{U}_1^{\log}(\gamma)=\mathcal{U}_\Gamma^{\log}(\gamma)$, for $\gamma=\partial_{i_1}\cdots\partial_{i_m}$, in particular

$$\mathcal{U}_1^{\log}(\gamma)=\mathcal{U}_1(\gamma)=\mathrm{HKR}(\gamma).$$

The first Taylor component of the pre-L_∞-morphism (9) equals the HKR quasi-isomorphism (8): thus, if we may prove that (9) has the L_∞-property, it will be automatically an L_∞-quasi-isomorphism.

6.5. **The L_∞-property for $\mathcal{U}^{\log}$.** We now want to prove that the well-defined pre-L_∞-morphism is actually an L_∞-morphism from $T_{\mathrm{poly}}(X)$ to $D_{\mathrm{poly}}(X)$. The L_∞-property for $\mathcal{U}^{\log}$ form Subsection 2.1 simplifies considerably, because the L_∞-structures on $T_{\mathrm{poly}}(X)$ and $D_{\mathrm{poly}}(X)$ are particularly simple: actually, aside from the change of propagator, $\mathcal{U}^{\log}$ is constructed as $\mathcal{U}$ in [12, Subsection 6.3] and [5, Appendix A], hence we may repeat *verbatim* all computations at the beginning of the proof of the Theorem in [12, Subsection 6.4] until the point where the validity of the L_∞-relations for $\mathcal{U}^{\log}$ is re-written in terms of suitable quadratic relations for the integral weights.

Observe that we have used a slightly different construction for $\mathcal{U}_\Gamma$ than in [12, Subsection 6.3]: in fact, we may re-write

$$\left(\mathcal{U}^{\log}_\Gamma)(\gamma_1,\dots,\gamma_n)\right)(a_1\otimes\cdots\otimes a_m) = \pm\varpi^{\log}_\Gamma\,(\mathcal{U}_\Gamma)(\gamma_1,\dots,\gamma_n))\,(a_1\otimes\cdots\otimes a_m),$$

where $\varpi^{\log}_\Gamma$ is as in (7), and $\mathcal{U}_\Gamma$ denotes the multi-differential operator from [12, Subsection 6.3].

Therefore, to prove the L_∞-relations for $\mathcal{U}^{\log}$, we have to prove that the quadratic relations between the standard integral weights ϖ_Γ from [12, subsection 6.2] holds true for the logarithmic integral weights $\varpi^{\log}_\Gamma$. The strategy of the proof is the same, but instead of the standard Stokes Theorem we use Theorem 3.1 for manifolds with corners and forms with poles of order 1.

First of all, let us consider an admissible weight Γ of type (n,m) such that $|E(\Gamma)| = 2n+m-3$. The first important observation is that $\omega^{\log}_\Gamma$ is closed by construction, whence $d\omega^{\log}_\Gamma = 0$ extends to an element of $\Omega^{2n+m-2}(\overline{C}^+_{n,m})$.

Proposition 5.3 implies that $\omega^{\log}_\Gamma$, for Γ in $\mathcal{G}_{n,m}$ with $E(\Gamma) = 2n+m-3$, belongs to $\Omega^{2n+m-3}_1(\overline{C}^+_{n,m})$; furthermore, the final part of the proof of Proposition 5.3 implies that $\omega^{\log}_\Gamma$ satisfies Property *ii)* in Theorem 3.1.

It thus remains to compute the regularization of $\omega^{\log}_\Gamma$ along any boundary stratum of $\overline{C}^+_{n,m}$ of codimension 1.

We recall that the boundary strata of codimension 1 of $\overline{C}^+_{n,m}$ are of two types according to Subsubsection 3.3.2, namely

i) there is a subset A of $[n]$ of cardinality $2\le |A|\le n$, such that the corresponding stratum is isomorphic to

$$\partial_A\overline{C}^+_{n,m} \cong \overline{C}_A\times \overline{C}^+_{([n]\smallsetminus A)\sqcup\{\bullet\},m},$$

ii) or there are a subset A of $[n]$ of cardinality $0\le |A|\le n$ and a subset B of $[m]$ consisting of consecutive points and of cardinality $0\le |B|\le m$, such that the corresponding stratum is isomorphic to

$$\partial_{A\sqcup B}\overline{C}^+_{n,m} \cong \overline{C}^+_{A,B}\times \overline{C}^+_{([n]\smallsetminus A),([m]\smallsetminus B)\sqcup\{\bullet\}}.$$

For Γ, A and B as above, we denote by Γ_A or $\Gamma_{A,B}$ the subgraph of Γ, whose vertices are labeled by A or $A\sqcup B$ and whose edges have both endpoints in A of $A\sqcup B$. Similarly, we denote by Γ/Γ_A or $\Gamma/\Gamma_{A,B}$ the corresponding quotient graphs, *i.e.* the graphs obtained from Γ by contracting the subgraph Γ_A or $\Gamma_{A,B}$ to a vertex of the first or second type respectively.

