A STOKES THEOREM IN PRESENCE OF POLES AND LOGARITHMIC SINGULARITIES

A. ALEKSEEV, J. LÖFFLER, C. A. ROSSI, AND C. TOROSSIAN

Abstract. The aim of this paper is to present a version of Stokes Theorem on compact orientable manifolds with corners which applies to differential forms admitting poles of order 1 and logarithmic singularities. A central notion in our study is the regularization morphism along the boundary strata inspired by the work of F. Brown [3, Section 4]. As a direct application of the Stokes Theorem, we re-prove the Kontsevich Vanishing Lemma which is one of the central pieces in the construction of the L_∞-quasi-isomorphism in [10, Section 6].

1. Introduction

The Stokes Theorem states that for a compact manifold with boundary M and for a smooth differential form ω of degree $\dim M - 1$ one has an equality of integrals

$$\int_M d\omega = \int_{\partial M} \omega.$$

This statement is one of the cornerstones of Analysis, and extending it to new situations is of importance for both advancing our conceptual understanding of the subject and for various applications.

In this paper, we will present several versions of the Stokes Theorem which apply to manifolds with corners and to differential forms which admit poles of order 1 along the boundary strata and whose coefficients might have logarithmic singularities. As a first example, consider the Fundamental Theorem of Calculus which represents the Stokes Theorem in dimension one:

$$\int_a^b \frac{df(x)}{dx}\, dx = f(b) - f(a).$$

Let us apply this formula to the function $f(x) = x\log(x)$ on the segment $[0,1]$, where (a bit improperly) $\log(\bullet)$ denotes the natural logarithm, *i.e.* the inverse function of the exponential function $\exp(\bullet)$ on $\mathbb{R}$. The function $df(x)/dx = \log(x) + 1$ is integrable on this segment, so the left-hand side is well defined. Strictly speaking, the value $f(0)$ on the right-hand side is not well defined. Instead, we should replace it with the limit $\lim_{x\to 0} x\log(x)$ which is actually equal to zero. In the paper, we will develop a "regularization" procedure which automatically takes care of the limits (for a certain class of integrands). For instance, it turns out that in the example above we should consider $f(x)$ as a polynomial in two variables, x and $\log(x)$, and put both of them equal zero at the end point $x = 0$. Of course, this reproduces the result obtained by computing the improper integral.

The true motivation for this paper comes from the Kontsevich graphical calculus [10]. The graphical calculus applies to certain directed graphs with vertices of two types. To such a graph Γ, one associates an integral weight ϖ_Γ *via*

$$\varpi_\Gamma = \int_{C^+_{n,m}} \omega_\Gamma,\ \omega_\Gamma = \prod_{e\in E(\Gamma)} \omega_e,$$

where $E(\Gamma)$ is the set of edges of Γ, $C^+_{n,m}$ is the configuration space of n points in the complex upper half-plane $\mathbb{H}^+$ and m ordered points on $\mathbb{R}$ *modulo* rescalings and real translations, and ω is the differential form on $C^+_{2,0}$ called the *propagator* (the name comes from the analogy with the Feynman graphical calculus in Quantum Field Theory). With every directed edge e of Γ is associated a natural projection π_e from $C^+_{n,m}$ onto $C^+_{2,0}$ or $C^+_{1,1}$, and ω_e denotes the pull-back of ω with respect to π_e. The standard choice for the form ω is given by

$$\omega(z_1,z_2) = \frac{1}{2\pi} d\arg\left(\frac{z_1 - z_2}{\overline{z}_1 - z_2}\right).$$

In [9, Subsection 4.1, F)], Kontsevich proposed the *logarithmic* propagator

$$\omega^{\log}(z_1,z_2) = \frac{1}{2\pi i} d\log\left(\frac{z_1 - z_2}{\overline{z}_1 - z_2}\right).$$

Using this differential form as a building block in the graphical calculus leads to several technical problems. First of all, the convergence of integral weights ϖ^{log}_Γ is not immediate because the 1-form $\omega^{\log}$ does not extend to the compactified configuration space $\overline{C}^+_{2,0}$. Instead, it admits first order poles along the boundary. Moreover, one needs a well-suited version of Stokes Theorem on compact, oriented manifolds with corners (such as the configuration spaces in question) which allows for differential forms with poles of order 1 along the boundary.

The main goal of the paper is to provide a version of the Stokes Theorem suitable for applications. We will apply our Stokes formula to construction of the L_∞-quasi-isomorphism $\mathcal{U}^{\log}$ in [1]. Further applications will be found in forthcoming works on singular propagators, Drinfel'd associators and the Grothendieck–Teichmüller group and Lie algebra [11], and on Tsygan's Conjecture in the presence of the logarithmic propagator and, more generally, in the presence of singular propagators.

The structure of the paper is as follows: in Section 2, we introduce the notion of regularization and prove three versions of the Stokes Theorem applicable to manifolds with corners and to differential forms with poles of order 1 and logarithmic singularities. In Section 3, we illustrate these results by giving a new proof of the Kontsevich Vanishing Lemma in graphical calculus.

We make use of computations from [7], which were communicated to the third author at the final stages of the present work.

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2. Real analytic manifolds with corners

In the present section we review, following [3, Subsection 4.1], the theory of oriented, real analytic manifolds with corners: from now on, for the sake of simplicity, by a manifold with corners is meant an oriented, real analytic manifold with corners.

Definition 2.1. A (compact) manifold with corners X of dimension d is a (compact) differentiable manifold in the usual sense, whose local charts are diffeomorphic to $U_{p,q} = \mathbb{R}_+^p \times \mathbb{R}^q$, $p+q=d$, and $\mathbb{R}_+ = \{x \geq 0\}$, for some $p \geq 1$ and $p+q=d$. Thus, locally, a manifold with corners looks like $U_{p,q}$, for $p \geq 1$ and $p+q=d$.

The real analyticity of X enters into play when characterizing its transition functions: first, we consider analytic diffeomorphisms $\phi = (\phi_1, \dots, \phi_d)$ of $\mathbb{R}^d$ with positive Jacobian, which restrict to analytic diffeomorphisms of $U_{p,q}$ of the form

$$\phi_i = x_i \psi_i,\ \psi_i = 1 + x_i \widetilde{\psi}_i,\ i = 1, \dots, p,$$

for $\widetilde{\psi}_i$ real analytic. The particular form of the functions $\psi_i = 1 + x_i\widetilde{\psi}_i$ ensures that $\psi_i|_{x_i=0} = 1$, and this condition will be important for us.

There is a natural action of the symmetric group $\mathfrak{S}_p \times \mathfrak{S}_q$ on $U_{p,q}$, and we may compose maps ϕ described above with this $\mathfrak{S}_p \times \mathfrak{S}_q$-action. A self-map obtained in this way is called an analytic isomorphism of $U_{p,q}$. A manifold with corners X is locally modeled on $U_{p,q}$ with analytic isomorphisms in the above sense as transition functions.

Observe that the set $U_{p,q}$ admits a boundary stratification, *i.e.* $U_{p,q} \supseteq \partial U_{p,q} \supseteq \partial^2 U_{p,q} \supseteq \cdots$ into boundary strata of codimension 1, 2, *etc.*. In particular, the boundary of $U_{p,q}$ splits into a union of boundary strata of codimension 1 of the form

$$\partial U_{p,q} = \bigcup_{i=1}^{p} \left(\mathbb{R}_+^{i-1} \times \{0\} \times \mathbb{R}_+^{p-i} \times \mathbb{R}^q \right).$$

Here, $\mathbb{R}_+^{i-1} \times \{0\} \times \mathbb{R}_+^{p-i} \times \mathbb{R}^q$ is (up to permutations of coordinates) isomorphic to $U_{p-1,q} \subseteq \mathbb{R}^{d-1}$, which obviously admits a boundary $\partial(\partial U_{p,q}) = \partial^2 U_{p,q}$, *etc.*.

Analytic isomorphisms are easily verified to preserve the boundary stratification of $U_{p,q}$. Therefore, a manifold with corners X of dimension d admits a boundary stratification $X \supseteq \partial X \supseteq \partial^2 X \supseteq \cdots$. Finally, the orientation of X induces in a natural way orientations on the boundary strata of codimension 1.

2.1. Differential forms on manifold with corners.

Let us first consider the local model $U_{p,q}$ for a manifold with corners. We denote by $\mathcal{O}(U_{p,q})$ the algebra of complex-valued real analytic functions on $U_{p,q}$. It is easy to verify that analytic isomorphisms act on $\mathcal{O}(U_{p,q})$. Hence, one can define the algebra $\mathcal{O}(X)$ for a given manifold with corners X.

More generally, we consider the graded algebra

$$\Omega^\bullet_{\mathrm{p,log}}(U_{p,q}) = \mathcal{O}(U_{p,q}) \left[\log(x_1), \dots, \log(x_p), \frac{dx_1}{x_1}, \dots, \frac{dx_p}{x_p}, dx_1, \dots, dx_{p+q} \right]$$

of complex-valued, real analytic differential forms on $U_{p,q}$ with poles and logarithmic singularities along $\partial U_{p,q}$. We assign degree 0 to generators $\log(x_i)$ and degree 1 to dx_i and dx_i/x_i.

