A SIMPLE PROOF OF A RESULT OF FURUSHO

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DRAFT VERSION

Abstract. In this short note we recover, using material of [1], a important result of Furusho [3], the Drinfeld pentagonal equation implies the hexagonal and the symmetry relations.

1. Introduction

In [1] we introduced new methods to study the Kashiwara-Vergne problem and we connected it to Drinfeld's theory of associators. We defined a differential on the Lie algebra of tangential derivations and calculated the second and third cohomological spaces in [1] Theorem 3.1. As a by product of this result we recover results of [3].

2. Free Lie algebras and their derivations

2.1. **Free Lie algebras and derivations.** Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathfrak{lie}_n = \mathfrak{lie}(x_1, \ldots, x_n)$ be the degree completion of the graded free Lie algebra over $\mathbb{K}$ with generators $x_1, \ldots, x_n$ of degree one. We shall denote by $\mathfrak{der}_n$ the Lie algebra of derivations of $\mathfrak{lie}_n$. An element $u \in \mathfrak{der}_n$ is completely determined by its values on generators, $u(x_1), \ldots, u(x_n) \in \mathfrak{lie}_n$. The Lie algebra $\mathfrak{der}_n$ carries a grading induced by the one of $\mathfrak{lie}_n$.

***Definition* 1.** *A derivation $u \in \mathfrak{der}_n$ is called* tangential *if there exist $a_i \in \mathfrak{lie}_n, i = 1, \ldots, n$ such that $u(x_i) = [x_i, a_i]$.*

Tangential derivations form a Lie subalgebra $\mathfrak{tder}_n \subset \mathfrak{der}_n$. Elements of $\mathfrak{tder}_n$ are in one-to-one correspondence with n-tuples of elements of $\mathfrak{lie}_n$, $(a_1, \ldots, a_n)$, which verify the condition that a_k has no linear term in x_k for all k. By abuse of notations, we shall often write $u = (a_1, \ldots, a_n)$ and $\mathfrak{tder}_n$ is a graded Lie algebra. For two elements of $\mathfrak{tder}_n$, $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$, we have $[u, v]_{\mathfrak{tder}} = (c_1, \ldots, c_n)$ with

$$c_k = u(b_k) - u(a_k) + [a_k, b_k]_{\mathfrak{lie}}. \tag{1}$$

***Definition* 2.** *A derivation $u = (a_1, \ldots, a_n) \in \mathfrak{tder}_n$ is called* special *if $u(x) = \sum_i [x_i, a_i] = 0$ for $x = \sum_{i=1}^n x_i$.*

We shall denote the space of special derivations by $\mathfrak{sder}_n$. It is obvious that $\mathfrak{sder}_n \subset \mathfrak{tder}_n$ is a Lie subalgebra. Both $\mathfrak{tder}_n$ and $\mathfrak{sder}_n$ integrate to prouniportent groups denoted by $TAut_n$ and $SAut_n$, respectively. In more detail, $TAut_n$ consists of automorphisms of $\mathfrak{lie}_n$ such that $x_i \mapsto \mathrm{Ad}_{g_i} x_i = g_i x_i g_i^{-1}$, where $g_i \in \exp(\mathfrak{lie}_n)$. Similarly, elements of $SAut_n$ are tangential automorphisms of $\mathfrak{lie}_n$ with an extra property $x = \sum_{i=1}^n x_i \mapsto x$.

The family of Lie algebras $\mathfrak{tder}_n$ is equipped with simplicial Lie homomorphisms $\mathfrak{tder}_n \to \mathfrak{tder}_{n+1}$. For instance, for $u = (a, b) \in \mathfrak{tder}_2$ we define

$$\begin{array}{rcl} u^{1,2} & = & (a(x,y), b(x,y), 0)\,, \\ u^{2,3} & = & (0, a(y,z), b(y,z))\,, \\ u^{12,3} & = & (a(x+y,z), a(x+y,z), b(x+y,z)), \end{array}$$

and similarly for other simplicial maps. These Lie homomorphisms integrate to group homomorphisms of $TAut_n$ and $SAut_n$.

