REMARKS ON THE THREE-MANIFOLD INVARIANTS θ_p

C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel

URA CNRS n^0 758
Département de Mathématiques
Faculté des Sciences
44072 NANTES CEDEX 03
FRANCE

Introduction

In [BHMV1] we defined for each integer $p \geq 1$ an invariant θ_p of closed oriented three-manifolds. If M^3 is presented by surgery on a banded link L in S^3, then $\theta_p(M)$ is given by evaluating at a primitive $2p$-th root of unity the one variable Kauffman bracket of a certain cabling of L. This method of constructing invariants had previously been used by Lickorish [Li2, Li3, Li4] who had considered the case of evaluations at $4r$-th roots of unity ($r \geq 3$). Our invariant θ_{2r} is indeed equal to Lickorish's invariant except for a slightly different normalisation. In this note we will discuss the invariant θ_r for odd r. The main result is a formula relating it to θ_1, θ_2, and θ_{2r}.

In section 1, we generalise θ_p to an invariant of pairs (M, K) where K is a banded link in M (This generalisation was noticed already by Lickorish for his invariant.) The invariant $\theta_p(M, K)$ lies in $\mathbf{Z}[\frac{1}{p}, A]/\phi_{2p}(A)$ where $\phi_{2p}(A)$ is the $2p$-th cyclotomic polynomial. In section 2, we show:

Theorem 2.1. *If r is odd, then*

$$i_{2,2r}(\theta_2(M, K))\, j_{r,2r}(\theta_r(M, K)) = \theta_1(M, K)\, \theta_{2r}(M, K)$$

Here $i_{2,2r}$ and $j_{r,2r}$ are ring homomorphisms defined by $i_{2,2r}(A) = A^{r^2}$ and $j_{r,2r}(A) = A^{r^2+1}$. (Notice that $\theta_1(M, K) \in \mathbf{Z}$. In fact, $\theta_1(M, K) = (-2)^{\sharp K}$ where $\sharp K$ denotes the number of components of K.)

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We will give a self-contained proof of theorem 2.1, using only the results of [BHMV1] and a simple parity argument. However, this result and its proof were inspired by [KM2]. Indeed, $\theta_{2r}(M)$ when evaluated at $A = -e^{2\pi i/4r}$ is, up to normalisation, the same as the invariant $\tau_r(M)$ constructed by Reshetikhin-Turaev [RT] and Kirby-Melvin [KM1,KM2] from the representation theory of the quantum group $U_qSU(2)$ at $q = e^{2\pi i/r}$. (This corresponds to the fact that the Jones polynomial $V_L(t)$ of a link L in S^3 and the one variable Kauffman bracket $< D > \in \mathbf{Z}[A, A^{-1}]$ of a diagram D of L are related through $t = A^{-4}$ (and multiplication by a constant depending on the writhe of D.)) We will see that in the same way $\theta_r(M)$, for r odd, is related to the refined invariant $\tau'_r(M)$ of [KM2], and theorem 2.1, when evaluated at the chosen root of unity, becomes a formula of [KM2] involving their invariants $\tau_3(M)$, $\tau_r(M)$, and $\tau'_r(M)$ (precise formulas will be given in prop. 2.2.) Notice however that the systematic approach in [BHMV1] gave all invariants θ_p $(p \geq 1)$ at once, and on an equal footing.

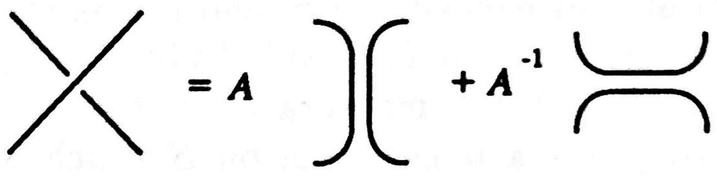
Of course, all these invariants are special cases of the so-called Jones-Witten invariants, whose existence was postulated by Witten [Wi] using Quantum Field Theory, and first proven by Reshetikhin and Turaev [RT], and Kirby and Melvin for the $U_qSU(2)$-case. The invariant θ_r for odd r seems to be related to the group $SO(3)$ in the same way as the invariant θ_{2r} is to $SU(2)$ (see remark 2.11.)

