Quantum Invariants of 3-manifolds

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1 Introduction

The idea to derive topological invariants of smooth manifolds from partitions functions of certain action functionals was suggested by A. Schwarz (1978) and highlighted by E. Witten (1988). Witten interpreted the Jones polynomial of links in the 3-sphere S^3 as a partition function of the Chern-Simons field theory. Witten conjectured the existence of mathematically defined topological invariants of 3-manifolds generalizing the Jones polynomial (or rather its values in complex roots of unity) to links in arbitrary closed oriented 3-manifolds. A rigorous construction of such invariants was given by N. Reshetikhin and V. Turaev (1989) using the theory of quantum groups. The Witten-Reshetikhin-Turaev invariants of 3-manifolds, called also the *quantum invariants*, extend to a Topological Quantum Field Theory (TQFT) in dimension 3.

2 Ribbon and modular categories

The Reshetikhin-Turaev approach begins with fixing suitable algebraic data which are best described in terms of monoidal categories. Let $\mathcal{C}$ be a monoidal category (i.e., a category with associative tensor product and unit object $\mathbb{1}$). A *braiding* in $\mathcal{C}$ assigns to any objects $V, W \in \mathcal{C}$ an invertible morphism $c_{V,W} : V \otimes W \to W \otimes V$ such that for any $U, V, W \in \mathcal{C}$,

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W),$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}).$$

A *twist* in $\mathcal{C}$ assigns to any object $V \in \mathcal{C}$ an invertible morphism $\theta_V : V \to V$ such that for any $V, W \in \mathcal{C}$,

$$\theta_{V\otimes W} = c_{W,V}\, c_{V,W}\, (\theta_V \otimes \theta_W).$$

A *duality* in $\mathcal{C}$ assigns to any object $V \in \mathcal{C}$ a "dual" object $V^* \in \mathcal{C}$ and evaluation and co-evaluation morphisms $d_V : V^* \otimes V \to \mathbb{1}$, $b_V : \mathbb{1} \to V \otimes V^*$ such that

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V,$$

$$(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}.$$

The category $\mathcal{C}$ with duality, braiding and twist is *ribbon* if for any $V \in \mathcal{C}$,

$$(\theta_V \otimes \mathrm{id}_{V^*})\, b_V = (\mathrm{id}_V \otimes \theta_{V^*})\, b_V .$$

For an endomorphism $f : V \to V$ of an object $V \in \mathcal{C}$, we define its *trace* $\mathrm{tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ to be the following composition:

$$\mathrm{tr}(f) = d_V\, c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*})\, b_V : \mathbb{1} \to \mathbb{1}.$$

This trace shares a number of properties of the standard trace of matrices, in particular, $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ and $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$. For an object $V \in \mathcal{C}$, set

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V) = d_V\, c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*})\, b_V .$$

Ribbon categories nicely fit the theory of knots and links in S^3. A *link* $L \subset S^3$ is a closed 1-dimensional submanifold of S^3. (A manifold is *closed* if it is compact and has no boundary). A link is *oriented* (resp. *framed*) if all its components are oriented (resp. provided with a homotopy class of non-singular normal vector fields). Given a framed oriented link $L \subset S^3$ whose components are labeled with objects of a ribbon category $\mathcal{C}$, one defines a tensor $\langle L \rangle \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. To compute $\langle L \rangle$, present L by a plane diagram with only double transversal crossings such that the framing of L is orthogonal to the plane. Each double point of the diagram is an intersection of two branches of L, going over and under, respectively. Associate with such a crossing the tensor $(c_{V,W})^{\pm 1}$ where $V, W \in \mathcal{C}$ are the labels of these two branches and ± 1 is the sign of the crossing determined by the orientation of L. We also associate certain tensors with the points of the diagram where the tangent line is parallel to a fixed axis on the plane. These tensors are derived from the evaluation and co-evaluation morphisms and the twists. Finally, all these tensors are contracted into a single element $\langle L\rangle \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. It does not depend on the intermediate choices and is preserved under isotopy of L in S^3. For the trivial knot $O(V)$ with framing 0 and label $V \in \mathcal{C}$, we have $\langle O(V)\rangle = \dim(V)$.

