FLAT CONNECTION AND TRIVALENT GRAPHS

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1. Introduction

This note is an informal complement of [2] and [3]. This is not for a publication. We describe our results in terms of trivalent graphs. We refer to [6] for more elaborated results.

2. Trivalent trees and special derivations

2.1. **Trivalent trees.** A *tree* is a connected graph without loop, see Fig. 1. External vertices of a tree are the ones of valency 1, the other one are internal vertices. A *trivalent tree*, is a tree with internal vertices of valency 3. So for trivalent tree

$$\sharp\{interior\ vertices\} + 2 = \sharp\{extremal\ vertices\} \quad \text{and} \quad \sharp arrows = 2\sharp vertices - 1$$

Each interior vertex admits a cyclic order on the set of 3 arrows linked the vertex. We allow trivalent tree without internal vertex : it's an arrow with two extremal vertices.

Arrows are not oriented and vertices are not labeled. Extremal arrow are connected to an extremal vertex, and interior arrows connect two interior vertices.

In a similar way you define *binary rooted tree*. Arrows are then oriented, starting from the root. The cyclic order at each interior vertex defines an order on the set of 2 outcoming arrows e_2, e_3. We follow the convention that $e_2 < e_3$ if the cyclic order is (e_1, e_2, e_3) with e_1 the incoming arrow at an interior vertex.

Consider $\mathcal{T}$ the vector space generated by trivalent trees. We consider the two relations on $\mathcal{T}$: the antisymmetry (AS) and the (IHX) relations (see Fig. 2). The (AS) relation corresponds to $\Gamma = -\Gamma^{opp}$ where Γ^{opp} is the trivalent tree Γ with one reverse cyclic order, while the (IHX) relation has a interpretation in terms of Jacobi relation. Let's denote by $\mathcal{I}$ the quotient $\mathcal{T}/ < (AS) + (IHX) >$.

Consider $\mathcal{R}$ the vector space of rooted binary trees with a cyclic order at interior vertices and at the root modulo the antisymmetry and the Jacobi relations.

If the extremal vertices are colored by $X_1, \ldots, X_n$ we will denote those spaces $\mathcal{T}(X_1, \ldots, X_n)$ etc..

2.2. **Special derivation associated to a colored trivalent tree.** We recall definition of [4]. For $X_1, \ldots, X_n$ generators of $\mathfrak{lie}_n$, Drinfeld defined $\mathcal{F}(X_1, \ldots, X_n)$ as the quotient $\mathfrak{lie}_n \otimes \mathfrak{lie}_n$ by the subspace generated by $x \otimes y - y \otimes x$ and $[x, y] \otimes z - x \otimes [y, z]$. The image of $x \otimes y$ in $\mathcal{F}(X_1, \ldots, X_n)$ we denote (x, y).

Date: July 23, 2009.

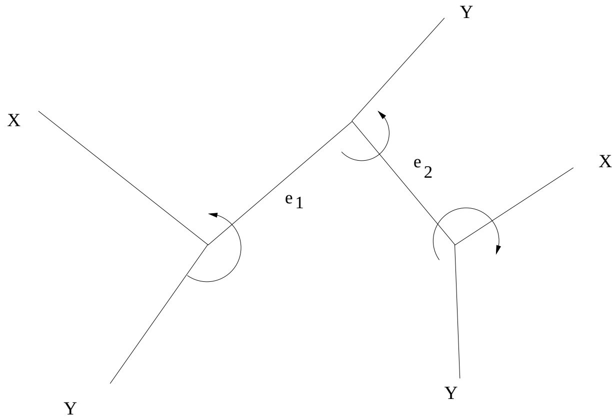

FIGURE 1. A trivalent tree with extremal vertices labelled by letters X, Y.

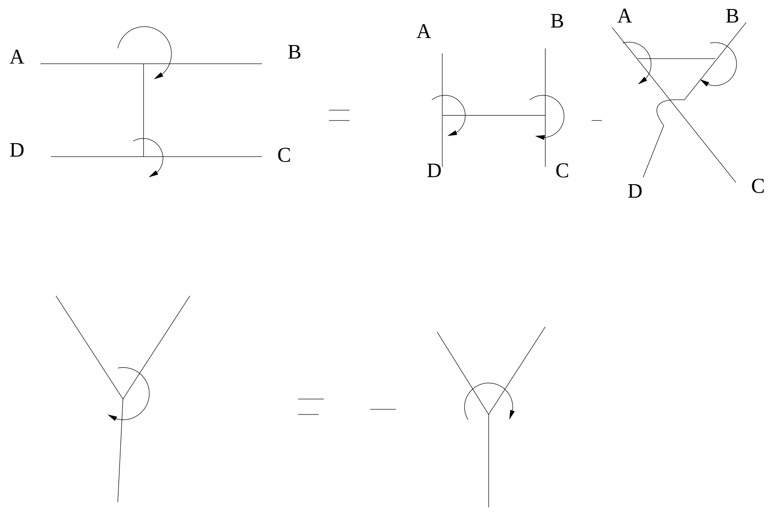

FIGURE 2. The two relation : (IHX) and (AS)

