Chapter 2

Combinatorial topology and discrete Morse theory

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Motivations

In the first draft of the program for this Summer School in Marrakech, we planned to expose some standard material in differential topology, and some recent developments in low dimensional topology. We had then a discussion with the organizers, and it appeared that a unifying view on the program of this school was to establish bridges between differential matter on one hand, and discrete or computational geometry on the other hand. So we decided to focus on combinatorial topology and include an introduction to Robin Forman *discrete Morse theory*. In this chapter, we start with some classical combinatorial topology, including piecewise linear manifolds. Then we define discrete Morse functions on CW complexes and show that we get the usual theorems of Morse theory. Everything here should sound rather familiar for people knowing classical theory. However the relation between differentiable objects, such that gradient vector fields, and their discrete counterpart has still to be explored. The generic questions below could give starting points for interesting research.

- What is a discrete counterpart of a given differentiable notion ?
- What is the interplay between the differentiable and discrete notions for a given manifold ? In particular in which way the discrete notion could approach the smooth one ?

- Is it computable ?

The subject of *Computational Topology* is the interrelationship between topology and algorithmic problems. Motivations come either from algorithmic questions involving topological matter, or from combinatorial problems supporting topological methods. Studying discrete Morse theory could be a first step towards this new domain. We will give at the end of this exposition some applications of classical Morse theory in low dimensional topology. Most of them can be obtained easily via the discrete theory. Some deeper ones, for example those using Cerf theory, are not so obvious. That would be interesting to revisit them using discrete methods and hopefully to go further towards new interesting results.

2.1 Introduction to combinatorial topology

The purpose of this section is to define basic objects in combinatorial topology involving simplicial complexes, triangulations, piecewise linear manifolds and CW complexes.

2.1.1 Simplicial complexes

Definition 2.1 *A n-dimensional simplex is the convex hool of $n+1$ points (the vertices) affinely independant in an real affine space.*

Points in the simplex $\sigma^n = (\alpha_0, \ldots, \alpha_n)$ are

$$x = \sum_{j=0}^{n} x_j \alpha_j \ , \ \forall j \ x_j \geq 0 \ , \ \sum_{j=0}^{n} x_j = 1 \ .$$

A face of σ^n is a simplex obtained with a subset of vertices. In particular, the $(n-1)$-dimensional faces (called facets) are the simplexes

$$\sigma_i^{n-1} = (\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n) = \{x = \sum_{j \neq i} x_j \alpha_j \ , \ \forall j \ x_j \geq 0 \ , \ \sum_{j \neq i} x_j = 1\} \ .$$

Definition 2.2 *A (locally finite) simplicial complex in a real affine space is a set F of simplices such that*

(i) all faces of simplices in F are in F,
(ii) F is locally finite i.e. each point in some simplex of F has a neighbourhood meeting only a finite number of simplices in F,
(iii) intersection of 2 simplices in F is either a common face or emptyset.

If S is the set of vertices (0-dimensional faces), we will use the notation $K = (S, F)$. The polyhedron of K is the union of all its simplices; we denote it by $|K|$.

Definition 2.3 *A triangulation of a topological space is the data given by a simplicial complex K, and an homeomorphism between $|K|$ and X.*

Definition 2.4 *A simplicial complex $K' = (S', F')$ is a subdivision of $K = (S, F)$ if and only if each simplex in F' is contained in a simplex in F, and each simplex in F is a union of simplices in F'.*

The barycentric subdivision of $K = (S, F)$ is obtained from the set of vertices S' formed with all barycenters of faces in F. Faces are obtained recursively. For each n dimensional simplex $\sigma \in F$, we add in F' cones from the barycenter of σ to all simplices in F' which are contained in the faces of σ.

Definition 2.5 *Let $K = (S, F)$ be a simplicial complex. A map from the polyhedron $|K|$ to a real affine space is said to be simplicial if and only if the restriction to each simplex is linear.*

Definition 2.6 *Two simplicial complexes $K = (S, F)$ and $K' = (S', F')$ are isomorphic if and only if there exists a bijection between S and S' inducing a bijection between F and F' (such a map induces a simplicial homeomorphism between $|K|$ and $|K'|$).*

Definition 2.7 *Two simplicial complexes are combinatorialy equivalent if and only if they have isomorphic subdivisions. The triangulations $t : |K| \to X$, $t' : |K'| \to X$ are combinatorialy equivalent if and only if there exist subdivisions L and L' such that $t'^{-1} \circ t$ defines a simplicial isomorphism between L and L'.*

2.1.2 PL manifolds

Definition 2.8 *A simplicial complex $K = (S, F)$ is a piecewise linear manifold if and only if each point in $|K|$ has a neighbourhood combinatorialy equivalent to a n dimensional simplex.*

Note that the property above only depends on the combinatorial equivalence class of K.

Definition 2.9 *A PL structure on a topological space X is a combinatorial equivalence class of triangulations by simplicial complexes which are piecewise linear manifolds.*

A PL manifold is a topological manifold with boundary.

Definition 2.10 *Let M, M' be PL manifolds. A map $f : M \to M'$ is PL if and only if there exist triangulations*

$$h : |K| \to M, \text{ et } h' : |K'| \to M'$$

such that ${h'}^{-1} \circ f \circ h$ is a simplicial map.

Collar neighbourhood

Definition 2.11 *Let $i : N \to M$ be a PL embedding between polyedra or between PL manifolds. A collar is an embedding $j : [0,1] \times N \to M$ such that*

$\forall x \in N,\ j(0,x) = i(x)$,
$j([0,1[\times N)$ *is an open neigbourhood of* $i(N)$.

A local collar at $x \in N$ is a collar for the restriction of i to some neighbourhood of x. Theorem below is proved in [11, chapitre 2]. The statement can be extended to the non compact case.

Theorem 2.12 (Collar neighbourhood) *If N is compact and $i : N \to M$ admits local collars, then there exists a collar.*

Corollary 2.13 *If M is a compact PL manifold, then its boundary admits a collar.*

Regular neighbourhood

Definition 2.14 *Let P be a polyhedron, and $Q \subset P$ a sub-polyhedron.*
a) We say that Q is an elementary collapse of P if and only if the closure $\sigma = \overline{P - Q}$ is a simplex with exactly one facet not included in Q. b) We say that P collapse onto Q if and only is there exits a sequence of sub-polyhedra $Q = Q_0 \subset Q_1 \cdots \subset Q_r = P$ such that Q_{i-1} is an elementary collapse of Q_i, for $1 \leq i \leq r$.

Definition 2.15 *Let M be a PL manifold, and $N \subset M$ be the image of a PL embedding of a compact polyhedron. A regular neighbourhood of N is a neighbourhood V such that,*

(i) V is a compact PL submanifold,
(ii) there exists a triangulation

$$t : (|K'|, |K|) \to (V, N)$$

and a collapse of $|K'|$ onto $|K|$.

Definition 2.16 *A PL ambiant isotopy (rel. N) is an homeomorphism $H : [0,1] \times M \to [0,1] \times M$ preserving the level and such that $\forall x\ H(0,x) = (0,x)$ ($\forall t,\ \forall x \in N,\ H(t,x) = (t,x)$).*

The following theorem is proved in [11, Chapter 3]

Theorem 2.17 *a) For any compact polyhedron N in a PL manifold M, there exists a regular neighbourhood.*
b) Two regular neighbourhood V et V' are ambiant isotopic rel. N.

Ambiant isotopy and isotopy by moves for knots

For PL embeddings, there exists a couple of different notions of isotopy that will not be developed here (see for example [6]). For knot theory one can show results stating *equivalence of equivalences.* We give below an example of such result.

Definition 2.18 *a) A (polygonal) knot is an PL embedding of S^1 in $\mathbb{R}^3$.*
b) Two knots are (PL) isotopic if and only if they correspond by an ambiant (PL) isotopy.
c) Two knots are equivalent by moves if and only if they correspond by a sequence of moves consisting in replacing in a knot K a segment AB by two segments BC and CA in such a way that ABC the triangle ABC intersects K along $ABC \cap K = AB$, or the converse.

Theorem 2.19 *For two PL knots, the statements below are equivalent. a) K and K' corresponds by a PL homeomorphism.*
b) K and K' are ambiant (PL) isotopic PL.
c) K and K' are equivalent by moves.

A proof can be found in [2].

2.1.3 Triangulation of manifolds

Triangulation of C^1 manifolds has been established by Cairns in the years 1930. Then Whitehead has proved that a C^1 manifold has a canonical PL structure, which is represented by a C^1 triangulation.

In dimension lower than 3, any topological manifold supports a smooth structure, and C^1 manifolds are diffeomorphic if and only if they are homeomorphic (resp. PL homeomorphic).

One says that Top, PL and Diff classifications coincide.

In dimension 4, PL and Diff classifications coincide, but there exist compact topological manifolds without PL structure, and homeomorphic C^1 manifolds which are not diffeomorphic.

A list of results and unsolved problems can be found in [1]. Moise's book [9] gives a detailled exposition in dimension 2 and 3. Triangulation of smooth manifolds is exposed in [10, chapitre II]; one can find here the following theorems.

Definition 2.20 *Let M be a C^r manifolds, $r \geq 1$. A triangulation $f : |K| \to M$ is C^r if and only if the restriction of f to each simplex in K is C^r and has maximal rank.*

Theorem 2.21 *Let $f : |K| \to M$, $f' : |K'| \to M$ two C^r triangulations C^r ($r \geq 1$) of the C^r manifold M. Then K and K' have isomorphic subdivisions.*

Theorem 2.22 *Any C^r ($r \geq 1$) manifold M support a C^r triangulation C^r. Moreover any C^r triangulation of the boundary ∂M can be extended.*

2.2 CW complexes

All these notions on CW complexes can be found in [7].

2.2.1 Combinatorial cell complexes

A *closed Euclidean n-cell* E^n is a homeomorphic image of the Euclidean n-cube $\mathbf{I}^n$, the cartesian product of n copies of the closed unit interval $\mathbf{I} = \{t \in \mathbb{R} | 0 \leq t \leq 1\}$. We note $\dot{E}^n$ the boundary of E^n.

Definition 2.23 *Let X be a set. A* cell structure *on X is a pair (X, Φ) where Φ is a collection of maps of closed Euclidean cells into X satisfying the following conditions.*

1. *If $\phi \in \Phi$ and ϕ has domain E^n, then ϕ is injective on $E^n - \dot{E}^n$.*
2. *The images $\{\phi(E^n - \dot{E}^n) | \phi \in \Phi\}$ partition X.*
3. *If $\phi \in \Phi$ has domain E^n, then $\phi(\dot{E}^n) \subset \bigcup\{\psi(E^k - \dot{E}^k) | \psi \in \Phi$ has domain $E^k, k \leq (n-1)\}$.*

If $\phi \in \Phi$ and ϕ has domain E^n, the image $\phi(E^n) = \overline{\sigma}^{(n)}$ is called an *closed n-cell* of (X, Φ). $\phi(\dot{E}^n) = \dot{\sigma}^{(n)})$ is called the *boundary* of $\sigma^{(n)}$, and $\phi(E^n - \dot{E}^n)$ is called its *interior*. If $n > 0$, $\phi(E^n - \dot{E}^n)$ is called an *open n-cell* and we

note that $\sigma^{(n)}$.
We call ϕ a *characteristic map* for the n-cell $\sigma^{(n)}$. Then Φ is the set of characteristic maps for the cells of (X, Φ).
Remark. When we talk of *cell*, this always means *open cell.*

The union $\bigcup\{\psi(E^k - \dot{E}^k)|\psi \in \Phi$ has domain $E^k, k \leq (n-1)\} = X^{n-1}$, which appears in condition *3.* is called the $(n-1)$-*skeleton* of the cell structure. Thus $\forall n \leq 1,\ \dot{\sigma}^{(n)} = \phi(\dot{E}^n) \subset X^{n-1}$.

We say that two cell structures (X, Φ) and (X, Φ') are *strictly equivalent* if there is a one-to-one correspondance between Φ and Φ' such that a characteristic function with domain E^n corresponds to a characteristic function with domain E^n, and corresponding functions differ only by a reparametrization of their domain. That is, if ϕ and ϕ' are corresponding functions of Φ and Φ', respectively, then $\phi' = \phi \circ h$, where $h : (E^n, \dot{E}^n) \to (E^n, \dot{E}^n)$ is a homeomorphism of pairs. One can check that it is an equivalence relation on the collection of cell structures on the set X.
If (X, Φ) is a cell structure, let S_Φ consists of all pairs $(\sigma^{(n)}, [\phi])$, where $\sigma^{(n)} = \phi(E^n - \dot{E}^n)$ and $[\phi]$ is a strict equivalence class of $\phi \in \Phi$.