Further constructions need the notion of a tangle. An (oriented) *tangle* is a compact (oriented) 1-dimensional submanifolds of $\mathbf{R}^2 \times [0,1]$ with endpoints on $\mathbf{R} \times 0 \times \{0,1\}$. Near each its endpoint, an oriented tangle T is directed either down or up and acquires thus a sign $+1$ or -1. One can view T as a morphism from the sequence of $\pm 1's$ associated with its bottom ends to the sequence of $\pm 1's$ associated with its top ends. Tangles can be composed by putting one on the top of the other. This defines a category of tangles $\mathcal{T}$ whose objects are finite sequences of $\pm 1's$ and whose morphisms are isotopy classes of framed oriented tangles. Given a ribbon category $\mathcal{C}$, we can consider $\mathcal{C}$-labeled tangles, i.e., (framed oriented) tangles whose components are labeled with objects of $\mathcal{C}$. They form a category $\mathcal{T}_{\mathcal{C}}$. Links appear here as tangles without endpoints, i.e., as morphisms $\emptyset \to \emptyset$. The link invariant $\langle L\rangle$ generalizes to a functor $\langle\cdot\rangle : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$.

To define 3-manifold invariants, we need *modular categories* ([Tu]). Let $\mathbf{k}$ be a field. A monoidal category $\mathcal{C}$ is $\mathbf{k}$-*additive* if its Hom sets are $\mathbf{k}$-vector spaces, composition and tensor product of morphisms are bilinear, and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathbf{k}$. An object $V \in \mathcal{C}$ is *simple* if $\mathrm{End}_{\mathcal{C}}(V) = \mathbf{k}$. A modular category is a $\mathbf{k}$-additive ribbon category $\mathcal{C}$ with a finite family of simple objects $\{V_\lambda\}_\lambda$ such that (i) for any object $V \in \mathcal{C}$ there is a finite expansion $\mathrm{id}_V = \sum_i f_i g_i$ for certain morphisms $g_i : V \to V_{\lambda_i}, f_i : V_{\lambda_i} \to V$ and (ii) the S-*matrix* $(S_{\lambda,\mu})$ is invertible over $\mathbf{k}$ where $S_{\lambda,\mu} = \mathrm{tr}(c_{V_\lambda,V_\mu} c_{V_\mu,V_\lambda})$. Note that $S_{\lambda,\mu} = \langle H(\lambda,\mu)\rangle$ where $H(\lambda,\mu)$ is the oriented Hopf link with framing 0, linking number $+1$ and labels V_λ, V_μ.

Axiom (i) implies that every simple object in $\mathcal{C}$ is isomorphic to exactly one of V_λ. In most interesting cases (when there is a well defined direct summation in $\mathcal{C}$) this axiom may be rephrased by saying that $\mathcal{C}$ is finite semisimple, i.e., $\mathcal{C}$ has a finite set of isomorphism classes of simple objects and all objects of $\mathcal{C}$ are direct sums of simple objects. A weaker version of the axiom (ii) yields *premodular* categories (A. Bruguières, M. Müger).

The invariant $\langle\cdot\rangle$ of links and tangles extends by linearity to the case where labels are finite linear combinations of objects of $\mathcal{C}$ with coefficients in $\mathbf{k}$. Such a linear combination $\Omega = \sum_\lambda \dim(V_\lambda)V_\lambda$ is called the *Kirby color*. It has the following sliding property: For any object $V \in \mathcal{C}$, the two tangles in Figure 1 yield the same morphism $V \to V$. Here the dashed line represents an arc on the closed component labeled by Ω. This arc can be knotted or linked with other components of the tangle (not shown on the picture).

