Axiomatic Approach to Topological Quantum Field Theory

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1 Introduction

The idea of topological invariants defined via path integrals was introduced by A.S. Schwartz (1977) in a special case and by E. Witten (1988) in its full power. To formalize this idea, Witten [Wi] introduced a notion of a Topological Quantum Field Theory (TQFT). Such theories, independent of Riemannian metrics, are rather rare in quantum physics. On the other hand, they admit a simple axiomatic description first suggested by M. Atiyah [At]. This description was inspired by G. Segal's [Se] axioms for a 2-dimensional conformal field theory. The axiomatic formulation of TQFTs makes them suitable for a purely mathematical research combining methods of topology, algebra and mathematical physics. Several authors explored axiomatic foundations of TQFTs (see Quinn [Qu], Turaev [Tu]).

2 Axioms of a TQFT

A $(n+1)$-*dimensional TQFT* $(\mathbf{V}, \tau)$ over a scalar field $\mathbf{k}$ assigns to every closed oriented n-dimensional manifold X a finite dimensional vector space $\mathbf{V}(X)$ over $\mathbf{k}$ and assigns to every cobordism (M, X, Y) a $\mathbf{k}$-linear map

$$\tau(M) = \tau(M, X, Y) : \mathbf{V}(X) \to \mathbf{V}(Y).$$

Here a *cobordism* (M, X, Y) between X and Y is a compact oriented $(n+1)$-dimensional manifold M endowed with a diffeomorphism $\partial M \approx \overline{X} \amalg Y$ (the overline indicates the orientation reversal). All manifolds and cobordisms are supposed to be smooth. A TQFT must satisfy the following axioms.

1. (Naturality) Any orientation-preserving diffeomorphism of closed oriented n-dimensional manifolds $f : X \to X'$ induces an isomorphism $f_\sharp : \mathbf{V}(X) \to \mathbf{V}(X')$. For a diffeomorphism g between the cobordisms (M, X, Y) and (M', X', Y'), the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{V}(X) & \xrightarrow{(g_{|X})_\sharp} & \mathbf{V}(X') \\ {\scriptstyle \tau(M)}\downarrow & & \downarrow{\scriptstyle \tau(M')} \\ \mathbf{V}(Y) & \xrightarrow{(g_{|Y})_\sharp} & \mathbf{V}(Y') \end{array} .$$

2. (Functoriality) If a cobordism (W, X, Z) is obtained by gluing two cobor-

disms (M, X, Y) and (M', Y', Z) along a diffeomorphism $f : Y \to Y'$, then the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{V}(X) & \xrightarrow{\tau(W)} & \mathbf{V}(Z) \\ {\scriptstyle \tau(M)}\downarrow & & \uparrow{\scriptstyle \tau(M')} \\ \mathbf{V}(Y) & \xrightarrow{f_\sharp} & \mathbf{V}(Y') \ . \end{array}$$

3. (Normalization) For any n-dimensional manifold X, the linear map

$$\tau([0,1] \times X) : \mathbf{V}(X) \to \mathbf{V}(X)$$

is identity.

4. (Multiplicativity) There are functorial isomorphisms

$$\mathbf{V}(X \amalg Y) \approx \mathbf{V}(X) \otimes \mathbf{V}(Y) \ ,$$

$$\mathbf{V}(\emptyset) \approx \mathbf{k} \ ,$$

such that the following diagrams are commutative.

$$\begin{array}{ccc} \mathbf{V}((X \amalg Y) \amalg Z) & \approx & (\mathbf{V}(X) \otimes \mathbf{V}(Y)) \otimes \mathbf{V}(Z) \\ \downarrow & & \downarrow \\ \mathbf{V}(X \amalg (Y \amalg Z)) & \approx & \mathbf{V}(X) \otimes (\mathbf{V}(Y) \otimes \mathbf{V}(Z)) \end{array} \ ,$$

$$\begin{array}{ccc} \mathbf{V}(X \amalg \emptyset) & \approx & \mathbf{V}(X) \otimes \mathbf{k} \\ \downarrow & & \downarrow \\ \mathbf{V}(X) & = & \mathbf{V}(X) \end{array} \ .$$

Here $\otimes = \otimes_{\mathbf{k}}$ is the tensor product over $\mathbf{k}$. The vertical maps are respectively the ones induced by the obvious diffeomorphisms, and the standard isomorphisms of vector spaces.

5. (Symmetry) The isomorphism

$$\mathbf{V}(X \amalg Y) \approx \mathbf{V}(Y \amalg X)$$

induced by the obvious diffeomorphism corresponds to the standard isomorphism of vector spaces

$$\mathbf{V}(X) \otimes \mathbf{V}(Y) \approx \mathbf{V}(Y) \otimes \mathbf{V}(X) \ .$$

Given a TQFT $(\mathbf{V}, \tau)$, we obtain an action of the group of diffeomorphisms of a closed oriented n-dimensional manifold X on the vector space $\mathbf{V}(X)$. This action can be used to study this group.

An important feature of a TQFT $(\mathbf{V}, \tau)$ is that it provides numerical invariants of compact oriented $(n+1)$-dimensional manifolds without boundary. Indeed, such a manifold M can be considered as a cobordism between two copies of $\emptyset$ so that $\tau(M) \in Hom_{\mathbf{k}}(\mathbf{k}, \mathbf{k}) = \mathbf{k}$. Any compact oriented $(n+1)$-dimensional manifold M can be considered as a cobordism between $\emptyset$ and ∂M; the TQFT assigns to this cobordism a vector $\tau(M)$ in $Hom_{\mathbf{k}}(\mathbf{k}, \mathbf{V}(\partial M)) = \mathbf{V}(\partial M)$ called the *vacuum vector*.