In his talk at this conference, the third author gave a survey of the construction of the invariants θ_p [BHMV1] and showed how the invariants lead to an elementary construction of a "Topological Quantum Field Theory" [BHMV2]. He also gave a short proof that the invariant θ_p can be obtained as a "thermodynamic limit" in the sense of Wenzl [We2]. This proof is the content of section 3.

1. The invariant $\theta_p(M, K)$

In this section, we extend the invariant θ_p of [BHMV1] to the case of banded links in oriented closed 3-manifolds. We first recall some results of [BHMV1].

Let M be a compact, oriented 3-manifold, possibly with boundary. A *banded link* in M is an oriented submanifold homeomorphic to a disjoint union of annuli $S^1 \times \mathbf{I}$ in M. Let $\mathbf{Z}[A, A^{-1}]$ denote the ring of Laurent polynomials in the indeterminate A. The *Jones-Kauffman module* $K(M)$ is the $\mathbf{Z}[A, A^{-1}]$-module generated by the set of isotopy classes of banded links in M, quotiented by the following *Kauffman relations*:



$$L \cup O = \delta L$$

Here $\delta = -A^2 - A^{-2}$.

Given a banded link $L \subset S^3$, its "value" in $K(S^3)$ is called the Kauffman bracket of L, and denoted $< L >$. It is a well known fact [Ka] that $K(S^3) \cong \mathbf{Z}[A, A^{-1}]$. We fix an isomorphism by the convention that the bracket of the empty link is equal to 1.

Notice that changing the writhe of a component of L by $+1$ multiplies $< L >$ by $-A^3$. (Recall that for a banded *knot* $K \subset S^3$, its writhe $w(K)$ is defined as the linking number of its two boundary components. Here one of the two components can be oriented arbitrarily, but the other must be oriented in the same way.)

Let $\mathcal{B}$ denote the Jones-Kauffman module of the standard solid torus $S^1 \times \mathbf{I} \times \mathbf{I}$. Gluing two solid tori together so as to get a third endowes $\mathcal{B}$ with a multiplication. Let $z \in \mathcal{B}$ be represented by a standard band, e. g. by $S^1 \times J \times pt$, where $J \subset \mathbf{I}$ is a proper subinterval. Then z^n means n parallel standard bands, and it is well known [T1] that $\mathcal{B}$ is isomorphic to the polynomial algebra $\mathbf{Z}[A, A^{-1}][z]$. It has a basis of monic polynomials e_i of degree i which satisfy $e_0 = 1$, $e_1 = z$ and $ze_j = e_{j+1} + e_{j-1}$ (they are related to the Jones idempotents [J] in the Temperley-Lieb algebra , see also [We1] [Li4].) Let t be the self-map of $\mathcal{B}$ induced by one positive twist. Then

$$t(e_i) = \mu_i e_i$$

where $\mu_i = (-1)^i A^{i^2+2i}$. For $p \geq 1$, define Ω_p by $\Omega_1 = 1$, $\Omega_2 = 1 + \frac{z}{2}$, and $\Omega_p = \sum_{i=0}^{n-1} < e_i > e_i$ for $p \geq 3$, where $n = [(p-1)/2]$.

The standard embedding $S^1 \times I \times I \to S^3$ yields a linear form $< >: \mathcal{B} \to K(S^3) = Z[A, A^{-1}]$. Notice that the Jones-Kauffman module of a disjoint union of n solid tori is the n-fold tensor product $\mathcal{B}^{\otimes n}$. Given a n-component banded link in S^3, the *meta-bracket* $< ,\ldots, >_L$ is defined to be the n-linear form on $\mathcal{B}$ given by replacing the components of L by elements of $\mathcal{B}$, and taking the bracket of the resulting linear combination of banded links in S^3. (This corresponds to the map Φ_D in [Li4].) The metabracket of a 1-component unknot with writhe zero yields a linear form on $\mathcal{B}$ which will be denoted $< >$.

If $L \subset S^3$ is a banded link, we obtain a 4-manifold W_L by attaching to the four-ball D^4 a 2-handle along each component of L. Let $M_L = \partial W_L$; one says it is the 3-manifold obtained from S^3 by surgery on L. It is well known [Li1] that up to oriented diffeomorphism, any oriented compact closed 3-manifold is obtained in this way.