3 Invariants of closed 3-manifolds

Given an embedded solid torus $g : S^1 \times D^2 \hookrightarrow S^3$ where D^2 is a 2-disk and $S^1 = \partial D^2$, we can build a 3-manifold as follows. Remove from S^3 the interior of $g(S^1 \times D^2)$ and glue back the solid torus $D^2 \times S^1$ along $g|_{S^1 \times S^1}$. This process is known as *surgery*. The resulting 3-manifold depends only on the isotopy class of the framed knot represented by g. More generally, a surgery on a framed link $L = \cup_{i=1}^m L_i$ in S^3 with m components yields a closed oriented 3-manifold M_L. A theorem of W. Lickorish and A. Wallace asserts that any closed connected oriented 3-manifold is homeomorphic to M_L for some L. R. Kirby proved that two framed links give rise to homeomorphic 3-manifolds if and only if these links are related by isotopy and a finite sequence of geometric transformations called *Kirby moves*. There are two Kirby moves: adjoining a distant unknot O^ε with framing $\varepsilon = \pm 1$ and sliding a link component over another one as in Figure 1.

Let $L = \cup_{i=1}^m L_i \subset S^3$ be a framed link and let $(b_{i,j})_{i,j=1,\dots,m}$ be its linking matrix: for $i \neq j$, $b_{i,j}$ is the linking number of L_i, L_j and $b_{i,i}$ is the framing number of L_i. Denote by e_+ (resp. e_-) the number of positive (resp. negative) eigenvalues of this matrix. The sliding property of modular categories implies the following theorem. In its statement a knot K with label Ω is denoted $K(\Omega)$.

Theorem 3.1 *Let $\mathcal{C}$ be a modular category with Kirby color Ω. Then $\langle O^1(\Omega)\rangle \neq 0, \langle O^{-1}(\Omega)\rangle \neq 0$ and the expression*

$$\tau_{\mathcal{C}}(M_L) = \langle O^1(\Omega)\rangle^{-e_+}\langle O^{-1}(\Omega)\rangle^{-e_-}\langle L_1(\Omega), \dots, L_m(\Omega)\rangle$$

is invariant under the Kirby moves on L. This expression yields therefore a well-defined topological invariant $\tau_{\mathcal{C}}$ of closed connected oriented 3-manifolds.

In the literature there are several competing normalizations of $\tau_{\mathcal{C}}$. Here we normalize so that $\tau_{\mathcal{C}}(S^3) = 1$ and $\tau_{\mathcal{C}}(S^1 \times S^2) = \sum_\lambda (\dim(V_\lambda))^2$. The invariant $\tau_{\mathcal{C}}$ extends to 3-manifolds with a framed oriented $\mathcal{C}$-labeled link K inside by

$$\tau_{\mathcal{C}}(M_L, K) = \langle O^1(\Omega)\rangle^{-e_+}\langle O^{-1}(\Omega)\rangle^{-e_-}\langle L_1(\Omega), \dots, L_m(\Omega), K\rangle.$$

4 3-dimensional TQFT's

A 3-dimensional TQFT $\mathbf{V}$ assigns to every closed oriented surface X a finite dimensional vector space $\mathbf{V}(X)$ over a field $\mathbf{k}$ and assigns to every cobordism (M, X, Y) a linear map $\mathbf{V}(M) = \mathbf{V}(M, X, Y) : \mathbf{V}(X) \to \mathbf{V}(Y)$. Here a *cobordism* (M, X, Y) between surfaces X and Y is a compact oriented 3-manifold M with $\partial M = (-X) \amalg Y$ (the minus sign indicates the orientation reversal). A TQFT has to satisfy axioms which can be expressed by saying that $\mathbf{V}$ is a monoidal functor from the category of surfaces and cobordisms to the category of vector spaces over $\mathbf{k}$. Homeomorphisms of surfaces should induce isomorphisms of the corresponding vector spaces compatible with the action of cobordisms. From the definition, $\mathbf{V}(\emptyset) = \mathbf{k}$. Every compact oriented 3-

manifold M is a cobordism between $\emptyset$ and ∂M so that $\mathbf{V}$ yields a "vacuum" vector $\mathbf{V}(M) \in \mathrm{Hom}(\mathbf{V}(\emptyset), \mathbf{V}(\partial M)) = \mathbf{V}(\partial M)$. If $\partial M = \emptyset$, then this gives a numerical invariant $\mathbf{V}(M) \in \mathbf{V}(\emptyset) = \mathbf{k}$.