Lemma 6.3. *For Γ in $\mathcal{G}_{n,m}$ with $|E(\Gamma)| = 2n+m-3$ and a subset A of $[n]$ of cardinality 2, we obtain*

$$\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^t_\Gamma) = \mathsf{vol}_{S^1}\ \omega^{\log}_{\Gamma/\Gamma_A}, \tag{10}$$

if $|E(\Gamma_A)| = 1$. Otherwise, $\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^t_\Gamma)$ vanishes. (Here, vol_{S_1} denotes the normalized volume form of $S^1 = \overline{C}_2$.)

Observe that, for Γ_A with $|E(\Gamma_A)| = 2$, $\omega^{\log}_{\Gamma/\Gamma_A}$ is a form of top degree on $C^+_{n-1,m}$: Proposition 5.2 implies that $\omega^{\log}_{\Gamma/\Gamma_A}$ extends to an element of $\Omega^{2n+m-4}(\overline{C}^+_{n-1,m})$.

Proof. As Γ_A is a subgraph of Γ with only two vertices, it must have 0, 1 or 2 edges.

First assume that Γ_A has no edges, whence $\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma)$ equals simply the restriction of $\omega^{\log}_\Gamma$ to $C_2\times C^+_{n-1,m}$ to the stratum $\{\rho_A = 0\}$. The computations in Proposition 5.2 imply that the angle coordinate φ_A is always annihilated by the restriction, as it always appears multiplied by ρ_A: in particular, $\omega^{\log}_{\Gamma/\Gamma_A}$ is a form of degree $2n+m-3$ over $C^+_{n-1,m}$, hence vanishes for degree reasons.

If Γ_A has two edges, it must be a 2-cycle graph. Let us choose coordinates near the stratum of the form $z_{a_1} = z_A$, $z_{a_2} = z_A + w_A$, $w_A = \rho_A e^{i\varphi_A}$: the computations in the proof of Proposition 5.2 imply that the regularization morphism on $\omega^{\log}_\Gamma$ acts simply as the restriction morphism on $\omega^{\log}_{\Gamma/\Gamma_A}$, while it acts in a non-trivial way on $\omega^{\log}_{\Gamma_A}$. Let us denote by $\{e_1,e_2\}$ the edges of Γ_A, then

$$\omega^{\log}_{e_1} = \frac{1}{2\pi i}\frac{d\rho_A}{\rho_A} + \frac{d\varphi_A}{2\pi} + \cdots,\ \omega^t_{e_2} = \frac{1}{2\pi i}\frac{d\rho_A}{\rho_A} + \frac{d\varphi_A}{2\pi} + \cdots,$$

where $\cdots$ denotes complex-valued, real analytic 1-forms, which are equal when restricted to $\{\rho_A = 0\}$. Thus, the regularization morphism on $\omega^{\log}_{\Gamma_A}$ yields the square of a 1-form, hence vanishes.

Finally, let us consider $\Gamma_A = e$; we use the same coordinate and notation as in the previous computations. In this case, the regularization morphism acts as restriction to $\{\rho_A = 0\}$ on $\omega^{\log}_{\Gamma/\Gamma_A}$ and in a non-trivial way on $\omega^{\log}_{\Gamma_A}$ again: in this case, we get

$$\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma) = \mathrm{Reg}_{\{\rho_A=0\}}(\omega^{\log}_{\Gamma_A})\ \omega^{\log}_{\Gamma/\Gamma_A}\big|_{\{\rho_A=0\}} = \left(\frac{d\varphi_A}{2\pi}+\cdots\right)\ \omega^{\log}_{\Gamma/\Gamma_A}\big|_{\{\rho_A=0\}} = \mathsf{vol}_{S^1}\ \omega^{\log}_{\Gamma/\Gamma_A}.$$

In the middle expression, $\cdots$ denotes a complex-valued, real analytic 1-form on $C^+_{n-1,m}$, and in the last summand, by abuse of notation, we have denoted by $\omega^{\log}_{\Gamma/\Gamma_A}$ the restriction to $\{\rho_A = 0\}$ of the form. Observe that $\omega^{\log}_{\Gamma/\Gamma_A}$ has top degree on $\{\rho_A = 0\} = C^+_{n-1,m}$, thus the product of $\omega^{\log}_{\Gamma/\Gamma_A}$ with a complex-valued, real analytic 1-form on $C^+_{n-1,m}$ vanishes by degree reasons. It is also clear that $d\varphi_A/2\pi$ is the normalized volume form of S^1. □

We further consider the case of a boundary stratum $\partial_A\overline{C}^+_{n,m}$ labeled by a subset A of $[n]$ with $|A| \geq 3$: this case requires a bit more care.