Theorem 1.1. [BHMV1] ([Li3],[Li4] in the case $p \equiv 0 \bmod 2$) *The expression*

$$\theta_p(L) = \frac{< \Omega_p, \ldots, \Omega_p >_L}{< t(\Omega_p) >^{b_+(L)} < t^{-1}(\Omega_p) >^{b_-(L)}}$$

defines an element of $Z[\frac{1}{p}, A]/\phi_{2p}(A)$ which is an invariant , denoted by $\theta_p(M_L)$, of the oriented 3-manifold M_L obtained by surgery on L.

(Here $b_+(L)$ and $b_-(L)$ are the number of positive and negative eigenvalues of the linking matrix of L, and ϕ_d denotes the d-th cylotomic polynomial in the indeterminate A.)

We now give a generalisation of this theorem to include the case of banded links. (This generalisation was noticed already by Lickorish [Li3, Li4] for his invariant.) Notice that if $K \subset M_L$ is a banded link, then by general position we may isotope it to lie in $S^3 - L$.

Theorem 1.2. *The expression*

$$\theta_p(L, K) = \frac{< \Omega_p, \ldots, \Omega_p, z, \ldots, z >_{L \cup K}}{< t(\Omega_p) >^{b_+(L)} < t^{-1}(\Omega_p) >^{b_-(L)}}$$

defines an element of $\mathbf{Z}[\frac{1}{p}, A]/\phi_{2p}(A)$ *which is an invariant , denoted by* $\theta_p(M_L, K)$, *of the pair* (M_L, K).

Notice that $< \Omega_p, \ldots, \Omega_p, z, \ldots, z >_{L \cup K}$ simply means the bracket of the linear combination of banded links obtained from $L \cup K$ by cabling the components of L by Ω_p, leaving the components of K unchanged .

This theorem is a straightforward generalisation of theorem 1.1 and we will only sketch the proof. It uses the following lemma which was shown in [BHMV1].

Lemma 1.3. *Let* $< \, , \, >$ *denote the symmetric bilinear form on* $\mathcal{B}$ *given by the meta-bracket of the banded Hopf link where each component has writhe zero. Let* $p \geq 1$. *Then for* $\epsilon = \pm 1$, *and all* $b \in \mathcal{B}$, *one has*

$$< t^\epsilon(\Omega_p), t^\epsilon(b) >=< t^\epsilon(\Omega_p) >< b >$$

in $\mathbf{Z}[A]/\phi_{2p}(A)$.

Proof of theorem 1.2. We first quickly review the proof of theorem 1.1. Kirby's calculus [K] as refined by Fenn-Rourke [FR] implies that an invariant of (isotopy classes of) banded links L in S^3 depends only on the 3-manifold M_L if and only if it is invariant under the following move K_ϵ ($\epsilon = \pm 1$) (and its inverse):
A K_ϵ-move consists of adding to L an unknotted component with writhe ϵ, and giving a full ϵ-twist to the part of L passing through the new component.

Suppose L has components $L_1, \ldots, L_k$, and let L' be the result of an K_ϵ-move on L, the new component being L'_{k+1}. Then Lemma 1.3 implies that for all $b_1, \ldots, b_k \in \mathcal{B}$, one has

$$< b_1, \ldots, b_k, \Omega_p >_{L'} =< t^\epsilon(\Omega_p) > < b_1, \ldots, b_k >_L$$

in $\mathbf{Z}[A]/\phi_{2p}(A)$. Setting all b_i equal to Ω_p, we see that

$$\theta_p(L) = \frac{< \Omega_p, \ldots, \Omega_p >_L}{< t(\Omega_p) >^{b_+(L)} < t^{-1}(\Omega_p) >^{b_-(L)}}$$

is invariant (modulo $\phi_{2p}(A)$) under K_ϵ-moves, and theorem 1.1. follows.