The manifold $[0, 1] \times X$, considered as a cobordism from $\overline{X} \amalg X$ to $\emptyset$ induces a non singular pairing

$$\mathbf{V}(\overline{X}) \otimes \mathbf{V}(X) \to \mathbf{k} \ .$$

We obtain a functorial isomorphism $\mathbf{V}(\overline{X}) = \mathbf{V}(X)^* = Hom_{\mathbf{k}}(\mathbf{V}(X), \mathbf{k})$.

We now outline definitions of several important classes of TQFTs.

If the scalar field $\mathbf{k}$ has a conjugation and all the vector spaces $\mathbf{V}(X)$ are equipped with natural non degenerate hermitian forms, then the TQFT $(\mathbf{V}, \tau)$

is *hermitian*. If $\mathbf{k} = \mathbf{C}$ is the field of complex numbers and the hermitian forms are positive definite, then the TQFT is *unitary*.

A TQFT $(\mathbf{V}, \tau)$ is *non-degenerate* or *cobordism generated* if for any closed oriented n-dimensional manifold X, the vector space $\mathbf{V}(X)$ is generated by the vacuum vectors derived as above from the manifolds bounded by X.

Fix a Dedekind domain $D \subset \mathbf{C}$. A TQFT $(\mathbf{V}, \tau)$ over $\mathbf{C}$ is *almost D-integral* if it is non-degenerate and there is $d \in \mathbf{C}$ such that $d\,\tau(M) \in D$ for all M with $\partial M = \emptyset$. Given an almost integral TQFT $(\mathbf{V}, \tau)$ and a closed oriented n-dimensional manifold X, we define $S(X)$ to be the D-submodule of $\mathbf{V}(X)$ generated by all the vacuum vectors. This module is preserved under the action of self-diffeomorphisms of X and yields a finer "arithmetic" version of $\mathbf{V}(X)$.

The notion of an $(n+1)$-dimensional TQFT over $\mathbf{k}$ can be reformulated in the categorical language as a symmetric monoidal functor from the category of n-manifolds and $(n+1)$-cobordisms to the category of finite dimensional vector spaces over $\mathbf{k}$. The source category is called the *$(n+1)$-dimensional cobordism category*. Its objects are closed oriented n-dimensional manifolds. Its morphisms are cobordisms considered up to the following equivalence: cobordisms (M, X, Y) and (M', X, Y) are equivalent if there is a diffeomorphism $M \to M'$ compatible with the diffeomorphisms $\partial M \approx \overline{X} \amalg Y \approx \partial M'$.

3 TQFTs in low dimensions

TQFTs in dimension $0 + 1 = 1$ are in one-to-one correspondence with finite dimensional vector spaces. The correspondence goes by associating with a 1-dimensional TQFT $(\mathbf{V}, \tau)$ the vector space $\mathbf{V}(pt)$ where pt is a point with positive orientation.

Let $(\mathbf{V}, \tau)$ be a 2-dimensional TQFT. The linear map τ associated with a pair of pants (a 2-disc with two holes considered as a cobordism between two circles $S^1 \amalg S^1$ and one circle S^1) defines a commutative multiplication on the vector space $\mathcal{A} = \mathbf{V}(S^1)$. The 2-disc, considered as a cobordism between S^1 and $\emptyset$, induces a non-degenerate trace on the algebra $\mathcal{A}$. This makes $\mathcal{A}$ into a commutative Frobenius algebra (also called a symmetric algebra). This algebra completely determines the TQFT $(\mathbf{V}, \tau)$. Moreover, this construction defines a one-to-one correspondence between equivalence classes of 2-dimensional TQFTs and isomorphism classes of finite dimensional commutative Frobenius algebras, see [Ko].

The formalism of TQFTs was to a great extent motivated by the 3-dimensional case, specifically, Witten's Chern-Simons TQFTs. A mathematical definition of these TQFTs was first given by Reshetikhin and Turaev using the theory of quantum groups. The Witten-Reshetikhin-Turaev 3-dimensional TQFTs do not satisfy exactly the definition above: the naturality and the functoriality axioms only hold up to invertible scalar factors called *framing anomalies*. Such TQFTs are said to be *projective*. In order to get rid of the framing anomalies, one has to add extra structures on the 3-dimensional cobordism category. Usually one endows surfaces X with Lagrangians (maximal isotropic subspaces in $H_1(X; \mathbf{R})$). For 3-cobordisms, several competing - but essentially equivalent - additional structures are considered in the literature: 2-framings ([At]), p_1-structures ([BHMV]), numerical weights (K. Walker, V. Turaev).

Large families of 3-dimensional TQFTs are obtained from so-called modular categories. The latter are constructed from quantum groups at roots of unity or from the skein theory of links. See the article *Quantum Invariants of 3-manifolds* in this Encyclopedia.

4 Additionnal structures

The axiomatic definition of a TQFT extends in various directions. In dimension 2 it is interesting to consider so-called open-closed theories involving 1-manifolds formed by circles and intervals and 2-dimensional cobordisms with boundary (G. Moore, G. Segal). In dimension 3 one often considers cobordisms including framed links and graphs whose components (resp. edges) are labeled with objects of a certain fixed category $\mathcal{C}$. In such a theory, surfaces are endowed with finite sets of points labeled with objects of $\mathcal{C}$ and enriched with tangent directions. In all dimensions one can study manifolds and cobordisms endowed with homotopy classes of mappings to a fixed space (Homotopy Quantum Field Theory in the sense of Turaev). Additional structures on the tangent bundles - spin structures, framings, etc - may be also considered provided the gluing is well defined.

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