First of all, let us consider local coordinates for $\overline{C}^+_{n,m}$ near the stratum $\partial_A\overline{C}^+_{n,m} \cong C_A \times C^+_{([n]\smallsetminus A)\sqcup\{\bullet\},m}$: we choose the section of the G_3-bundle Conf_A over C_A which identifies C_A with the trivial S^1-bundle $S^1 \times \mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$. Writing $A = \{a_1,\dots,a_p\}$, $3 \leq |A| = p \leq n$, we may write $z_a = z_\bullet + w_A y_a$, where $w_A = \rho_A e^{i\varphi_A}$ and $y_a = 0$ for $a = a_1$, $y_a = 1$ for $a = a_2$, and the remaining y_a are complex coordinates for $\mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$; $z_\bullet$ is a complex coordinate for a point in $\mathbb{H}^+$, and ρ_A is small enough.

Lemma 6.4. *For Γ in $\mathcal{G}_{n,m}$ and $|E(\Gamma)| = 2n+m-3$ and a subset A of $[n]$ of cardinality $|A| \geq 3$, we obtain*

$$\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma) = 0. \tag{11}$$

Proof. We compute $\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma)$ with respect to coordinates near the boundary stratum as Formula (5), where we choose the section of C_A which identifies it with $S^1 \times \mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$.

Recalling previous notation and computations, we have

$$\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma) = \mathrm{Reg}_{\{\rho_A=0\}}(\omega^{\log}_{\Gamma_A})\, \omega^{\log}_{\Gamma/\Gamma_A}\big|_{\{\rho_A=0\}}.$$

Let us consider an edge $e = (s(e), t(e))$ of Γ_A: then, with respect to the coordinates of C_A as at the end of Subsection 3.2, we find near the given boundary stratum

$$\begin{aligned}\omega^t_e &= \frac{1}{2\pi i} d\log(w_A(y_{s(e)} - y_{t(e)})) - \frac{1}{2\pi i} d\log(\overline{z}_A - z_A + \overline{w}_A \overline{y}_{s(e)} - w_A y_{t(e)}) = \\ &= \frac{1}{2\pi i}\frac{d\rho_A}{\rho_A} + \mathsf{vol}_{S^1} + \frac{1}{2\pi i} d\log(y_{s(e)} - y_{t(e)}) + \cdots = \frac{1}{2\pi i}\frac{d\rho_A}{\rho_A} + \theta_e + \dots,\end{aligned}$$

where $\dots$ denotes a complex-valued, real analytic 1-form. Therefore, we obtain

$$\mathrm{Reg}_{\partial_A\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma) = \prod_{e\in E(\Gamma_A)} (\theta_e + \widetilde{\omega}^{\log}_A)\, \omega^{\log}_{\Gamma/\Gamma_A}\big|_{\{\rho_A=0\}}.$$

Let us inspect more closely the first factor on the right-hand side of the previous equality: the fact that it must be a form of top degree on $S^1 \times \mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$ forces either $|E(\Gamma_A)| = 2|A| - 3$ or $|E(\Gamma_A)| = 2|A| - 2$.

Let us consider the second case: because of degree reasons we find

$$\prod_{e\in E(\Gamma_A)} (\theta_e + \widetilde{\omega}^{\log}_A) = \widetilde{\omega}^{\log}_A \left(\sum_{e\in E(\Gamma_A)} (-1)^{e-1} \prod_{e'\neq e} \theta_{e'}\right).$$

Borrowing previous notation for the coordinates on $S^1 \times \mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$, it is easy to verify, by skew-symmetry of the product of 1-forms, that the second factor on the right-hand side of the previous identity vanishes.

We are thus left with the case $|E(\Gamma_A)| = 2|A| - 3$. In this situation, we only have to consider the form

$$\prod_{e\in E(\Gamma_A)} \theta_e$$

of top degree on $S^1 \times \mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$. Since $|A| - 2 \geq 1$, $\mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$ is a complex manifold and θ_e depends holomorphically on $\mathrm{Conf}_{|A|-2}(\mathbb{C}\smallsetminus\{0,1\})$, the above form obviously vanishes. □

Let us finally consider a boundary stratum $\partial_{A,B}\overline{C}^+_{n,m}$ labeled by a subset $A \sqcup B$ of $[n] \sqcup [m]$.

Lemma 6.5. *For Γ in $\mathcal{G}_{n,m}$ with $|E(\Gamma)| = 2n+m-3$ and a subset $A \sqcup B$ of $[n] \sqcup [m]$ as above, we obtain*

$$\mathrm{Reg}_{\partial_{A,B}\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma) = \omega^{\log}_{\Gamma_{A,B}}\, \omega^{\log}_{\Gamma/\Gamma_{A,B}}. \tag{12}$$

Observe that Identity (12) is non-trivially satisfied only if both $\Gamma_{A,B}$ and $\Gamma/\Gamma_{A,B}$ are admissible and $|E(\Gamma_{A,B})| = 2|A|+|B|-2$, $|E(\Gamma/\Gamma_{A,B})| = 2n+m-2|A|-|B|-1$: in this case, Proposition 5.2 implies that $\omega^{\log}_{\Gamma_{A,B}}$ and $\omega^{\log}_{\Gamma/\Gamma_{A,B}}$ belongs to $\Omega^{2|A|+|B|-2}(\overline{C}^+_{A,B})$ and $\Omega^{2n-2|A|+m-|B|-1}(\overline{C}^+_{[n]\smallsetminus A,([m]\smallsetminus B)\sqcup\{\bullet\}})$.