Definition 2.24 *A cell complex on a set X or a* cellular decomposition *of a set X is an equivalence class of cell structures (X, Φ) under the equivalence relation of strict equivalence. A cell complex on X will be denoted by a pair (X, S), where $S = S_\Phi$ for some representative cell structure (X, Φ). The set S is called the set of (open) cells of (X, S).*

Definition 2.25 *A* subcomplex *(A, T) of a cell complex (X, S), which we denote by $(A, T) \subset (X, S)$, is a cell complex such that $A \subset X$ et $T \subset S$.*

Subsequently, we'll need the following proposition which characterizes the cells of a subcomplex.

Proposition 2.26 *Let $(A, T) \subset (X, S)$ and σ be a cell of S. Then σ is a cell of T if and only if $\sigma \cap A \neq \emptyset$.*

Proof: If $\sigma \in T$ then $\sigma \subset A$ and so $\sigma \cap A \neq \emptyset$.
Conversely, suppose $\sigma \cap A \neq \emptyset$. Choose characteristic maps Φ for (X, S) and Ψ for (A, T) such that $\Psi \subset \Phi$. If $y \in \sigma \cap A$, then $y \in \phi_\sigma(E_\sigma - \dot{E}_\sigma)$ for $\phi_\sigma \in \Phi$. Since the open cells partition X and so A, in order that $y \in A$ we must have $\phi_\sigma \in \Psi$. Thus $\sigma \in T$. □

Definition 2.27 *A cell complex (X, S) is finite or countable if S is a finite or countable set.*

A cell is called a regular cell if it has one (and hence every equivalent) characteristic map bijective on all E^n. A cell complex is closure finite if each n-cell meets only a finite number of open cells $\sigma^{(p)}$ with $p < n$.
A cell complex is regular if all its cells are regular.
A cell complex is normal if, for each closed cell $\overline{\sigma}$, the subset $\overline{\sigma}$ carries the structure of a subcomplex.
The cell complex (X, S) has dimension n if it has no cells $\sigma^{(p)}$ of dimension p greater than n, and at least one cell of dimension n.

2.2.2 CW Complexes

A CW complex is a cell structure on a set X together with a topology given by characteristic maps. Morse precisely we have:

Definition 2.28 *A (Hausdorff) space X is a* CW complex *with respect to a family of cells S provided:*

(i) the pair (X, S) is a cell complex such that each cell $\sigma \in S$ has a continuous characteristic function.

(ii) the space X has the weak topology with respect to the closed cells $\overline{S}$.

(iii) the cell complex (X, S) is closure finite.

If (X, S) is a cell complex such that X satisfies *(i) (ii) (iii)* with respect to S we say that X is a CW complex with cells S.

From the definition, we obtain a lemma which links the topology of X with the topology of Euclidean spaces and a proposition which characterizes continuous map from a CW complex to another topological space.

Lemma 2.29 *Let X be a CW complex with cells S. Then*

(i) each cell $\overline{\sigma} \in S$ is a closed subset of X.

(ii) for each cell $\sigma \in S$, the restriction of the characteristic map ϕ_σ to $E^n - \dot{E}^n$ is a homeomorphism onto σ.

Proof: The characteristic map ϕ_σ for $\sigma \in S$ is a closed map. For if $K \subset E_\sigma$ any closed set, K is compact because E_σ is. Since ϕ_σ is continuous, $\phi_\sigma(K)$ is compact in X (X is Hausdorff). Taking $K = \dot{E}_\sigma$ we get part (i). Since ϕ_σ is closed, its restriction $\phi_\sigma|(E_\sigma - \dot{E}_\sigma)$ is a closed continuous bijection onto σ hence a homeomorphism. □

Proposition 2.30 *Let (X, S) be a CW complex. If Y is a topological space and if $f : X \to Y$ is a function, then f is continuous if and only if $f|_{\overline{\sigma}}$ is continuous $\forall \sigma \in S$.*

Proof: Suppose that $f : X \to Y$ is continuous and $C \subset Y$ is closed. Then $f^{-1}(C)$ is closed in X and $(f|_{\overline{\sigma}})^{-1}(C) = f^{-1}(C) \cap \overline{\sigma}$ is closed in $\overline{\sigma}$ for each cell $\forall \sigma \in S$. Thus $f|_{\overline{\sigma}}$ is continuous $\forall \sigma \in S$.
Conversely, suppose that $f : X \to Y$ is such that $f|_{\overline{\sigma}}$ is continuous for each cell σ and $C \subset Y$ is a closed set. Then $(f|_{\overline{\sigma}})^{-1}(C)$ is closed in $\overline{\sigma}$ for each σ, and therefore $f^{-1}(C)$ is closed in X. □

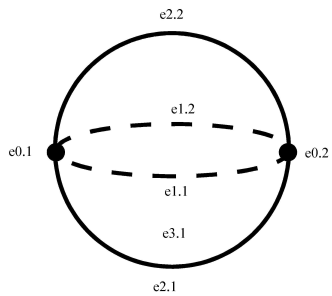

Figure 2.2.1: Regular cellular decomposition of the unit ball in $\mathbb{R}^3$

Remark. We must be carefull with this topology. For example it is not true in general that an open cell is open in (X, S) (consider the regular cellular decomposition of the unit ball in $\mathbb{R}^3$ in 7 cells (2 of dimension 0, 2 of dimension 1, 2 of dimension 2 and one of dimension 3, cf figure 0.2.1).

2.2.3 Regular and normal CW complexes

Definition 2.31 *A CW complex X is* regular *if each closed cell is homeomorphic to a a closed Euclidean n-cell. A CW complex is* normal *if each closed cell is a subcomplex.*

We point out that if (X, S) is regular, then we can choose a cell structure on X in which each characteristic map is a homeomorphism and for which the cells are the same subset of X as those of the cells of S. Before proving that a regular cell complex is normal we need a definition and some preliminary results.

Definition 2.32 *If (X, S) is a cell complex, the* carrier *of $A \subset X$ is the intersection of all subcomplexes of (X, S) whose underlying set contains A. The carrier of A will be denoted by $(C(A), S|A)$, or just by $C(A)$ if the cellular decomposition is fixed.*

We remark that (X, S) can be equipped with a topology where the closed sets are the $C(A), A \in \mathcal{P}(X)$. As the union of any subcomplex is still a subcomplex, the union of any family of closed sets is still a closed set.

Lemma 2.33 *If (X, S) is a cell complex, $A \subset X$, and $C(A)$ the carrier of A, then*

1. $C(\emptyset) = \emptyset$;
2. $\forall A \subset X, A \subset C(A)$;
3. $\forall A \subset X, C(C(A)) = C(A)$;
4. $\forall A \subset X$ *et* $B \subset X, C(A \cup B) = C(A) \cup C(B)$;
5. $\forall A \subset X$ *et* $B \subset X$, *si* $A \subset B$, *alors* $C(A) \subset C(B)$.

Proof: Since the carrier topology is a topology, we have the first four properties. The fifth follows from the first four. □

Lemma 2.34 *IF (X, S) is a cell complex and $A \subset X$ then*

$$C(A) = \bigcup \{C(\overline{\sigma}) | \sigma \in S \text{ and } \sigma \cap A \neq \emptyset\}.$$

Proof: Since the sets σ partition X, we see

$$A \subset \{\overline{\sigma} | \sigma \in \ S \text{ and } \sigma \cap A \neq \emptyset\}.$$

By lemma 0.33 we get

$$C(A) \subset \bigcup \{C(\overline{\sigma}) | \sigma \in S \text{ and } \sigma \cap A \neq \emptyset\}.$$

On the other hand, if $\sigma \in S$ is such that $\sigma \cap A \neq \emptyset$, then σ meets any subcomplex of (X, S) which contains A. By proposition 0.26, σ must be contained in such a complex, so that $\overline{\sigma} \subset C(A)$. Thus $C(\overline{\sigma}) \subset C(C(A)) = C(A)$, and we obtain

$$\bigcup \{C(\overline{\sigma}) | \sigma \in S \text{ and } \sigma \cap A \neq \emptyset\} \subset C(A).$$

□

Lemma 2.35 *If (X, S) is a cell complex and $\sigma \in S$, then*

$$C(\overline{\sigma}) = C(\dot{\sigma}) \cup \sigma.$$

Proof: We have $C(\dot{\sigma}) \subset C(\overline{\sigma})$ by lemma 0.33, and since $\sigma \subset \overline{\sigma}$, we have $\sigma \cup C(\dot{\sigma}) \subset C(\overline{\sigma})$.
Conversely, $\overline{\sigma} = \sigma \cup \dot{\sigma} \subset \sigma \cup C(\dot{\sigma})$, and $\sigma \cup C(\dot{\sigma})$ is a subcomplex of (X, S). Thus from the definition we have $C(\overline{\sigma}) \subset C(\dot{\sigma}) \cup \sigma$. □

We need more results before proving that a regular CW complex is normal.

Lemma 2.36 *A CW complex X is normal if and only if for each cell σ of X, $C(\dot{\sigma}) \subset \overline{\sigma}$.*

Proof: X is normal if and only if each closed cell $\overline{\sigma}$ is a subcomplex, which is equivalent to $\overline{\sigma} = C(\overline{\sigma}) = \sigma \cup \; C(\dot{\sigma})$ for each cell σ. □

Lemma 2.37 *A CW complex X is normal if and only of for each cell σ of X, the boundary $\dot{\sigma}$ is a subcomplex of X.*

Proof: IF X is normal, then $\overline{\sigma}^{(n)}$ and X^{n-1} are subcomplexes of X, so that $\dot{\sigma}^{(n)} = \overline{\sigma}^{(n)} \cap X^{n-1}$ is a subcomplex of X.
On the other hand, if for each cell σ of X, $\dot{\sigma}$ is a subcomplex of X, then $\overline{\sigma} = \sigma \cup \dot{\sigma} = \sigma \cup C(\dot{\sigma}) = C(\overline{\sigma})$, and $\overline{\sigma}$ is a subcomplex. Thus X is normal. □

Proposition 2.38 *If A is a subcomplex of a CW regular (resp. normal) CW complex X, then A is a regular (resp. normal) CW complex.*

The proof if immediate. □

We now cite without proving it the theorem of invariance of domain and one of its corollary.