2.2. **Braid Lie algebra.** Consider $t = (y, x) \in \mathfrak{sder}_2$. By composing various simplicial maps we obtain $n(n-1)/2$ elements of $t_{i,j} = t_{j,i} \in \mathfrak{tder}_n$ with non-vanishing components x_i at the jth place and x_j at the ith place.

Elements $t_{i,j} \in \mathfrak{sder}_n$ span a Lie subalgebra isomorphic to the quotient of the free Lie algebra with $n(n-1)/2$ generators by the following relations,

$$[t_{i,j}, t_{k,l}] = 0 \tag{2}$$

for $k, l \neq i, j$, and

$$[t_{i,j} + t_{i,k}, t_{j,k}] = 0 \tag{3}$$

for all triples of distinct indices i, j, k. We denote by $\mathfrak{t}_n$ the Lie algebra defined by relations (2) and (3). This is the pure braid Lie algebra. Recall the following result: the element $\sum_{i<j} t^{i,j}$ belongs to the center of $\mathfrak{sder}_n$.

2.3. **Cyclic words.** Let $Ass_n^+ = \prod_{k=1}^{\infty} Ass^k(x_1, \ldots, x_n)$ be the graded free associative algebra (without unit) with generators $x_1, \ldots, x_n$. Every element $a \in Ass_n^+$ admits a unique decomposition of the form $a = \sum_{i=1}^n (\partial_i a) x_i$, where $a_i \in Ass_n$ (Ass_n is a free associative algebra with unit).

We define the graded vector space cy_n as a quotient

$$\mathrm{cy}_n = Ass_n^+ / \langle (ab - ba); a, b \in Ass_n \rangle.$$

Here $\langle (ab-ba); a, b \in Ass_n \rangle$ is the subspace of Ass_n^+ spanned by commutators. The multiplication map of Ass_n^+ does not descend to cy_n which only has a structure of a graded vector space. We shall denote by $\mathrm{tr} : Ass_n^+ \to \mathrm{cy}_n$ the natural projection. By definition, we have $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in Ass_n$ imitating the defining property of trace. In general, graded components of cy_n are spanned by words of a given length modulo cyclic permutations.

***Example* 1.** *The space* cy_1 *is isomorphic to the space of formal power series in one variable without constant term,* $\mathrm{cy}_1 \cong xk[[x]]$. *This isomorphism is given by the following formula,*

$$f(x) = \sum_{k=1}^{\infty} f_k x^k \mapsto \sum_{k=1}^{\infty} f_k \mathrm{tr}(x^k).$$

2.4. **Divergence.** Let $u = (a_1, \ldots, a_n) \in \mathfrak{tder}_n$. We define the divergence as

$$\mathrm{div}(u) = \sum_{i=1}^{n} \mathrm{tr}(x_i(\partial_i a_i)).$$

It is a 1-cocycle of $\mathfrak{tder}_n$ with values in cy_n (see Proposition 3.6 in [1]). Actually this

We define $\mathfrak{krv}_n \subset \mathfrak{sder}_n \subset \mathfrak{tder}_n$ as the Lie algebra of special derivation with vanishing divergence. Hence, $u = (a_1, \ldots, a_n) \in \mathfrak{krv}_n$ is a solution of two equations:

$\sum_{i=1}^n [x_i, a_i] = 0$ and $\sum_{i=1}^n \mathrm{tr}(x_i(\partial_i a_i)) = 0$. We shall denote by $KRV_n = \exp(\mathfrak{krv}_n)$ the corresponding prounipotent group.

In [1] we have defined a differential $\mathrm{d} : \mathfrak{tder}_n \to \mathfrak{tder}_{n+1}$ and computed some cohomology group. Recall the formula

(4) $\mathrm{d}\, u = u^{2,3,\ldots,n+1} - u^{12,\ldots,n,n+1} + \cdots + (-1)^n u^{1,2,\ldots,n-1,n(n+1)} + (-1)^{n+1} u^{1,2,\ldots,n}$.

***Example* 2.** *For* $u \in \mathfrak{tder}_2$ *we get* $\mathrm{d}\, u = u^{2,3} - u^{12,3} + u^{1,23} - u^{1,2}$. *For* $u \in \mathfrak{tder}_3$ *we obtain* $\mathrm{d}\, u = u^{2,3,4} - u^{12,3,4} + u^{1,23,4} - u^{1,2,34} + u^{1,2,3}$.

