FORMALITY FOR THE LOG PROPAGATOR

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This notes are informal but an extended version is proposed in a paper with C. Rossi, J. Loffler and T. Willwacher.

1. Introduction

In '97, Kontsevich proved the formality theorem. As consequence any Poisson structure on smooth manifold can be formally quantized, that is any Poisson structure is associated to a formal star product. Kontsevich's proof uses a explicit formula in case of $\mathbb{R}^d$. This formula is expressed with graphs, hyperbolic propagator, coefficients and bi-differential operators associated to graphs.

In case of a finite dimensional Lie algebra $\mathfrak{g}$, this construction gives rise to a new expression for the Campbell-Hausdorff formula and to a homotopy between standard product and Lie product. This homotopy was used to solve the Kashiwara-Vergne conjecture [1].

Using deformation theory, we introduced in [5] connections $\Omega_n, \Omega_n^\infty$ on configurations spaces $C_{n,0}$ and C_n and we proved that they are flat :

$$\mathrm{d}\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n] = 0.$$

We defined a even real associator in sense of [3] Φ_{AT} as parallel transport of Ω_3^∞, which is proved recently to be a Drinfeld associator [13].

In this article we consider the Kontsevich's construction [10] of a L^∞-quasi isomorphism with the use of the log propagator instead of the hyperbolic propagator. These considerations were introduced by Kontsevich in [11] as a remark and developed by Merkulov in [12]

2. Deformation quantization

Many sources are now available on the Kontsevich formula for quantization of Poisson brackets (see *e.g.* [8]). For convenience of the reader, we briefly recall the main ingredients of [10] for $\mathbb{R}^d$ and the construction [14] of the connection ω_n.

2.1. **Kontsevich construction.**

2.1.1. *Configurations spaces.* We denote by $C_{n,m}$ the configuration space of n distincts points in the upper half plane and m points on the real line modulo the diagonal action of the group $z \mapsto az + b$ ($a \in \mathbb{R}_+, b \in \mathbb{R}$). In [10], Kontsevich constructed compactifications of spaces $C_{n,m}$ denoted by $\overline{C}_{n,m}$. These are manifolds with corners of dimension $2n-2+m$. We denote by $\overline{C}^+_{n,m}$ the connected component of $\overline{C}_{n,m}$ with real points in the standard order ($id.$ $\overline{1} < \overline{2} < \cdots < \overline{m}$).

The compactified configuration space $\overline{C}_{2,0}$ (the "Kontsevich eye") is shown on Fig. 1. The upper and lower eyelids correspond to one of the points (z_1 or z_2) on the real line, left and right corners of the eye are configurations with $z_1, z_2 \in \mathbb{R}$ and $z_1 > z_2$ or $z_1 < z_2$. The boundary of the iris takes into account configurations where z_1 and z_2 collapse inside the complex plane. The angle along the iris keeps track of the angle at which z_1 approaches z_2.

FIGURE 1. Variety $\overline{C}_{2,0}$.

Recall the compactification of $C_{n,m}$. First we start with C_n (n distincts points in $\mathbb{C}^n$ modulo the group $az+b$) which is identified to the space of general configurations $c = (z_i)_{1\leq i\leq n}$ in standard position that is mean $\underline{c} = \frac{1}{n}\sum z_i = 0$ and variance $||c||^2 = \frac{1}{n}\sum |z_i|^2 = 1$. Consider the compact sphere

$$\widetilde{C_n} = \{c = (z_i)_{1\leq i\leq n}, z_i \in \mathbb{C}, \underline{c} = 0; , ||c||^2 = 1\},$$

and write $\widetilde{C_J}$ if the set of indices is replaced by J. Then we consider the natural map

$$C_n \longrightarrow \prod_{\sharp J\geq 2} \widetilde{C_J}$$

and take the closure. This is exactly the definition of the compactification $\overline{C}_n$ used by Kontsevich. This is a manifold with corners. The canonical stratification is described by means of trees labeled by $1, 2, \ldots, n$, and one has local charts with a precise description [8] Appendice A.