Interesting TQFT's are often defined for surfaces and 3-cobordisms with additional structure. The surfaces X are usually endowed with Lagrangians, i.e., with maximal isotropic subspaces in $H_1(X; \mathbf{R})$. For 3-cobordisms, several additional structures are considered in the literature: 2-framings (M. Atiyah), p_1-structures ([BHMV]), numerical weights (K. Walker, V. Turaev). All these choices are equivalent. The TQFT's requiring such additional structures are said to be *projective* since they provide projective linear representations of the mapping class groups of surfaces.

Every modular category $\mathcal{C}$ with ground field $\mathbf{k}$ and simple objects $\{V_\lambda\}_\lambda$ gives rise to a projective 3-dimensional TQFT $\mathbf{V}_\mathcal{C}$. It depends on the choice of a square root $\mathcal{D}$ of $\sum_\lambda (\dim(V_\lambda))^2 \in \mathbf{k}$. For a connected surface X of genus g,

$$\mathbf{V}_\mathcal{C}(X) = \mathrm{Hom}_\mathcal{C}(\mathbb{1}, \bigoplus_{\lambda_1,\ldots,\lambda_g} \bigotimes_{r=1}^{g} (V_{\lambda_r} \otimes V_{\lambda_r}^*)).$$

The dimension of this vector space enters the *Verlinde formula*

$$\dim_\mathbf{k}(\mathbf{V}_\mathcal{C}(X)) \cdot 1_\mathbf{k} = \mathcal{D}^{2g-2} \sum_\lambda (\dim(V_\lambda))^{2-2g}$$

where $1_\mathbf{k} \in \mathbf{k}$ is the unit of the field $\mathbf{k}$. If $\mathrm{char}(\mathbf{k}) = 0$, then this formula computes $\dim_\mathbf{k}(\mathbf{V}_\mathcal{C}(X))$. For a closed connected oriented 3-manifold M with numerical weight zero, $\mathbf{V}_\mathcal{C}(M) = \mathcal{D}^{-b_1(M)-1} \tau_\mathcal{C}(M)$ where $b_1(M)$ is the first Betti number of M.

The TQFT $\mathbf{V}_\mathcal{C}$ extends to a vaster class of surfaces and cobordisms. Surfaces may be enriched with a finite set of marked points, each labeled with an object of $\mathcal{C}$ and endowed with a tangent direction. Cobordisms may be enriched with ribbon (or fat) graphs whose edges are labeled with objects of $\mathcal{C}$ and whose vertices are labeled with appropriate intertwiners. The resulting TQFT, also denoted $\mathbf{V}_{\mathcal{C}}$, is *non-degenerate* in the sense that for any surface X, the vacuum vectors in $\mathbf{V}(X)$ determined by all M with $\partial M = X$ span $\mathbf{V}(X)$. A detailed construction of $\mathbf{V}_{\mathcal{C}}$ is given in [Tu].

The 2-dimensional part of $\mathbf{V}_{\mathcal{C}}$ determines a *modular functor* in the sense of G. Segal, G. Moore and N. Seiberg.