2.5. **General projection.** Let $\mathfrak{krv}_n \subset \mathfrak{sder}_n \subset \mathfrak{tder}_n$ the Lie algebra of special derivation with divergence 0. Let denote $\widehat{\mathfrak{krv}_n}$ the extended Lie algebra of special derivations u, such that it exists $f \in xk[[x]]$ such that

$$\mathrm{div}(u) = \mathrm{tr}\big(f(x_1) + f(x_2) \ldots + f(x_n) - f(x_1 + \ldots + x_n)\big).$$

Such f are necessarily odd by [1] Proposition 4.1 and $\mathfrak{krv}_n$ is an ideal of $\widehat{\mathfrak{krv}_n}$.

Let's define π_n the map from $\mathfrak{sder}_n$ to $\mathfrak{sder}_{n-1}$ by

$$\pi_n(a_1, \ldots, a_n)|_{(x_1,\ldots,x_{n-1})} = (a_1 - a_n, a_2 - a_n, \cdots, a_{n-1} - a_n)|_{(x_1,\ldots,x_n)}$$

with $x_n = -x_1 - \ldots - x_{n-1}$.

It is well known that derivation of degree one in $\mathfrak{sder}_n$ are linear combination of $t_{i,j} \in \mathfrak{krv}_n$. We get for example $\pi_n(t_{i,j}) = t_{i,j}$ for $i, j \neq n$ and

$$\pi_n(t_{i,n}) = -\sum_{k=1}^{n-1} t_{k,i} \in \mathfrak{krv}_{n-1}.$$

***Lemma* 1.** *The map* π_n *is a Lie map from* $\mathfrak{sder}_n$ *to* $\mathfrak{sder}_{n-1}$ *and* $\pi_n(\widehat{\mathfrak{krv}_n}) \subset \widehat{\mathfrak{krv}_{n-1}}$.

Proof : The Lie property is left to the reader. Actually π_n is not a Lie map on $\mathfrak{tder}_n$. Let's verify that π_n restricted to $\widehat{\mathfrak{krv}_n}$ is valued to $\widehat{\mathfrak{krv}_{n-1}}$.

The first equation (special derivation) is easy because

$$[x_1, a_1 - a_n] + \ldots + [x_{n-1}, a_{n-1} - a_n] = [x_1, a_1] + \ldots + [x_{n-1}, a_{n-1}] + [x_n, a_n] = 0.$$

Let's now consider the divergence equation and $\deg(u) > 1$. Indeed, if $\deg(u) = 1$ then u is a linear combinaison of $t_{i,j}$ and clearly $\pi_n(t_{i,j}) \in \mathfrak{krv}_{n-1}$.

Suppose

$$\mathrm{div}(a_1, \ldots, a_n)|_{(x_1,\ldots,x_n)} = \mathrm{tr}\Big(f(x_1) + f(x_2) + \ldots + f(x_n) - f(x_1 + x_2 + \cdots + x_n)\Big).$$

Let $a \in \mathfrak{lie}_n$ with $a = \sum_{i=1}^n \partial_i(a) x_i$, then at $(x_1, x_2, \ldots, x_{n-2}, -x_1 - x_2 \ldots - x_{n-1})$ we get

$$a(x_1, x_2, \ldots, x_{n-2}, -x_1 - x_2 \ldots - x_{n-1}) = \sum_{i=1}^{n-1} (\partial_i a - \partial_n a) x_i = \sum_{i=1}^{n-1} (\partial'_i a) x_i.$$

We deduce the corresponding relation after substitution $x_n \mapsto -x_1 - x_2 \ldots - x_{n-1}$,

$$\partial'_i a = (\partial_i a - \partial_n a)_{|(x_1, x_2, \ldots, x_{n-2}, -x_1 - x_2 \cdots - x_{n-1})}.$$