Lemma 2.2. *The group of analytic isomorphisms of $U_{p,q}$ acts on the graded algebra $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$.*

Proof. A general analytic isomorphism ϕ of $U_{p,q}$ can be written as a composition of a permutation in $\mathfrak{S}_p \times \mathfrak{S}_q$ with an analytic self-map of the form

$$\phi = (\phi_1, \phi_2, \dots, \phi_{p+q}),\ \phi_i = x_i \psi_i,\ \psi_i|_{x_i=0} \equiv 1,\ i = 1, \dots, p.$$

We have

$$\log(\phi_i) = \log(x_i) + \log(\psi_i),\ \frac{d\phi_i}{\phi_i} = \frac{dx_i}{x_i} + \frac{d\psi_i}{\psi_i},$$

and by previous assumptions it is clear that $\log(\psi_i)$ belongs to $\mathcal{O}(U_{p,q})$ and that $d\psi_i/\psi_i$ is a complex-valued, real analytic 1-form on $U_{p,q}$, $i = 1, \dots, p$, whence the claim follows. □

Similarly, one can define the graded subspaces $\Omega^\bullet(U_{p,q})$, $\Omega^\bullet_{\log}(U_{p,q})$, $\Omega^\bullet_1(U_{p,q})$ of complex-valued, real analytic forms, of forms with only logarithmic singularities, and of forms with poles of order 1 along $\partial U_{p,q}$ simply *via*

$$\begin{aligned}
\Omega^\bullet(U_{p,q}) &= \mathcal{O}(U_{p,q})\,[dx_1, \dots, dx_{p+q}],\\
\Omega^\bullet_{\log}(U_{p,q}) &= \mathcal{O}(U_{p,q})\,[\log(x_1), \dots, \log(x_p), dx_1, \dots, dx_{p+q}],\\
\Omega^\bullet_1(U_{p,q}) &= \mathcal{O}(U_{p,q})\,[dx_1, \dots, dx_{p+q}]\left\langle \frac{dx_1}{x_1}, \dots, \frac{dx_p}{x_p}\right\rangle,
\end{aligned}$$

where the third graded subspace is spanned by the logarithmic differentials dx_i/x_i, $i = 1, \dots, p$, over the algebra $\Omega^\bullet(U_{p,q})$.

It is clear that $\Omega^\bullet(U_{p,q}) \subseteq \Omega^\bullet_{\log}(U_{p,q}) \subseteq \Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ and $\Omega^\bullet(U_{p,q}) \subseteq \Omega^\bullet_1(U_{p,q}) \subseteq \Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$.

Lemma 2.3. *The group of analytic isomorphisms of $U_{p,q}$ acts on $\Omega^\bullet(U_{p,q})$, $\Omega^\bullet_{\log}(U_{p,q})$ and $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$.*

The proof follows along the same lines as the proof of Lemma 2.2. In particular, it follows that, for any manifold with corners X, we may consider the graded vector spaces $\Omega^\bullet(X)$, $\Omega^\bullet_{\log}(X)$ and $\Omega^\bullet_{\mathrm{p,log}}(X)$.

It will be convenient for us to introduce one more graded subspace $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(U_{p,q}) \subset \Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$, whose elements are differential forms with poles of order 1 and logarithmic singularities along $\partial U_{p,q}$ which look as follows:

$$\omega = \sum_{i=1}^p \frac{dx_i}{x_i}\omega_i + \eta,\ \omega_i \in \mathcal{O}(U_{p,q})[x_1 \log(x_1)^{r^i_1}, \dots, \log(x_i), \dots, x_p \log(x_p)^{r^i_p}, dx_1, \dots, dx_d],\ r^i_j \geq 0,\ j \in \{1, \dots, p\} \smallsetminus \{i\},$$

$$\eta \in \mathcal{O}(U_{p,q})[x_1 \log(x_1)^{s_1}, \dots, x_p \log(x_p)^{s_p}, \log(x_1)^{t_1} dx_1, \dots, \log(x_p)^{t_p} dx_p],\ s_i, t_i \geq 0,\ i = 1, \dots, p.$$

In other words, an element $\omega \in \mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ has the following properties: *i*) ω may have poles of order 1 of the form dx_i/x_i, *ii*) the coefficient ω_i corresponding to the pole dx_i/x_i has only logarithmic singularities, and, for $j \neq i$, whenever $\log(x_j)$ or any non-negative power thereof appears in ω_i, it is always accompanied by x_j or some non-negative power thereof, and *iii*) η has only logarithmic singularities and whenever $\log(x_i)$ or any non-negative power thereof appears, it is always accompanied either by x_i or a non-negative power thereof or by dx_i, $i = 1, \dots, p$.

Lemma 2.4. *The group of analytic isomorphisms of $U_{p,q}$ acts on $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$.*

Proof. Consider

$$\phi^*(x_j \log(x_j)) = \phi_j \log(\phi_j) = \psi_j x_j (\log(x_j) + \log(\psi_j)) = \psi_j (x_j \log(x_j)) + x_j \psi_j \log(\psi_j).$$

The fact that $\psi_j \equiv 1$ when $x_j = 0$ implies that the function $\psi_j \log(\psi_j)$ is real analytic. Then, the computation above together with the binomial expansion implies that $x_j^{p_j} \log(x_j)^{q_j}$, $j \neq i$, $p_j \geq 1$, $q_j \geq 0$, is mapped by an analytic isomorphism ϕ of $U_{p,q}$ to a sum of terms of the same form.

Next, we have

$$\phi^*(\log(x_i)^{t_i} dx_i) = (\log(x_i) + \log(\psi_i))^{t_i} (dx_i \psi_i + x_i d\psi_i)$$

which shows that the logarithmic terms appear in combination either with x_i or dx_i.

Finally, consider the 1-form

$$\phi^*\left(\frac{dx_i}{x_i} \log(x_i)^{r_i}\right) = \left(\frac{dx_i}{x_i} + \frac{d\psi_i}{\psi_i}\right)\Big(\log(x_i) + \log(\psi_i)\Big)^{r_i}.$$

Since $d\psi_i/\psi_i$ is real analytic, the only singular terms which appear are of the form $\log(x_i)^r dx_i$ and $\log(x_i)^r dx_i/x_i$, as required. □

As a consequence, it makes sense to define the graded vector space $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(X)$ for a general manifold with corners X.

2.2. **Regularization morphism.** We now discuss regularization morphisms for $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ in a way similar to what has been done in [3, Section 4], from which we borrow notation and some conventions.

Definition 2.5. The regularization of an element ω of $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ along the boundary stratum $D = \{x_1 = \cdots = x_l = 0\}$ of codimension l, $1 \le l \le p$, is defined *via*

$$\mathrm{Reg}_D(\omega) = \omega\left(x_i = \log(x_i) = dx_i = \frac{dx_i}{x_i} = 0,\ i = 1,\dots,l\right).$$

In other words, we first regard ω in $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ as a polynomial in the graded variables $\log(x_i)$, $i = 1,\dots,p$ (of degree 0) and dx_i/x_i, $i = 1,\dots,p$, dx_i, $i = 1,\dots,p+q$ (of degree 1), with coefficients in the ring $\mathcal{O}(U_{p,q})$: the regularization of ω along D is defined simply by setting $x_i = 0$, $i = 1,\dots,l$, and formally setting to 0 the graded variables $\log(x_i)$ and dx_i/x_i, for $i = 1,\dots,l$.

The following Lemma is a generalization of [3, Definition-Proposition 4.4].

Lemma 2.6. *For D a boundary stratum of $U_{p,q}$ of codimension l as above, the assignment $\omega \mapsto \mathrm{Reg}_D(\omega)$ is compatible with the action by analytic isomorphisms on both sides.*

Proof. We need to check that the ideal of $\Omega^\bullet_{\mathrm{p,log}}(U_{p,q})$ generated by the elements $x_i, \log(x_i), dx_i/x_i$ for $i = 1,\dots,l$ is stable under the action of analytic isomorphisms. Note that $dx_i = x_i(dx_i/x_i)$, so we do not need to to treat it as an independent generator. As in the previous calculations, assume that the action of the permutation group is trivial and compute,

$$\phi^*(x_i) = \phi_i = x_i\psi_i$$

which shows that the image of x_i is contained in the same ideal. Furthermore,

$$\phi^*\log(x_i) = \log(x_i) + \log(\psi_i) = \log(x_i) + \log(1 + x_i\widetilde{\psi}_i) = \log(x_i) + \sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{k}x_i^k\widetilde{\psi}_i^k$$

which establishes this property for generators $\log(x_i)$. Finally, for the generators dx_i/x_i we have

$$\phi^*\left(\frac{dx_i}{x_i}\right) = \frac{dx_i}{x_i} + \frac{d\psi_i}{\psi_i} = \frac{dx_i}{x_i} + \frac{dx_i\widetilde{\psi}_i + x_i d\widetilde{\psi}_i}{1 + x_i\widetilde{\psi}_i}$$

which completes the proof.