We now generalise this as follows. Consider banded links of the form

$$L \cup K = L_1 \cup \ldots L_k \cup K_1 \cup \ldots K_r$$

and let a $\tilde{K}_\epsilon$-move consist of adding to L an unknotted component with writhe ϵ, and giving a full ϵ-twist to the part of $L \cup K$ passing through the new component. Then the above reasoning shows that

$$\theta_p(L, K) = \frac{< \Omega_p, \ldots, \Omega_p, z, \ldots, z >_{L \cup K}}{< t(\Omega_p) >^{b_+(L)} < t^{-1}(\Omega_p) >^{b_-(L)}}$$

is invariant (modulo $\phi_{2p}(A)$) under $\tilde{K}_\epsilon$-moves. Notice that this almost proves theorem 1.2. Indeed, the only remaining point is to show that $\theta_p(L, K)$ is independent of the isotopy class of K in M_L. This point is settled as follows. We define a β-move on $L \cup K$ to consist of replacing a component of K by a band-connected sum with a pushoff of a component of L. (This is one of the moves appearing in Kirby's original theorem [K]. Notice however that in [K] all components are surgery instructions, whereas here only the components of L are.) Now it is not hard to see that banded links in $S^3 - L$ are isotopic in M_L if and only if they are related by a sequence of isotopies in $S^3 - L$ and β-moves (and their inverses.) But as shown by Fenn and Rourke, a β-move may be replaced by a sequence of $\tilde{K}_\epsilon$-moves (and their inverses). This completes the proof.

1.4. Remarks.

1) The invariant θ_p has the following properties.

(i) $\theta_p(S^3) = 1, \quad \theta_p(S^3, K) = < K > \in \mathbb{Z}[A]/\phi_{2p}(A)$

(ii) $\theta_p(M \sharp M', K \cup K') = \theta_p(M, K)\theta_p(M', K')$

(iii) $\theta_p(-M, K) = \overline{\theta_p(M, K)}$ (Here, the conjugation on $\mathbb{Z}[A, \frac{1}{p}]/\phi_{2p}(A)$ is defined by $\overline{A} = A^{-1}$, and $-M$ denotes M with reversed orientation.)

(iv) $\theta_p(S^1 \times S^2) = < \Omega_p > = < t\Omega_p >< t^{-1}\Omega_p >$.

2) If $p \in \{1, 3, 4\}$, then $\Omega_p = 1$, hence $\theta_p(M, K) = < K > \in \mathbb{Z}[A]/\phi_{2p}(A)$. Moreover, one has $< K > = \delta^{\sharp K}$ in these three cases,

where $\sharp K$ denotes the number of components of K. (This follows easily from the Kauffman relations.) In particular

$$\theta_1(M,K) = (-2)^{\sharp K}$$

3) Notice that instead of Ω_p we could have used $\lambda\Omega_p$, where λ is some invertible scalar, since this would not affect the validity of Lemma 1.3. The effect on $\theta_p(M,K)$ would merely be multiplication by $\lambda^{b_1(M)}$, where $b_1(M)$ denotes the first Betti number of M. This follows since the first Betti number $b_1(M_L)$ is equal to the number $b_0(L)$ of zero eigenvalues of the linking matrix of L, and $b_+(L) + b_-(L) + b_0(L)$ is equal to the number of components of L.

The uniqueness result of [BHMV1] can be sharpened as follows: Set $\Lambda'_p = \mathbf{Z}[\frac{1}{p}, A]/\phi_{2p}(A)$ for $p \notin \{1,3,4,6\}$, $\Lambda'_p = \mathbf{Z}[A]/\phi_{2p}(A)$ for $p \in \{1,3,4\}$, and $\Lambda'_6 = \mathbf{Z}[\frac{1}{2}, A]/\phi_{12}(A)$.

Proposition 1.5. *Let Λ be an integral domain containing a homomorphic image of $\mathbf{Z}[A,A^{-1}]$, and let $\Omega \in \mathcal{B} \otimes \Lambda = \Lambda[z]$ such that $< t^\epsilon(\Omega) >$ is invertible in Λ for $\epsilon = \pm 1$. Suppose that for each pair (L,K) of banded links in S^3, the expression*

$$\frac{< \Omega, \ldots, \Omega, z, \ldots, z >_{L \cup K}}{< t(\Omega) >^{b_+(L)} < t^{-1}(\Omega) >^{b_-(L)}}$$