5 Constructions of modular categories

The universal enveloping algebra $U\mathfrak{g}$ of a (finite dimensional complex) simple Lie algebra $\mathfrak{g}$ admits a deformation $U_q\mathfrak{g}$ which is a quasitriangular Hopf algebra. The representation category $\mathrm{Rep}(U_q\mathfrak{g})$ is $\mathbf{C}$-linear and ribbon. For generic $q \in \mathbf{C}$, this category is semisimple. (The irreducible representations of $\mathfrak{g}$ can be deformed to irreducible representations of $U_q\mathfrak{g}$.) For q an appropriate root of unity, a certain subquotient of $\mathrm{Rep}(U_q\mathfrak{g})$ is a modular category with ground field $\mathbf{k} = \mathbf{C}$. For $\mathfrak{g} = sl_2(\mathbf{C})$, this was pointed out by Reshetikhin and Turaev; the general case (H. Andersen and J. Paradowski, A. Kirillov, Jr.) involves the theory of tilting modules. The corresponding 3-manifold invariant τ is denoted $\tau_q^{\mathfrak{g}}$. For example, if $\mathfrak{g} = sl_2(\mathbf{C})$ and M is the Poincaré homology sphere (obtained by surgery on a left-hand trefoil with framing -1), then [Le]

$$\tau_q^{\mathfrak{g}}(M) = (1-q)^{-1} \sum_{n \geq 0} q^n (1-q^{n+1})(1-q^{n+2}) \cdots (1-q^{2n+1}).$$

The sum here is finite since q is a root of unity.

There is another construction [Le] of a modular category associated with a simple Lie algebra $\mathfrak{g}$ and certain roots of unity q. The corresponding quantum invariant of 3-manifolds is denoted $\tau_q^{P\mathfrak{g}}$. (Here it is normalized so that $\tau_q^{P\mathfrak{g}}(S^3) =$

1). Under mild assumptions on the order of q, we have $\tau_q^{\mathfrak{g}}(M) = \tau_q^{\mathfrak{g}}(M)\,\tau'(M)$ for all M where $\tau'(M)$ is a certain Gauss sum determined by $\mathfrak{g}$, the homology group $H = H_1(M)$ and the linking form $\mathrm{Tors}H \times \mathrm{Tors}H \to \mathbf{Q}/\mathbf{Z}$.

A different construction derives modular categories from the category of framed oriented tangles $\mathcal{T}$. Given a ring K, we can consider a bigger category $K[\mathcal{T}]$ whose morphisms are linear combinations of tangles with coefficients in K. Both $\mathcal{T}$ and $K[\mathcal{T}]$ have a natural structure of a ribbon monoidal category.

The skein method builds ribbon categories by quotienting $K[\mathcal{T}]$ by local "skein" relations which appear in the theory of knot polynomials (the Alexander-Conway polynomial, the Homfly polynomial, the Kauffman polynomial). In order to obtain a semisimple category, one completes the quotient category with idempotents as objects (the Karoubi completion). Choosing appropriate skein relations, one can recover the modular categories derived from quantum groups of series A, B, C, D. In particular, the categories determined by the series A arise (C. Blanchet) from the Homfly skein relation shown in Figure 2 where $a, s \in K$. The categories determined by the series B, C, D arise from the Kauffman skein relation (V. Turaev and H. Wenzl, A. Beliakova and C. Blanchet).

The quantum invariants of 3-manifolds and the TQFT's associated with sl_N can be directly described in terms of the Homfly skein theory avoiding the language of ribbon categories (W. Lickorish, C. Blanchet, N. Habegger, G. Masbaum, P. Vogel for sl_2 and Y. Yokota for all sl_N).

6 Unitarity

From both physical and topological viewpoints, one is mainly interested in Hermitian and unitary TQFT's (over $\mathbf{k} = \mathbf{C}$). A TQFT $\mathbf{V}$ is *Hermitian* if the vector space $\mathbf{V}(X)$ is endowed with a non-degenerate Hermitian form

$\langle .,. \rangle_X : \mathbf{V}(X) \otimes_{\mathbb{C}} \mathbf{V}(X) \to \mathbb{C}$ such that:

- The form $\langle .,. \rangle_X$ is natural with respect to homeomorphisms and multiplicative with respect to disjoint union;

- For any cobordism (M, X, Y) and any $x \in \mathbf{V}(X), y \in \mathbf{V}(Y)$,

$$\langle \mathbf{V}(M, X, Y)(x), y \rangle_Y = \langle x, \mathbf{V}(-M, Y, X)(y) \rangle_X.$$

If $\langle .,. \rangle_X$ is positive definite for every X, then the Hermitian TQFT is *unitary.* Note two features of Hermitian TQFT's. If $\partial M = \emptyset$ then $\mathbf{V}(-M) = \overline{\mathbf{V}(M)}$. The group of self-homeomorphisms of any X acts in $\mathbf{V}(X)$ preserving the form $\langle .,. \rangle_X$. For a unitary TQFT, this gives an action by unitary matrices.