□

Observe that, for D as above, the regularization morphism along D on $\Omega^\bullet(U_{p,q})$ obviously coincides with the restriction morphism induced by the natural injection $D \hookrightarrow U_{p,q}$. Lemma 2.6 implies that, for any manifold with corners X and a given boundary stratum D of ∂X, there is a well-defined regularization morphism

$$\mathrm{Reg}_D : \Omega_{\mathrm{p,log}}(X) \to \Omega_{\mathrm{p,log}}(D).$$

As a consequence, the regularization morphism restricts to $\Omega^\bullet_{\log}(X)$, $\Omega^\bullet_1(X)$ and $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(X)$.

2.3. **Variants of Stokes' Theorem.** Now with all technical tools ready, we can state and prove several versions of Stokes' Theorem for manifolds with corners. The first one deals with elements of $\mathcal{F}_1\Omega^\bullet_{\mathrm{p,log}}(X)$.

Theorem 2.7. *Let X be a compact manifold with corners of dimension $d \ge 2$, and let $\omega \in \mathcal{F}_1\Omega^{d-1}_{\mathrm{p,log}}(X)$ satisfying the following assumptions:*

- *i) its exterior derivative $d\omega$ belongs to $\Omega^d_{\log}(X)$, and*
- *ii) the regularization morphism $\mathrm{Reg}_{\partial X}(\omega)$ of ω along the boundary strata ∂X of codimension 1 of X belongs to $\Omega^{d-1}_{\log}(\partial X)$.*

Then, the integrals of $d\omega$ over X and of $\mathrm{Reg}_{\partial X}(\omega)$ over ∂X exist and the following identity holds true:

$$\int_X d\omega = \int_{\partial X}\mathrm{Reg}_{\partial X}(\omega). \tag{1}$$

Proof. First of all, assumptions *i)* and *ii)* on ω imply that the expressions on the left- and the right-hand side of Identity (1) are well-defined: *e.g.* by the results of [3, Subsection 4.4], forms with logarithmic singularities on a manifold with corners are absolutely integrable.

We first prove Identity (1) in the local setting of $X = U_{p,q}$. Let us introduce, for $\varepsilon > 0$ small enough, the regularized integration domain $U_{p,q}(\varepsilon) = [\varepsilon,\infty)^p \times \mathbb{R}^q \subset U_{p,q}$. Let us further consider ω in $\mathcal{F}_1\Omega^{d-1}_{\mathrm{p,log}}(U_{p,q})$ and a smooth function ϕ with compact support on $U_{p,q}$.

The fact that ω belongs to $\mathcal{F}_1\Omega^{\bullet}_{\mathrm{p,log}}(U_{p,q})$ implies that

$$\omega|_{\{x_i=\varepsilon\}} = \mathrm{Reg}_{\{x_i=0\}}(\omega) + \mathcal{O}(\varepsilon)\left(\sum_{j\neq i}\frac{dx_j}{x_j}\widetilde{\omega}_j + \widetilde{\eta}\right),\ i = 1,\dots,p,$$

where $\widetilde{\omega}_j$, $j \neq i$, and $\widetilde{\eta}$ may have logarithmic singularities along $\{x_i = 0\}$. This is because the forms ω_j for $j \neq i$ and the form η always have a factor of x_i in combination with any non-vanishing power of $\log(x_i)$. By multiplying ω by ϕ smooth with compact support, the previous identity yields

$$(\phi\ \omega)|_{\{x_i=\varepsilon\}} = \phi|_{\{x_i=\varepsilon\}}\ \mathrm{Reg}_{\{x_i=0\}}(\omega) + \mathcal{O}(\varepsilon)\ \phi|_{\{x_i=\varepsilon\}}\left(\sum_{j\neq i}\frac{dx_j}{x_j}\widetilde{\omega}_j + \widetilde{\eta}\right),\ i = 1,\dots,p.$$

We now get the following chain of equalities:

$$\int_{U_{p,q}} d(\phi\ \omega) = \lim_{\varepsilon\to 0}\int_{U_{p,q}(\varepsilon)} d(\phi\ \omega) = \lim_{\varepsilon\to 0}\int_{\partial U_{p,q}(\varepsilon)}(\phi\ \omega) = \int_{\partial U_{p,q}}\mathrm{Reg}_{\partial U_{p,q}}(\phi\ \omega),$$

ϕ being a general smooth function with compact support on $U_{p,q}$. Improperly, we denote also by ω, ϕ their restrictions to the boundary.

The first equality is the definition of the improper integral of $d(\phi\ \omega)$ over $U_{p,q}$. The second equality is the standard (local) Stokes Theorem for the complex-valued, smooth, compactly supported form $\phi\ \omega$ of top degree on $U_{p,q}(\varepsilon)$.

For the third equality, choose $1 \leq i \leq p$ and consider the corresponding regularized integral contribution

$$\int_{\partial_i U_{p,q}(\varepsilon)}(\phi\ \omega) = \int_{\partial_i U_{p,q}(\varepsilon)}\left(\phi|_{\{x_i=\varepsilon\}}\ \mathrm{Reg}_i(\omega)\right) + \mathcal{O}(\varepsilon)\left(\sum_{j\neq i}\int_{\partial_i U_{p,q}(\varepsilon)}\left(\frac{dx_j}{x_j}\ \phi|_{\{x_i=\varepsilon\}}\ \widetilde{\omega}_j\right) + \int_{\partial_i U_{p,q}(\varepsilon)}\left(\phi|_{\{x_i=\varepsilon\}}\ \widetilde{\eta}\right)\right).$$

As ε approaches 0, the first term on the right-hand side exists because of the assumption that $\mathrm{Reg}_{\{x_i=0\}}(\omega)$ only has logarithmic singularities and ϕ has compact support, so that it is possible to exchange the limit with the integral. Observe that $\mathrm{Reg}_{\{x_i=0\}}(\omega)$ does not depend on ε, only $\phi|_{\{x_i=\varepsilon\}}$ does, and its limit as ε approaches 0 equals simply the restriction of ϕ to the stratum $\{x_i = 0\}$. Finally, the regularization morphism, as already observed, equals the standard restriction morphism on complex-valued, real analytic functions, hence also on smooth functions.

The integrands in the other two integrals are compactly supported, as they are multiplied by $\phi|_{\{x_i=\varepsilon\}}$, and may have poles of order 1 and logarithmic singularities. Due to the fact that $\phi|_{\{x_i=\varepsilon\}}$ has compact support, these integrals can be estimated from above by powers of integrals of the form

$$\int_{\varepsilon}^{R} x^m|\log(x)|^n dx,\ m \geq -1,\ n \geq 0,\ R \text{ finite},$$

which behave as $\varepsilon^r|\log(\varepsilon)|^s$, for $r \geq 0$ and $s \geq 0$. Hence, for ε approaching 0, the term $\mathcal{O}(\varepsilon)$ dominates the growth of the logarithmic factors and lets them tend to zero.

Observe that multiplication of a local form ω satisfying the assumptions of Theorem 2.7 by a smooth function with compact support spoils the property that $d(\phi\ \omega)$ only has logarithmic singularities. However, the previous computations imply that the improper integral of $d(\phi\ \omega)$ over $U_{p,q}$ exists and it is equal to the integral of $\mathrm{Reg}_{\partial U_{p,q}}(\phi\ \omega)$ over $\partial U_{p,q}$, which exists because the integrand has only logarithmic singularities.

The previous local result yields the global result in view of the assumption on the compactness of X, as we may choose a finite partition of unity in order to apply the previous local version of Stokes' theorem to any local chart and glue the local results together again to a global one; observe that the insertion of a finite partition of unity produces on each open subset associated with the chosen covering of X a product of a form $\mathcal{F}_1\Omega^{d-1}_{\mathrm{p,log}}(U_{p,q})$ by a smooth function with compact support, whence the above arguments apply to the local situation. □

Remark 2.8. We observe that **locally** any element of $\mathcal{F}_1\Omega^{\bullet}_{\mathrm{p,log}}(X)$ satisfying Assumptions $i)$ and $ii)$ of Theorem 2.7 can be re-written in the following form:

$$\omega = \sum_{i=1}^{p}\frac{dx_i}{x_i}\widetilde{\omega}_i + \widetilde{\eta},\ \widetilde{\omega}_i = \sum_{k_i\geq 0}\widetilde{\omega}_{i,k_i}\ \log(x_i)^{k_i},\ d\widetilde{\omega}_{i,k_i} = 0,$$

and $\widetilde{\omega}_{i,k_i}$ does not depend on x_i; finally, $\widetilde{\eta}$ has only logarithmic singularities. Furthermore, it is clear that $d\omega = d\widetilde{\eta}$; also, one proves that

$$\mathrm{Reg}_{\{x_i=0\}}(\omega) = \widetilde{\eta}|_{\{x_i=0\}},\ \forall i = 1,\dots,p.$$

These computations permit to produce a different proof of Theorem 2.7, where one eliminates from the beginning poles on both sides of Identity 1: still, we prefer to stick to the proof presented here, because it does not rely on local arguments. In fact, one checks immediately that the form $\widetilde{\eta}$ does not transform well with respect to the action of analytic isomorphisms.