Proof. The computations in the middle of Subsection 4.2 imply immediately that for the boundary stratum $\partial_{A,B}\overline{C}^+_{n,m}$ that the regularization morphism on $\omega^{\log}_\Gamma$ along $\partial_{A,B}\overline{C}^+_{n,m}$ coincides with the restriction morphism, whence the claim follows as in [12, Subsubsection 6.4.2].

Observe that the boundary conditions of $\omega^{\log}$ imply that the restriction to the stratum of $\omega^{\log}_{\Gamma/\Gamma_{A,B}}$ vanishes, if $\Gamma/\Gamma_{A,B}$ has a "bad edge", *i.e.* an edge joining a vertex in $A\sqcup B$ to its complement in $[n]\sqcup[m]$: in particular, a non-trivial contribution from $\Gamma/\Gamma_{A,B}$ forces it to be admissible. □

By the results of Subsection 6.4, the first Taylor component $\mathcal{U}^{\log}_1$ of the pre-L_∞-morphism $\mathcal{U}^{\log}$ constructed in Subsection 6.3 identifies with the HKR quasi-isomorphism.

Furthermore, Proposition 5.2 implies that $\mathcal{U}^{\log}$ is well-defined, *i.e.* the integral weights converge. Proposition 5.3, together with Lemmata 6.3, 6.4 and 6.5, imply that we may apply Theorem 3.1 to forms of the type $\omega^{\log}_\Gamma$, for Γ admissible of type (n,m) and $E(\Gamma)=2n+m-3$: in particular, we find

$$\int_{\partial_{A,B}\overline{C}^+_{n,m}}\mathrm{Reg}_{\partial_{A,B}\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma)=\varpi^{\log}_{\Gamma_{A,B}}\varpi^{\log}_{\Gamma^{A,B}},\quad \int_{\partial_{A}\overline{C}^+_{n,m}}\mathrm{Reg}_{\partial_{A}\overline{C}^+_{n,m}}(\omega^{\log}_\Gamma)=\begin{cases}\int_{\overline{C}^+_{([n]\smallsetminus A)\sqcup\{\bullet\},m}}\omega^{\log}_{\Gamma^A}=\varpi^{\log}_{\Gamma^A}, & |A|=2\\ 0, & |A|\geq 3.\end{cases}$$

These identities yield the quadratic relations among the logarithmic integral weights of $\mathcal{U}^{\log}$, implying finally the L_∞-property.

We may summarize all these arguments into the following Theorem.

Theorem 6.6. *The well-defined logarithmic pre-L_∞-morphism $\mathcal{U}^{\log}$ satisfies the L_∞-property and its first Taylor component identifies with the HKR quasi-isomorphism: in particular $\mathcal{U}^{\log}$ is an L_∞-quasi-isomorphism.*

6.6. **Globalization results.** The present Subsection deals with the problem of extending the logarithmic formality L_∞-quasi-isomorphism $\mathcal{U}^{\log}$ to any real or complex manifold or smooth algebraic variety X from the local result of Theorem 6.6.

For this purpose, we observe that, being the logarithmic integral weights in $\mathcal{U}^{\log}$ in the local formulation for $X=\mathbb{R}^d$ complex numbers, we have to restrict ourselves to real or complex manifolds or algebraic varieties over a ground field $\mathbb{K}$ of characteristic 0, such that $\mathbb{C}\subseteq\mathbb{K}$.

In order to do this, and given the fact that $\mathcal{U}^{\log}$ is constructed exactly as $\mathcal{U}$ as in [12] by simply replacing the standard integral weights by the logarithmic ones, to perform the globalization procedure along the lines of [17, Sections 6-9] we have to prove that the local logarithmic L_∞-quasi-isomorphism $\mathcal{U}^{\log}$ satisfies certain additional properties.

For $\mathbb{K}$ a field of characteristic 0 such that $\mathbb{C}\subseteq\mathbb{K}$, we consider the algebra $F=\mathbb{K}[[x_1,\dots,x_d]]$ of formal power series in d variables over $\mathbb{K}$.

With F, we may associate the dg Lie algebras $(T_{\mathrm{poly}}(F),0,[\bullet,\bullet])$ and $(D_{\mathrm{poly}}(F),\mathrm{d_H},[\bullet,\bullet])$ of formal poly-vector fields and formal poly-differential operators on F respectively. The definition of $T_{\mathrm{poly}}(F)$ and $D_{\mathrm{poly}}(F)$ is similar to the previous definition of $T_{\mathrm{poly}}(X)$ and $D_{\mathrm{poly}}(X)$ in Subsection 6.1: *e.g.* $T_{\mathrm{poly}}(F)$ is the graded vector space (with shifted degree) of skew-symmetric polyderivations of F, viewed as a free F-module. Similarly, $D_{\mathrm{poly}}(F)$ is the graded vector space (with shifted degree) of multidifferential operators on F: it may be regarded as the continuous Hochschild cochain complex of F, F being endowed with the adic topology.