defines an invariant $\theta_\Omega(M_L, K) \in \Lambda$. Then there is an integer $p \geq 1$ and a unit $\lambda \in \Lambda$ such that the map $\mathbf{Z}[A,A^{-1}] \to \Lambda$ factors through a homomorphism $f : \Lambda'_p \to \Lambda$, and such that

$$\theta_\Omega(M,K) = \lambda^{b_1(M)} f(\theta_p(M,K))$$

Proof. The hypothesis implies that for all $b \in \mathcal{B}$ and $\epsilon = \pm 1$, one has $< t^\epsilon(\Omega), t^\epsilon(b) > = < t^\epsilon(\Omega) >< b >$ as elements of Λ. Hence, the result follows from prop. 6.10 of [BHMV1].

2. A relation between θ_1, θ_2, θ_r, and θ_{2r} (r odd)

In this section, we will give a formula relating θ_1, θ_2, θ_r, and θ_{2r}, assuming that r is odd. As in [BHMV1], we will use the notation

$\Lambda_p = \mathbf{Z}[A, A^{-1}]/\phi_{2p}(A)$, where ϕ_d denotes the d-th cylotomic polynomial in the indeterminate A. Thus the invariant $\theta_p(M, K)$ lies in $\Lambda_p[\frac{1}{p}]$.

Notation. Let $r \geq 1$ be odd. Notice that in Λ_{2r}, we have $\phi_4(A^{r^2}) = 0$ and $\phi_{2r}(A^{r^2+1}) = 0$ (because $gcd(r^2, 4r) = r$ and $gcd(r^2+1, 4r) = 2$ as is easily verified.) Hence we may define ring homomorphisms

$$i_{2,2r} : \Lambda_2 \to \Lambda_{2r}, \quad A \mapsto A^{r^2}$$

$$j_{r,2r} : \Lambda_r \to \Lambda_{2r}, \quad A \mapsto A^{r^2+1}$$

Theorem 2.1. *If r is odd, then*

$$i_{2,2r}(\theta_2(M, K))\, j_{r,2r}(\theta_r(M, K)) = \theta_1(M, K)\, \theta_{2r}(M, K)$$

Remark. Notice in particular that if $K = \emptyset$, then $i_{2,6}(\theta_2(M)) = \theta_6(M)$ (because $\theta_3(M) = 1$.)

Here is the relationship with the invariants $\tau_r(M)$ and $\tau'_r(M)$ of [KM2]. Let $b_1(M)$ denote the first Betti number of M.

Proposition 2.2. *For $r \geq 3$*

$$\tau_r(M) = \left(\frac{2 \sin \frac{\pi}{r}}{\sqrt{2r}}\right)^{b_1(M)} \theta_{2r}(M)\big|_{A=-e^{2\pi i/4r}}$$

If moreover r is odd, then

$$\tau'_r(M) = \left(\frac{2 \sin \frac{\pi}{r}}{\sqrt{r}}\right)^{b_1(M)} \theta_r(M)\big|_{A=e^{2\pi i(1+r^2)/4r}}$$

This will be shown at the end of this section. (Strictly speaking, we verified the formula for τ'_r only in the case where $\theta_2(M) \neq 0$.)

Remark. Notice that in particular

$$\sqrt{2}^{b_1(M)} \tau_3(M) = \theta_6(M)\big|_{A=-e^{2\pi i/12}} = \theta_2(M)\big|_{A=i}$$

Setting $K = \emptyset$ and $A = -e^{2\pi i/4r}$, theorem 2.1 becomes Kirby and Melvin's formula

$$\tau_r(M) = \begin{cases} \tau_3(M)\ \tau_r'(M) & \text{if } r \equiv 3 \bmod 4 \\ \overline{\tau_3(M)}\ \tau_r'(M) & \text{if } r \equiv 1 \bmod 4 \end{cases}$$

Remark. It was shown in [KM2] that $\tau_3(M)$ (hence $\theta_2(M)$) is zero if and only if there exist $\alpha \in H^1(M; \mathbf{Z}/2)$ with $\alpha^3 \neq 0$. One can show that $\theta_2(M, K) = 0$ if and only if there exist $\alpha \in H^1(M; \mathbf{Z}/2)$ with $< \alpha^3, [M] > \neq < \alpha, [K] >$ (where $[M]$ is the fundamental class of M, and $[K]$ is the value of K in $H_1(M; \mathbf{Z}/2)$.)