Then

(5)
$$x_1\partial_1'(a_1-a_n)+\ldots+x_{n-1}\partial_{n-1}'(a_{n-1}-a_n) = x_1(\partial_1 a_1-\partial_n a_1)+\ldots+x_{n-1}(\partial_{n-1}a_{n-1}-\partial_n a_{n-1})$$
$$+\, x_1(-\partial_1 a_n+\partial_n a_n)+\cdots+x_{n-1}(-\partial_{n-1}a_n+\partial_n a_n)$$
$$= x_1\partial_1 a_1+\ldots+x_{n-1}\partial_{n-1}a_{n-1} - x_1\partial_n a_1 - \cdots - x_{n-1}\partial_n a_{n-1} - x_n\partial_n a_n$$
$$- x_1\partial_1 a_n - \ldots - x_{n-1}\partial_{n-1}a_n$$

Use the relation $[x_1,a_1]+\ldots+[x_n,a_n]=0$, we get

$$0=\partial_n([x_1,a_1]+\ldots+[x_n,a_n]) = x_1\partial_n(a_1)+\ldots x_n\partial_n(a_n)-a_n.$$

By (5) and $\mathrm{tr}(x_1\partial_1 a_n+\ldots+x_{n-1}\partial_{n-1}a_n)=\mathrm{tr}(a_n-x_n\partial_n(a_n))$ and after evaluating at $(x_1,x_2,\ldots,x_{n-2},-x_1-x_2\ldots-x_{n-1})$ we get

(6) $\mathrm{div}(\pi_n(u)) = \mathrm{tr}(x_1\partial_1 a_1+\ldots+x_{n-1}\partial_n a_n)-2\mathrm{tr}(a_n) =$
$$\mathrm{tr}\Big(f(x_1)+\ldots+f(x_{n-1})+f(x_n)\Big) = \mathrm{tr}\Big(f(x_1)+\ldots+f(x_{n-1})-f(x_1+\ldots+x_{n-1})\Big)$$
because f is odd and $\deg(a_n)>1$. ■

Our maps extend to formal groups KRV_n. In particular the kernel of π_3 is a sub-group of KRV_3. Set $KRV_3^*=\ker(\pi_3)$. Any Drinfeld associators lives in KRV_3^* because $\pi_3(\Phi(t_{1,2},t_{2,3}))=\Phi(t_{1,2},-t_{1,2})=1$.

2.6. **Symmetries of order $n+1$ in $\mathfrak{krv}_n$.** Let $u\in\mathfrak{krv}_n$ and define
$$S_{n+1}(u)=\pi_{n+1}(u^{2,3,\ldots,n+1}).$$
For $u\in\widehat{\mathfrak{krv}_n}$ the previous computation ensures that $S_{n+1}(u)\in\widehat{\mathfrak{krv}_n}$ with the same f in the divergence's formula.

***Lemma* 2.** *Suppose $u=(a,b)\in\mathfrak{tder}_2$, then $\mathrm{d}\,u\in\mathfrak{krv}_3$, if and only if $u=cr+v$, with $r=(y,0)$ and $v\in\widehat{\mathfrak{krv}_2}$.*

Proof : Indeed $f(x,y)=u(x+y)=[x,a]+[y,b]$ verifies $\delta f=(\mathrm{d}\,u)(x+y+z)=0$, so by [1] Theorem 3.1, $f=0$ if $\deg(u)>1$, else $u=c(y,0)+\alpha t$ with $t=(y,x)$. Now put $g(x,y)=\mathrm{div}(u)$, then $\delta(g)=\mathrm{div}(\mathrm{d}\,u)=0$ and $g=\delta(h)$ by [1], Theorem 2.1. ■

***Lemma* 3.** *The map S_{n+1} is a Lie map of order $n+1$.*

Proof : It's an easy verification. Our map is a transformation corresponding to the companion matrix
$$\begin{pmatrix} 0 & 0 & 0 & \ldots & -1\\ 1 & 0 & 0 & \ldots & -1\\ 0 & 1 & 0 & \ldots & -1\\ & & \vdots & & \\ 0 & 0 & \ldots & 1 & -1\end{pmatrix}$$
■

3. Furusho's result

We now recover Furusho's result [3] as consequence of [1] Theorem 3.1.