If (.,.) is a invariant nondegenerated scalar product on a Lie algebra $\mathfrak{g}$, then any element in $\mathcal{F}(X_1, \ldots, X_n)$ can be regarded as a function on $\mathfrak{g} \times \mathfrak{g}$.

Actually any $A \in \mathfrak{lie}_n$ defines a element of $\mathcal{R}(X_1, \ldots, X_n)$. For $f \in \mathcal{F}(X_1, \ldots, X_n)$ consider $A, B \in \mathfrak{lie}_n$, such $f = (A, B)$.

To any $f = \sum_i (A_i, B_i) \in \mathcal{F}(X_1, \ldots, X_n)$, with A_i, B_i Lie words, we associate a element of $\mathcal{I}(X_1, \ldots, X_n)$ by adding an arrow between the roots of A_i, B_i. It's easy to verify that Γ doesn't depend of the choice of A, B modulo relations (AS) and (IHX). Reciprocally let $\Gamma \in \mathcal{I}(X_1, \ldots, X_n)$ a trivalent tree with extremal vertices colored by the X_i, then by deleting any arrow, we define two Lie words A, B and $f_\Gamma = (A, B) \in \mathcal{F}(X_1, \ldots, X_n)$. The AS and IHX relations map to 0 in $\mathcal{F}(X_1, \ldots, X_n)$

For example, you get for Fig.1 by deleting arrow e_1 or e_2

$$f_\Gamma = ([X,Y],[[X,Y],Y]) = ([Y,[X,Y]],[X,Y]).$$

2.2.1. *Tangential dérivation u_Γ associated to* Γ. The definition is from [4]. Let Γ a trivalent tree. If $e = (T,U)$ is an extremal arrow with U an extremal vertex, we write $\partial_e(\Gamma)$ the rooted tree obtained by deleting e and U. The root of $\partial_e(\Gamma)$ is T. Because of the cyclic ordering we associate a Lie word $\partial_e(\Gamma)(X_1, X_2, \ldots, X_n)$. We define

$$a_i = \partial_i(\Gamma) = \sum_{\substack{e=(-,U) \\ U \text{ labeled by } X_i}} \partial_e(\Gamma),$$

and $u_\Gamma = (a_1, \ldots, a_n)$. On extends the definition by linearity to $\mathcal{T}(X_1, \ldots, X_n)$, obviously $< (AS)+(IHX) >$ is mapped to 0. By linearity we define u_f a tangential derivation for any $f \in \mathcal{F}(X_1, \ldots, X_n)$.

The following proposition is proved in [4] and seems not to be well known by Lie theory specialists.

***Proposition* 1.** *The tangential derivation u_Γ is a special derivation and any special derivation is uniquely obtained as as u_f with $f \in \mathcal{F}(X_1, \ldots, X_n)$.*

This proposition solved the first homogeneous Kashiwara-Vergne equation. Indeed any solution of $[x_1, a_1] + \ldots + [x_n, a_n] = 0$ is uniquely determined by $a_i = \partial_i(f)$ with $f \in \mathcal{F}(X_1, \ldots, X_n)$.

Let's just prove the direct implication, the opposite implication is the Lemma after Prop. 6.1 in [4].