FIGURE 2. A stratum of codimension 7 in $\overline{C}_{10,2}$ and the associated tree $\mathcal{S}_T = C_{2,2} \times C_3 \times C_2 \times C_2 \times C_{1,0} \times C_{1,1} \times C_2 \times C_{1,2}$

By construction $C_p \times C_{n-p+1} \subset \partial\overline{C}_n$. Near this stratum, the analytic germ is $C_p \times C_{n-p+1,2} \times \mathbb{R}_+$. Let's define the local chart from to $C_p \times C_{n-p+1} \times \mathbb{R}_+$. Let $\xi = (\xi_J) \in \overline{C}_n$, with ξ_J normalized configurations and $\xi_{[1,n]} = (z_1, \ldots, z_n) \in \widetilde{C_n} \subset$

$\overline{C}_n \subset \prod_{\sharp J \geq 2} \widetilde{C_J}$ a normalized configuration, define $z_0 = \frac{1}{p}(z_1 + \ldots + z_p)$ and

$$s_1 = \frac{||(z_1, \ldots, z_p)||_2}{||(z_0, z_{p+1}, \ldots, z_n)||_2}.$$

Let's define c_2 as the normalized configuration associated to $(z_0, z_{p+1}, \ldots, z_n)$. Then the map $\Psi(\xi) = (\xi_{[1,p]}, c_2, s_1)$ define the local chart near the stratum $C_p \times C_{n-p+1}$.

2.1.2. *Graphs.* A graph Γ is a collection of vertices V_Γ and oriented edges E_Γ. Vertices are ordered, and the edges are ordered in a way compatible with the order of the vertices. We denote by $G_{n,2}$ the set of graphs with $n+2$ vertices and $2n$ edges verifying the following properties:

i - There are n vertices of the first type $1, 2, \cdots, n$ and 2 vertices of the second type $\overline{1}, \overline{2}$
ii - Edges start from vertices of the first type, 2 edges per vertex.
iii - Source and target of an edge are distinct.
iv - There are no multiple edges (same source and target).

2.1.3. *The angle map and Kontsevich weights.* Let p and q be two points on the upper half plane. Consider the hyperbolic angle map on $C_{2,0}$:

$$\phi_h(p, q) = \arg\left(\frac{q - p}{q - \overline{p}}\right) \in \mathbb{T}^1. \tag{1}$$

This function admits a continuous extension to the compactification $\overline{C}_{2,0}$.

Consider a graph $\Gamma \in G_{n,2}$, and draw it in the upper half plane with vertices of the second type on the real line. By restriction, each edge e defines an angle map ϕ_e on $\overline{C}^+_{n,2}$. The ordered product

$$\Omega_\Gamma = \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e \tag{2}$$

is a regular $2n$-form on $\overline{C}^+_{n,2}$ (which is a $2n$-dim compact space).

***Definition* 1.** *The Kontsevich weight of* Γ *is given by the following formula,*

$$w_\Gamma = \frac{1}{(2\pi)^{2n}} \int_{\overline{C}^+_{n,2}} \Omega_\Gamma. \tag{3}$$

3. Log propagator

We consider the propagator $\mathrm{d}\omega_{\log}$ on $C_{2,0}$

$$\mathrm{d}\omega_{\log}(z_i, z_j) = \frac{1}{2i\pi} \mathrm{d}\log\left(\frac{z_i - z_j}{\overline{z_i} - z_j}\right).$$

This propagator is well defined on $C_{2,0}$, admits a regular extension along the eyelid, but is singular along the Iris.

4. Kontsevich L_∞-quasi isomorphism for the log propagator

Let's consider the log propagator $\mathrm{d}\omega_{\log}$ on $C_{2,0}$

$$\mathrm{d}\omega_{\log}(z_i, z_j) = \frac{1}{2i\pi}\mathrm{d}\log\left(\frac{z_i - z_j}{\overline{z_i} - z_j}\right),$$

and consider the Kontsevich's formula for quantization with the use of this propagator.