Theorem 2.39 (Invariance of domain) *Let A be a subset of the topological n-manifold X, let B be a subset of the topological n-manifold Y, and let $f : A \to B$ be a homeomorphism. Then if A is open in X, then B is open in Y.*

Corollary 2.40 *If $m < n$ and A be a nonempty open subset of the topological n-manifold X, then A is not homeomorphic to any open subset of a topological m-manifold Y.*

Here is the theorem:

Theorem 2.41 *If X is a regular CW complex, then X is a normal CW complex.*

Proof: By lemma 0.37, it will suffice ti prove that for any n-cell $\sigma^{(n)}$ of X, $\dot{\sigma}$ is a subcomplex. Then we must prove that if $\tau^{(q)}$ is a cell of X and $\tau^{(q)} \cap \dot{\sigma}^{(n)} \neq \emptyset$ then $\overline{\tau}^{(q)} \subset \dot{\sigma}^{(n)}$. Since $\dot{\sigma}^{(n)} \subset X^{n-1}$, we know that $\tau^{(q)} \cap \dot{\sigma}^{(n)} \neq \emptyset$, then $q \leq n-1$. We prove the proposition by descending induction on q.
Let τ be an $(n-1)$-cell of X such that $\tau \cap \dot{\sigma}^{(n)} \neq \emptyset$. Then $\tau \cap \dot{\sigma}^{(n)}$ is closed in τ. Since X is a regular CW complexe, $\dot{\sigma}^{(n)}$ is homeomorphic to an $(n-1)$-sphere $\mathbf{S}^{n-1}$. Also τ is homeomorphic to Euclidean space $\mathbb{R}^{n-1}$. By invariance of domain the open subset $\tau \cap \dot{\sigma}^{(n)}$ of $\dot{\sigma}^{(n)}$ is homeomorphic to an open subset of τ. Thus $\tau \cap \dot{\sigma}^{(n)}$ is both open and closed in the connected space τ, and since $\tau \cap \dot{\sigma}^{(n)} \neq \emptyset$ we must have $\tau \subset \dot{\sigma}^{(n)}$. Finally, since $\dot{\sigma}^{(n)}$ is closed in X, we have $\overline{\tau} \subset \dot{\sigma}^{(n)}$.
Now let $C = \bigcup\{\overline{\tau} | \tau$ is an $(n-1)$-cell of X and $\overline{\tau} \subset \dot{\sigma}^{(n)}\} \subset \dot{\sigma}^{(n)}$. We have shown that $\dot{\sigma}^{(n)} \subset C \cup X^{n-2}$. Suppose that $\dot{\sigma}^{(n)} \subset C \cup X^{q}$, where $q \leq (n-2)$, and suppose that $\tau^{(q)} \subset X^{q} - C$. Since $\tau^{(q)}$ is open in X^{q}, $\tau^{(q)}$ is open in $X^{q} - C$, which is open in $X^{q} \cup C$, so that $\tau^{(q)}$ is open in $X^{q} \cup C$. By hypothesis, $\dot{\sigma}^{(n)} \subset C \cup X^{q}$, so that $\tau^{(q)} \cap \dot{\sigma}^{(n)}$ is open in $\dot{\sigma}^{(n)}$. But $q \leq (n-2)$, $\tau^{(q)}$ is homeomorphic to $\mathbb{R}^{d}$, and $\dot{\sigma}^{(n)}$ is homeomorphic to $\mathbf{S}^{n-1}$. By the corollary 0.40 to invariance of domain, we must have $\tau^{(q)} \cap \dot{\sigma}^{(n)} = \emptyset$. Thus $\dot{\sigma}^{(n)} \subset C \cup X^{q-1}$ and, by induction, $\dot{\sigma}^{(n)} \subset C \cup X^{-1} = C$ (with the convention $X^{-1} = \emptyset$) so that we have $\dot{\sigma}^{(n)} = C$. Also, if τ is a q-cell of X such that $\tau \cap \dot{\sigma}^{(n)} \neq \emptyset$, then $\overline{\tau} \subset C = \dot{\sigma}^{(n)}$, and $\dot{\sigma}^{(n)}$ is a subcomplex of X. □

2.2.4 Still some more results and definitions...

From now on, when we talk of *cell* we are meaning *open cells*. let M be a CW complex and let **K**denote the set of open cells of M, with $\mathbf{K}_p$ the cells of dimension p. The notation $\sigma^{(p)}$ will indicate that σ is a cell of dimension p. We define a relationship between cells as follows: we write $\tau > \sigma$ (or $\sigma < \tau$) if $\tau \neq \sigma$ and $\sigma \subset \overline{\tau}$, where $\overline{\tau}$ is the closure of τ. In this case we say σ is a face of τ. We write $\sigma \leq \tau$ if either $\tau = \sigma$ or $\sigma < \tau$.
Suppose $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$. Let **B** be a closed euclidian ball of dimension $p+1$ et $h : \mathbf{B} \to M$ be the characteristic map of τ. Then

Definition 2.42 *Say $\sigma^{(p)}$ is a* regular face *of $\tau^{(p+1)}$ if*

1. *$h : h^{-1}(\sigma) \to \sigma$ is a homeomorphism,*
2. *$\overline{h^{-1}(\sigma)}$ is a closed p-ball.*

otherwise we say σ is an irregular face *of τ.*

Remark. Let M be a regular CW complex, then all faces are regular.

Proof: Let $\sigma^{(p)}$ be a face of $\tau^{(p+1)}$ and h the characteristic map of τ. Since h is a homeomorphism an $\sigma \subset \overline{\tau}$ we get $h : h^{-1}(\sigma) \to \sigma$ which is a homeomorphism for the induce topology. We also saw that $\overline{\sigma} \subset \overline{\tau}$ and h is a homeomorphism so

$$h^{-1}(\overline{\sigma}) \xrightarrow{\simeq} \overline{\sigma} \xrightarrow{\simeq} \mathbf{B}^p.$$

Finally σ is a regular face of τ. □

We get the following property. Choose an orientation for each cell in M and suppose $\sigma^{(p)}$ is a regular face of $\tau^{(p+1)}$. Then we consider σ and τ as elements of the cellular chain groups $C_p(M, \mathbb{Z})$ and $C_{p+1}(M, \mathbb{Z})$ respectively. Then

$$< \partial\tau, \sigma >= \pm 1 \tag{2.2.1}$$

where $< \partial\tau, \sigma >$ is the incidence number of τ and σ (for the link between the incidence number and cellular homology see [4, chap.2], for the formula see Corollary V.3.6 de [7]). Thus we have the following property for CW complex:

Theorem 2.43 *Suppose $\tau^{(p+1)} > \sigma^{(p)} > \nu^{(p-1)}$ then one of the following property is true.*

(i) σ is a regular face of τ.

(ii) ν is a regular face of σ.

(iii) There is a p-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \nu$.

Proof: Suppose neither (i) nor (ii) is true. Choose an orientation for each cell of M. Since σ is a regular face of τ, (0.2.1) holds so that

$$\partial\tau = \pm\sigma + \sum_{\substack{\widetilde{\sigma}^{(p)} \neq \sigma \\ \widetilde{\sigma} < \tau}} c_{\widetilde{\sigma}} \widetilde{\sigma}$$

where $c_{\widetilde{\sigma}} \in \mathbb{Z}$. In the same way, since ν is a regular face of σ,

$$\partial\sigma = \pm\nu + \sum_{\widetilde{\nu} \neq \nu,\ \nu < \sigma} c_{\widetilde{\nu}} \widetilde{\nu}.$$

Therefore,

$$0 = \partial^2\tau = \pm\partial\sigma + \sum_{\tau > \widetilde{\sigma} \neq \sigma} c_\sigma \partial\widetilde{\sigma} = \mp\nu + \sum_{\tau > \widetilde{\sigma} \neq \sigma} c_{\widetilde{\sigma}} \partial\widetilde{\sigma} \pm \sum_{\widetilde{\nu} \neq \nu,\ \nu < \sigma} c_{\widetilde{\nu}} \widetilde{\nu}.$$

For this equation to hold, there must be some $\widetilde{\sigma}$, with $\tau > \widetilde{\sigma} \neq \sigma$ satisfying

$$\partial\widetilde{\sigma} = c\nu + (\text{sum de } (p-1)\text{-cells other than } \nu)$$

with $c \neq 0$ (we look at the the coefficient of $\widetilde{\sigma}$ corresponding to ν). So $\widetilde{\sigma} > \nu$ et ainsi $\tau > \widetilde{\sigma} > \nu$ as desired. □

We now give an essential definition.

Definition 2.44 *Let M be a CW complex and $\sigma^{(p)} < \tau^{(p+1)}$ two cells of M which satisfy*

(i) σ is a regular face of τ

(ii) σ is not a face of any other cell.

Let $N = M - (\sigma \cup \tau)$. We say M collapse onto *N σ is called a* free face *of τ.*
More generally, we say M collapse onto N if N is obtained from M by a finite sequence of such operations. We write $M \searrow N$.

Then we have the following property.

Theorem 2.45 *Let M and N be two regular CW complexes. If $M \searrow N$ then N is a deformation retract of M.*

Proof: We consider σ and τ as in definition 0.44. Since σ is a regular face of τ, we get $\overline{h^{-1}(\sigma)} \subset \partial\mathbf{B}^{p+1}$ and $\overline{h^{-1}(\sigma)}$ is homeomorphic to $\mathbf{B}^p$. So, $\overline{h^{-1}(\sigma)}$ is a closed connected subset of $\partial\mathbf{B}^{p+1}$ but nonequal to $\partial\mathbf{B}^{p+1}$ (the cell is regular so the characteristic map is bijective on the closure of this one). With an appropriate homeomorphism of $\mathbf{B}^{p+1}$ we can set this subset to be $\mathbf{I}^p \times \{0\}$. Then one must be convinced (we don't explicit the homotopy) that the frontier of $\mathbf{I}^{p+1}$ without an open face is a strong deformation retract of $\mathbf{I}^{p+1}$. □
Example. Here are two illustrations of collapsings (figures 0.2.2 and 0.2.3).
Remark. The proof of theorem 0.41 show that if τ has dimension $p+1$ and M is a regular CW complex then the p-cells are dense in $\overline{\tau} - \tau$, i.e.

$$\overline{\bigcup_{\sigma^{(p)}<\tau} \sigma} = \overline{\tau} - \tau$$

Theorem 2.46 *Suppose M is a regular CW complex, and fr some p and $r \geq 1$ we have $\tau^{(p+r)} > \nu^{(p-1)}$. Then there are two $(p+r-1)$-cells $\sigma^{(p+r-1)}$ and $\widetilde{\sigma}^{(p+r-1)}$ such that $\widetilde{\sigma} \neq \sigma$ and*

$$\tau > \sigma > \nu, \qquad \tau > \widetilde{\sigma} > \nu.$$

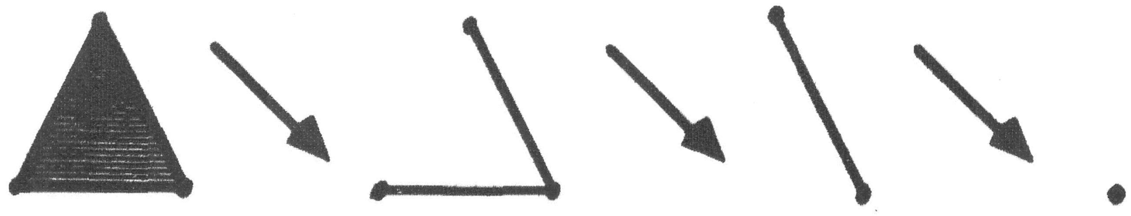

Figure 2.2.2: Collapsing a 2-simplex onto a vertex

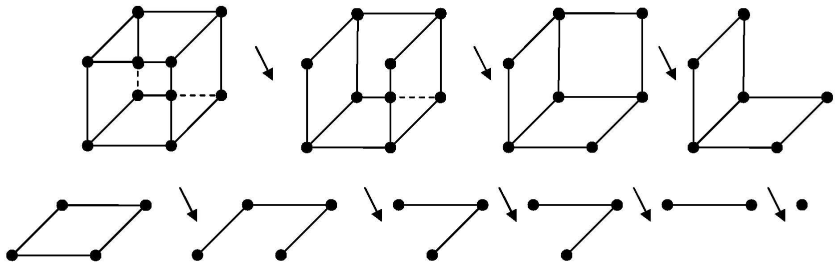

Figure 2.2.3: Collapsing the boundary of the cube but one face, onto a vertex

Proof: The proof is by induction on r. Suppose $r = 1$, that is $\tau^{(p+1)} > \sigma^{(p-1)}$. Since M is regular, the p-cells in $\overline{\tau} - \tau$ are dense in $\overline{\tau} - \tau$ (remark 0.2.4), i.e.

$$\overline{\bigcup_{\sigma^{(p)}<\tau} \sigma} = \overline{\tau} - \tau.$$

Thus there exists a p-cell $\sigma^{(p)}$ with $\tau > \sigma$ such that $\sigma > \nu$. Theorem 0.43 guarantees the existence of $\widetilde{\sigma}^{(p)} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \nu$.
For general r, we again have the $(p+r-1)$-cells dense in $\overline{\tau} - \tau$. Thus we can find a $(p+r-1)$-cell σ with $\tau > \sigma > \nu$. Continuing in this fashion, we can find a $(p+r-2)$-cell $\widetilde{\nu}$ such that $\sigma > \widetilde{\nu} > \nu$. Applying theorem 0.43 to the triple $\tau > \sigma > \widetilde{\nu}$ we learn there is a $(p+r-1)$-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \widetilde{\nu}$. Then, the cells σ and $\widetilde{\sigma}$ satisfy the desired properties. □

2.3 Discrete Morse theory

2.3.1 Discrete Morse function

In this section, we introduce the main definitions. Let M be a finite CW complexe.