***Theorem* 1.** *Let $\varphi \in \mathfrak{lie}_2$ a solution of the Drinfeld's pentagonal equation. Then φ is in $\mathfrak{grt}_1$.*

Proof : Recall that φ solves the tangential pentagone equation if

$$\varphi(t_{2,3},t_{3,4})+\varphi(t_{1,2}+t_{1,3},t_{4,2}+t_{4,3})+\varphi(t_{1,2},t_{2,3}) = \varphi(t_{1,3}+t_{2,3},t_{3,4})+\varphi(t_{1,2},t_{2,3}+t_{2,4}).$$

Let's consider $\phi = \varphi(t_{1,2},t_{2,3}) \in \mathfrak{krv}_3$. Then

$$\text{(7)} \quad \mathrm{d}\phi = \varphi(t_{2,3},t_{3,4}) + \varphi(t_{1,2}+t_{1,3},t_{4,2}+t_{4,3}) + \varphi(t_{1,2},t_{2,3}) - \varphi(t_{1,3}+t_{2,3},t_{3,4}) - \varphi(t_{1,2},t_{2,3}+t_{2,4}) = 0$$

by the pentagonal equation with d the differential define on $\mathfrak{tder}_n$ [1]. Then by Lemma1 π_4 is a Lie map, we get

$$\pi_4(\mathrm{d}\phi) = \varphi(t_{1,2},t_{2,3}) + \varphi(t_{2,3},t_{1,2}),$$

because as element of $\mathfrak{t}_3 \subset \mathfrak{sder}_3$ (modulo $c = t_{1,2}+t_{1,3}+t_{2,3}$ a central element in $\mathfrak{sder}_3$) $\pi_4(t_{3,4}) = -t_{1,3}-t_{2,3} = t_{1,2}$, $\pi_4(t_{4,2}+t_{4,3}) = -t_{3,2} = t_{1,3}+t_{1,2}$, $\pi_4(t_{3,4}) = -t_{1,3}-t_{2,3}$ and $\pi_4(t_{2,3}+t_{2,4}) = t_{2,3}-t_{2,1}-t_{2,3} = -t_{1,2}$. So φ is antisymmetric and $\phi^{1,2,3}+\phi^{3,2,1} = 0$.

By [1] Theorem 3.1, there exists a unique $f \in \mathfrak{tder}_2$ such that $\mathrm{d}\, f = \phi$, because ϕ has no terms of degree less than 2. We have $(\mathrm{d}\, f)^{3,2,1} = -\mathrm{d}(f^{21})$, because $\mathrm{d}\, f = f^{2,3} - f^{12,3} + f^{1,23} - f^{1,2}$. So by uniqueness we get $f = f^{2,1}$. The hexagonal equation is then perfectly automatic :

$$\text{(8)} \quad (\mathrm{d}\, f)^{1,2,3} + (\mathrm{d}\, f)^{2,3,1} + (\mathrm{d}\, f)^{3,1,2} = \psi(t_{1,2},t_{2,3}) + \psi(t_{2,3},t_{3,1}) + \psi(t_{3,1},t_{1,2}) = 0.$$

Indeed suppose $f = (a,b)$ then

$$\mathrm{d}\, f = -(a^{1,2},b^{1,2},0) - (a^{12,3},a^{12,3},b^{12,3}) + (0,a^{2,3},b^{2,3}) + (a^{1,23},b^{1,23},b^{1,23})$$
$$(\mathrm{d}\, f)^{3,1,2} = -(b^{3,1},0,a^{3,1}) - (a^{31,2},b^{31,2},a^{31,2}) + (a^{1,2},b^{1,2},0) + (b^{3,12},b^{3,12},a^{3,12})$$
$$(\mathrm{d}\, f)^{2,3,1} = -(0,a^{2,3},b^{2,3}) - (b^{23,1},a^{23,1},a^{23,1}) + (b^{3,1},0,a^{3,1}) + (b^{2,31},a^{2,31},b^{2,31}),$$

and it's easy to verify the hexagon equation. ■

***Lemma* 4.** *Reciprocally suppose* $\mathrm{d}\, f$ *solve the hexagone equation (8) then* $f = f^{2,1}$ *if* f *is of degree at least* 3.