The 3-dimensional TQFT derived from a modular category $\mathcal{V}$ is Hermitian (resp. unitary) under additional assumptions on $\mathcal{V}$ which we briefly discuss. A *conjugation* in $\mathcal{V}$ assigns to each morphism $f : V \to W$ in $\mathcal{V}$ a morphism $\overline{f} : W \to V$ so that:

$\overline{\overline{f}} = f$, $\overline{f+g} = \overline{f} + \overline{g}$ for any $f, g : V \to W$;

$\overline{f \otimes g} = \overline{f} \otimes \overline{g}$ for any morphisms f, g in $\mathcal{C}$;

$\overline{f \circ g} = \overline{g} \circ \overline{f}$ for any morphisms $f : V \to W, g : W \to V$.

One calls $\mathcal{V}$ *Hermitian* if it is endowed with conjugation such that

$$\overline{\theta_V} = (\theta_V)^{-1}, \overline{c_{V,W}} = (c_{V,W})^{-1},$$

$$\overline{b_V} = d_V c_{V,V^*} (\theta_V \otimes \mathbf{1}_{V^*}),$$

$$\overline{d_V} = (\mathbf{1}_{V^*} \otimes \theta_V^{-1})\, c_{V^*,V}^{-1} b_V,$$

for any objects V, W of $\mathcal{V}$. A Hermitian modular category $\mathcal{V}$ is *unitary* if $tr(f\overline{f}) \geq 0$ for any morphism f in $\mathcal{V}$. The 3-dimensional TQFT derived from a Hermitian (resp. unitary) modular category has a natural structure of a Hermitian (resp. unitary) TQFT.

The modular category derived from a simple Lie algebra $\mathfrak{g}$ and a root of unity q is always Hermitian. It may be unitary for some q. For simply-laced $\mathfrak{g}$, there are always such roots of unity q of any given sufficiently big order (H. Wenzl). For non-simply-laced $\mathfrak{g}$, this holds under certain divisibility conditions on the order of q.

7 Integral structures in TQFT's

The quantum invariants of 3-manifolds have one fundamental property: up to an appropriate rescaling they are algebraic integers. This was first observed by H. Murakami, who proved that $\tau_q^{sl_2}(M)$ is an algebraic integer provided the order of q is an odd prime and M is a homology sphere. This extends to an arbitrary closed connected oriented 3-manifold M and an arbitrary simple Lie algebra $\mathfrak{g}$ as follows ([Le]): For any sufficiently big prime integer r and any primitive r-th root of unity q,

$$\tau_q^{P\mathfrak{g}}(M) \in \mathbf{Z}[q] = \mathbf{Z}[\exp(2\pi i/r)]. \qquad (1)$$

This inclusion allows to expand $\tau_q^{P\mathfrak{g}}(M)$ as a polynomial in q. A study of its coefficients leads to the Ohtsuki invariants of rational homology spheres and further to perturbative invariants of 3-manifolds due to T. Le, J. Murakami and T. Ohtsuki, see [Oh]. Conjecturally, the inclusion (1) holds for non-prime (sufficiently big) r as well. Connections with algebraic number theory (specifically modular forms) were studied by D. Zagier and R. Lawrence.