We will extend the L_∞ property for this log propagator. To do that we need three ingredients. First the convergence of all coefficients associated to any graph Γ, then the Kontsevich's annulation Lemma 6.6 and finally the Stokes formula.

4.1. **Convergence for the integrals.** Let Γ be a graph with n-vertices of the first type and m vertices of second type on the real line and $(2n+m-2)$ arrows. Because we are looking towards a L_∞-quasi isomorphism, at vertices of the first type start any number of arrows. We consider the corresponding $(2n+m-2)$-form $\Omega_\Gamma^{\log}$. This is a regular form in $C^+_{n,m}$. As noticed in [12], this form admits continuous extension along the interior of the real strata because the propagator himself admits a continuous extension.

We look first at co-dimension 1-strata of complex type $C_p \times C^+_{n-p+1,m}$. These strata correspond to p-vertices gluing together at a complex position.

Suppose the set of points $(z_i)_{i\in[1,p]}$ are gluing to w. Set $z_i, z_j \to w = x + iy$, and write $z_i = w + \epsilon u_i$, with $(u_i)_{i\in[1,p]}$ a normalized complex configuration, and $w, (z_k)_{k\in[p+1,n]}, \overline{1}, \overline{2}, \dots, \overline{m}$ a normalized real configuration. Remark that $s_1 = \epsilon$ and modulo regular terms in $\mathrm{d}\epsilon$ or form with 0 value at $\epsilon = 0$, we get

$$\mathrm{d}\omega_{\log}(z_i, z_j) \underset{\epsilon\mapsto 0}{\sim} \frac{1}{2i\pi}\left(\mathrm{d}\log(u_i - u_j) + \mathrm{d}\log(\epsilon) - \mathrm{d}\log y\right).$$

$$\mathrm{d}\omega_{\log}(z_k, z_j) \underset{\epsilon\mapsto 0}{\sim} \mathrm{d}\omega_{\log}(z_k, w).$$

Then $\mathrm{d}\omega_{\log}(z_i, z_j)$ admits a value (regularized) along the border $C_p \times C^+_{n-p+1,m}$, which is $\mathrm{d}\log(u_i - u_j) - \mathrm{d}\log y$, because the singular term $\mathrm{d}\log\epsilon$ is in a normal direction for the stratum. The term $\mathrm{d}\log(u_i - u_j)$ is a 1-form on C_p and the term $\mathrm{d}\log y$ is a regular 1-form on $C^+_{n-p+1,m}$ (because the concentration takes place at a complex position $y > 0$).

***Lemma* 1.** *If $p \geq 3$, the product of the $2p-3$ forms $\mathrm{d}\log(u_i - u_j)$ is 0 on C_p.*

Proof : Recall $C_p \subset \mathbb{S}^{2p-3}$ the $2p-3$-dimensional sphere. The product above is the restriction at $\mathbb{S}^{2p-3}$ of the $2p-3$-form $\bigwedge \mathrm{d}\omega_{\log}(z_i - z_j)$. This form is holomorphic on $\mathbb{C}^{2p-2}$ which is impossible if $2p - 3 > p - 1$. ∎

***Lemma* 2.** *The $(2n+m-2)$-form Ω_Γ admits a continuous extension near the stratum $C_p \times C^+_{n-p+1,m}$.*

Proof : Arrows which are concentrated define the *interior graph* Γ_{int}, and the other ones define the *exterior graph* when you collapse the interior vertices to a single vertex. In local coordinates, near the stratum $C_p \times C^+_{n-p+1,m}$ the form is as following, modulo regular terms at $\epsilon = 0$

$$\left(\bigwedge^{\sharp int} \mathrm{d}\log(u_i - u_j) + (\frac{\mathrm{d}\epsilon}{\epsilon} - \frac{\mathrm{d}y}{y}) \wedge \left(\sum \bigwedge^{\sharp int-1} \mathrm{d}\log(u_i - u_j)\right)\right) \wedge A(\epsilon)$$

with $\sharp int$ the number of concentrated arrows.