Proof : Take $f = (a,b)$ and $g = a - b^{21}$. Then you get by hexagonal equation $g^{12,3} + g^{13,2} - g^{1,23} = 0$ id.

$$g(x+y,z) - g(x,y+z) + g(x+z,y) = 0.$$

Put $x = 0$ you get g antisymmetric. Put $x = y$, you get $g(2x,z) = 2g(x,x+z)$, then $g(2^n x,z) = 2^n g(x,(2^n-1)x+z)$ and by $x \mapsto 2^{-n}x$ you get $g(x,z) = 2^n g(2^{-n}x,(1-2^{-n})x+z)$. Put $n \mapsto \infty$, you get g is degree one in x and by antisymmetry you get $g(x,y) = [x,y]$. If f is of degree at least 3, $g = 0$ and $f^{21} = f$.■

4. Symmetries of order 3

In this section we investigate some properties of our symmetries.

4.1. **Symmetries and differential.**

***Proposition* 1.** *For $f \in \widehat{\mathfrak{krv}_2}$ and $\phi \in \mathfrak{krv}_3$ we have*

$$S_4(\mathrm{d}\, f) + \mathrm{d}\,(S_3(f)) = (S_3 f - f)^{2,3} \tag{9}$$

$$S_5(\mathrm{d}\phi) + \mathrm{d}\,(S_4(\phi)) = (S_4\phi + \phi)^{2,3,4} \tag{10}$$

Proof : We first prove (9). Take $f = (a, b)$, then $\mathrm{d}\, f = f^{2,3} - f^{12,3} + f^{1,23} - f^{1,2}$, i.e.

$$\mathrm{d}\, f = (0, a, b)_{y,z} - (a, a, b)_{x+y,z} + (a, b, b)_{x,y+z} - (a, b, 0)_{x,y}$$

and $S_3(f) = (-b, a - b)_{y,-x-y}$. For $x + y + z + t = 0$ you get then

$$\mathrm{d}\, S_3(f) = (0, -b, a-b)_{z,x+t} - (-b, -b, a-b)_{z,t} + (-b, a-b, a-b)_{y+z,t} - (-b, a-b, 0)_{y,z+t}.$$

By definition $S_4(\mathrm{d}\, f) = \pi_4((\mathrm{d}\, f)^{2,3,4})$ so

$$S_4(\mathrm{d}\, f) = (-b, -b, a - b)_{z,t} - (-b, a - b, a - b)_{y+z,t} + (-b, a - b, 0)_{y,z+t} - (0, a, b)_{y,z}$$

you deduce

$$S_4(\mathrm{d}\, f) + \mathrm{d}\,(S_3(f)) = (0, -b, a - b)_{z,x+t} - (0, a, b)_{y,z} = (S_3 f - f)^{2,3}.$$

We prove now (10). Take $\phi = (a, b, c)_{x,y,z}$. Then $S_4(\phi) = (-c, a-c, b-c)_{y,z,-x-y-z}$, and

$$\mathrm{d}\, \phi = (0, a, b, c)_{y,z,t} - (a, a, b, c)_{x+y,z,t} + (a, b, b, c)_{x,y+z,t} - (a, b, c, c)_{x,y,z+t} + (a, b, c, 0)_{x,y,z}.$$

For $x + y + z + t + w = 0$ you get

$$\begin{aligned}\mathrm{d}\, S_4(\phi) = (0, -c, a-c, b-c)_{z,t,x+w} - (-c, -c, a-c, b-c)_{z,t,w} + (-c, a-c, a-c, b-c)_{y+z,t,w} \\ - (-c, a - c, b - c, b - c)_{y,z+t,w} + (-c, a - c, b - c)_{y,z,w+t}\end{aligned}$$

and

$$\begin{aligned}S_5(\mathrm{d}\, \phi) = (-c, -c, a-c, b-c)_{z,t,w} - (-c, a-c, a-c, b-c)_{y+z,t,w} + (-c, a-c, b-c, b-c)_{y,z+t,w} \\ - (-c, a - c, b - c, 0)_{y,z,t+w} + (0, a, b, c)_{y,z,t}.\end{aligned}$$