Definition 2.47 *A* discrete Morse function *on M is a function $f : \mathbf{K} \to \mathbb{R}$ satisfying for all $\sigma \in \mathbf{K}_p$*

(i) If σ is an irregular face of $\tau^{(p+1)}$ then $f(\tau) > f(\sigma)$. Moreover,

$$\#\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\} \leq 1.$$

(ii) Si $\nu^{(p-1)}$ is an irregular face of σ then $f(\nu) < f(\sigma)$. Moreover,

$$\#\{\nu^{(p-1)} < \sigma | f(\nu) \geq f(\sigma)\} \leq 1.$$

Definition 2.48 *Let f a discrete Morse function on M. We say $\sigma \in \mathbf{K}_p$ is a* critical point of index p *if*

(i) $\#\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\} = 0.$

(ii) $\#\{\nu^{(p-1)} < \sigma | f(\nu) \geq f(\sigma)\} = 0.$

Example. Consider the CW complex below:

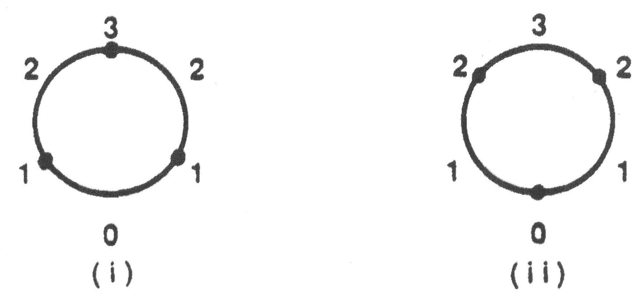

Figure 2.3.1: Two discrete functions on a CW complex

On one hand, we see that the function in figure 0.3.1 (i) is not a discrete Morse function. More precisely the edge $f^{-1}(0)$ violates rule (ii) of definition 0.47, and the vertex $f^{-1}(3)$ violates rule (i) of definition 0.47. On the other hand,the function of figure 0.3.1 (ii) is a discrete Morse function.
Example. definitions 0.47 and 0.48 imply that if M is a regular CW complexe, then the minimum of f must occur n a vertex, which must be a critical point of index 0.
Indeed, if $p \geq 1$, then each p-cell has at least 2 $(p-1)$-faces (consequence of theorem 0.46).

It follows from definition 0.48 that a p-cell σ is not critical if and only if either of the following conditions holds

(i) $\exists \tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$.

(ii) $\exists \nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

Lemma 2.49 *Conditions (i) and (ii) cannot both be true.*

Proof: Condition (ii) requires $p \geq 1$ which we now assume.
Suppose (i) is true. Then σ must be a regular face of τ (since f respect partial order on irregular faces). From condition (i) of dÃľfinition 0.47, if $\widetilde{\sigma} \neq \sigma$ is another p-face of τ, we must have

$$f(\widetilde{\sigma}) < f(\tau). \tag{2.3.1}$$

Therefore $f(\widetilde{\sigma}) < f(\sigma)$
Now suppose condition (ii) is true. Then ν must also be a regular face of σ. By theorem 0.43, there exists a p-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \nu$.
From condition (ii) of dÃľfinition 0.47, $f(\nu)$ cannot be $\geq$ both $f(\sigma)$ and $f(\widetilde{\sigma})$. Thus $f(\nu) < f(\widetilde{\sigma})$. Combining this with (0.3.1) we obtain

$$f(\sigma) \leq f(\nu) < f(\widetilde{\sigma}) < f(\tau) \leq f(\sigma)$$

which is a contradiction. □

2.3.2 Morse theorems for regular CW complexes

In this section, we will demonstrate Morse theorems in the case of finite regular CW complexes. All along this section, M will be a finite regular CW complex and f a discrete Morse function on M.

Definition 2.50 *For $c \in \mathbb{R}$, we define*

$$M(c) = \bigcup_{\substack{\sigma \in \mathbf{K} \\ f(\sigma) \leq c}} \bigcup_{\tau \leq \sigma} \tau.$$

That is $M(c)$ is the set of all cells on which f is $\leq c$, as well as all of their faces. In particular M is a subcomplex of M.
Here is a lemma which gives us a criteria to see if σ such that $f(\sigma) > c$ lies in $M(c)$. To see that, it suffices to find τ with $\sigma < \tau$ and $f(\tau) \leq c$ and more precisely the lemma tells us to consider only τ with $dim\ \tau = dim\ \sigma + 1$.

Lemma 2.51 *Let σ be a p-cell of M and suppose $\tau > \sigma$. Then there exists a $(p+1)$-cell $\widetilde{\tau}$ with $\sigma < \widetilde{\tau} \leq \tau$ and $f(\widetilde{\tau}) \leq f(\tau)$.*

Proof: We prove this by induction on the dimension of τ.
Since $\tau > \sigma$, we have $dim\ \tau > dim\ \sigma$. If $dim\ \tau = p+1$, we can consider $\widetilde{\tau} = \tau$.
Suppose

$$dim\ \tau = p + r, \qquad r > 1$$

by theorem 0.46, there exists $(p+r-1)$-cells ν_1 and ν_2 satisfying $\tau > \nu_1 > \sigma$ and $\tau > \nu_2 > \sigma$. Condition (ii) of definition 0.47 gives either $f(\nu_1) < f(\tau)$ or $f(\nu_2) < f(\tau)$. By induction on one of this two cells for which the inequality is true we obtain the result. □

Here comes the theorems of discrete Morse theory.

Theorem 2.52 *Let M be a finite regular CW complexe and f a discrete Morse function on M. If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $M(b) \searrow M(a)$.*

Proof: First of all, we will demonstrate that we can perturb f to obtain a $1-1$ function on $\mathbf{K}$. If $\tau^{(p+1)} > \sigma^{(p)}$ satisfies $f(\tau) \leq f(\sigma)$ then we perturb f by replacing $f(\tau)$ by $f(\tau) - \varepsilon$, or $f(\sigma)$ by $f(\sigma) + \varepsilon$ for $\varepsilon \geq 0$ small enough, without changing which cells are critical. In the same way if $\sigma^{(p)}$ satisfies $f(\tau^{(p+1)}) \neq f(\sigma) \neq f(\nu^{(p-1)})$ for each $\tau^{(p+1)} > \sigma > \nu^{(p-1)}$, then we may perturb f by changing $f(\sigma)$ by $f(\sigma) \pm \varepsilon$, for $\varepsilon \leq 0$ small enough, without changing which cells are critical. Perturbing f this way, we don't change neither $M(a)$, nor $M(b)$. Thus

$$f : \mathbf{K} \to \mathbb{R}$$

is $1-1$.
If $f^{-1}([a, b]) = \emptyset$ Then $M(a) = M(b)$ and there is nothing to prove. Otherwise, by partitioning $[a, b]$ into smaller intervals if necessary, we may assume there is a single noncritical cell σ with

$$f(\sigma) \in [a, b].$$

By lemma 0.49, exactly one of the following holds:

(i) $\exists \tau^{(p+1)} > \sigma$ with $f(\tau) \leq f(\sigma)$.

(ii) $\exists \nu^{(p-1)} < \sigma$ with $f(\nu) \geq f(\sigma)$.

In case (i), we must have $f(\tau) < a$ (for τ and σ are two noncritical cells in $[a,b]$). Thus $\tau \subseteq M(a)$. Since σ is a face of τ we have $\sigma \subseteq M(a)$ and consequently

$$M(a) = M(b).$$

Again, there is nothing to prove.

Suppose (ii) is true. Lemma 0.49 tells us that $\forall \tau^{(p+1)} > \sigma$, $f(\tau) > f(\sigma)$. In particular, $f(\tau) > b$ (otherwise σ and τ are in $[a,b]$ which contradicts the hypothesis). By lemma 0.51, we have $\forall \tau > \sigma$, $f(\tau) > b$. Thus

$$\sigma \cap M(a) = \emptyset.$$

Since (ii) is true, there exists $\nu^{(p-1)} < \sigma$ with $f(\nu) > f(\sigma)$, and in particular $f(\nu) > b$. If $\widetilde{\nu}^{(p-1)} \neq \nu$ is another $(p-1)$-face of σ, then $f(\widetilde{\nu}) < f(\sigma)$ (cf definition 0.47 (ii)). Therefore

$$f(\widetilde{\nu}) < a.$$

Thus $\widetilde{\nu}$ and all its faces are contained in $M(a)$.

Let $\widetilde{\sigma}^{(p)} \neq \sigma$ be another p-face of M with $\widetilde{\sigma} > \nu$. Condition (i) of definition 0.47 implies

$$f(\widetilde{\sigma}) > f(\nu) > b.$$

By lemma 0.51, if $\widetilde{\sigma}$ is any cell such that $\widetilde{\sigma} > \nu$, thus $f(\widetilde{\sigma}) > b$. So

$$\nu \cap M(a) = \emptyset.$$

It follows that $M(b)$ can be expressed as a disjoint union

$$M(b) = M(a) \cup \sigma \cup \nu,$$

where ν is a free face of σ. Finally $M(b) \searrow M(a)$. □

Theorem 2.53 *Let M be a finite regular CW complexe and f a discrete Morse function on M. Suppose $\sigma^{(p)}$ is a critical cell of index p with*

$$f(\sigma) \in [a,b]$$

and $f^{-1}([a,b])$ contains no other critical points. Then $M(b)$ est homotopy equivalent to

$$M(a) \bigcup_{\dot{e}^p} e^p,$$

where e^p denotes a p-dimensional cell with boundary $\dot{e}^p$.

Proof: As in proof of theorem 0.52 we may assume f is $1-1$. Thus we can find a' and b' with $a \leq a' < b' \leq b$ with

$$\sigma = f^{-1}([a', b']).$$

By theorem 0.52 $M(b) \searrow M(b')$ and $M(a') \searrow M(a)$, so it suffices to prove that $M(b')$ homotopy equivalent to

$$M(a') \bigcup_{\dot{e}^p} e^p.$$

Since σ is a critical cell, if $\tau^{(p+1)} > \sigma$ then $f(\tau) > f(\sigma)$ and so $f(\tau) > b'$. Lemma 0.51 tells us that if τ is a cell of M such that $\tau > \sigma$ then $f(\tau) > b'$. Therefore

$$\sigma \cap M(a') = \emptyset.$$

Since σ is a critical cell of M, then for any cell $\nu^{(p-1)} < \sigma$ we have $f(\nu) < f(\sigma)$ and so

$$f(\nu) < a'.$$

Consequently $\nu \subseteq M(a')$ and $\dot{\sigma} \subseteq M(a')$. Therefore

$$M(b') = M(a') \bigcup_{\dot{\sigma}} \sigma.$$

Since σ is homeomorphic to e^p, the proof is complete. □

Let $m_p(f)$ (or simply m_p if it will not cause confusion) be the number of critical cells of index p. The m_p's are called the *Morse number of f*. A consequence of theorems 0.52 and 0.53 is

Corollary 2.54 *Let M be a finite regular CW complexe and f a discrete Morse function on M. Then M is homotopy equivalent to a CW complex with exactly $m_p(f)$ cells of dimension p.*

2.3.3 Morse inequalities

References for this sections are [8] (p.28 to p.31) for the theorem and [3] (chap.I) for results on homology.
We first need some definitions and preliminary results.

Definition 2.55 *Let S be a function from certains pairs of spaces to the integers. S is* subadditive *(respectively* additive*) if $\forall\ X \supset Y \supset Z$ we have $S(X, Z) \leq S(X, Y) + S(Y, Z)$ (respectively $=$).*

Here are two important examples:
Example 1. $b_\lambda(X, Y) = \lambda$th *Betti number* of (X, Y) = rank over a field $\mathbf{F}$ of $H_\lambda(X, Y)$, for any pair (X, Y) such that this rank is finite.
b_λ is subadditive by examining the following portion of the exact sequence for the triple $X \supset Y \supset Z$

$$\ldots \to H_\lambda(Y, Z) \to H_\lambda(X, Z) \to H_\lambda(X, Y) \to \ldots$$

(for a proof see theorem 10.2 p.25 de [3]).
Example 2. If the integers $(b_\lambda(X, Y))$ are all zero but a finite number, then we define the *Euler characteristic* by

$$\chi(X, Y) = \sum_{\lambda=0}^{+\infty} (-1)^\lambda b_\lambda(X, Y).$$

The Euler characteristic is additive. As a matter of fact, if two at least of following Euler characteristic are defined, then the third is, and

$$\chi(X) = \chi(Y) + \chi(X, Y).$$

This comes from the exactness axiom (cf Axiom 4 p.11 [3]) and the fact that the exact sequence is finite.