The following useful version of Stokes' Theorem is an immediate corollary of Theorem 2.7.

Theorem 2.9. *Let X be a compact manifold with corners of dimension $d \geq 2$, and let $\omega \in \Omega_1^{d-1}(X)$ with two additional properties:*

i) the exterior derivative $d\omega$ belongs to $\Omega^d(X)$,
ii) the image of the regularization morphism $\mathrm{Reg}_{\partial X}(\omega)$ for boundary strata ∂X of codimension 1 belongs to $\Omega^{d-1}(\partial X)$.

Then, the integrals over X of $d\omega$ and the integral over ∂X of $\mathrm{Reg}_{\partial X}(\omega)$ exist and the following identity holds true:

$$\int_X d\omega = \int_{\partial X} \mathrm{Reg}_{\partial X}(\omega). \tag{2}$$

One more version of Stokes' Theorem has been stated and proved in a slightly different form in [3, Theorem 4.11].

Theorem 2.10. *Let X be a compact manifold with corners of dimension $d \geq 1$, and let $\omega \in \Omega_{\log}^{d-1}(X)$ such that $d\omega \in \Omega_{\log}^d(X)$.*

Then, the integrals of $d\omega$ over X and of $\mathrm{Reg}_{\partial X}(\omega)$ over ∂X exist and the following identity holds true:

$$\int_X d\omega = \int_{\partial X} \mathrm{Reg}_{\partial X}(\omega). \tag{3}$$

Proof. Note that the assumption $d\omega \in \Omega_{\log}^d(X)$ is exactly Assumption $i)$ in Theorem 2.7. Furthermore, since ω has only logarithmic singularities along ∂X, its regularization $\mathrm{Reg}_{\partial X}(\omega)$ has only logarithmic singularities as well, whence also Assumption $ii)$ in Theorem 2.7 is verified.

It remains to prove that ω as in statement belongs to $\mathcal{F}_1\Omega_{\mathrm{p,log}}^{d-1}(X)$. This can be done by computations in local charts $U_{p,q}$. Since, ω has only logarithmic singularities, it suffices to show that whenever $\log(x_i)$ appears in the expression it is accompanied either by dx_i or by some positive power of x_i.

We follow closely the proof of [3, Lemma 4.10]. Since ω is of degree $d-1$, it can be written as

$$\omega = \sum_{i=1}^{d} (-1)^{i-1} \omega_i dx_1 \cdots \widehat{dx_i} \cdots dx_d, \text{ where } \omega_i = \sum_{k \geq 0} \log(x_i)^k F_{i,k},\ i = 1, \dots, p,$$

and $F_{i,k}$ is real analytic with respect to x_i, but may admit logarithmic singularities with respect to x_j, $j \neq i$, $j = 1, \dots, p$.

On the one hand, for $i = 1, \dots, p$, $F_{i,k}$ are the function coefficients of the only summand in ω which does not contain dx_i. On the other hand, consider the form $d\omega$. By assumption, it is has only logarithmic singularities along $\partial U_{p,q}$, and therefore it is of the form $d\omega = f dx_1 \cdots dx_d$ where f is a function on $U_{p,q}$ with only logarithmic singularities with respect to the coordinates $\{x_1, \dots, x_p\}$. Using previous notation, we have $f = \sum_{i=1}^d \partial_{x_i} F_i$. Then, the form $d\omega$ may have a pole of order 1 with respect to x_i. The corresponding contribution in $d\omega$ is given by

$$\frac{1}{x_i} \sum_{k \geq 1} k \log(x_i)^{k-1} F_{i,k}(x_1, \dots, x_i = 0, \dots, x_d).$$

But by assumptions $d\omega$ has at most logarithmic singularities. Hence, $F_{i,k}(x_1, \dots, x_i = 0, \dots, x_d) = 0$ for $i = 1, \dots, p$ and $F_{i,k}$ is proportional to a positive power of x_i for all $k \geq 1$. This implies $\omega \in \mathcal{F}_1\Omega_{\mathrm{p,log}}^{d-1}(U_{p,q})$, as required.

We conclude that all assumptions of Theorem 2.7 are satisfied and the claim follows for $d \geq 2$; the case $d = 1$ can be checked by a simple computations (similar to the example treated in the Introduction). □

3. Examples and applications

In this Section, we give some examples for the variants of Stokes's Theorem stated and proved in Subsection 2.3.

We begin by discussing an easy example in dimension $d = 1$ of Theorem 2.7. We then consider a more interesting example in dimension $d = 2$: the 1-form is the simplest instance of a form with poles and logarithmic singularities which appears in the proof of the famous Kontsevich's Vanishing Lemma from [10, Subsection 6.6]. Finally, guided by the previous 2-dimensional example, we re-prove Kontsevich's Vanishing Lemma [10, Subsection 6.6] by means of Stokes' Theorem 2.7.

3.1. Example in dimension $d = 1$.

We consider the compact manifold with boundary $X = [0, 1]$ of dimension 1, endowed with its natural orientation, so as the boundary strata $\{1\}$ and $\{0\}$ have positive and negative orientation, respectively. We denote by x a global coordinate on X.

We consider the 0-form $\omega = x^k \ \log(x)^l$, where k is a strictly positive integer and l is a non-negative integer. By its very definition, it is clear that ω belongs to $\mathcal{F}_1\Omega^0_{\mathrm{p,log}}(X)$ with a logarithmic singularity at $x = 0$. Furthermore, a direct computation implies that

$$d\omega = \frac{dx}{x} \ \left(lx^k \log(x)^{l-1}\right) + kx^{k-1}\log(x)^l \ dx.$$

Since $k \geq 1$, we conclude that $d\omega$ has only logarithmic singularities. Finally, we have

$$\mathrm{Reg}_{\{x=0\}}(\omega) = 0, \ \mathrm{Reg}_{\{x=1\}}(\omega) = \begin{cases} 1, & l = 0 \\ 0, & l \geq 1. \end{cases}$$

We may then apply Stokes' Theorem 2.7 to obtain

$$\int_X d\omega = \mathrm{Reg}_{\{x=1\}}(\omega) - \mathrm{Reg}_{\{x=0\}}(\omega) = \begin{cases} 1, & l = 0, \\ 0, & l \geq 1. \end{cases}$$

This result agrees with the standard computation of the improper integral of $d\omega$ using integration by parts.

3.2. **Example in dimension** $d = 2$**.** Consider $X = \mathbb{C} \setminus \{0,1\}$, and let $\overline{X}$ be the compactification of X where we attach three copies of the circle S^1 to compactify the neighborhoods of $0, 1$ and ∞. In more detail, for the neighborhood of $z = 0$ we use polar coordinates $z = \rho e^{i\phi}$ and we attach a circle at $\rho = 0$. For the neighborhood of $z = 1$, use the parametrization $z = 1 + \rho e^{i\phi}$ and again attach a circle at $\rho = 0$. Finally, for the neighborhood of $z = \infty$ use the polar coordinates $z^{-1} = \rho e^{i\phi}$ and as before attach a circle at $\rho = 0$.

Consider the 1-form $\omega = \log(|z|) d\log(|z-1|)$ with $d\omega = d\log(|z|) d\log(|z-1|)$. We claim that ω extends to an element of $\mathcal{F}_1\Omega^1_{\mathrm{p,log}}(\overline{X})$, which satisfies Properties $i)$ and $ii)$ of Theorem 2.7. Let us write

$$\omega = \log(|z|) d\log(|z-1|) = \frac{1}{2}\log(|z|)\frac{dz}{z-1} + \frac{1}{2}\log(|z|)\frac{d\overline{z}}{\overline{z}-1}.$$

In the parametrization $z = \rho e^{i\phi}$ it takes the form

$$\omega = \frac{1}{2}\left(\frac{e^{i\varphi}}{\rho e^{i\varphi}-1} + \frac{e^{-i\varphi}}{\rho e^{-i\varphi}-1}\right)\log(\rho)d\rho + \frac{1}{2}\left(\frac{ie^{i\varphi}}{\rho e^{i\varphi}-1} - \frac{ie^{-i\varphi}}{\rho e^{-i\varphi}-1}\right)\rho\log(\rho)d\varphi.$$

The functions $1/(\rho e^{i\varphi} - 1)$, $1/(\rho e^{-i\varphi} - 1)$ are both regular at $\rho = 0$. Hence, near the boundary stratum which corresponds to $z = 0$ we only have logarithmic singularities. It is easy to see that the same applies to $d\omega$: the only potential source of poles in ρ is the differential of $\log(\rho)$ but it is always accompanied either by $d\rho$ or by ρ.

In the parametrization $z = 1 + \rho e^{i\phi}$, we obtain

$$\omega = \log(|1 + \rho e^{i\varphi}|)\frac{d\rho}{\rho}.$$

Since the function $\log(|1 + \rho e^{i\varphi}|)$ vanishes at $\rho = 0$, ω is actually regular at the boundary stratum corresponding to $z = 1$. This implies that $d\omega$ is regular near this stratum as well.