Theorem 6.7. *There is an L_∞-quasi-isomorphism*

$$\mathcal{U}^{\log}:(T_{\mathrm{poly}}(F),0,[\bullet,\bullet])\to(D_{\mathrm{poly}}(F),\mathrm{d_H},[\bullet,\bullet]),$$

enjoying the following properties:

i) *$\mathcal{U}^{\log}$ is $GL(d,\mathbb{K})$-equivariant.*

ii) *The first Taylor coefficient of $\mathcal{U}^{\log}$ coincides with the natural extension of the HKR quasi-isomorphism*

$$\mathrm{HKR}(\partial_{i_1}\cdots\partial_{i_p})=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}(-1)^\sigma\partial_{i_{\sigma(1)}}\otimes\cdots\otimes\partial_{i_{\sigma(p)}}$$

of complexes from $(T_{\mathrm{poly}}(F),0)$ to $(D_{\mathrm{poly}}(F),d_{\mathrm{H}})$.

iii) *If $n\geq 2$, and γ_i, $i=1,\dots,n$, are elements of $T^0_{\mathrm{poly}}(F)$, then*

$$\mathcal{U}^{\log}_n(\gamma_1,\dots,\gamma_n)=0.$$

iv) *If $n\geq 2$, γ_1 is a linear vector field on F (i.e. an element of $\mathfrak{gl}(d,\mathbb{K})$), γ_i, $i=2,\dots,n$ are general elements of $T_{\mathrm{poly}}(F)$, then*

$$\mathcal{U}^{\log}_n(\gamma_1,\gamma_2,\dots,\gamma_n)=0.$$

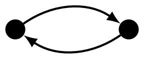

FIGURE 3. The only non-trivial admissible graph in $\mathcal{G}_{2,0}$.

Proof. It is easy to prove that $\mathcal{U}^{\log}$ extends in a natural way to an L_∞-morphism from $T_{\mathrm{poly}}(F)$ to $D_{\mathrm{poly}}(F)$: namely, the basic ingredient in the construction of $\mathcal{U}^{\log}$, besides the logarithmic integral weights, is the endomorphism τ of $T_{\mathrm{poly}}(X)^{\otimes 2}$, which obviously extends to a continuous endomorphism τ of $T_{\mathrm{poly}}(F)^{\otimes 2}$.

The $GL(d,\mathbb{K})$-equivariance follows easily from the $GL(d,\mathbb{K})$-invariance of τ, which proves Property $i)$. Property $ii)$ is obvious, recalling that $T_{\mathrm{poly}}(F)$ is a free F-module with basis $\{\partial_{i_1}\cdots\partial_{i_p}\}$, $1\leq i_1<\cdots<i_p\leq d$.

It remains to prove Properties $iii)$ and $iv)$.

The proof of Property $iii)$ follows along the same lines of the arguments in [12, Subsubsubsection 7.3.1.1]. In fact, let us consider an admissible graph Γ of type (n,m), $n\geq 2$, with which we associate

$$\left(\mathcal{U}_\Gamma^{\log}(\gamma_1,\dots,\gamma_n)\right)(a_1\otimes\cdots\otimes a_m).$$

Because of the construction of the multidifferential operator $\mathcal{U}_\Gamma^{\log}$ and of the fact that γ_i is an element of $T^0_{\mathrm{poly}}(F)$, it follows that from each vertex of the first type of Γ must depart exactly one edge; on the other hand, for the corresponding logarithmic integral weight to be well-defined, the identity $E(\Gamma)=2n+m-2$ must hold true. In summary, we have the equality $2n+m-2=n$, whence $n+m=2$. It follows that $n=2$ and $m=0$, and the only non-trivial admissible graph of type $(2,0)$ is depicted in Figure 3.

Lemma 6.8. *The logarithmic integral weight $\varpi_\Gamma^{\log}$ associated with the admissible graph Γ in Figure 3 equals* 0.

Proof. The logarithmic integral weight $\varpi_\Gamma^{\log}$ can be written explicitly as

$$\varpi_\Gamma^{\log}=\int_{\overline{C}^+_{2,0}}\omega_{\log}(z_1,z_2)\omega_{\log}(z_2,z_1)=\int_{\mathbb{H}^+\smallsetminus\{i\}}\omega_{\log}(i,z)\omega_{\log}(z,i),$$

where we have denoted loosely by $\mathbb{H}^+\smallsetminus\{i\}$ the compactification of it in the form of the Eye, see Subsection 4.1.