Lemma 2.56 *Let S be subadditive and $X_0 \subset X_1 \subset \ldots \subset X_n$. Then*

$$S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}).$$

If S is additive then the equality holds.

Proof: We prove this by induction on n. For $n = 1$, we have the equality and if $n = 2$, it's only the definition of subadditivity.
If the result is true for $n - 1$, then $S(X_{n-1}, X_0) \leq \sum_{i=1}^{n-1} S(X_i, X_{i-1})$. Therefore,

$$S(X_n, X_0) \leq S(X_{n-1}, X_0) + S(X_n, X_{n-1}) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$$

and the result is true for n.
We prove the result the same way for S additive. □

Let $S(X, \emptyset) = S(X)$. Taking $X_0 = \emptyset$ in lemma 0.56 we get

$$S(X_n) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}) \tag{2.3.2}$$

with equality if S is additive.

Let M be a finite regular CW complex and f a discrete Morse function on $\mathbf{K}$ with isolated critical points. Let $a_1 < \ldots < a_k$ be such that M^{a_i} contains exactly i critical cells, and $M^{a_k} = M$. By theorem 0.53 we have

$$\begin{aligned} H_*(M^{a_i}, M^{a_{i-1}}) &= H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) \\ &= H_*(e^{\lambda_i}, \dot{e}^{\lambda_i}) \\ &= \begin{cases} \text{coefficient group in dimension } \lambda_i \text{ i.e. } \mathbf{F} \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

where λ_i is the index of the critical cell (for a proof see theorem 16.4 p.45 [3] and excision axiom for a homology).

Applying (0.3.2) to $\emptyset = M^{a_0} \subset \ldots M^{a_k} = M$ with $S = b_\lambda$ we have

$$b_\lambda(M) \leq \sum_{i=1}^{k} b_\lambda(M^{a_i}, M^{a_{i-1}}) = m_\lambda;$$

where m_λ is the Morse number of index λ (as a matter of fact $dim_{\mathbf{F}} H_{\lambda_i}(e^{\lambda_i}, \dot{e}^{\lambda_i}) = 1$).
Applying this formula to the case $S = \chi$ we obtain

$$\chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = m_0 - m_1 + m_2 - \ldots \pm m_{dim\ M}$$

Thus we have proven:

Theorem 2.57 (Weak Morse Inequalities) *Let M be a finite regular CW complex and f a discrete Morse function on M. If m_λ denotes the Morse number of index λ of f and b_λ the λth Betti number, then*

$$b_\lambda(M) \leq m_\lambda \tag{2.3.3}$$

$$\sum_{\lambda=0}^{+\infty} (-1)^\lambda b_\lambda(M) = \sum_{\lambda=0}^{+\infty} (-1)^\lambda m_\lambda \tag{2.3.4}$$

Slightly sharper inequalities can be proven by the following argument.

Lemma 2.58 *The function S_λ is subadditive where*

$$S_\lambda(X, Y) = b_\lambda(X, Y) - b_{\lambda-1}(X, Y) + b_{\lambda-2}(X, Y) - \ldots \pm b_0(X, Y)$$

Proof: Given an exact sequence of vector space

$$\xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \ldots \longrightarrow D \longrightarrow 0$$

Then the rank of h plus the rank of i is equal to the rank of A. Therefore,

$$\begin{aligned} rank(h) &= rank(A) - rank(i) \\ &= rank(A) - rank(B) + rank(j) \\ &= rank(A) - rank(B) + rank(C) - rank(k) \\ &\ldots \\ &= rank(A) - rank(B) + rank(C) - \ldots \pm rank(D). \end{aligned}$$

Hence the last expression is ≥ 0. Now consider the homology exact sequence of a triple $X \supset Y \supset Z$. Then with

$$H_{\lambda+1}(X,Y) \xrightarrow{\partial} H_\lambda(Y,Z)$$

we obtain

$$rank(\partial) = b_\lambda(Y,Z) - b_\lambda(X,Z) + b_\lambda(X,Y) - b_{\lambda-1}(Y,Z) + \ldots \geq 0.$$

Collecting terms, $S_\lambda(Y,Z) - S_\lambda(X,Z) + S_\lambda(X,Y) \geq 0$, which complete the proof. □

Applying this subadditive function to the spaces

$$\emptyset \subset M^{a_1} \subset \ldots \subset M^{a_k}$$

we obtain:

Theorem 2.59 (Strong Morse Inequalities) *Let M be a finite regular CW complex and f a discrete Morse function on M. If m_λ denotes the Morse number of index λ of f and b_λ the λth Betti number, then*

$$\forall\ \lambda \geq 0, \qquad b_\lambda(M) - b_{\lambda-1}(M) + \ldots \pm b_0(M) \leq m_\lambda - m_{\lambda-1} + \ldots \pm m_0. \tag{2.3.5}$$

Remark. These inequalities are sharper than the previous ones.
In fact, adding (0.3.5) for λ and $\lambda - 1$ we obtain (0.3.3).
Comparing (0.3.5) for λ and $\lambda - 1$ where $\lambda > dim(M)$ we obtain (0.3.4).

2.3.4 Examples of discrete Morse functions

Every CW complex M has a discrete Morse function. For example, define a Morse function f by setting, for each $\sigma \in \mathbf{K}$,

$$f(\sigma) = dim(\sigma).$$

We see that every cell is critical. Corollary 0.54 in the case of regular CW complexes tells us that a CW complex with m_p faces of dimension p is homotopy equivalent to a CW complex with m_p cells of dimension p.

We now examine ways in which a Morse function on one CW complex may induce a Morse function on another CW complex

Lemma 2.60 *Let M be a CW complex and $N \subseteq M$ be a subcomplex. Then any discrete Morse function on M restricts to a discrete Morse function on N. If $\sigma \subseteq N$ is a critical cell for f (the original function), then it is critical for the restriction.*

Proof: This follows directly from definitions 0.47 and 0.48. □

Here is a lemma which gives us a converse of the previous lemma.

Lemma 2.61 *Let M be a (finite) CW complex and $N \subseteq M$ be a subcomplex. Then any discrete Morse function on N can be extended to a discrete Morse function on M i.e. if f is a discrete Morse function on N, then there is a discrete Morse function g on M such that*

$$\forall \sigma \subseteq N,\ g(\sigma) = f(\sigma).$$

Proof: Let $c = max_{\sigma \subseteq N} f(\sigma)$. Define g on M by setting,

$$g(\sigma) = \begin{cases} f(\sigma) & si\ \sigma \subseteq N \\ c + dim(\sigma) & si\ \sigma \nsubseteq N \end{cases}$$

One easily check that g is a discrete Morse function on M that extends f. □

The Morse function constructed in the above proof may be very insufficient. In particular, every cell of $M - N$ is a critical cell. There may exist extensions to M with many fewer critical cells. In the following lemma, we will see it is possible when M collapses onto N.

Lemma 2.62 *Let M be a (finite) CW complex and $N \subseteq M$ be a subcomplex such that $M \searrow N$. Let f be a discrete Morse function on N and $c = max_{\sigma \subseteq N} f(\sigma)$. Then f can be extended to a Morse function on M with*

$$N = M(c)$$

and such there are no critical cells in $M - N$.

Proof: The proof is by induction on the number of elementary collapses and so it suffices to prove this when M collapses onto N by a single elementary collapse. Supposes τ is a cell of M with a free face $\sigma < \tau$ such that M is a disjoint union

$$M = N \cup \sigma \cup \tau.$$

Define a Morse function g on M by

$$\begin{aligned} g(\nu) &= f(\nu), \quad \nu \neq \sigma, \tau \\ g(\tau) &= c+1 \\ g(\sigma) &= c+2 \end{aligned}$$

Then g satisfies the required properties. □

Corollary 2.63 *Let Δ^n be a n-simplex with its standard triangulation, and $\dot{\Delta}^n$ its boundary. Then*

(i) Δ^n has a Morse function with exactly 1 critical point.

(ii) $\dot{\Delta}^n$ has a Morse function with exactly 2 critical points.

Proof: (i) follows from lemma 0.62 and the fact that Δ^n collapses onto one of its vertex.
(ii) follows from lemmas 0.61 and 0.62, and also the fact that for any $(n-1)$-cell σ of $\dot{\Delta}^n$, $\dot{\Delta}^n - \sigma$ collapses onto one of its vertex. □

2.3.5 The discrete gradient vector field and the associated flow

We will define in a natural way the notion of discrete gradient vector fields thanks to the discrete Morse function. To do this, we will draw one's inspiration from the definition of gradient vector field on smooth manifold.

Let M be a general CW complex (not necessary regular) and f be a discrete Morse function on $\mathbf{K}$ the open cells of M.

We try to define an object $V_f : \mathbf{K} \to \mathbf{K}$ which is a discrete analog of $-\nabla f$ the gradient vector field, and also Φ_f the associated flow. For now, we assume f a discrete Morse function has been fixed, and we write V and Φ (we forget the suffix f).

To give a natural definition of the discrete vector field, we first consider the vertices of M. Let $v \in \mathbf{K}_0$. If v is a critical cell, we wish its value to be zero as in smooth Morse theory. So we set

$$V(v) = 0.$$

If v is not a critical cell, then there is a unique edge $e > v$ with $f(e) \leq f(v)$. The edge e specifies the unique direction in which f is not increasing

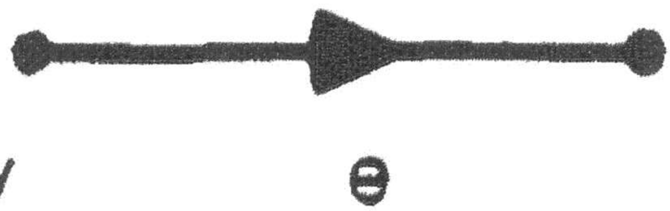

Figure 2.3.2: Direction of the gradient vector field

(cf Fig. 0.3.2). SO $-\nabla f(v)$ should be e. We must now precise this. We wish to think of $V(v)$ as a discrete tangent vector field pointing away from v. That is $V(v)$ is e with a chosen orientation.

Thus we introduce the chain complex of M. Fix an orientation for each cell $\sigma \in \mathbf{K}$. Let $C_i(M, \mathbb{Z})$ be the free abelian group generated over $\mathbb{Z}$ by these oriented cells of dimension i of M. We identify $-\sigma$ with σ given with the opposite orientation. Let ∂ denote the usual boundary operator.

$$\partial : C_p(m, \mathbb{Z}) \to C_{p-1}(M, \mathbb{Z}).$$

Then

$$\partial\sigma = \sum_{\nu^{(p-1)} < \sigma} \epsilon(\sigma, \nu)\nu,$$

where the ϵ's are integers called the incidence numbers.
It is convenient to introduce an inner product $<,>$ on C_* by declaring the cells of M to be an orthonormal basis. Thus we obtain

$$\partial\sigma = \sum_{\nu^{(p-1)} < \sigma} < \partial\sigma, \nu > \nu.$$

Thanks to this, we complete our definition of the discrete gradient vector field on the vertices. If $v \in \mathbf{K}_0$ is not a critical cell, and $e \in \mathbf{K}_1$ verifies $e > v$, $f(e) \leq f(v)$, we set $V(v) = \pm e$, where the sign is determined so that

$$< \partial(V(v)), v >= -1.$$

now we define the (discrete time) flow Φ also in a natural way. If v is a critical cell, then $V(v) = 0$ and so v is fixed under the gradient flow, i.e.

$$\Phi(v) = v.$$

If v is not critical, and $V(v) = \pm e$, then v should flow to the "other end" of e. That is,

$$\Phi(v) = v + \partial(V(v)).$$

Note that this formula holds for all vertices, wether critical or not. Moreover, if the cell is not critical, then it has actually moved since $< \Phi(v), v >= 0$.

We extend V and Φ linearly to maps on chains

$$V : C_0(M,\mathbb{Z}) \longrightarrow C_1(M,\mathbb{Z})$$

$$\Phi : C_0(M,\mathbb{Z}) \longrightarrow C_0(M,\mathbb{Z}).$$

We now extend V to higher dimension cells.