We will now give a proof of theorem 2.1. We recall the definition of the quotient algebras V_p (see [BHMV1]). Let $p \geq 1$ be an integer. Let $\mathcal{B}_p$ denote the Jones-Kauffman module of the solid torus $S^1 \times \mathbf{I} \times \mathbf{I}$ with coefficients in Λ_p, i.e. $\mathcal{B}_p = \mathcal{B} \otimes \Lambda_p = \Lambda_p[z]$. As in lemma 1.3, let $<\ ,\ >$ denote the symmetric bilinear form on $\mathcal{B}$ given by the meta-bracket of the banded Hopf link where each component has writhe zero. Set $V_p = \mathcal{B}_p / N_p$ where N_p denotes the kernel of the induced bilinear form on $\mathcal{B}_p$ with values in Λ_p. It was shown in [BHMV1] that N_p is a principal ideal generated by a polynomial $Q_{n(p)}$ of degree $n(p) = [(p-1)/2]$ if $p \geq 3$, and $n(p) = p$ if p is 1 or 2. Hence V_p is a finite-dimensional algebra with basis (the images of) $e_0 = 1, e_1, \ldots e_{n(p)-1}$. Moreover, the twist map t induces a self-map of V_p, and for any k-component banded link $L \subset S^3$, the induced meta-bracket, when viewed with values in Λ_p, factors through $V_p^{\otimes k}$. Hence we view

$$\Omega_p = \sum_{i=1}^{n(p)-1} < e_i > e_i$$

as an element of V_p (except for $p = 2$ where $\Omega_2 = 1 + \frac{z}{2} \in V_2 \otimes \mathbf{Z}[\frac{1}{2}]$.)

The idea of the proof is to observe that if $r = 2m + 1 \geq 3$ is odd, then $e_{m+i} = e_{m-1-i}$ in V_r (see [BHMV1].) Hence in that case we may write

$$\Omega_r = \sum_{i=0}^{m-1} < e_{2i} > e_{2i}$$

which looks like the even part of Ω_{2r}. Indeed, this makes sense, because if $p = 2r$ is even, then we have a natural decomposition

$$V_{2r} = V_{2r}^{(0)} \oplus V_{2r}^{(1)}$$

into even and odd parts which is induced by the $Z/2$-grading on $B = K(S^1 \times \mathrm{I} \times \mathrm{I}) = \mathbf{Z}[A, A^{-1}][z]$ given by the parity in z of a polynomial. This follows from the fact that if $p = 2r$ is even, then $Q_{n(p)}$ is homogeneous with respect to the $Z/2$-grading (in fact, if $r \geq 2$, it is equal to e_{r-1}, which is even or odd according to whether $r-1$ is even or odd, and if $r = 1$, it is equal to $z^2 - 4$, which is even.) Let

$$\Omega_{2r} = \Omega_{2r}^{(0)} + \Omega_{2r}^{(1)}$$

correspond to this decomposition. By the multilinearity of the meta-bracket, we can write

$$< \Omega_{2r}, \ldots, \Omega_{2r}, z, \ldots z >_{L \cup K} =$$
$$\sum_{L' \subset L} < \Omega_{2r}^{(\epsilon_1(L'))}, \ldots, \Omega_{2r}^{(\epsilon_l(L'))}, z, \ldots, z >_{L \cup K}$$

where L' runs through the sublinks of $L = L_1 \cup \ldots \cup L_l$, and $\epsilon_i(L')$ is 1 or 0 according to whether $L_i \subset L'$ or not. The following proposition shows how to replace the $\Omega_{2r}^{(1)}$'s by $\Omega_{2r}^{(0)}$'s.

Proposition 2.3. *Let $r \geq 1$ be odd. Then*

$$< \Omega_{2r}^{(\epsilon_1(L'))}, \ldots, \Omega_{2r}^{(\epsilon_l(L'))}, z, \ldots, z >_{L \cup K} =$$
$$A^{r^2(L' \cdot L' + 2L' \cdot K)} < \Omega_{2r}^{(0)}, \ldots, \Omega_{2r}^{(0)}, z, \ldots, z >_{L \cup K}$$

Here the dot $\cdot$ denotes the total linking number. Actually, this is only defined for banded links whose cores are oriented, but in the formula above the orientations do not enter. This is because A^{r^2} is a fourth root of unity in Λ_{2r}, and the values modulo 4 of $L' \cdot L'$ and $2L' \cdot K$ are independent of the orientation of the cores of the components.