Proof : Consider $f_\Gamma \in \mathcal{F}(X_1, \ldots, X_n)$, $f_\Gamma = (A,B)$ and the linear term in ϵ for

$$f_\Gamma(X_1 + \epsilon[Y,X_1], X_2 + \epsilon[Y,X_2], \ldots, X_n + \epsilon[Y,X_n]).$$

We get for any $Y \in \mathfrak{lie}_n$

$$(1) \quad (A(X_1 + \epsilon[Y,X_1], \ldots, X_n + \epsilon[Y,X_n]), B(X_1 + \epsilon[Y,X_1], \ldots, X_n + \epsilon[Y,X_n])) = \\ ([Y,A],B) + (A,[Y,B]) = 0 = \\ ([Y,X_1], \partial_1\Gamma) + \ldots + ([Y,X_n], \partial_n\Gamma) = (Y, [X_1, \partial_1\Gamma] + \ldots + [X_n, \partial_n\Gamma])$$

Then you get $[X_1, \partial_1\Gamma] + \ldots + [X_n, \partial_n\Gamma] = 0$. ■

2.3. **Operations on trivalent trees.** We define now the connecting operation. Let Γ_1 and Γ_2 two connected trivalent graphs colored by the X_i. Consider the fundamental graph K_i with 1 interior vertex and 3 extremal vertices colored by X_i, and 3 arrows (e_1, e_2, e_3).

We define $\Gamma_1 \circ_i \Gamma_2$ as the sum of trivalent trees, where we connect each extremal arrow colored by X_i in Γ_1 and Γ_2 by the use of K_i, with the convention that e_1 replace the extremal arrow in Γ_1 and e_2 the one in Γ_2.

We define

$$\Gamma_1 \circ \Gamma_2 = \sum_{i=1}^{n} \Gamma_1 \circ_i \Gamma_2,$$

and extend the definition by linearity.

Proposition 2. *We have* $u_{\Gamma_1 \circ \Gamma_2} = [u_{\Gamma_1}, u_{\Gamma_2}]$.

Proof : Left to the reader . ∎

2.3.1. *Divergence.* Let's now consider trivalent wheels, id. trivalent graphs with 1 oriented wheel (cycle). Let's denote $\mathcal{W}$ the free vector space generated by those wheel type graphs and $\mathcal{J} = \mathcal{W}/ < (IHX) + (AS) >$.

If you connect $\Gamma \in \mathcal{T}(X_1, \ldots, X_n)$ with himself by the use of K_i you get a sum of simple graph of wheel type denoted $\mathrm{tr}_i(\Gamma)$. By convention the wheel is oriented in the direction $e_1 \to e_2$. We write $\mathrm{tr}(\Gamma) = \sum_{i=1}^{n} \mathrm{tr}_i(\Gamma) \in \mathcal{W}$.

Modulo the relation (AS) and (IHX) this defines an element of cy_n (see Fig. 3). We still write $\mathrm{tr}(\Gamma)$ this element if there is no confusion. Note that the orientation of the wheel is important to define this element.