Definition 2.64 *Let σ be a p-cell de M (with a fixed orientation). If there is $\tau^{(p+1)} > \sigma$ with $f(\tau) \leq f(\sigma)$, we set*

$$V(\sigma) = - < \partial\tau, \sigma > \tau.$$

(note that σ must be a regular face of τ (for f respects the partial order on irregular faces of τ) and so $< \partial\tau, \sigma >= \pm 1$).
If there is no such τ, then we set

$$V(\sigma) = 0.$$

For each p, we extend V linearly to a map

$$V : C_p(M,\mathbb{Z}) \longrightarrow C_{p+1}(M,\mathbb{Z}).$$

Remark. Note that $V(\sigma) = 0$ without σ fixed under the flow.
In figure 0.3.3 we have $V(e_1) = 0$. Nevertheless e_1 must not be fixed under the flow Φ. The boundary of e_1 (i.e. its two vertices) "move downwards". This illustrates that there are a tangent and a transversal component of the gradient $-\nabla f(v)$. $V(\sigma)$ is the transversal component and $V(\partial\sigma)$ is the tangent one (here we catch the movement of the vertices).

Definition 2.65 *For any oriented face σ we define the (discrete time) gradient flow Φ by*

$$\Phi(\sigma) = \sigma + \partial V(\sigma) + V(\partial\sigma)$$

or, more succinctly,

$$\Phi = 1 + \partial V + V\partial.$$

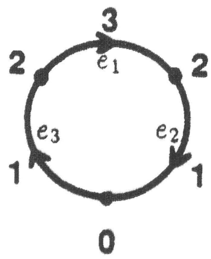

Figure 2.3.3: Cell for which the gradient is null without being fixed under the flow

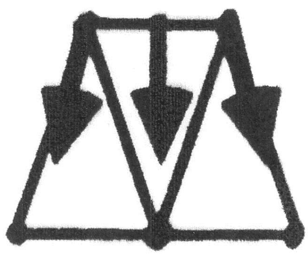

Figure 2.3.4: Gradient vector field of the upper edge

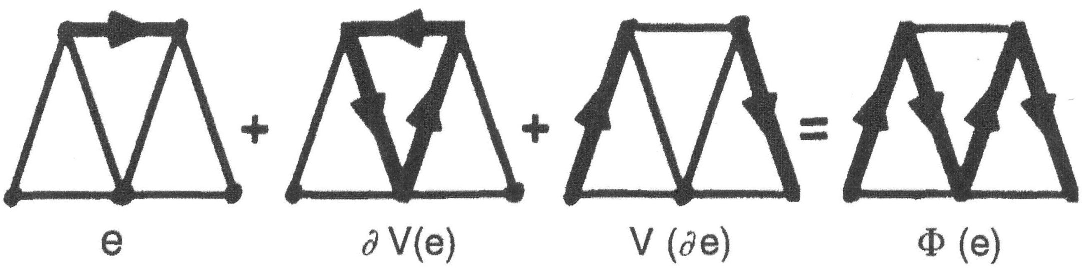

Figure 2.3.5: Expression of the flow $\Phi(e)$

Example Figures 0.3.4 and 0.3.5 illustrate the gradient vector field and the flow for a cell e oriented from the left to the right.

The main properties of V and Φ are contained in the following theorems.

Theorem 2.66 *(1)* $V \circ V = 0$.

(2) If σ is an oriented p-cell of M, then

$$\#\{\nu^{(p-1)} | V(\nu) = \pm\sigma\} \leq 1.$$

(3) If σ is an oriented p-cell of M, then

$$\sigma \text{ is critical} \Leftrightarrow [\sigma \notin Im(V) \text{ and } V(\sigma) = 0].$$

Proof: (1) If $V(\nu^{(p-1)}) = \pm\sigma^{(p)}$ then $\nu < \sigma$ and $f(\nu) \geq f(\sigma)$. By lemma 0.49, there is no $\tau > \sigma$ with $f(\tau) \leq f(\sigma)$. Thus $V(\sigma) = 0$.
(2) If $V(\nu^{(p-1)}) = \pm\sigma^{(p)}$ then $\nu < \sigma$ and $f(\nu) \geq f(\sigma)$. By condition (ii) of definition 0.47, ν is unique.
(3) From definition 0.48, σ is critical if and only if

(i) $\nexists\ \nu^{(p-1)} < \sigma$, $f(\nu) \geq f(\sigma)$ and

(ii) $\nexists\ \tau^{(p+1)} > \sigma$, $f(\tau) \leq f(\sigma)$.

These conditions are equivalent to

(i) $\nexists\ \nu^{(p-1)}$, $V(\nu) = \pm\sigma$ and

(ii) $\nexists\ \tau^{(p+1)}$, $V(\sigma) = \pm\tau$.

i.e.

(i) $\sigma \notin Im(V)$ and

(ii) $V(\sigma) = 0$.

□

Theorem 2.67 *(1) $\Phi\partial = \partial\Phi$.*
Let $\sigma_1, \dots, \sigma_r$ denote the p-cells of M with a chosen orientation. Write

$$\Phi(\sigma_i) = \sum_j a_{ij}\sigma_j.$$

(2) $\forall\ i$, $a_{ii} = 0$ or 1, and $a_{ii} = 1$ if and only if σ_i is a critical cell.

(3) If $i \neq j$, then $a_{ij} \in \mathbb{Z}$. If $i \neq j$ and $a_{ij} \neq 0$ then $f(\sigma_j) < f(\sigma_i)$.

Proof: (1) Since $\Phi = 1 + \partial V + V\partial$, we have

$$\Phi\partial = (1 + \partial V + V\partial)\partial = \partial + \partial V\partial + V\partial^2 = \partial + \partial V\partial$$

$$\partial\Phi = \partial(1 + V\partial + \partial V) = \partial + \partial V\partial + \partial^2 V = \partial + \partial V\partial.$$

We prove (2) and (3) simultaneously. first, since ∂ and V both map integer chains to integer chains, each $a_{ij} \in \mathbb{Z}$.
By theorem 0.66, each cell σ^p satisfies exactly one of the following properties:

(i) σ is a critical cell

(ii) $\pm\sigma \in Im(V)$

(iii) $V(\sigma) \neq 0$.

We examine each case independently to prove the theorem.
(i) If σ is critical then $V(\sigma) = 0$, so

$$\Phi(\sigma) = \sigma + V(\partial\sigma) = \sigma + \sum_{\nu^{(p-1)}<\sigma} < \partial\sigma, \nu > V(\nu).$$

Since σ is critical, $\forall\ \nu^{(p-1)} < \sigma$, $f(\nu) < f(\sigma)$. For each such ν, either $V(\nu) = 0$ or $V(\nu) = \widetilde{\sigma}^{(p)}$ with

$$f(\widetilde{\sigma}) \leq f(\nu) < f(\sigma).$$

Thus $\Phi(\sigma) = \sigma + \sum a_{\widetilde{\sigma}}\widetilde{\sigma}$ where $a_{\widetilde{\sigma}} \neq 0 \Rightarrow f(\widetilde{\sigma}) < f(\sigma)$.
(ii) Suppose $\pm\sigma \in Im(V) \subseteq Ker(V)$ (and so $V(\sigma) = 0$). Then

$$\Phi(\sigma) = \sigma + V(\partial\sigma) = \sigma + \sum_{\nu^{(p-1)}<\sigma} < \partial\sigma, \nu > V(\nu).$$

By theorem 0.66 (2), there is exactly one $(p-1)$-cell $\widetilde{\nu} < \sigma$ with $V(\widetilde{\nu}) = \pm\sigma$ (for $\sigma \in Im(V)$) and $< \partial\sigma, \widetilde{\nu} > V(\widetilde{\nu}) = -\sigma$ (cf definition 0.64). Moreover, if $\widetilde{\nu} \neq \nu < \sigma$ then either $V(\nu) = 0$ or $V(\nu) = \widetilde{\sigma}$ with $f(\widetilde{\sigma}) \leq f(\nu) < f(\sigma)$. Thus

$$\Phi(\sigma) = \sum_{\widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}}\widetilde{\sigma},$$

oÃź $a_{\widetilde{\sigma}} \neq 0 \Rightarrow f(\widetilde{\sigma}) < f(\sigma)$.
(iii) Suppose $V(\sigma) = - < \partial\tau, \sigma > \tau \neq 0$. Then

$$\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial\sigma).$$

Since $V(\sigma) \neq 0$, $\pm\sigma \notin Im(V)$. Thus $\forall\ \nu^{(p-1)} < \sigma$, either $V(\nu) = 0$ or $V(\nu) = \pm\widetilde{\sigma}$ avec $f(\widetilde{\sigma}) \leq f(\nu) < f(\sigma)$. Moreover

$$\partial(V(\sigma)) = - < \partial\tau, \sigma > \partial\tau = - < \partial\tau, \sigma >^2 \sigma + \sum b_{\widetilde{\sigma}}\widetilde{\sigma} = -\sigma + \sum b_{\widetilde{\sigma}}\widetilde{\sigma}$$

oÃź $b_{\widetilde{\sigma}} \neq 0 \Rightarrow f(\widetilde{\sigma}) < f(\tau) \leq f(\sigma)$.
This complete the proof. □

One can say (by theorem 0.67) that Φ decreases f, and $\sigma \subseteq \Phi(\sigma)$ if and only if σ is critical. For example consider Fig. 0.3.3 then

$$\Phi(e_1) \supseteq e_1$$

$$\Phi(e_2) \subseteq e_2$$
$$\Phi(e_3) \subseteq e_3$$

More precisely,

$$\Phi(e_1) = e_1 + e_2 + e_3$$
$$\Phi(e_2) = \Phi(e_3) = 0.$$

2.4 The Morse complex and invariant chains

Let M be a finite CW complex. Let $C_p^\Phi(M,\mathbb{Z})$ denote the Φ-invariant p-chains of M, that is

$$C_p^\Phi(M,\mathbb{Z}) = \{c \in C_p(M,\mathbb{Z}) | \Phi(c) = c\}.$$

From theorem 0.67, the boundary operator ∂ maps C_p^Φ to C_{p-1}^Φ. Thus we have a differential complex

$$\mathcal{C}^\Phi : 0 \longrightarrow C_n^\Phi(M,\mathbb{Z}) \xrightarrow{\partial} C_{n-1}^\Phi(M,\mathbb{Z}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0^\Phi(M,\mathbb{Z}) \longrightarrow 0. \quad (2.4.1)$$

The complex $\mathcal{C}^\Phi$ is called the *Morse complex.*
The goal of this section is to prove that the homology of $\mathcal{C}^\Phi$ is precisely the cell homology of M. The first step is to investigate the stabilization map $\Phi^\infty : C_* \longrightarrow C_*$ given by

$$\Phi^\infty = \lim_{N\to\infty} \Phi^N$$

More precisely, we will prove that there is $N \in \mathbb{N}$ large enough such that $\Phi^\infty = \Phi^N$.

Lemma 2.68 *Let $c \in C_p^\Phi(M,\mathbb{Z})$ and write $c = \sum_{\sigma\in\mathbf{K}_p} a_\sigma \sigma$. Let*

$$\sigma^* = \textit{any maximizer of } \{f(\sigma) | a_\sigma \neq 0\}.$$

Then σ^* is a critical cell of f.

Proof: Since c is Φ-invariant, $c = \Phi(c) = \sum_{\sigma\in\mathbf{K}_p} a_\sigma \Phi(\sigma)$. Therefore,

$$a_{\sigma^*} =< c, \sigma^* >= \sum_{\sigma\in\mathbf{K}_p} a_\sigma < \Phi(\sigma), \sigma^* > .$$

from theorem 0.67 (3), if $\sigma \neq \sigma^*$ et $f(\sigma) \leq f(\sigma^*)$ then

$$< \Phi(\sigma), \sigma^* >= 0.$$

Thus $0 \neq a_{\sigma^*} = a_{\sigma^*} < \Phi(\sigma^*), \sigma^* >$. So $< \Phi(\sigma^*), \sigma^* >\neq 0$. Theorem 0.67 (2) implies that σ^* is a critical cell. □

Theorem 2.69 $\exists N \in \mathbb{N}$ *large enough* $\Phi^N = \Phi^{N+1} = \dots = \Phi^\infty$.