First write $A(\epsilon) = \mathrm{d}\epsilon \wedge B(\epsilon) + C(\epsilon)$, with $C(0) = \bigwedge^{\sharp ext} \mathrm{d}\omega_{\log}$ a form on $C^+_{n-p+1,m}$. If $\sharp ext > 2n-2p+m = \dim(C^+_{n-p+1,m})$, then $C(0) = 0$ because of dimension reason and the singular term in ϵ disappears. So $\sharp ext \leq 2n - 2p + m$ and $\sharp int \geq 2p - 2$. The previous lemma implies $p = 2$ and $\sharp int = 2$, else there is obviously no singular part in ϵ. In that case, because the log propagator is symmetric,

$$\lim_{\epsilon \mapsto 0} \mathrm{d}\omega_{\log}(z_i, z_j) \wedge \mathrm{d}\omega_{\log}(z_j, z_i) = 0$$

and the singular term in ϵ disappears.

■

Lemma 3. *The $(2n + m - 2)$-form Ω_Γ admits a continuous extension at $\overline{C}^+_{n,m}$, and the integral $\int_{C_{n,m}} \Omega_\Gamma$ is convergent.*

Proof : Consider any stratum associated to a tree T, see Fig.2 for an example, and the corresponding analytic germ isomorphic to $S_T \times \mathbb{R}^{codim}_+$. Each codimensional coordinate corresponds to a scaling (magnification in [10]). The propagator $\mathrm{d}\omega_{\log}$ admits continuous extension along real concentrations and if $(z_i)_{i \in I}$ concentrate to a real point x, $z_i = x + \epsilon v_i$, with $v_i \in C^+_{p,q}$ we get

$$\mathrm{d}\log \frac{z_i - z_j}{\overline{z_i} - z_j} \sim \mathrm{d}\log \frac{v_i - v_j}{\overline{v_i} - v_j}.$$

We deduce that no singularities appear in this co-dimensional coordinate ϵ. Singularities appear only when you concentrate any points at a complex position in the upper half plane as explained in [12]. They correspond to black interior vertices in the tree, (see Fig.2 where we get 4 potential singular coordinates at $\epsilon_i = 0$). By applying separately the previous lemma for each complex concentration, we get the result.

Idée : appliquer successivement le cas précédent, dans les amas. C'est un peu pénible à écrire, mais j'ai un brouillon.

■

4.2. **Stokes' formula.** The main ingredient for the L^∞ property is to consider Γ a graph with n vertices of the first type and m vertices of second type on the real line and $(2n + m - 3)$ arrows. We consider the corresponding closed $(2n + m - 3)$-form $\Omega^{\log}_\Gamma$ on a $(2n + m - 2)$-dimensional space $C^+_{n,m}$. Such forms are singular along codimension 1 strata (for example the propagator himself is singular near the iris C_2) but tangential value is well defined as we will see now; either we take the residue of the form or the standard restriction value.

Lemma 4. *Tangential restricted value of $\Omega^{\log}_\Gamma$ to dimension 1 real-strata is well defined except if you concentrate one arrow and two points, in that case the residue value is well defined.*

The codimension 1 border are of two types : complex $C_p \times C^+_{n-p+1,m}$ or real $C_{n-n_1,m-m_1+1} \times C^+_{n_1,m_1}$. The form $\Omega^{\log}_\Gamma$ admits a restricted value along the real strata $\Omega^{\log}_{\Gamma^{ext}} \wedge \Omega^{\log}_{\Gamma^{int}}$. Along complex strata, the situation is more delicate. Arguments in lemma 2 proved that $\sharp int \geq 2p-3$ else restricted value exists equal 0. By lemma 1, $p \leq 3$. There are several subcases :