Finally, in the parametrization $z^{-1} = \rho e^{i\phi}$ we get

$$\omega = \log(\rho)\frac{d\rho}{\rho} - \frac{1}{2}\left(\frac{e^{i\phi}}{\rho e^{i\phi}-1} + \frac{e^{-i\phi}}{\rho e^{-i\phi}-1}\right)\log(\rho)d\rho + \frac{1}{2}\left(\frac{ie^{i\phi}}{\rho e^{i\phi}-1} - \frac{ie^{-i\phi}}{\rho e^{-i\phi}-1}\right)\rho\log(\rho)d\phi.$$

Note that near the boundary stratum corresponding to $z = \infty$ the form ω does have a pole of order 1. Note that $d\omega$ only has logarithmic singularities: the term $\log(\rho)d\rho/\rho$ is closed and in the other terms $\log(\rho)$ is again accompanied either by $d\rho$ or by ρ.

These explicit formulæ show that $\omega = \log(|z|)d\log(|z-1|)$ is indeed an element of $\mathcal{F}_1\Omega^1_{\mathrm{p,log}}(\overline{X})$. Since $d\omega$ has at most logarithmic singularities at the boundary, Condition $i)$ of Theorem 2.7 is verified. By direct inspection, the regularizations along all boundary strata vanish, whence Condition $ii)$ of Theorem 2.7 is also verified.

Then, we may safely apply Theorem 2.7 to obtain

$$\int_X d\omega = \int_{\partial\overline{X}} \mathrm{Reg}_{\partial\overline{X}}(\omega) = 0.$$

3.3. **Configuration spaces.** Before entering into the details of the 2-dimensional example, where we apply Theorem 2.7, we need a review of some configuration spaces introduced in [10, Section 5].

The (open) configuration space C_A is defined as

$$C_A = \mathrm{Conf}_A/G_3 = \left\{p \in \mathbb{C}^A \,|\, p(a) \neq p(a') \text{ if } a \neq a'\right\}/G_3,$$

where G_3 is the semi direct product $\mathbb{R}^+ \ltimes \mathbb{C}$, which acts diagonally on $\mathbb{C}^A$ *via*

$$(\lambda,\mu)p = \lambda p + \mu,\ \ \lambda \in \mathbb{R}^+,\ \ \mu \in \mathbb{C}.$$

It is clear that G_3 is a real Lie group of dimension 3, whose action on Conf_A is free precisely when $2|A| - 3 \geq 0$: then, C_A is a smooth real manifold of dimension $2|A| - 3$. For $A = \{1,\dots,n\}$, we use the simpler notation C_n.

We consider C_n, for $n \geq 2$. We then define a global section of C_n *via*

$$C_n \ni [(z_1,\dots,z_n)] \mapsto \left(0, \frac{z_2 - z_1}{|z_2 - z_1|}, \frac{z_3 - z_1}{|z_2 - z_1|}, \dots, \frac{z_n - z_1}{|z_2 - z_1|}\right) \in \mathrm{Conf}_n.$$

By forgetting the first point in the n-tuple on the right-hand side of the previous assignment, we may identify the image of C_n with respect to this section with

$$\left\{(e^{i\varphi}, u_1, \dots, u_{n-2}) \in S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C} \smallsetminus \{0\}) :\ w_i \neq e^{i\varphi}\right\},$$

and finally we get the useful identification

$$C_n \cong S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C} \smallsetminus \{0,1\})$$

via the diffeomorphism

$$C_n \ni (e^{i\varphi}, u_1, \dots, u_{n-2}) \mapsto (e^{i\varphi}, e^{-i\varphi}u_1, \dots, e^{-i\varphi}u_{n-2}) \in S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C} \smallsetminus \{0,1\}).$$

It is not difficult to see that, as a consequence of the previous computations, Conf_A admits the structure of a trivial principal bundle over C_A due to the existence of smooth global sections. In particular, C_A inherits an orientation from the obvious orientations of Conf_A and G_3.

3.3.1. *The compactified configuration space* $\overline{C}_A$. We briefly recall the explicit construction of the compactified configuration space $\overline{C}_A$.

We denote by $\mathbb{S}_0^A(1)$, for a finite set A as above, the subset of $\mathbb{C}^A$ cut out by the equations $\sum_{a\in A} z_a = 0$ and $\sum_{a\in A}|z_a|^2 = 1$: it is obviously a real smooth, compact submanifold of $\mathbb{C}^A$ of dimension $2|A| - 3$ and defines in fact a smooth section of the G_3-bundle Conf_A over C_A. Observe that $\mathbb{S}_0^A(1)$ contains all diagonals in $\mathbb{C}^A$, except the smallest diagonal subset, where all components of a tuple in $\mathbb{C}^A$ are equal.

For any subset $\widetilde{A} \subset A$ of cardinality $2 \leq |\widetilde{A}| \leq |A| - 1$, there is a natural projection from C_A onto $C_{\widetilde{A}}$. Then, the compactified configuration space $\mathcal{C}_A$ is defined as the closure of the natural embedding

$$C_A \hookrightarrow \prod_{\substack{\widetilde{A}\subseteq A\\ 2\leq|\widetilde{A}|\leq|A|}} \mathbb{S}_0^{\widetilde{A}}(1),$$

where we have used the above section of Conf_A as a G_3-bundle over C_A to construct embeddings $C_{\widetilde{A}} \hookrightarrow \mathbb{S}_0^{\widetilde{A}}(1)$.

The main property of $\overline{C}_A$ consists in the fact that it is a manifold with corners in the sense of Definition 2.1: we will sketch a construction of local coordinates near boundary strata of any given codimension. This will be particularly important for later computations.

3.3.2. *The boundary stratification of* $\overline{C}_A$. Let us first describe the boundary stratification of $\overline{C}_A$ from a combinatorial point of view.

From a combinatorial point of view, a general boundary strata of codimension $1 \leq p \leq |A|-2$ of $\overline{C}_A$ are in one-to-one correspondence with nested families $\{A_1,\dots,A_p\}$ of subsets of A of cardinality $2 \leq |A_i| \leq |A| - 1$, $i = 1,\dots,p$, *i.e.* families of p subsets of A, such that either $A_i \cap A_j = \varnothing$ or $A_i \subset A_j$ or $A_i \supset A_j$.

For a given nested family $\{A_1,\dots,A_p\}$ as above, the star of A_i, denoted $\mathrm{star}(A_i)$, for $i = 1,\dots,p$, is defined as the subfamily of $\{A_1,\dots,A_p\}$ of subsets of A_i which are maximal with respect to the partial order $\subset$; we set $A_0 = A$, and accordingly we may define its star $\mathrm{star}(A_0)$. By its very definition, $\mathrm{star}(A_0)$ is not empty, while $\mathrm{star}(A_i)$ may be empty, for $i = 1,\dots,p$. Furthermore, for $i \neq j$, $\mathrm{star}(A_i) \cap \mathrm{star}(A_j) = \varnothing$.

For a nested family $\{A_1,\dots,A_p\}$ of subsets of A, the corresponding stratum $\partial_{A_1,\dots,A_p}\overline{C}_A$ of codimension p is isomorphic to the product of compactified configuration spaces

$$\partial_{A_1,\dots,A_p}\mathcal{C}_A \cong \prod_{i=0}^{p} \mathcal{C}_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}, \tag{4}$$

where $A_i \smallsetminus \mathrm{star}(A_i)$ is a short-hand notation for the complement of the elements of $\mathrm{star}(A_i)$ inside A_i, and $\{\bullet\}^{\mathrm{star}(A_i)}$ denotes a set of cardinality $|\mathrm{star}(A_i)|$.

We now illustrate how to construct a structure of manifold with corners on $\overline{C}_A$.

Let us first introduce the notation

$$A_i = \left\{a_1^{A_i}, \dots, a_{|A_i|}^{A_i}\right\},\ A_i \smallsetminus \mathrm{star}(A_i) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|} = \left\{a_{i_k}^{A_i} \in A_i \smallsetminus \mathrm{star}(A_i);\ a_{A_j},\ A_j \in \mathrm{star}(A_i)\right\}.$$

We consider a general element a of A, which is exhausted by a chain of nested elements of $\{A_1, \dots, A_p\}$ as follows: by construction, either a in $A_0 \smallsetminus \mathrm{star}(A_0)$ or a in $A_{i_1}^a$, for $A_{i_1}^a$ in $\mathrm{star}(A_0)$. In the second case, a is either in $A_{i_1}^a \smallsetminus \mathrm{star}(A_{i_1}^a)$ or in $A_{i_2}^a$, for $A_{i_2}^a$ in $\mathrm{star}(A_{i_1}^a)$. Once again, in the second case, we may proceed as before, until a is in $A_{i_{q_a}}^a \smallsetminus \mathrm{star}(A_{i_{q_a}}^a)$, for some $A_{i_{q_a}}^a$. The corresponding chain of nested elements is denoted by $\{A_{i_1}^a, \dots, A_{i_{q_a}}^a\}$, where $A_{i_1}^a \supset \cdots \supset A_{i_{q_a}}^a$, where we adopt the convention that, if $q_a = 0$, then $A_{i_0} = A_0 = A$. Observe that, by its very construction, a uniquely determines its exhausting chain $\{A_{i_1}^a, \dots, A_{i_{q_a}}^a\}$.