Proof: The proof is by induction on $r = \#\{\widetilde{\sigma} \in \mathbf{K} | f(\widetilde{\sigma}) < f(\sigma)\}$.
Suppose $r = 0$, then by theorem 0.67, either $\Phi(\sigma) = \sigma$ or $\Phi(\sigma) = 0$. In either case $\Phi(\sigma) = \Phi^{\infty}(\sigma)$.
Suppose the property holds for $r - 1$. Then for r we have two cases:
(i) σ is not critical. Then by theorem 0.67

$$\Phi(\sigma) = \sum_{f(\widetilde{\sigma})<f(\sigma)} a_{\widetilde{\sigma}} \widetilde{\sigma}.$$

By induction, $\exists N_{\widetilde{\sigma}} \in \mathbb{N}$ such that $\Phi^{N_{\tilde{\sigma}}}(\widetilde{\sigma})$ is Φ-invariant whenever $f(\widetilde{\sigma}) < f(\sigma)$. Therefore $\Phi^{Max\{N_{\tilde{\sigma}}\}}$ is Φ-invariant ($Max\{N_{\widetilde{\sigma}}\}$ exists for M is finite).
(ii) σ is critical. Let $c = V(\partial\sigma)$. Then

$$\Phi^m(\sigma) = \sigma + c + \Phi(c) + \ldots + \Phi^{m-1}(c).$$

It follows that $\Phi^m(\sigma)$ is Φ-invariant if and only if $\exists N \in \mathbb{N}$, $\Phi^N(c) = 0$. A seen in proof of theorem 0.67, c is the sum of p-cells $\widetilde{\sigma}$ such that $f(\widetilde{\sigma}) < f(\sigma)$. By induction, $\exists \widetilde{N} \in \mathbb{N}$, $\Phi^{\widetilde{N}}(c)$ is Φ-invariant.
We now observe that $c \in Im(V)$ and $Im(V)$ is Φ-invariant, since

$$\Phi V = (1 + \partial V + V\partial)V = V(1 + \partial V)$$

(cf theorem 0.66 (1)). Thus $\Phi^{\widetilde{N}} \in Im(V)$. By theorem 0.66 (3), the image of V is orthogonal to the critical faces and therefore $\Phi^{\widetilde{N}}(c)$ is a p-chain Φ-invariant which is orthogonal to the critical faces. By lemma 0.68, $\Phi^{\widetilde{N}}(c) = 0$.
□
Example. The figure 0.4.1 illustrates that N is finite for large enough (here N=2 for the cell e).

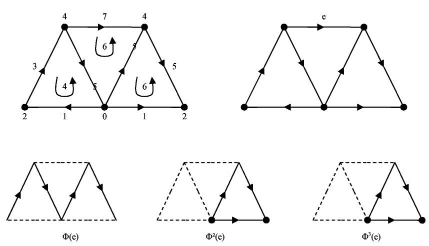

Figure 2.4.1: For the cell e, $N = 2$

By theorem 0.69, there is an $N \in \mathbb{N}$ large enough so that every chain c,

$$\Phi^N(c) = \Phi^{N+1}(c) = \Phi^{N+2}(c) = \ldots.$$

Let $\Phi^\infty(c)$ denote this Φ-invariant chain. Then for each $p \in \mathbb{N}$, we have maps

$$\Phi^\infty : C_p(M,\mathbb{Z}) \longrightarrow C_p^\Phi(M,\mathbb{Z})$$

$$i : C_p^\Phi(M,\mathbb{Z}) \hookrightarrow C_p(M,\mathbb{Z})$$

where i is the natural inclusion. Note that $\Phi^\infty \circ i$ is the identity on $C_p^\Phi(M,\mathbb{Z})$. We have now the following theorem:

Theorem 2.70 *Let C_*^Φ denote the Morse complex (0.4.1). Then $\forall p \in \mathbb{N}$*

$$H_p(C_*^\Phi) \cong H_p(M,\mathbb{Z}).$$

Proof: Consider the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_n(M,\mathbb{Z}) & \stackrel{\partial}{\longrightarrow} & C_{n-1}(M,\mathbb{Z}) & \stackrel{\partial}{\longrightarrow} & \dots & \stackrel{\partial}{\longrightarrow} & C_0(M,\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow_{\Phi^\infty} \uparrow^i & & \downarrow_{\Phi^\infty} \uparrow^i & & & & \downarrow_{\Phi^\infty} \uparrow^i & & \\ 0 & \longrightarrow & C_n^\Phi(M,\mathbb{Z}) & \stackrel{\partial}{\longrightarrow} & C_{n-1}^\Phi(M,\mathbb{Z}) & \stackrel{\partial}{\longrightarrow} & \dots & \stackrel{\partial}{\longrightarrow} & C_0^\Phi(M,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Let

$$\Phi_*^\infty : H_*(M,\mathbb{Z}) \longrightarrow H_*(C_*^\Phi)$$

$$i_* : H_*(C_*^\Phi) \longrightarrow H_*(M,\mathbb{Z})$$

denote the induced maps on homology. Our goal is to show that i_* and Φ_*^∞ are isomorphisms. In fact, we will that they are inverses of each other. Since $\Phi^\infty \circ i = 1$ we have

$$1 = (\Phi^\infty \circ i)_* = \Phi_*^\infty \circ i_*$$

To see that $i_* \circ \Phi_*^\infty = 1$, it is sufficient to find an operator

$$L : C_*(M,\mathbb{Z}) \longrightarrow C_{*-1}(M,\mathbb{Z})$$

such that

$$1 - i \circ \Phi^\infty = \partial L + L\partial.$$

As a matter of fact, closed forms would be mapped to exact forms by $1 - i \circ \Phi^\infty$, and this map would be the zero map on homology. Since i is the identity map on chais and $\Phi^\infty = \Phi^N$ for some N large enough,

$$\begin{aligned} 1 - i \circ \Phi^\infty &= 1 - \Phi^N = (1-\Phi)(1+\Phi+\Phi^2+\ldots+\Phi^{N-1}) \\ &= (-\partial V - V\partial)(1+\Phi+\Phi^2+\ldots+\Phi^{N-1}) \\ &= \partial[-V(1+\Phi+\Phi^2+\ldots+\Phi^{N-1})] + [-V(1+\Phi+\Phi^2+\ldots+\Phi^{N-1})]\partial \end{aligned}$$

(we used $\Phi\partial = \partial\Phi$).
$L = [-V(1+\Phi+\Phi^2+\ldots+\Phi^{N-1})]$ is as desired. □

2.4.1 The Morse complex and critical points

Let M be a finite CW complex. The goal of this section is to prove that the Morse complex can be expressed thanks to critical cells. We first prove that the space of Φ-invariant chains is canonically isomorphic to the span of the critical cells. Then, we will find an explicit expression of the boundary operator of the Morse complex thanks to the critical cells and the notion of gradient path.

For each $p \in \mathbb{N}$, let $\mathcal{M}_p$ denote the span of the critical p-cells, i.e.

$$\mathcal{M}_p = \{\sum_{\sigma \in \mathbf{K}_p} a_\sigma \sigma | a_\sigma \in \mathbb{Z} \text{ and } a_\sigma \neq 0 \Longrightarrow \sigma \text{ is critical}\}.$$

By restricting the map Φ^∞ defined in the previous section, we get a map

$$\Phi^\infty : \mathcal{M}_p \longrightarrow C_p^\Phi(M, \mathbb{Z}). \tag{2.4.2}$$

Fix an orientation for each p-cell σ and identify $-\sigma$ with σ given the opposite orientation.

Lemma 2.71 *Let σ be a critical p-cell. If $\widetilde{\sigma} \neq \sigma$ is critical, then*

$$< \Phi^\infty(\sigma), \widetilde{\sigma} >= 0.$$

Proof: As seen in the proof of lemma 0.68

$$\Phi^\infty(\sigma) = \sigma + c,$$

where $c \in Im(V) \subseteq \mathcal{M}_p^\perp$. □

Theorem 2.72 *The map 0.4.2 is an isomorphism.*

Proof: (onto) Suppose $c \in C_p^\Phi(M, \mathbb{Z})$, and let

$$\widetilde{c} = \sum_{\sigma \text{ critical}} < c, \sigma > \sigma \in \mathcal{M}_p.$$

We shall see that $\Phi^\infty(\widetilde{c}) = c$. From lemma 0.71, for any critical cell σ

$$< \Phi^\infty(\widetilde{c}), \sigma >=< c, \sigma > .$$

So $\Phi^\infty(\widetilde{c}) - c$ is a Φ-invariant chain such that for any critical cell σ

$$< \Phi^\infty(\widetilde{c}) - c, \sigma >= 0.$$

By lemma 0.68 we have $\Phi^\infty(\widetilde{c}) - c = 0$.
(1-1) Suppose $c \in \mathcal{M}_p$ satisfies $\Phi^\infty(c) = 0$. Then, for any critical cell σ,

$$< \Phi^\infty(c), \sigma >= 0.$$

We use again the equality $\Phi^\infty(c) = c + c_V$, where $c_V \in Im(V) \subseteq \mathcal{M}_p^\perp$. Therefore, for any critical cell σ,

$$< c, \sigma >= 0,$$

which implies $c = 0$. □

Theorem 0.72 implies that the Morse complex is isomorphic to

$$\mathcal{M} : 0 \longrightarrow \mathcal{M}_n \xrightarrow{\widetilde{\partial}} \mathcal{M}_{n-1} \xrightarrow{\widetilde{\partial}} \dots \xrightarrow{\widetilde{\partial}} \mathcal{M}_0 \to 0 \tag{2.4.3}$$

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_p & \xrightarrow{\widetilde{\partial}} & \mathcal{M}_{p-1} \\ \Phi^\infty \downarrow & & \Phi^\infty \downarrow \\ C_p^\Phi & \xrightarrow{\partial} & C_{p-1}^\Phi \end{array}$$

We obtain, for $c \in \mathbf{M}_p$, $(\Phi^\infty)^{-1}(\partial \Phi^\infty(c)) = \widetilde{\partial} c$. Thanks to the beginning of the proof of theorem 0.72, we have

$$\widetilde{\partial} c = \sum_{\widetilde{\sigma} \in \mathbf{K}_{p-1} \ and \ critical} < \partial \Phi^\infty(c), \widetilde{\sigma} > \widetilde{\sigma}.$$

So, if $c \in \mathcal{M}_p$, σ is a critical $(p-1)$-face

$$< \widetilde{\partial} c, \sigma >=< \partial \Phi^\infty c, \sigma >=< \Phi^\infty \partial c, \sigma > . \tag{2.4.4}$$

Since $H_*(\mathcal{M}) \cong H_*(M, \mathbb{Z})$, we learn from the Universal Coefficient Theorem that for any field $\mathbf{F}$,

$$H_*(\mathcal{M} \otimes \mathbf{F}) \cong H_*(\mathcal{M}) \otimes \mathbf{F} \cong H_*(M, \mathbb{Z}) \otimes \mathbf{F} \cong H_*(M, \mathbf{F}).$$

Thus $\mathcal{M} \otimes \mathbf{F}$ is a differential complex of vector spaces over $\mathbf{F}$ with the same homology as M. Moreover, $dim_{\mathbf{F}} \mathcal{M}_p \otimes \mathbf{F} = m_p(f)$, it follows that

Corollary 2.73 *If M is a finite CW complex, f a discrete a discrete Morse Function on M and $\mathbf{F}$ is any coefficient field, then the Strong Morse Inequalities and hence the Weak Morse Inequalities hold.*

Now we look for a suitable expression for $\widetilde{\partial}$. We will show that for $\tau^{(p+1)}$ and $\sigma^{(p)}$ two critical cells, $< \widetilde{\partial}\tau, \sigma >$ can be expressed in term of *gradient paths* from $\partial\tau$ to σ.

Definition 2.74 *A gradient path of dimension p is a sequence γ of p-cells of M*

$$\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$$

such that for every $i = 0, \ldots, r-1$,

(i) if $V(\sigma_i) = 0$ then $\sigma_{i+1} = \sigma_i$.

(ii) if $V(\sigma_i) \neq 0$ then $\sigma_{i+1} < V(\sigma_i)$ and $\sigma_{i+1} \neq \sigma_i$.