- If $p = 2$ and $\sharp int = 2$, because the log propagator is symmetric,
$$\lim_{\epsilon \mapsto 0} \mathrm{d}\omega_{\log}(z_i, z_j) \wedge \mathrm{d}\omega_{\log}(z_j, z_i) = 0$$
and the restricted value along the strata is 0.
- If $p = 3$ and $\sharp int = 3$, there are two subcases.
  - The concentrated graph admits a cycle of order 2, we recover the previous case.
  - The concentrated graph is a cycle of order 3 and the form is

$$\Big(\bigwedge^3 \mathrm{d}\log(u_i - u_j) + \big(\frac{\mathrm{d}\epsilon}{\epsilon} - \frac{\mathrm{d}y}{y}\big) \wedge \Big(\mathrm{d}\log(u_1 - u_2) \wedge \mathrm{d}\log(u_2 - u_3) + \mathrm{d}\log(u_2 - u_3) \wedge \mathrm{d}\log(u_3 - u_1) + d\log(u_3 - u_1) \wedge \mathrm{d}\log(u_1 - u_2)\Big)\Big) \wedge A(\epsilon). \tag{4}$$

Lemma 1 and Arnold's relation

$$\mathrm{d}\log(u_1 - u_2) \wedge \mathrm{d}\log(u_2 - u_3) + \mathrm{d}\log(u_2 - u_3) \wedge \mathrm{d}\log(u_3 - u_1) + \mathrm{d}\log(u_3 - u_1) \wedge \mathrm{d}\log(u_1 - u_2) = 0, \tag{5}$$

forces the restricted value to 0[1].

So the only contribution to restricted value for complex strata is $p = 2$ and $\sharp int = 1$, In that case, when $\epsilon \mapsto 0$, modulo terms regular in $\mathrm{d}\epsilon$ or with 0-value at $\epsilon = 0$, the form is
$$\left(\mathrm{d}\log(u_i - u_j) + \frac{\mathrm{d}\epsilon}{\epsilon} - \frac{\mathrm{d}y}{y}\right) \wedge A(\epsilon),$$
with $A(\epsilon) = \mathrm{d}\epsilon \wedge B(\epsilon) + C(\epsilon)$, $C(0) = \Omega^{\log}_{\Gamma^{ext}}$ a $(2n + m - 4)$ form on $C^+_{n-1,m}$. So by dimension reason $\frac{\mathrm{d}y}{y} \wedge C(0) = 0$ and modulo terms regular in $\mathrm{d}\epsilon$ or with 0-value at $\epsilon = 0$, we get
$$\left(\mathrm{d}\log(u_i - u_j) + \frac{\mathrm{d}\epsilon}{\epsilon}\right) \wedge \Omega^{\log}_{\Gamma^{ext}}.$$

The tangential restricted value cannot be taken as $(\mathrm{d}\log(u_i - u_j)) \wedge \Omega^{\log}_{\Gamma^{ext}}$ because the $\frac{\mathrm{d}\epsilon}{\epsilon}$ term make it a non intrinsic expression. Instead we will consider the residue value of the singular part; actually locally only the complex part $\frac{\mathrm{d}(z_i - z_j)}{(z_i - z_j)} \wedge \Omega^{\log}_{\Gamma^{ext}}$ contributes and the residue value on $\int_{C^+_{n-1,m}} \Omega^{\log}_{\Gamma^{ext}}$ is well defined.

Remark normal behavior of the connection near singular strata as a logarithmic divergence, as we know for the standard KZ-connection.

Now we apply a Stokes' type theorem for forms with singularities.

■

[1]This remark has been noticed to us by Merkulov

***Lemma* 5.** *The Stokes formula is valid*

$$0=\int_{C^+_{n,m}}\mathrm{d}\Omega^{\log}_\Gamma=\sum\int_{C_{n-n_1,m-m_1+1}\times C^+_{n_1,m_1}}\Omega^{\log}_\Gamma+\sum\int_{C^+_{n-1,m}}\Omega^{\log}_{\Gamma^{ext}}$$

.