We consider the open configuration space $C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$, for which we choose a global section, *e.g.* the one described at the beginning of Subsection 3.1: the corresponding set of local coordinates is denoted, according to the previously introduced notation, by

$$\{z_{i_k}^{A_i};\ z_{A_j},\ A_j \in \mathrm{star}(A_i)\} \in C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}.$$

With any A_i, we additionally associate a parameter ρ_{A_i}, which ranges in an interval $[0, \varepsilon_i)$, for ε_i sufficiently small; we finally set $\rho_{A_0} = \rho_A = 1$.

Then, a set of local coordinates for $\mathcal{C}_A$ near the boundary stratum $\partial_{A_1, \dots, A_p} \mathcal{C}_A$ is specified by the Formula

$$\prod_{i=1}^{p} [0, \varepsilon_i) \times \prod_{i=0}^{p} C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}} \ni \prod_{i=0}^{p} \left(\rho_{A_i}; \underline{z}_{i_k}^{A_i}; \underline{z}_{A_j}\right) \mapsto \left\{ A \ni a \mapsto z_a = \sum_{j=1}^{q_a} \left(\prod_{k=1}^{j-1} \rho_{A_{i_k}^a} \right) z_{A_{i_j}^a} \right\} \in \mathcal{C}_A. \tag{5}$$

In Formula (5), we have used the short-hand notation $\underline{z}_{i_k}^{A_i}$ for the corresponding tuple of local coordinates, and similarly for $\underline{z}_{A_j}$; when $j = q_a$ in the sum on the rightmost expression in Formula (5), then $z_{A_{i_{q_a}}^a} = z_{i_a}$, for the unique index i_a such that in $A_{i_{q_a}}^a$, $a = a_{i_a}$. According to the convention that, if $q_a = 0$, then $A_{i_0} = A$, we set $z_a = z_{i_a}^{A_0}$ for the unique index i_a such that in $A \smallsetminus \mathrm{star}(A)$, $a = a_{i_a}$. Finally, we have used the convention that a product with a negative number of factors is 1.

There is an important *caveat* to be made at this point: for ease of later computations and of notation, the $2|A| - 3$-tuple on the right-hand side of Formula (5) is meant to be an equivalence class of the corresponding tuple in $\mathbb{C}^A$ with respect to the action of G_3. Hence, coordinates *stricto sensu* are obtained by choosing a section of Conf_A, which we have sort of "hidden" in the notation.

It will be convenient for later computations to choose the global section for any $C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$ as at the beginning of Subsection 3.1: this means that one of the coordinates among $(\underline{z}_{i_k}^{A_i}; \underline{z}_{A_j})$ is set to be 0, another one is is set to be 1 and all of them are multiplied by an element of S^1. With this in mind, the expression on the right-hand side of Formula (5) can be re-written as

$$A \ni a \mapsto z_a = \sum_{j=1}^{q_a} \left(\prod_{k=1}^{j-1} \rho_{A_{i_k}^a} \right) z_{A_{i_j}^a} = \sum_{j=1}^{q_a} \left(\prod_{k=1}^{j-1} w_{A_{i_k}^a} \right) z_{A_{i_j}^a},$$

where $w_{A_{i_k}^a} = \rho_{A_{i_k}^a} e^{i\varphi_{A_{i_k}^a}}$, and the angle coordinate $\varphi_{A_{i_k}^a}$ is associated with $A_{i_k}^a$ by means of the chosen global section of $C_{(A_i \smallsetminus \mathrm{star}(A_i)) \sqcup \{\bullet\}^{|\mathrm{star}(A_i)|}}$, and the parameter $\rho_{A_{i_k}^a}$ is viewed as the other corresponding polar coordinate with respect to $\varphi_{A_{i_k}^a}$.

3.4. **Kontsevich's Vanishing Lemma.** Let us consider a graph Γ with n vertices and $2n - 4$ un-directed edges, which has neither multiple edges nor short loops (*i.e.* there is at most one edge connecting two distinct vertices of Γ and there is no edge, whose endpoints coincide). The set $E(\Gamma)$ of edges of Γ consists therefore of $2n - 4$ distinct subsets of $\{1, \dots, n\}$ of cardinality 2. We also implicitly assume that the set of vertices of Γ is endowed with a total order, as well as the set $E(\Gamma)$. We make the additional assumption that all vertices of Γ, except the first two, are at least tri-valent, and that no edge of Γ connects the first two vertices; we further assume the first two vertices to be at least bi-valent.

In Figure 1, *i*), is depicted an example of a graph Γ with 6 vertices and 8 un-directed edges with all above properties: observe that the lowest and the highest vertices of Γ are the first and second with respect to the total order on the set of vertices of Γ.

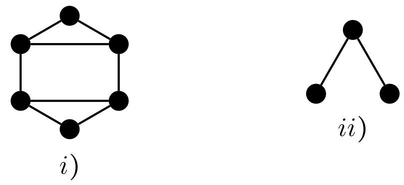

FIGURE 1. *i*) A graph Γ with 6 vertices and 8 edges and only tri-valent vertices and *ii*) the triangle graph.

With such a graph Γ, we associate a differential form of degree $2n-4$ on $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ *via* the assignment

$$\int_{\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})}\prod_e d\log(|z_{t(e)}-z_{s(e)}|). \tag{6}$$

We observe that the total order on $E(\Gamma)$ is necessary to make the integrand well-defined, because it is a product of 1-forms.

Remark 3.1. Let us consider $n=3$: the simplest graph Γ as above without short loops and double edges with 3 vertices and 2 un-directed edges is the graph depicted in Figure 1, *ii*): choosing a total order on the set $E(\Gamma)=\{e_1,e_2\}$ of its edges, the corresponding integral equals

$$\int_{\mathrm{Conf}_1(\mathbb{C}\smallsetminus\{0,1\})} d\log(|z|)d\log(|z-1|).$$

As already observed in Subsection 3.2 for the case $n=3$, the identification $C_n=\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ induces, *via* the compactification $\overline{C}_n$ of C_n described in Subsection 3.3, a compactification $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ of $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$.

We describe the boundary stratification of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ and begin by its boundary strata of codimension 1:

i) there exists a subset A of $[n-2]$, $1\leq|A|\leq n-2$, such that the corresponding boundary stratum describes the collapse of the points in $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ labeled by A to 0;

ii) there exists a subset A of $[n-2]$, $1\leq|A|\leq n-2$, such that the corresponding boundary stratum describes the collapse of the points in $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ labeled by A to 1;

iii) there exists a subset A of $[n-2]$, $2\leq|A|\leq n-2$, such that the corresponding boundary stratum describes the collapse of the points in $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ labeled by A to a single point in $\mathbb{C}\smallsetminus\{0,1\}$;

iv) there exists a subset A of $[n-2]$, $1\leq|A|\leq n-2$, such that the corresponding boundary stratum describes the situation, where points in $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ labeled by A tend to infinity in $\mathbb{C}$.

It is not difficult to put these four boundary strata into relationship with the corresponding boundary strata of codimension 1 of $\overline{C}_n$: namely, the boundary stratum in *i*) of $\overline{C}_{n-2}(\mathbb{C})$ corresponds to the boundary stratum of $\overline{C}_n$ labeled by the subset $A\sqcup\{1\}$, $2\notin A$, where i denotes the corresponding coordinate in $\overline{C}_n$, $i=1,2$. Similarly, the boundary stratum in *ii*) of $\overline{C}_{n-2}(\mathbb{C})$ corresponds to the boundary stratum of $\overline{C}_n$ labeled by the subset $A\sqcup\{2\}$, $1\notin A$. The boundary stratum in *iii*) corresponds to the boundary stratum of $\overline{C}_n$ labeled by the subset A, $1,2\notin A$. Finally, the boundary stratum in *iv*) corresponds to the boundary stratum of $\overline{C}_n$ labeled by $[n-2]\smallsetminus A$, where the points labeled by the subset $[n-2]\smallsetminus A$, which contains both 1, 2, collapse together.

From this dictionary, it is not difficult to figure out the shape of a general boundary stratum of codimension p of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ from the corresponding one of $\overline{C}_n$. Local coordinates for these boundary strata can be read directly from Subsubsection 3.3.2: boundary strata corresponding to points collapsing together in $\mathbb{C}\smallsetminus\{0,1\}$ to 0 or 1, are described by local coordinates nearby as in Subsubsection 3.3.2, whereas the description of local coordinates near boundary strata at infinity in $\mathbb{C}$ needs a bit more care.