(The total linking number is defined as follows. If K and K' are disjoint oriented knots in S^3, $K \cdot K'$ denotes their linking number, and if K is a banded knot, $K \cdot K$ denotes its writhe. If $K = \bigcup K_i$ and $K' = \bigcup K'_j$ are banded links whose cores are oriented, then $K \cdot K'$ is defined as $\sum K_i \cdot K'_j$.)

The proof of prop. 2.3 will be given later in this section. Notice the following corollary.

Corollary 2.4.

$$< \Omega_{2r}, \dots, \Omega_{2r}, z, \dots z >_{L \cup K} =$$
$$< \Omega_{2r}^{(0)}, \dots, \Omega_{2r}^{(0)}, z, \dots z >_{L \cup K} \sum_{L' \subset L} A^{r^2(L' \cdot L' + 2L' \cdot K)}$$

Next, we need two facts about the Kauffman bracket.

Lemma 2.5. *Let K be a banded link in S^3, with $\sharp K$ components. Then*

(i) $< K >$ *is homogeneous of degree* $d(K) = 2\sharp K - K \cdot K$ *with respect to the natural* $\mathbf{Z}/4$*-grading of* $\mathbf{Z}[A, A^{-1}]$.(This simply means that $< K >= A^{d(K)} f(A^4)$ for some Laurent polynomial f.)

(ii) the image of $< K >$ *in* $\Lambda_2 = \mathbf{Z}[A]/\phi_4(A)$ *is given by*

$$< K >= 2^{\sharp K} A^{K \cdot K}$$

Proof. This follows easily from the Kauffman relations.

Corollary 2.6.

$$(i) \qquad i_{2,2r}(< \Omega_2, \dots, \Omega_2, z, \dots z >_{L \cup K}) =$$
$$2^{\sharp K} A^{r^2 K \cdot K} \sum_{L' \subset L} A^{r^2(L' \cdot L' + 2L' \cdot K)}$$
$$(ii) \qquad j_{r,2r}(< \Omega_r, \dots, \Omega_r, z, \dots z >_{L \cup K}) =$$
$$(-1)^{\sharp K} A^{-r^2 K \cdot K} < \Omega_{2r}^{(0)}, \dots, \Omega_{2r}^{(0)}, z, \dots z >_{L \cup K}$$

Proof. i) Since $\Omega_2^{(0)} = 1$, cor. 2.4 implies

$$< \Omega_2, \dots, \Omega_2, z, \dots z >_{L \cup K} =< K > \sum_{L' \subset L} A^{L' \cdot L' + 2L' \cdot K}$$

The result now follows from lemma 2.5 (ii).

(ii) If $r = 1$, the result follows from remark 1.4.2) and lemma 2.5 (ii). Now consider $r = 2m + 1 \geq 3$ and let

$$\tilde{\Omega}_{2r}^{(0)} = \sum_{i=0}^{m-1} < e_{2i} > e_{2i} \in \mathbf{Z}[A, A^{-1}][z]$$

Using lemma 2.5 (i), it is not hard to see that

$$< \tilde{\Omega}_{2r}^{(0)}, \ldots, \tilde{\Omega}_{2r}^{(0)}, z, \ldots z >_{LUK} = A^{2\sharp K - K \cdot K} f(A^4)$$

for some Laurent polynomial f. (The components of L don't contribute, because they are replaced by something even.) The image of $\tilde{\Omega}_{2r}^{(0)}$ in V_{2r} is $\Omega_{2r}^{(0)}$, and since $e_{m+i} = e_{m-1-i}$ in V_r , the image of $\tilde{\Omega}_{2r}^{(0)}$ in V_r is Ω_r. Also observe that $j_{r,2r}(A^4) = A^4$. Thus

$$\begin{aligned} j_{r,2r}(< \Omega_r, \ldots, \Omega_r, z, \ldots z >_{LUK}) &= j_{r,2r}(A^{2\sharp K - K \cdot K} f(A^4)) \\ &= A^{(1+r^2)(2\sharp K - K \cdot K)} f(A^4) \\ &= A^{r^2(2\sharp K - K \cdot K)} \\ &\quad < \Omega_{2r}^{(0)}, \ldots, \Omega_{2r}^{(0)}, z, \ldots z >_{LUK} \end{aligned}$$

Since $A^{2r^2} = -1$, the result follows.