It is important to obtain similar integrality results for TQFT's. Following P. Gilmer, fix a Dedekind domain $D \subset \mathbf{C}$ and call a TQFT $\mathbf{V}$ *almost D-integral* if it is non-degenerate and there is $d \in \mathbf{C}$ such that $d\,\mathbf{V}(M) \in D$ for all M with

$\partial M = \emptyset$. Given an almost integral TQFT $\mathbf{V}$ and a surface X, we define $S(X)$ to be the D-submodule of $\mathbf{V}(X)$ generated by all vacuum vectors for X. This module is preserved under the action of self-homeomorphisms of X. It turns out that $S(X)$ is a finitely generated projective D-module and $\mathbf{V}(X) = S(X) \otimes_D \mathbf{C}$. A cobordism (M, X, Y) is *targetet* if all its connected components meet Y along a non-empty set. In this case $\mathbf{V}(M)(S(X)) \subset S(Y)$. Thus applying S to surfaces and restricting τ to targetet cobordisms we obtain an "integral version" of $\mathbf{V}$. In many interesting cases the D-module $S(X)$ is free and its basis may be described explicitly (P. Gilmer, G. Masbaum). A simple Lie algebra $\mathfrak{g}$ and a primitive r-th (in some cases $4r$-th) root of unity q with sufficiently big prime r give rise to an almost D-integral TQFT for $D = \mathbf{Z}[q]$ (Q. Chen and T. Le).

8 State-sums invariants

Another approach to 3-dimensional TQFT's is based on the theory of $6j$-symbols and state sums on triangulations of 3-manifolds. This approach introduced by V. Turaev and O. Viro is a quantum deformation of the Ponzano-Regge model for the three-dimensional lattice gravity. The quantum $6j$-symbols derived from representations of $U_q(sl_2\mathbf{C})$ are $\mathbf{C}$-valued rational functions of the variable $q_0 = q^{1/2}$

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \tag{2}$$

numerated by 6-tuples of non-negative integers i, j, k, l, m, n. One can think of these integers as of labels sitting on the edges of a tetrahedron, see Figure 3. The $6j$-symbol admits various equivalent normalisations and we choose the one which has full tetrahedral symmetry. Let now $q_0 \in \mathbf{C}$ be a primitive $2r$-th root of unity with $r \geq 2$. Set $I = \{0, 1, ..., r-2\}$. Given a labeled tetrahedron T as in Figure 3 with $i, j, k, l, m, n \in I$, we can evaluate the $6j$-symbol (2) at q_0 and obtain a complex number denoted $|T|$. Consider a closed 3-dimensional manifold M with triangulation t. (Note that all 3-manifolds can be triangulated.) A *coloring* of M is a mapping φ from the set $\mathrm{Edg}(t)$ of the edges of t to I. Set

$$|M| = (\sqrt{2r}/(q_0 - q_0^{-1}))^{-2a} \sum_{\varphi} \prod_{e \in \mathrm{Edg}(t)} \langle \varphi(e) \rangle \prod_{T} |T^{\varphi}|$$

where a is the number of vertices of t, $\langle n \rangle = (-1)^n (q_0^n - q_0^{-n})/(q_0 - q_0^{-1})$ for any integer n, T runs over all tetrahedra of t and T^{φ} is T with the labeling induced by φ. Theorem: $|M|$ does not depend on the choice of t and yields thus a topological invariant of M.

The invariant $|M|$ is closely related to the quantum invariant $\tau_q^{\mathfrak{g}}(M)$ for $\mathfrak{g} = sl_2(\mathbf{C})$. Namely, $|M|$ is the square of the absolute value of $\tau_q^{\mathfrak{g}}(M)$, i.e., $|M| = |\tau_q^{\mathfrak{g}}(M)|^2$ (V. Turaev, K. Walker). This computes $|\tau_q^{\mathfrak{g}}(M)|$ inside M without appeal to surgery. No such computation of the phase of $\tau_q^{\mathfrak{g}}(M)$ is known.