We say γ is a gradient path *from σ_0 to σ_r. The* length *of γ, denoted $|\gamma|$ is equal to r.*

Thus we have the two following properties

Lemma 2.75 *(i) If $\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$ is a gradient path then for each $i = 0, \ldots, r-1$ either $\sigma_i = \sigma_{i+1}$ or $f(\sigma_i) > f(\sigma_{i+1})$.*

(ii) If $\gamma_1 = \sigma_0, \sigma_1, \ldots, \sigma_r$ and $\gamma_2 = \sigma_{r+1}, \sigma_1, \ldots, \sigma_{r+s}$ are two sequences of p-cells, then

$$\sigma_0, \sigma_1, \ldots, \sigma_{r+1}, \ldots, \sigma_{r+s}$$

is a gradient path if and only if γ_1 and γ_2 are gradient paths.

We now introduce the notion of multiplicity of a gradient path to "measure" the way orientation is carried along a gradient path.

Suppose$\sigma \neq \widetilde{\sigma}$ are two p-cells of M and τ is a $(p+1)$-cell with $\sigma < \tau$ and $\widetilde{\sigma} < \tau$ and both are regular faces of τ. Then an orientation on σ induces an orientation on $\widetilde{\sigma}$ in the following way. AN orientation on σ induces an orientation on τ so that $< \partial\tau, \sigma >= -1$. Given the orientation on τ, we choose the orientation on $\widetilde{\sigma}$ so that $< \partial\tau, \widetilde{\sigma} >= 1$. Equivalently, fixing an orientation on σ and τ, an orientation is induced on $\widetilde{\sigma}$ so that

$$< \partial\tau, \sigma >< \partial\tau, \widetilde{\sigma} >= -1.$$

One can say that we induce an orientation on$\widetilde{\sigma}$ by "sliding" σ across τ to $\widetilde{\sigma}$. Figure 0.4.2 presents this process, with a fixed orientation of σ.

Moreover, if $\sigma = \widetilde{\sigma}$, then the orientation induced on $\widetilde{\sigma}$ must be the same orientation on σ. Thus, if $\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$ is a gradient path, σ_0 induces an orientation by degrees on σ_i an in particular on σ_r. Since we have fixed an

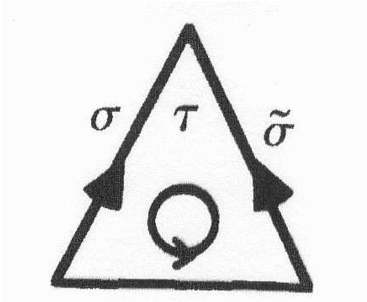

Figure 2.4.2: Induced orientation

orientation for each cell of M, we set $m(\gamma)$ by $m(\gamma) = 1$ if the induced orientation by σ_0 on σ_r is the orientation of this one, and $m(\gamma) = -1$ otherwise. Equivalently, we set

$$m(\gamma) = \prod_{\substack{i=0 \\ V(\sigma_i)\neq 0}}^{r-1} < \partial V(\sigma_i), \sigma_{i+1} > . \tag{2.4.5}$$

We can use this formula to define the multiplicity of any gradient path.

Definition 2.76 *Let $\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$ be a gradient path of dimension p. We define the* multiplicity *of γ, $m(\gamma)$ by the formula (0.4.5).*

We note that this notion is compatible with the composition of gradient paths. More precisely, if $\gamma_0 = \sigma_0, \sigma_1, \ldots, \sigma_r$ and $\gamma_1 = \sigma_{r+1}, \ldots, \sigma_{r+s}$ are two gradient paths of same dimension, then

$$m(\gamma_1)m(\gamma_0) = m(\gamma_1 \circ \gamma_0) \tag{2.4.6}$$

where $\gamma_1 \circ \gamma_0 = \sigma_0, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}$.
For p-cells σ and $\widetilde{\sigma}$, let $\Gamma_r(\sigma, \widetilde{\sigma})$ the set of all gradient paths from σ to $\widetilde{\sigma}$ of length r.

We now show that if $\tau^{(p+1)}$ and $\sigma^{(p)}$ are two critical cells, then

$$< \widetilde{\partial}\tau, \sigma >= \sum_{\widetilde{\sigma}^{(p)} < \tau} < \partial\tau, \widetilde{\sigma} > \sum_{\gamma \in \Gamma_N(\widetilde{\sigma}, \sigma)} m(\gamma)$$

for any N large enough.

Definition 2.77 *Define a* reduced flow $\widetilde{\Phi} : C_p(M,\mathbb{Z}) \to C_p(M,\mathbb{Z})$ *by*

$$\widetilde{\Phi} = 1 + \partial V$$

Though $\widetilde{\Phi}$ differs from Φ, we will prove that we can replace Φ by $\widetilde{\Phi}$ in equation (0.4.4). The gain is that $\widetilde{\Phi}$ is easier than Φ to compute.

Lemma 2.78 *For any critical cells $\tau^{(p+1)}$ and $\sigma^{(p)}$*

$$< \widetilde{\partial}\tau, \sigma >=< \widetilde{\Phi}^{\infty}\partial\tau, \sigma > .$$

Proof: It is sufficient to prove that for every $r \geq 0$

$$< \widetilde{\Phi}^{r}\partial\tau, \sigma >=< \Phi^{r}\partial\tau, \sigma > .$$

This follows from the observation that for every chain c and every $r \geq 0$

$$\Phi^r(c) - \widetilde{\Phi}^r(c) \in Im(V) \subseteq \mathcal{M}_*^{\perp}.$$

We prove this by induction on r.
If $r = 0$ then there is nothing to prove.
Suppose the property is true for $r-1$. For r,

$$\Phi^r(c) = \Phi(\Phi^{r-1}(c)) = \Phi(\widetilde{\Phi}^{r-1}(c) + V(\widetilde{c}))$$

for some chain $\widetilde{c}$ (by the inductive hypothesis).

$$\begin{aligned} \Phi^r(c) &= (\widetilde{\Phi} + V\partial)(\widetilde{\Phi}^{r-1}(c) + V(\widetilde{c})) \\ &= \widetilde{\Phi}^r(c) + \widetilde{\Phi}(V(\widetilde{c})) + V\partial\widetilde{\Phi}^{r-1}(c) + V\partial V(\widetilde{c}) \\ &= \widetilde{\Phi}^r(c) + V(\widetilde{c} + \partial\widetilde{\Phi}^{r-1}(c) + \partial V(\widetilde{c})) \end{aligned}$$

where the last equality follows from $\widetilde{\Phi}V = V + \partial V^2 = V$. □

Lemma 2.79 $\forall(\sigma_1^{(p)}, \sigma_2^{(p)}) \in (\mathbf{K}_p)^2$,

$$< \widetilde{\Phi}\sigma_1, \sigma_2 >= \sum_{\gamma\in\Gamma_1(\sigma_1,\sigma_2)} m(\gamma). \tag{2.4.7}$$

Proof: We distinguish elementary cases to prove this.
First, suppose $V(\sigma_1) = 0$. Then

$$< \widetilde{\Phi}\sigma_1, \sigma_2 >=< \sigma_1, \sigma_2 >= \begin{cases} 1, & si\ \sigma_1 = \sigma_2 \\ 0, & si\ \sigma_1 \neq \sigma_2 \end{cases}$$

On the other hand, the only gradient path of length 1 beginning at σ is the trivial one $\gamma = \sigma_1, \sigma_1$ if $\sigma_2 = \sigma_1$ and so $m(\gamma) = 1$. If $\sigma_1 \neq \sigma_2$, $\Gamma_1(\sigma_1, \sigma_2) = \emptyset$ so

$$\sum_{\gamma \in \Gamma_1(\sigma_1, \sigma_2)} m(\gamma) = 0.$$

We have the equality between the two expressions.
Now, suppose $V(\sigma_1) \neq 0$. If $\sigma_1 = \sigma_2$, we calculate the left hand side of (0.4.7) to find

$$< \widetilde{\Phi}\sigma_1, \sigma_1 >=< \sigma_1, \sigma_1 > + < \partial V(\sigma_1), \sigma_1 >= 1 - 1 = 0.$$

We calculate the right hand side of (0.4.7). Since $V(\sigma_1) \neq 0$, there is no gradient path of length 1 from σ_1 to σ_1 so $\Gamma_1(\sigma_1, \sigma_1) = \emptyset$ and

$$\sum_{\gamma \in \Gamma_1(\sigma_1, \sigma_1)} m(\gamma) = 0.$$

Now, suppose $\sigma_1 \neq \sigma_2$ then

$$< \widetilde{\Phi}\sigma_1, \sigma_2 >=< \sigma_1, \sigma_2 > + < \partial V(\sigma_1), \sigma_2 >=< \partial V(\sigma_1), \sigma_2 > .$$

If σ_2 is not a face of $V(\sigma_1)$, then $< \widetilde{\Phi}\sigma_1, \sigma_2 >= 0$. In this case there are no gradient paths of length 1 from σ_1 to σ_2 so that

$$\sum_{\gamma \in \Gamma_1(\sigma_1, \sigma_2)} m(\gamma) = 0.$$

If σ_2 is a face of $V(\sigma_1)$, then there exactly one gradient path of length from σ_1 to σ_2: $\gamma = \sigma_1, \sigma_2$. So

$$m(\gamma) =< \partial V(\sigma_1), \sigma_2 >$$

as desired. □

Theorem 2.80 $\forall (\tau^{(p+1)}, \sigma^{(p)}) \in (\mathbf{K}_{p+1} \times \mathbf{K}_p)$ *which are critical cells*

$$< \widetilde{\partial}\tau, \sigma >= \sum_{\widetilde{\sigma}^{(p)} < \tau} < \partial\tau, \widetilde{\sigma} > \sum_{\gamma \in \Gamma_N(\widetilde{\sigma}, \sigma)} m(\gamma)$$

for N large enough.

Proof: By lemma 0.78, for N large enough

$$< \widetilde{\partial}\tau, \sigma >=< \widetilde{\Phi}^N \partial\tau, \sigma > .$$

Since $\partial\tau = \sum_{\widetilde{\sigma}^{(p)}<\tau} < \partial\tau, \widetilde{\sigma} > \widetilde{\sigma}$, We find

$$< \widetilde{\partial}\tau, \sigma >= \sum_{\widetilde{\sigma}^{(p)}<\tau} < \partial\tau, \widetilde{\sigma} >< \widetilde{\Phi}^N \widetilde{\sigma}, \sigma >$$

for N large enough. We prove by induction on $r \geq 1$

$$< \widetilde{\Phi}^r \widetilde{\sigma}, \sigma >= \sum_{\gamma \in \Gamma_r(\widetilde{\sigma}, \sigma)} m(\gamma).$$

The case $r = 1$ is lemma 0.79.
Suppose the property is true for $r - 1$.

$$\begin{aligned}
< \widetilde{\Phi}^r \widetilde{\sigma}, \sigma > &= < \widetilde{\Phi}(\widetilde{\Phi}^{r-1}\widetilde{\sigma}), \sigma > \\
&= \sum_{\sigma'^{(p)}} < \widetilde{\Phi}^{r-1}\widetilde{\sigma}, \sigma' >< \widetilde{\Phi}\sigma', \sigma > && (\widetilde{\Phi}^{r-1}\widetilde{\sigma} = \sum_{\sigma'^{(p)}} < \widetilde{\Phi}^{r-1}\widetilde{\sigma}, \sigma' > \sigma') \\
&= \sum_{\sigma'} \sum_{\gamma \in \Gamma_{r-1}(\widetilde{\sigma}, \sigma')} m(\gamma) < \widetilde{\Phi}\sigma', \sigma > && \text{(by induction hypothesis)} \\
&= \sum_{\sigma'} \sum_{\gamma \in \Gamma_{r-1}(\widetilde{\sigma}, \sigma')} m(\gamma) \sum_{\gamma' \in \Gamma_1(\sigma', \sigma)} m(\gamma') && \text{(by lemma 0.78)} \\
&= \sum_{\gamma \in \Gamma_r(\widetilde{\sigma}, \sigma)} m(\gamma) && \text{(by lemma 0.75 (ii) and (0.4.6))}
\end{aligned}$$

So the property is true for r and the proof is complete. □

2.5 Applications and perspectives

2.5.1 h-cobordism

Triade cellulaire, rÃľarrangement d'anses.

Theorem 2.81 *Tout h-cobordisme PL, simplement connexe, entre variÃľtÃľs de dimension supÃľrieure ou Ãľgales Ãă 5 est Ãľquivalent Ãă un produit.*

2.5.2 Morse theory and low dimensional topology

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