So you get

$$S_5(\mathrm{d}\phi) + \mathrm{d}\,(S_4(\phi)) = (0, -c, a - c, b - c)_{z,t,-y-z-t} + (0, a, b, c)_{y,z,t} = (\phi + S_4\phi)^{2,3,4}.$$

■

4.2. **Σ_3-invariant action on associators.** We have described on $\widehat{\mathfrak{krv}_2}$, a symmetry of order 2 , $f \mapsto f^{2,1}$ and a symmetry of order 3, $f \mapsto S_3(f)$. The group generated by those symmetries is of order 6, isomorphic to the permutations group Σ_3. Similar symmetries appear in Drinfeld's seminal work [2].

***Lemma* 5.** *The Lie algebra $\mathfrak{grt}_1$ has the two symmetries.*

Proof : Recall that we constructed a Lie map from $\mathfrak{grt}_1$ into $\widehat{\mathfrak{krv}_2}$ by

$$\psi \in \mathfrak{grt}_1 \mapsto \Psi = (\psi(-x-y,x), \psi(-x-y,y)$$

and we proved that $\mathrm{d}\Psi = \psi(t_{1,2}, t_{2,3}) \in \mathfrak{krv}_3$. We have $\Psi = \Psi^{2,1}$ and

$$\pi_3(\mathrm{d}\Psi) = \Psi - S_3(\Psi) = \pi_3(\psi(t_{1,2}, t_{2,3})) = 0$$

because $\pi_3(t_{2,3}) = -t_{1,2}$. So Ψ is S_3-invariant. ∎

We investigate the action of $\widehat{KRV_2}^{\Sigma_3}$ on associators of KRV_3. Let's put $KRV_3^* = \ker(\pi_3)$.

***Proposition* 2.** *The group $\widehat{KRV_2}^{S_3}$ acts almost freely and transitively on Pentagon solution in KRV_3^*. The group $\widehat{KRV_2}^{\Sigma_3}$ acts almost freely and transitively on associators of KRV_3^*. Every Drinfeld's associator lives in KRV_3^*.*

Proof : Let's recall the Pentagonal equation for $\Phi \in KRV_3$:

$$\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{12,3,4}\Phi^{1,2,34}. \tag{11}$$

We know by Theorem 7.1 [1] that $\widehat{KRV_2}$ acts almost freely and transitively on Pentagon solution in KRV_3 and by Proposition 8.8 [1] that $\widehat{KRV_2}^{S_2}$ acts almost freely and transitively on associators of KRV_3.

For $\pi_3(\Phi) = 1$ then $\pi_4(\Delta^{1,2}\Phi) = \pi_4(\Phi^{12,3,4}) = \Delta^{1,2}\pi_3(\Phi) = 1$, because $\pi_4 \circ \Delta^{1,2} = \Delta^{1,2} \circ \pi_3$. Apply π_4 to equation (11), you get $S_4(\Phi)\Phi = 1$.[1] For Φ an associator $\Phi^{3,2,1}\Phi = 1$, $S_4(\Phi) = \Phi^{3,2,1}$ and S_4 acts on associators as a symmetry of order 2.

Take $g \in \widehat{KRV_2}$ and $\Phi \in KRV_3^*$. Then $g \cdot \Phi = g^{2,3}g^{1,23}\Phi(g^{1,2}g^{12,3})^{-1}$ and $\pi_3(g \cdot \Phi) = g^{-1}S_3(g)$. So $g \cdot \Phi \in KRV_3^*$ iff $S_3(g) = g$. We conclude $\widehat{KRV_2}^{S_3}$ acts almost freely and transitively on Pentagon solution in KRV_3^*. ∎

***Remark* 1.** *Recall that any associator in KRV_3 is solved by a Kashiwara-Vergne twist, that is $\Phi_F = F^{1,23}F^{2,3}(F^{12,3}F^{1,2})^{-1}$. Probably there is a matter to write down directly an Σ_3 action on F as we did for the Σ_2 action in* [1] *§8.*

[1] For the tangential version, you get $S_4(\varphi) + \varphi = 0$

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