Proof : We get $\mathrm{d}\Omega^{\log}_\Gamma=0$. The right hand side is convergent by previous lemmas. Let's prove it is equal to 0. We apply Stokes' formula for a manifold with corner and a form with (complex) singularities.

For dimension reason only codimension 1 strata has to be considered; indeed in case of $k>1$ concentrations of 2 points, the singular contribution looks like $\frac{\mathrm{d}\epsilon_1}{\epsilon_1}\wedge\ldots\wedge\frac{\mathrm{d}\epsilon_k}{\epsilon_k}\wedge\Omega^{\log}_{\Gamma^{ext}}$ where $\Omega^{\log}_{\Gamma^{ext}}$ is a $(2n+m-3-k)$-forms on $C^+_{n-k,m}$ which is of dimension $(2n+m-2-2k)$.

By the previous lemma restriction to co-dimension 1 real strata are well defined. For concentration of two points (says z_i,z_j) and one arrow the contribution the residue value contributes like $\Omega^{\log}_{\Gamma^{ext}}$ on $C^+_{n-1,m}$.

Trouver une référence pour cette affirmation ■

4.3. L_∞**-quasi-isomorphism.** We conclude this subsection by the L_∞-quasi-isomorphism formula with the log propagator.

***Theorem* 1.** *The Kontsevich construction with the* log*-propagator is completely valid.*

Proof : The convergence of integrals and Stokes' formula have been proved. The annulation Lemma 6.6 in [10] is a consequence of the existence of a restricted value of $\Omega^{\log}_\Gamma$ along the border and of the Lemma 1.

■

5. The Drinfeld associator Φ_{KZ}

We now extend results of [5], with the use of the log propagator. The coefficients $\int_{C_{n,2}}\Omega^{\log}_\Gamma$ are still convergent integrals if the 2 special vertices $\overline{1},\overline{2}$ are at general position in $C_{2,0}$. So results of [5] extend for this log propagator. We get a flat connection $\Omega^{\log}_2$ on $C_{2,0}$. A priori this connection admits a singularity in a normal direction along the Iris $C_2\subset\overline{C}_{2,0}$. These singularities are coming from an arrow between the vertices of second type. This only appears for a trivial extended graph, i.e. a graph with no vertex of the first type. This corresponds to the degree 1 connection terms in $\mathfrak{tder}_2$. So we can write at $\xi=(z_1,z_2)\in C_{2,0}$

$$\Omega^{\log}_2=\mathrm{d}\omega_{\log}(z_1,z_2)(y,0)+\mathrm{d}\omega_{\log}(z_2,z_1)(0,x)+(\Omega^{\log}_2)_{\geq 2},$$

with $(\Omega^{\log}_2)_{\geq 2}$ continuous on $\overline{C}_{2,0}$. We get same kind of results for the n-points connection.

We regularize the parallel transport in a similar way as Kontsevich integral for tangle.

The complex propagator at infinity is $\frac{1}{2i\pi}\mathrm{d}\log(z_i - z_j)$. The three point connection at infinity on C_3 is easy to compute because there is no antiholomorphic part in the form Ω_Γ^∞, except for trivalent trees with no interior vertex. So $(\Omega_3^{\log})_{\geq 2}$ is 0 at infinity and we get the connection on C_3 valued in $\mathfrak{t}_3 \subset \mathfrak{krv}_3$,where $\mathfrak{t}_3$ is the braid algebra ;

$$\frac{1}{2i\pi}\left(\frac{\mathrm{d}(z_2 - z_1)}{z_2 - z_1}(y, x, 0) + \frac{\mathrm{d}(z_3 - z_1)}{z_3 - z_1}(z, 0, x) + \frac{\mathrm{d}(z_3 - z_2)}{z_3 - z_2}(0, z, y)\right) = \Omega_{KZ}.$$

The connection on $[0, 1]$ is $\frac{1}{2i\pi}\left(\frac{\mathrm{d}u}{u}(y, x, 0) + \frac{\mathrm{d}(u-1)}{u-1}(0, z, y)\right)$. So $(\Omega_3^{\log})^\infty = \Omega_{KZ}$ is the KZ-connection which admits singularities in the normal direction along the stratum corresponding to the collapsing of two points.