For the sake of clarity, let us *e.g.* describe the local coordinates for a boundary stratum of codimension 1 of type *iv*): namely, we assume that the points labeled by A tend to infinity in $\mathbb{C}$. W.l.o.g. we may assume $A=[l]$, $1\leq l\leq n-2$, then

$$\begin{aligned}&(0,\infty)\times S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})\ni\left(R;e^{i\varphi},w_1,\dots,w_{l-1};z_1,\dots,z_{n-2-l}\right)\mapsto\\&\mapsto\left(Re^{i\varphi},Re^{i\varphi}w_1,\dots,Re^{i\varphi}w_{l-1},z_1,\dots,z_{n-2-l}\right)\in\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}).\end{aligned}$$

The stratum is realized as (formally) $R=\infty$. To describe local coordinates near a boundary stratum of codimension p of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$, where certain points in $\mathbb{C}\smallsetminus\{0,1\}$ tend to infinity "at different speeds", it is better to look at the "dual" picture, *i.e.* to look at the corresponding boundary stratum of $\overline{C}_n$. In particular, the key point for all subsequent computations when dealing with boundary strata of codimension p of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$, where some points are allowed to tend to infinity "at different speeds", is the following one: the strata "at infinity" are realized when certain variables in $\mathbb{R}^+$ are formally set to ∞. Recalling the global section of Conf_A over C_A from the beginning of Subsection 3.1, there is an angle coordinate for any subset in the chain labeling the boundary stratum under inspection: whenever we deal with a stratum "at infinity" labeled by a subset C_j, we make the following change of variables $R_{C_j}e^{i\varphi_{C_j}} \mapsto \rho_{C_j}e^{i\psi_{C_j}}$, $\rho_{C_j} = 1/R_{C_j}$, $\psi_{C_j} = -\varphi_{C_j}$, and the stratum is then realized as $\rho_{C_j} = 0$.

For Γ as above, we have

$$\prod_{e'\neq e} d\log\left(|z_{t(e')} - z_{s(e')}|\right) = d\left(\log\left(|z_{t(e')} - z_{s(e')}|\right) \prod_{e''\neq e,e'} d\log\left(|z_{t(e'')} - z_{s(e'')}|\right)\right) = d\omega_\Gamma$$

on $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$, where we have implicitly used a total order on $E(\Gamma)$.

Lemma 3.2. *For a graph Γ as above, the form ω_Γ extends to an element of $\mathcal{F}_1\Omega^{2n-5}_{\mathrm{p,log}}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$, satisfies Assumptions i) and ii) of Theorem 2.7 and the regularization $\mathrm{Reg}_{\partial\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})}(\omega_\Gamma)$ equals to 0.*

Proof. Without loss of generality, we may assume that, with respect to the chosen total order on $E(\Gamma)$, $e = (v_1, v_2)$ and $e' = (v_3, v_1)$, whence

$$\omega_\Gamma = \log(|z_1|) \prod_{e''\neq e,e'} d\log\left(|z_{t(e'')} - z_{s(e'')}|\right),$$

and there is at least one factor in ω_Γ of the form $d\log(|z_{t(e'')}-1|)$ for some edge e'' of Γ. More precisely, the assumption on Γ that it has vertices which are at least tri-valent, except the first two vertices, which are assumed to be at least bi-valent, forces at least three such factors and two factors of the form $d\log(|z_{i_\gamma}|)$ in ω_Γ.

First of all, we prove that $d\omega_\Gamma$ extends to an element of $\Omega^{2n-4}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$.

To analyze the behavior of $d\omega_\Gamma$ near a given boundary stratum of codimension p of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$, we use the previous description of the boundary stratification and the fact that, for any subset A_i of $[n-2]$, there is a complex coordinate w_{A_i} on $\mathbb{C}\smallsetminus\{0\}$.

Using the multiplicative property of the logarithm and Leibniz' rule, near the given boundary stratum of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$, the form $d\omega_\Gamma$ of top degree may admit singularities of the form

$$\frac{1}{2}\frac{dw_{A_i}}{w_{A_i}} + \frac{1}{2}\frac{d\overline{w}_{A_i}}{\overline{w}_{A_i}} = \frac{d\rho_{A_i}}{\rho_{A_i}},$$

for any subset A_i. Every such term appears at most once in every summand, and the remaining factors are easily verified to be complex-valued, real analytic. In particular, observe that for any $C_i = A_i$, $|A_i| \geq 2$, the singular terms dw_{A_i}/w_{A_i} and $d\overline{w}_{A_i}/\overline{w}_{A_i}$ appear in the same summand.

Since ω_Γ is of top degree on $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$, near the given boundary stratum every summand must be proportional to a form which contains all differentials: in particular, it must contain the 2-form $dw_{A_i}d\overline{w}_{A_i}$, for every A_i.

Now, if *e.g.* the singular term dw_{A_i}/w_{A_i} is paired with the differential $d\overline{w}_{A_i}$, the resulting 2-form gives a complex-valued, real analytic 2-form on the (real) blow-up of $\mathbb{C} = \mathbb{R}^2$ at the origin, provided no other singularity at $w_{A_i} = \overline{w}_{A_i} = 0$ is present. By the previous observations, such a singularity may only come if dw_{A_i}/w_{A_i} is paired with $d\overline{w}_{A_i}/\overline{w}_{A_i}$: since $d\omega_\Gamma$ is a product of logarithmic differentials and because of the previous expression for the singular terms labeled by A_i, for any summand containing a singular term as above there is exactly another summand, which looks the same except for the positions of dw_{A_i}/w_{A_i} and $d\overline{w}_{A_i}/\overline{w}_{A_i}$ being swapped. The skew-symmetry of these logarithmic differentials implies that both terms sum up to 0, hence every singular logarithmic differential dw_{A_i}/w_{A_i} is paired with $d\overline{w}_{A_i}$, and similarly for $d\overline{w}_{A_i}/\overline{w}_{A_i}$, and the remaining terms are real analytic.

We may use polar coordinates as well: in this case, the singular differentials are proportional to $d\rho_{A_i}/\rho_{A_i}$, for any A_i. Because of the fact that we consider the logarithm of the complex norm, the corresponding angle coordinate φ_{A_i} may appear only in a complex-valued, real analytic term as $\rho_{A_i}d\varphi_{A_i}$. Hence, whenever $d\rho_{A_i}/\rho_{A_i}$ appears, it is paired with $\rho_{A_i}d\varphi_{A_i}$ and the claim follows.

We next prove that ω_Γ belongs to $\mathcal{F}_1\Omega^{2n-5}_{\mathrm{p,log}}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$: it is better to use polar coordinates $w_{A_j} = \rho_{A_j}e^{i\varphi_{A_j}}$ for this purpose.

First of all, near a given boundary stratum of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ as above, ω_Γ can be written as a sum of forms, each of which contains at most a logarithmic singularity $\log(\rho_{A_i})$ by the multiplicative property of the logarithm. Further, since the degree of ω_Γ equals the dimension of $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ minus 1, we may apply a slight variation of the previous arguments for the proof that $d\omega_\Gamma$ extends to an element of $\Omega^{2n-4}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$ to verify that any summand of ω_Γ contains at most a pole of order 1 of the form $d\rho_{A_i}/\rho_{A_i}$.

To prove that ω_Γ belongs to $\mathcal{F}_1\Omega^{2n-5}_{\mathrm{p,log}}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$, it remains to show that a summand containing $\log(\rho_{A_i})$ and no differential $d\rho_{A_i}$ or $d\rho_{A_i}/\rho_{A_i}$ is proportional to ρ_{A_i}. Such a summand has degree equal to the dimension of $\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ minus 1, and by assumption does not contain the differential $d\rho_{A_i}$: therefore, it is proportional to the product of all other differentials, in particular it must contain $d\varphi_{A_i}$. Previous arguments related to the fact that we consider the logarithm of the complex norm imply that the differential $d\varphi_{A_i}$ comes from a 1-form factor containing $\rho_{A_i} e^{i\varphi_{A_i}}$: hence, the differential $d\varphi_{A_i}$ is multiplied by ρ_{A_i} and the claim follows.

We now verify Property *ii)* in Theorem 2.7. We consider again a boundary stratum as in the previous arguments: we have to show that the regularization of ω_Γ obtained by setting formally $\rho_{A_i} = \log(\rho_{A_i}) = d\rho_{A_i} = d\rho_{A_i}/\rho_{A_i} = 0$ has only logarithmic singularities. The arguments used to prove that ω_Γ belongs to $\mathcal{F}_1\Omega^{2n-5}_{\mathrm{p,log}}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$ apply as well, hence the regularization contains neither $\log(\rho_{A_i})$ nor $d\rho_{A_i}$. Observe that a general summand in the regularization may contain logarithmic singularities $\log(\rho_{A_j})$, $j\neq i$ as well as logarithmic differentials $d\rho_{A_j}/\rho_{A_j}$: since the regularization of ω_Γ has top degree on the corresponding stratum, degree reasons imply that such a summand must contain $d\varphi_{A_j}$. Previous arguments imply that the differential $d\varphi_{A_j}$ appears always paired with ρ_{A_j}.

Finally, let us consider the boundary strata $\partial\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ of codimension 1 of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$: it remains to compute the regularization of ω_Γ along $\partial\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ and prove its vanishing.