We now compute the integral explicitly. Using the complex coordinate z for $\mathbb{H}^+\smallsetminus\{i\}$, it is not difficult to prove that the rightmost integral may be re-written more explicitly as

$$\varpi_\Gamma^{\log}=\frac{2i}{(2\pi i)^2}\int_{\mathbb{H}^+\smallsetminus\{i\}}d\left(\frac{dz}{z^2+1}\log(i-\overline{z})\right),$$

where we have made a choice of complex logarithm log yielding a smooth, well-defined function on $\mathbb{H}^+$; here, we have chosen the branch of the complex logarithm for which $\log(i)=i\pi/2$.

We observe that the integrand in the expression on the right-hand side is complex-valued, real analytic on $\mathbb{H}^+\smallsetminus\{i\}$ and has a complex, simple pole of order 1 at i: it satisfies the assumptions of Theorem 3.1, whence

$$\varpi_\Gamma^{\log}\propto\int_{\partial(\mathbb{H}^+\smallsetminus\{i\})}\mathrm{Reg}_{\partial(\mathbb{H}^+\smallsetminus\{i\})}\left(\frac{dz}{z^2+1}\log(i-\overline{z})\right)=\int_{\mathbb{R}}\frac{\log(i-x)}{x^2+1}dx-\pi\log(2i).$$

The first observation is that the integrand extends continuously to the real axis $\mathbb{R}=\overline{C}^+_{1,1}$ and to the "half-circle" at infinity $\overline{C}^+_{1,1}$ in $\mathbb{H}^+$, hence the regularization along these two boundary strata of codimension 1 equals the corresponding restriction: it is not difficult to prove by direct computation that the restriction of the integrand to the real axis equals the first integrand in the rightmost expression in the above chain of equalities, while its restriction to the "half-circle" at infinity vanishes.

It remains to consider the regularization along the boundary stratum $\overline{C}_2$ corresponding to the approach of z to i: once again, we use polar coordinates $z=i+\rho e^{i\varphi}$, where ρ approaches 0 and φ in $[0,2\pi)$. Therefore, we find

$$\frac{dz}{z^2+1}\log(i-\overline{z})=\frac{dz}{z-i}\frac{\log(i-\overline{z})}{z+i}=\left(\frac{d\rho}{\rho}+id\varphi\right)\frac{\log(2i-\rho e^{-i\varphi})}{2i+\rho e^{i\varphi}},$$

whence, by direct computation,

$$\mathrm{Reg}_{\partial(\mathbb{H}^+\smallsetminus\{i\})}\left(\frac{dz}{z^2+1}\log(i-\overline{z})\right)=\frac{1}{2}\log(2i)d\varphi.$$

The corresponding boundary stratum is isomorphic to a copy of S^1, and integration thereof yields the desired result.

It remains to compute the integral

$$\int_{\mathbb{R}}\frac{\log(i-x)}{x^2+1}dx=\frac{1}{2}\int_{\mathbb{R}}\frac{\ln(x^2+1)}{x^2+1}dx+i\int_{\mathbb{R}}\frac{\arg(i-x)}{x^2+1}dx,\tag{13}$$

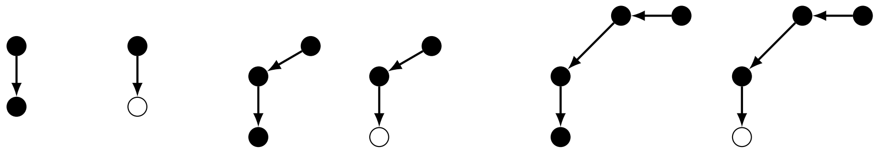

FIGURE 4. *i)* The 4 possible configuration of Γ at the vertex associated with γ_1; *ii)* the two graphs $\Gamma_{\bullet\bullet\bullet\bullet}$ used to in the proof of Lemma 6.9.

where by $\arg(\bullet)$ we have denoted the smooth, well-defined argument function from $\mathbb{H}^+$ to $(0,\pi)$, such that $\arg(i) = \pi/2$. The second integral on the right-hand side of (13) is easily computed and yields $i\pi^2/2$.

Let us consider the integral function

$$f(t) = \int_{\mathbb{R}} \frac{\ln(tx^2+1)}{x^2+1} dx, \ t \geq 0.$$

Standard calculus shows that $f(t)$ exists for $t \geq 0$, and that it is a differentiable function on $t > 0$: its derivative may be computed by exchanging the derivative with respect to t with the integral, whence

$$\frac{df}{dt} = \int_{\mathbb{R}} \frac{x^2}{(tx^2+1)(x^2+1)} dx = \frac{\pi}{\sqrt{t}(\sqrt{t}+1)}, \ t > 0,$$

where the last identity is a consequence of Theorem 3.1. Namely, Theorem 3.1 yields in this situation the Residue Theorem, whence the result. Thus, we have

$$f(t) = 2\pi \ln(\sqrt{t}+1) + f(0) = 2\pi \ln(\sqrt{t}+1),$$

whence

$$\frac{1}{2}\int_{\mathbb{R}} \frac{\ln(x^2+1)}{x^2+1} dx = \frac{1}{2} f(1) = \pi \ln(2),$$

and finally the right-hand side of (13) equals $\pi\ln(2) + i\pi^2/2 = \pi\log(2i)$, and the claim follows. $\square$