Let $T_{KZ}(\epsilon)$ be the parallel transport for $\mathrm{d}g = \Omega_2^{\log} g$ from the corner to the complex position $z_1 \sim i, z_2 \sim i$ with $z_2 - z_1 \sim \epsilon e^{i\theta}$, i.e. (z_1, z_2) is at position θ on the iris of $C_{2,0}$. Because the connection at infinity is equivalent to $\frac{\mathrm{d}\epsilon}{2i\pi\epsilon}(y, x) = \frac{\mathrm{d}\epsilon}{2i\pi\epsilon}t_{1,2}$, we get the asymptotic (integration is from $1 \to \epsilon$)

$$T_{KZ}(\epsilon) \sim \epsilon^{+\frac{t_{1,2}}{2i\pi}} C,$$

with $C \in TAut_2$.

***Definition* 2.** *Let define* $F_{KZ} = \lim_{\epsilon \mapsto 0} \epsilon^{-\frac{t_{1,2}}{2i\pi}} T_{KZ}(\epsilon)$.

***Theorem* 2.** *The associator corresponding to the* log *propagator is just the* Φ_{KZ} *associator. We get* $\Phi_{KZ} F_{KZ}^{12,3} F_{KZ}^{1,2} = F_{KZ}^{1,23} F_{KZ}^{2,3}$.

Proof : The argument is similar as [9] §2 and [5]. We consider the parallel transport for 3 points z_1, z_2, z_3 starting from a position $(0, s, 1)$ on the real axis and going to infinity. When $s \mapsto 0$ we get a solution of $\mathrm{d}g = \Omega_3^{\log} g$ with asymptotic

$$(6)\quad (z_3 - z_1)^{+\frac{t_{12,3}}{2i\pi}} F_{KZ}^{12,3} (z_2 - z_1)^{+\frac{t_{1,2}}{2i\pi}} F_{KZ}^{1,2} = (z_3 - z_1)^{+\frac{t_{12,3}}{2i\pi}} (z_2 - z_1)^{+\frac{t_{1,2}}{2i\pi}} F_{KZ}^{12,3} F_{KZ}^{1,2}.$$

Here in passing to the second equality we used that $t_{1,2}$ commute to $F_{KZ}^{12,3}$. When $s \mapsto 1$ we get a asymptotic

$$(7)\quad (z_3 - z_1)^{+\frac{t_{12,3}}{2i\pi}} F_{KZ}^{1,23} (z_3 - z_2)^{+\frac{t_{2,3}}{2i\pi}} F_{KZ}^{2,3} = (z_3 - z_1)^{+\frac{t_{1,23}}{2i\pi}} (z_3 - z_2)^{+\frac{t_{2,3}}{2i\pi}} F_{KZ}^{1,23} F_{KZ}^{2,3}.$$

As in [9] §2 set W_1 and W_2 the solutions of the KZ-equation $\mathrm{d}g = \Omega_{KZ} g$ with asymptotic $W_1 \sim (z_2 - z_1)^{+\frac{t_{1,2}}{2i\pi}} (z_3 - z_1)^{+\frac{t_{12,3}}{2i\pi}}$ and $W_2 \sim (z_3 - z_2)^{+\frac{t_{2,3}}{2i\pi}} (z_3 - z_1)^{+\frac{t_{1,23}}{2i\pi}}$. Then by construction ([9] §2) we get $W_1 = W_2 \Phi_{KZ}$.