For a boundary stratum of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ of codimension 1, labeled by a subset A of $[n-2]$ as in *i)*-*iv)* as above, we may assume without loss of generality that $A=[l]$, where $1\leq l\leq n-2$ in cases *i)*, *ii)* and *iv)*, and $2\leq l\leq n-2$ in case *iii)*. Here is a description of such strata by means of corresponding local coordinates:

i) the boundary stratum is isomorphic to $S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})$, and local coordinates nearby are specified *via*

$$\mathbb{R}^+\times\left(S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\right)\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})\ni\left(\rho;e^{i\varphi},w_1,\dots,w_{l-1};z_1,\dots,z_{n-2-l}\right)\mapsto$$
$$\mapsto\left(\rho e^{i\varphi},\rho e^{i\varphi}w_1,\dots,\rho e^{i\varphi}w_{l-1},z_1,\dots,z_{n-2-l}\right)\in\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\});$$

ii) the boundary stratum is isomorphic to $S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})$, and local coordinates nearby are specified *via*

$$\mathbb{R}^+\times\left(S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\right)\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})\ni\left(\rho;e^{i\varphi},w_1,\dots,w_{l-1};z_1,\dots,z_{n-2-l}\right)\mapsto$$
$$\mapsto\left(1+\rho e^{i\varphi},1+\rho e^{i\varphi}w_1,\dots,1+\rho e^{i\varphi}w_{l-1},z_1,\dots,z_{n-2-l}\right)\in\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\});$$

iii) the boundary stratum is isomorphic to $S^1\times\mathrm{Conf}_{l-2}(\mathbb{C}\smallsetminus\{0,1\})\times\mathrm{Conf}_{n-1-l}(\mathbb{C}\smallsetminus\{0,1\})$, and local coordinates nearby are specified *via*

$$\mathbb{R}^+\times\left(S^1\times\mathrm{Conf}_{l-2}(\mathbb{C}\smallsetminus\{0,1\})\right)\times\mathrm{Conf}_{n-1-l}(\mathbb{C}\smallsetminus\{0,1\})\ni\left(\rho;e^{i\varphi},w_1,\dots,w_{l-2};z_1,\dots,z_{n-1-l}\right)\mapsto$$
$$\mapsto\left(z_1,z_1+\rho e^{i\varphi},z_1+\rho e^{i\varphi}w_1,\dots,z_1+\rho e^{i\varphi}w_{l-2},z_2,\dots,z_{n-1-l}\right)\in\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\});$$

iv) the boundary stratum is isomorphic to $S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})$, and local coordinates nearby are specified *via*

$$\mathbb{R}^+\times\left(S^1\times\mathrm{Conf}_{l-1}(\mathbb{C}\smallsetminus\{0,1\})\right)\times\mathrm{Conf}_{n-2-l}(\mathbb{C}\smallsetminus\{0,1\})\ni\left(R;e^{i\varphi},w_1,\dots,w_{l-1};z_1,\dots,z_{n-2-l}\right)\mapsto$$
$$\mapsto\left(Re^{i\varphi},Re^{i\varphi}w_1,\dots,Re^{i\varphi}w_{l-1},z_1,\dots,z_{n-2-l}\right)\in\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}),$$

(once again, we recall that corresponding boundary stratum is recovered as $R=\infty$ formally: as before, we will use the involution $z\mapsto 1/z$ to obtain the corresponding boundary stratum as $\rho=1/R=0$).

We consider first a boundary stratum of type *i)* labeled by $A=[l]$. Near the corresponding boundary stratum, ω_Γ is, by its very definition, proportional to $\log(\rho)$: hence, its regularization vanishes automatically.

Let us then consider a boundary stratum of type *ii)* labeled by $A=[l]$: in this case, the regularization of ω_Γ vanishes because the function $\log(|1+\rho e^{i\varphi}|)$ vanishes as $\rho=0$.

We consider a boundary stratum of type *iii)*. We focus first on the point z_1: because of the assumption on Γ having vertices at least tri-valent, except the first two, which are assumed to be at least bi-valent, ω_Γ contains at least two factors of the type $d\log(|z_1-z_{s(e'')}|)$, for two distinct edges $e''\neq e,e'$ of Γ. If $z_{s(e'')}$ as before collapses to z_1, the corresponding 1-form $d\log(|z_1-z_{s(e'')}|)$ equals $d\rho/\rho$ with respect to the above coordinates: therefore $\widehat{\eta}_\Gamma$ is proportional to $d\rho/\rho$, whence its regularization vanishes. If not, the regularization of ω_Γ still vanishes, because it will be independent of S^1: this follows immediately from the fact that we deal with the logarithm of the complex norm, which is obviously S^1-invariant. This eliminates immediately the S^1-dependence from the factors of ω_Γ corresponding to edges joining two vertices corresponding to points collapsing to z_1, and the regularization eliminates the S^1-dependence from the remaining factors.

Finally, let us consider a boundary stratum of type *iv)*: in this case, using the involution $z\mapsto 1/z$, we see immediately that ω_Γ is proportional to $\log(\rho)$, $\rho=1/R$, hence its regularization also vanishes. □

As a consequence of Lemma 3.2, the integral

$$\int_{\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})} d\omega_\Gamma$$

converges, because $d\omega_\Gamma$ belongs to $\Omega^{2n-4}_{\log}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$. Further, we may compute the integral explicitly by means of Theorem 2.7, because the form ω_Γ belongs to $\mathcal{F}_1\Omega^{2n-5}_{\mathrm{p},\log}(\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\}))$ and satisfying Assumptions $i)$ and $ii)$ of the said Theorem

$$\int_{\mathrm{Conf}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})} d\omega_\Gamma = \int_{\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})} d\omega_\Gamma = \int_{\partial\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})} \mathrm{Reg}_{\partial\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})}(\omega_\Gamma) = 0,$$

being the regularization of ω_Γ trivial on every boundary stratum of $\overline{\mathrm{Conf}}_{n-2}(\mathbb{C}\smallsetminus\{0,1\})$ of codimension 1.

References

[1] Anton Alekseev, Johannes Löffler, Carlo A. Rossi, and Charles Torossian, *The logarithmic formality quasi-isomorphism*, (in preparation) (2012).

[2] Anton Alekseev and Charles Torossian, *Kontsevich deformation quantization and flat connections*, Comm. Math. Phys. **300** (2010), no. 1, 47–64, DOI 10.1007/s00220-010-1106-8. MR2725182 (2011j:53181)

[3] Francis C. S. Brown, *Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}$*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 3, 371–489 (English, with English and French summaries). MR2543329 (2010f:32013)

[4] Alberto S. Cattaneo, Giovanni Felder, and Lorenzo Tomassini, *Fedosov connections on jet bundles and deformation quantization*, Deformation quantization (Strasbourg, 2001), IRMA Lect. Math. Theor. Phys., vol. 1, de Gruyter, Berlin, 2002, pp. 191–202. MR1914788 (2003d:53161)

[5] ———, *From local to global deformation quantization of Poisson manifolds*, Duke Math. J. **115** (2002), no. 2, 329–352, DOI 10.1215/S0012-7094-02-11524-5. MR1944574 (2004a:53114)

[6] V. G. Drinfel′d, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* Gal($\overline{\mathbf{Q}}/\mathbf{Q}$), Algebra i Analiz **2** (1990), no. 4, 149–181 (Russian); English transl., Leningrad Math. J. **2** (1991), no. 4, 829–860. MR1080203 (92f:16047)

[7] A. G. Khovanskiĭ, *On a lemma of Kontsevich*, Funktsional. Anal. i Prilozhen. **31** (1997), no. 4, 89–91, DOI 10.1007/BF02466068 (Russian); English transl., Funct. Anal. Appl. **31** (1997), no. 4, 296–298 (1998). MR1608932 (99a:58077)

[8] Maxim Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 97–121. MR1341841 (96h:57027)

[9] ———, *Operads and motives in deformation quantization*, Lett. Math. Phys. **48** (1999), no. 1, 35–72, DOI 10.1023/A:1007555725247. Moshé Flato (1937–1998). MR1718044 (2000j:53119)

[10] ———, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216. MR2062626 (2005i:53122)

[11] Carlo A. Rossi and Thomas Willwacher, *Proof of Pavel Etingof conjecture about Drinfel'd associators*, (in preparation).

[12] Thomas Willwacher, *M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra* (2010), available at `arXiv:1009.1654`.

E-mail address: `Anton.Alekseev@unige.ch`

Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, c.p. 64, 1211 Genève 4 (Switzerland)

E-mail address: `johannes@mpim-bonn.mpg.de`

MPIM Bonn, Vivatsgasse 7, 53111 Bonn (Germany)

E-mail address: `crossi@mpim-bonn.mpg.de`

MPIM Bonn, Vivatsgasse 7, 53111 Bonn (Germany)

E-mail address: `torossian@math.dot.jussieu.dot.fr`

Université Denis-Diderot-Paris 7, UFR de mathḿatiques, Site Chevaleret, Case 7012 , 75205 Paris cedex 13 (France)