These constructions generalize in two directions. Firstly, they extend to manifolds with boundary. Secondly, instead of the representation category of $U_q(sl_2\mathbf{C})$, one can use an arbitrary modular category $\mathcal{C}$. This yields a 3-dimensional TQFT which associates to a surface X a vector space $|X|_{\mathcal{C}}$ and to a 3-cobordism (M, X, Y) a homomorphism $|M|_{\mathcal{C}} : |X|_{\mathcal{C}} \to |Y|_{\mathcal{C}}$, see [Tu]. When $X = Y = \emptyset$ this homomorphism is multiplication $\mathbf{C} \to \mathbf{C}$ by a topological invariant $|M|_{\mathcal{C}} \in \mathbf{C}$. The latter is computed as a state sum on a triangulation of M involving the $6j$-symbols associated with $\mathcal{C}$. In general, these $6j$-symbols are not numbers but tensors so that instead of their product one should use an appropriate contraction of tensors. The vectors in $V(X)$ are geometrically represented by trivalent graphs on X such that every edge is labeled with a simple object of $\mathcal{C}$ and every vertex is labeled with an intertwiner between the three objects labeling the incident edges. The TQFT $|\cdot|_{\mathcal{C}}$ is related to the TQFT

$\mathbf{V} = \mathbf{V}_\mathcal{C}$ by $|M|_\mathcal{C} = |\mathbf{V}(M)|^2$ (V. Turaev, K. Walker). Moreover, for any closed oriented surface X,

$$|X|_\mathcal{C} = \mathrm{End}(\mathbf{V}(X)) = \mathbf{V}(X) \otimes (\mathbf{V}(X))^* = \mathbf{V}(X) \otimes \mathbf{V}(-X)$$

and for any 3-dimensional cobordism (M, X, Y),

$$|M|_\mathcal{C} = \mathbf{V}(M) \otimes \mathbf{V}(-M) : \mathbf{V}(X) \otimes \mathbf{V}(-X) \to \mathbf{V}(Y) \otimes \mathbf{V}(-Y).$$

J. Barrett and B. Westbury introduced a generalization of $|M|_\mathcal{C}$ derived from so-called spherical monoidal categories (which we suppose to be semisimple with a finite set of isomorphism classes of simple objects). This class includes modular categories and a most interesting family of (unitary monoidal) categories arising in the theory of subfactors, see [EK], [KS]. Every spherical category $\mathcal{C}$ gives rise to a topological invariant $|M|_\mathcal{C}$ of a closed oriented 3-manifold M. (It seems that this approach has not yet been extended to cobordisms.)

Every monoidal category $\mathcal{C}$ gives rise to a *double* (or a *center* $Z(\mathcal{C})$) which is a braided monoidal category, see [Ma]. If $\mathcal{C}$ is spherical, then $Z(\mathcal{C})$ is modular (M. Müger). Conjecturally, $|M|_\mathcal{C} = \tau_{Z(\mathcal{C})}(M)$. In the case where $\mathcal{C}$ arises from a subfactor, this has been recently proven by Y. Kawahigashi, N. Sato, and M. Wakui.

The state sum invariants above are closely related to spin networks, spin foam models and other models of quantum gravity in dimension $2+1$, see [Ba], [Ca].

References

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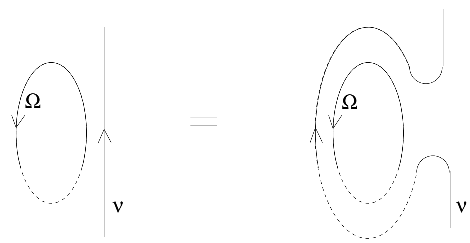

Figure 1: Sliding property

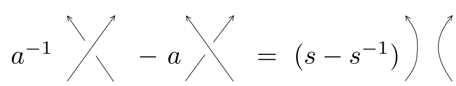

Figure 2: The Homfly relation

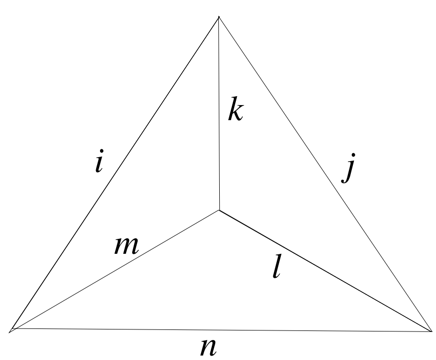

Figure 3: Labeled tetrahedron