The product $W_1 F_{KZ}^{12,3} F_{KZ}^{1,2}$ has asymptotic (6) and $W_2 F_{KZ}^{1,23} F_{KZ}^{2,3}$ has asymptotic (7). Because the parallel transport is trivial along the real axis, $W_1 F_{KZ}^{12,3} F_{KZ}^{1,2}$ and $W_2 F_{KZ}^{1,23} F_{KZ}^{2,3}$ are equal at infinity. We get

$$\Phi_{KZ} F_{KZ}^{12,3} F_{KZ}^{1,2} = F_{KZ}^{1,23} F_{KZ}^{2,3}.$$

■

***Corollary* 1.** *From results of* [4] F_{KZ} *has a nice and explicit expression. We get*

$$F_{KZ}(x) = \mathrm{Ad}(\Phi_{KZ}(x, -x - y)) \cdot x, \quad F_{KZ}(y) = \mathrm{Ad}(e^{-(x+y)/2} \Phi(y, -x - y)) \cdot y$$

6. From arg propagator to log propagator

This section is a conjectural description of a deformation of the log – arg propagator on $C_{2,0}$:

$$\mathrm{d}\Phi_t(z_i,z_j)=\frac{t}{2i\pi}\mathrm{d}\log\left(\frac{|z_i-z_j|}{|\overline{z_i}-z_j|}\right)+\frac{1}{2\pi}\mathrm{d}\arg\left(\frac{z_i-z_j}{\overline{z_i}-z_j}\right).$$

For $t=0$ we recover the standard Kontsevich angle function, for $t=1$ we get the propagator $\mathrm{d}\omega_{\log}$. The propagator $\mathrm{d}\Phi_t$ is symmetric : $\mathrm{d}\Phi_t(z_1,z_2)=\mathrm{d}\Phi_t(z_2,z_1)$.

Coefficients $\int_{C_{n,m}}\Omega_\Gamma$ are not obviously a convergent integral. This integral admits logarithmic divergence along the border and could be regularized with use of standard techniques [6, 7] (partition of unity, and local chart with rescaling parameters as positive real coordinates). Let write $\int^*_{C_{n,m}}\Omega_\Gamma$ for the regularized integral. At the moment there are no evidences for the generalization of the annulation Lemma [10] §6.6, so we state the following conjecture

***Conjecture* 1.** *Let Γ a graph with more than 3 vertices and 3 arrows and Ω_Γ a form obtained by mixing $\mathrm{d}\log(|z_i-z_j|)$ and $\mathrm{d}\arg(z_i-z_j)$ as propagator. Then $\int^*_{C_p}\Omega_\Gamma=0$.*

If this conjecture is true, we would apply results of Section [5] for regularized coefficients to define a connection $\Omega_{s,\xi}$ on $C_{2,0}\times[0,1]$. The propagator at infinity is

$$\mathrm{d}\Phi^\infty_s(z_i,z_j)=\frac{s}{2i\pi}\mathrm{d}\log\left(|z_i-z_j|\right)+\frac{1}{2\pi}\mathrm{d}\arg\left(z_i-z_j\right),$$

and the regularized value of coefficients associated to the wheel diagrams would be 0 because of the conjecture 1. This generalizes the annulation Lemma §6.6 in [10].

$\mathscr{C}$

By [3] the connection at infinity Ω^∞_s would have divergence

$$-W^{comp}_s(x)-W^{comp}_s(y)+W^{comp}_s(x+y),$$

with W^{comp}_s an odd function. By comparing Duflo's function for this propagator, (see [4] for the log case) we deduce the conjectural expression of Drinfeld generators in terms of diagrams, modulo bracket in $\mathfrak{krv}_2$:

$$\sum_{n\geq 1}\frac{\zeta(n)}{(2i\pi)^n n}\sigma_{2n+1}=-\int_0^1\Omega^\infty_s.$$

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