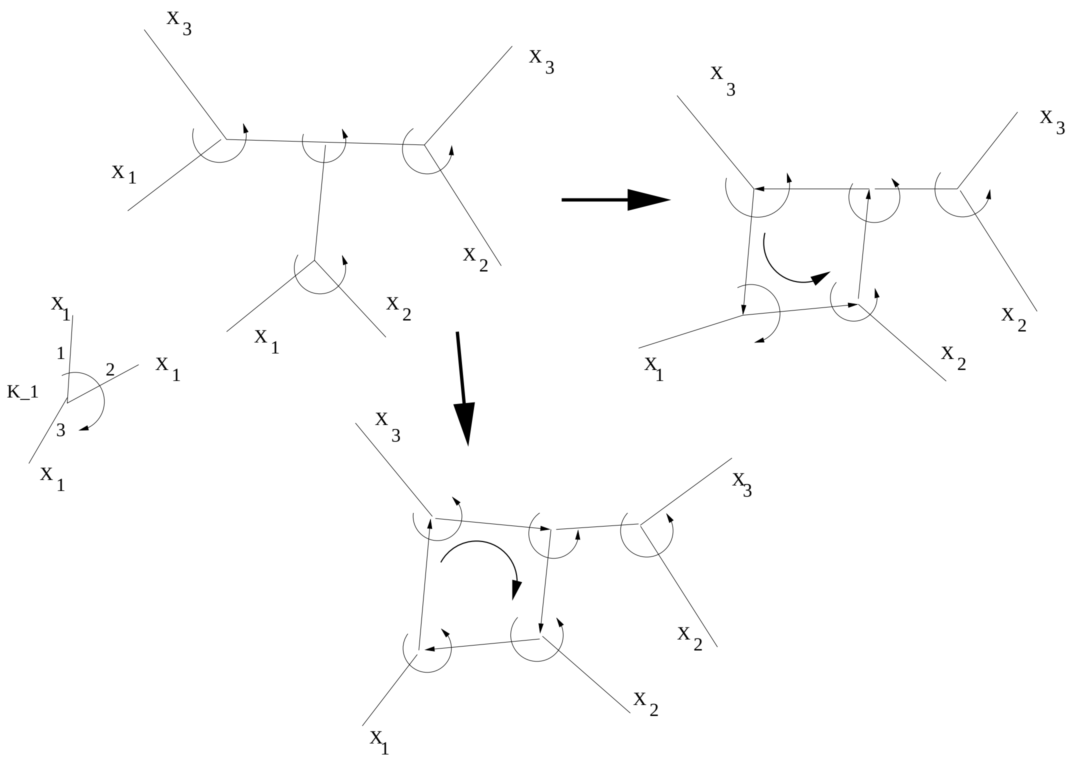

FIGURE 3. Example of a wheel type diagram

The next proposition has an interesting application for quadratic Lie algebras [3]. Indeed it proves that the Kashiwara-Vergne problem [5] in case of quadratic Lie algebras, is solved by the first equation.

Recall that the universal enveloping algebra of $\mathfrak{lie}_n$ is the free associative algebra Ass_n. The canonical anti-automorphism in $U(\mathfrak{lie}_n) = Ass_n$ is denoted by α. We consider the space quad_n of quadratic cyclic words, that is the quotient of space of cy_n (the cyclic words) by the space generated by $u - \alpha(u)$ for $u \in U(\mathfrak{lie}_n) = Ass_n$, that is words of type $a_1 a_2 \dots a_k - (-1)^k a_k a_{k-1} \dots a_1$ with $a_i \in Lie_n$ and $k \geq 1$.

Define the quadratic trace qtr as the natural projection $Ass_n \to cy_n \to \mathrm{quad}_n$. The quadratic trace is the universal trace in case of quadratic Lie algebras, id. Lie algebras with a non degenerated invariant bilinear form.

Note for $a \in \mathfrak{lie}_n$, $\mathrm{qtr}(a^{2n+1}) = 0$. For $u \in \mathfrak{tder}_n$ we define the quadratic divergence as

$$\mathrm{div}_{quad}(u) = \mathrm{qtr}(\mathrm{div}(u)).$$

The Lie algebra $\mathfrak{qkrv}_n$ is the Lie algebra of special derivations with quadratic divergence 0.

Results of [1] extends easily to the quadratic case. In particular we define quadratic associator.

Proposition 3. *We have* $\mathrm{div}(u_\Gamma) = \mathrm{tr}(\Gamma) \in \mathrm{cy}_n$ *and* $\mathrm{div}_{quad}(u_\Gamma) = 0$

Proof : Left to the reader, we just illustrate the proposition with an example. Consider Fig. 3, you get $u_\Gamma = (a_1, a_2, a_3)$ with

$$a_1 = [[[X_1, X_2], [X_2, X_3]], X_3] + [X_2, [[X_2, X_3], [X_3, X_1]]].$$

Then

$$\mathrm{tr}_1(\Gamma) = -\mathrm{tr}(X_3[X_2, X_3]X_2X_1) + \mathrm{tr}(X_2[X_2, X_3]X_3X_1)$$

and Fig. 3 corresponds to $\mathrm{tr}(X_2[X_2, X_3]X_3X_1)$ while $\mathrm{tr}(X_3[X_2, X_3]X_2X_1)$ corresponds to the opposite orientation of the wheel. For quadratic Lie algebras, this is 0. If the cycle has length n then you get $\mathrm{tr}_i(\Gamma) = \mathrm{tr}(a_1 \dots a_n) + (-1)^{n-1}\mathrm{tr}(a_n \dots a_1)$. ■

References

[1] Alekseev, A.; Torossian, C.,– *The Kashiwara-Vergne conjecture and Drinfeld's associators.* arxiv: 0802.4300

[2] Alekseev, A.; Torossian, C.,– *Kontsevich deformation quantization and flat connections* Preprint 2007, arxiv0906.0187

[3] Alekseev, A.; Torossian, C.,–*On triviality of the Kashiwara-Vergne problem for quadratic Lie algebras* preprint 2009.

[4] Drinfeld, V. G.,– *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{Q}/Q)$. (Russian) Algebra i Analiz 2 (1990), no. 4, 149–181; translation in Leningrad Math. J. 2 (1991), no. 4, 829–860

[5] Kashiwara, M. ; Vergne, M.,– The Campbell-Hausdorff formula and invariant hyperfunctions. *Inventiones Math.* **47** (1978), 249–272.

[6] P. Severa and T. Willwacher, *Equivalence of formalities of the little discs operad*, preprint arXiv:0905.1789

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