As for Property *iv)*, let us consider as before an admissible graph Γ of type (n,m), with $n \geq 2$, to which we associate

$$\left(\mathcal{U}_\Gamma^{\log}(\gamma_1,\dots,\gamma_n)\right)(a_1\otimes\cdots\otimes a_m), \ \gamma_1 \in \mathfrak{gl}(d,\mathbb{K}).$$

Because of the construction of the multidifferential operator $\mathcal{U}_\Gamma^{\log}$ and that γ_1 is a linear vector field on $\mathbb{K}^d$, it follows that the vertex of the first type, with which we associate γ_1, must have exactly one departing edge and at most one arriving edge. The four possible configurations of Γ at the vertex of the first type corresponding to γ_1 are depicted in Figure 4, *i)*.

Lemma 6.9. *For an admissible graph Γ of type (n,m) with $n \geq 2$ and $E(\Gamma) = 2n+m-2$ admitting a vertex as in Figure 4, the corresponding logarithmic integral weight $\varpi_\Gamma^{\log}$ vanishes.*

Proof. (In the case of the standard argument propagator from [12], the main argument is due to the fourth author of the present paper: we have simply adapted the arguments to the case of the logarithmic propagator.)

We consider Γ as in the first and third configuration in Figure 4, *i)*. The proof of the claim for the other two configurations is similar.

The fact that $n \geq 2$ permits *e.g.* to use the global section of $C_{n,m}^+$ from Subsubsection 3.2.1, *iv)*, to fix a vertex of the first type different from the one associated with γ_1 to i.

If the configuration of Γ at the vertex of the first type associated with γ_1 is as the first picture in Figure 4, *i)*, since $n \geq 2$, we may *e.g.* consider the global section of $\mathrm{Conf}_{n,m}^+$ over $C_{n,m}^+$ which fixes to i a vertex of the first type of Γ different from the one corresponding to γ_1: this vertex carries integration over a 2-dimensional domain, while there is only one 1-form which is integrated thereover. Hence, the corresponding logarithmic integral weight vanishes.

Let us consider the subgraph $\Gamma_{\bullet\bullet\bullet}$ of Γ in the third picture in Figure 4, *ii)*: by means of Fubini's Theorem, we may isolate in the corresponding logarithmic weight the integral

$$\int_{\mathbb{H}^+\smallsetminus\{z_1,z_2\}} \omega_{\log}(z_1,z)\omega_{\log}(z,z_2), \ z_1 \neq z_2 \in \mathbb{H}^+.$$

More precisely, we consider the natural projection from $C^+_{3,0}$ onto $C^+_{2,0}$, $1 \le i < j \le 3$ which forgets the middle component. For x a general point of $C^+_{2,0}$, the previous integral defines a function on $C^+_{2,0}$, whose evaluation at x is

$$\int_{\mathbb{H}^+ \smallsetminus \{z_1, z_2\}} \omega_{\log}(z_1, z)\omega_{\log}(z, z_2) = \int_{C^+_{3,0}} \omega^{\log}_{\Gamma_{\bullet\bullet\bullet}},$$

where $[(z_1, z_2)] = x$ in $C^+_{2,0}$. Here, we have denoted by $C^+_{3,0}(x)$ the fiber over x of the said projection.

Let us then consider the first graph $\Gamma_{\bullet\bullet\bullet\bullet}$ in Figure 4, *ii*): borrowing previous notation, we associate with it the closed 3-form $\omega^{\log}_{\Gamma_{\bullet\bullet\bullet}}$ on $C^+_{4,0}$. We further consider the projection from $C^+_{4,0}$ to $C^+_{2,0}$ induced by $(z_1, z_2, z_3, z_4) \to (z_2, z_3)$ and the corresponding integration along the fiber; observe that the fiber of the said projection has dimension 4. We borrow previous notation for $C^+_{4,0}(x)$, for x a general point in $C^+_{2,0}$: the arguments of Subsection 3.3 can be translated almost *verbatim* to prove that $C^+_{4,0}(x)$ admits a compactification $\overline{C}^+_{4,0}(x)$, which is a manifold with corners in the sense of Subsection 4.1. The boundary stratification of $\overline{C}^+_{4,0}(x)$ can be read immediately from the boundary stratification of $\overline{C}^+_{4,0}$.

By slightly adapting the arguments of the proof of Proposition 5.3, it is not difficult to prove that, for a general x in $C^+_{2,0}$, $\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}$ belongs to $\Omega^3_1(\overline{C}^+_{4,0}(x))$. Moreover, the very same arguments together with the fact that $\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}$ is closed imply that we may safely apply Theorem 3.1 in order to compute explicitly the integral

$$0 = \int_{\overline{C}^+_{4,0}(x)} d\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}} = \int_{\partial \overline{C}^+_{4,0}(x)} \mathrm{Reg}_{\partial \overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}).$$

The boundary strata of codimension 1 of $\overline{C}^+_{4,0}(x)$ can be read directly from the boundary strata of codimension 1 of $\overline{C}^+_{4,0}$. Namely, since x belongs to $C^+_{2,0}$, either *i*) there is a subset $C = A$ of $[4]$, $2 \le |A| \le 3$, such that either $i \in A$, for a unique $i = 1, 4$, or $[2] \cap A = \varnothing$, or *ii*) there is a subset $C = A \sqcup \varnothing$, $0 \le |A| \le 4$, such that either $\{1,4\} \subseteq A$ or $\{1,4\} \cap A = \varnothing$, which labels the corresponding boundary stratum: we further have the identifications

$$\partial_C \overline{C}^+_{4,0}(x) \cong C_A \times C^+_{([4]\smallsetminus A)\sqcup\{\bullet\}),0}(x),\ C = A,\ 2 \le |A| \le 3,$$

in case *i*), and

$$\partial_C \overline{C}^+_{4,0}(x) \cong \begin{cases} C^+_{[4]\smallsetminus A,1} \times C^+_{A,0}(x), & \{1,4\} \subseteq A, \\ C^+_{A,0} \times C_{([4]\smallsetminus A),1}(x)^+, & \{1,4\} \cap A = \varnothing, \end{cases}$$

where notation is borrowed from above.

The computation of the regular part of $\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}$ along the previous boundary strata of codimension 1 can be copied almost *verbatim* from the computations in the proofs of Lemmata 6.3, 6.4 and 6.5.

For the case *i*), if $C = A$ obeys $\{1,4\} \cap A = \varnothing$, it follows immediately that $A = \{2,3\}$: Lemma 6.3 implies

$$\mathrm{Reg}_{\partial_C \overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}) = \mathsf{vol}_{S^1}\ \omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_A},$$

where $\Gamma_A = \Gamma_{\bullet\bullet}$. Observe that the quotient graph $\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}$ equals $\Gamma_{\bullet\bullet\bullet}$, the third graph in Figure 4, *i*), and a slight modification of the arguments of the proof of Proposition 5.2 imply that $\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}}$ belongs to $\Omega^2(\overline{C}^+_{3,0}(x))$.

If $1 \in A$, then $|A| = 2$ or $|A| = 3$. In the third case, $A = \{1,2,3\}$, and the arguments in the proof of Lemma 6.4 imply that

$$\mathrm{Reg}_{\partial_C \overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}) = 0.$$

If $|A| = 2$, then either $A = \{1,2\}$ or $A = \{1,3\}$ and again the computations of the proof of Lemma 6.3 imply

$$\mathrm{Reg}_{\partial_C \overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}) = \begin{cases} \mathsf{vol}_{S^1}\ \omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}}, & A = \{1,2\} \\ 0, & A = \{1,3\}. \end{cases}$$

Similarly, if $4 \in A$, we find

$$\mathrm{Reg}_{\partial_C \overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}) = \begin{cases} \mathsf{vol}_{S^1}\ \omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}}, & A = \{3,4\} \\ 0, & A = \{2,4\} \text{ or } A = \{2,3,4\}. \end{cases}$$

If $C = A \sqcup \varnothing$, and A is such that either $\{1,4\} \subseteq A$ or $\{1,4\} \cap A = \varnothing$, we may apply the arguments of the proof of Lemma 6.5 to compute the regular part of $\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}$: we observe that in all cases, there is a "bad edge", *i.e.* an edge from a vertex labeled by A to a vertex labeled by its complement in $[4]$. The boundary conditions for $\omega^{\log}$ from Subsection 4.2 imply therefore

$$\mathrm{Reg}_{\partial_C \overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}}) = 0,\ C = A \sqcup \varnothing.$$

Summarizing the previous computations, we find

$$0=\int_{\partial\overline{C}^+_{4,0}(x)}\mathrm{Reg}_{\partial\overline{C}^+_{4,0}(x)}(\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}})=\pm\int_{C_{3,0}(x)}\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}}\pm\int_{C_{3,0}(x)}\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}}\pm\int_{C_{3,0}(x)}\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}},$$

where the sum is only over the subsets $A=\{1,2\}$, $A=\{2,3\}$ and $A=\{3,4\}$, which implies that the integral

$$\int_{C^+_{3,0}(x)}\omega^{\log}_{\Gamma_{\bullet\bullet\bullet\bullet}/\Gamma_{\bullet\bullet}}=\int_{C^+_{3,0}(x)}\omega^{\log}_{\Gamma_{\bullet\bullet\bullet}}$$

vanishes, and the claim follows. □

All previous arguments, together with Lemmata 6.8 and 6.9, imply the main statement, hence the globalization techniques from [17, Sections 6, 8, 9] apply also to the L_∞-quasi-isomorphism $\mathcal{U}^{\log}$. □

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