Proof of theorem 2.1. By cor. 2.4 and 2.6, we have

$$\begin{aligned} i_{2,2r}(< \Omega_2, \ldots, \Omega_2, z, \ldots z >_{LUK})\, j_{r,2r}(< \Omega_r, \ldots, \Omega_r, z, \ldots z >_{LUK}) \\ = (-2)^{\sharp K} < \Omega_{2r}, \ldots, \Omega_{2r}, z, \ldots z >_{LUK} \end{aligned}$$

If we apply this to the case of a banded unknot with writhe ϵ, we find

$$i_{2,2r}(< t^\epsilon \Omega_2 >)\, j_{r,2r}(< t^\epsilon \Omega_r >) = < t^\epsilon \Omega_{2r} >$$

Hence

$$i_{2,2r}(\theta_2(M_L, K))\, j_{r,2r}(\theta_r(M_L, K)) = (-2)^{\sharp K}\, \theta_{2r}(M_L, K)$$

Since $\theta_1(M_L, K) = (-2)^{\sharp K}$ by remark 1.4, this implies theorem 2.1.

It remains to prove proposition 2.3. We will need the following two lemmas.

Lemma 2.7. *Let L be a banded link in $S^1 \times \mathbf{I} \times \mathbf{I}$. Then its value in $\mathcal{B} = K(S^1 \times \mathbf{I} \times \mathbf{I})$ is a homogeneous polynomial with respect to the $\mathbf{Z}/2$-grading, whose parity is equal to the image of L in $H_1(S^1 \times \mathbf{I} \times \mathbf{I}; \mathbf{Z}/2) = \mathbf{Z}/2$.*

Proof. Define the parity of a banded link in $S^1 \times \mathbf{I} \times \mathbf{I}$ to be its image in mod 2 homology. Since the Kauffman relations respect parity, this induces a $\mathbf{Z}/2$-grading on the Jones-Kauffman-module $\mathcal{B} = \mathbf{Z}[A, A^{-1}][z]$, which is precisely given by parity in the variable z.

Lemma 2.8. *Let $r \geq 1$ be odd. Then in Λ_{2r} we have*

$$< t^w \Omega_{2r}^{(1)}, z^i >= A^{r^2(w+2i)} < t^w \Omega_{2r}^{(0)}, z^i >$$

(Recall that $< \, , \, >$ denotes the metabracket of the banded Hopf link where both components have writhe zero.)

Proof. It was shown in [BHMV1, section 3] that the e_i are an eigenbasis of $\mathcal{B} = \mathbf{Z}[A, A^{-1}][z]$ for both the twist map t and the map c which satisfies $< c(u), v >= < u, zv >$ for all $u, v \in \mathcal{B}$, with eigenvalues $\mu_i = (-1)^i A^{i^2+2i}$ under t, and $\lambda_i = -(A^{2i+2} + A^{-2i-2})$ under c. For $r \neq 1$, we may write

$$\Omega_{2r}^{(0)} = \sum < e_k > e_k \ , \quad \Omega_{2r}^{(1)} = \sum < e_{r-2-k} > e_{r-2-k}$$

the sum being over all even k such that $0 \leq k \leq r-2$. Since

$$< t^w e_j, z^i >= \mu_j^w \lambda_j^i < e_j >$$

the result now follows upon observing that in Λ_{2r} we have $\mu_{r-2-k} = (-1)^k A^{r^2} \mu_k$, $\lambda_{r-2-k} = -\lambda_k$, $< e_{r-2-k} >= - < e_k >$, and $A^{2r^2} = -1$.

For $r = 1$ the proof is similar but easier.

Proof of proposition 2.3.
To shorten the notation, let us set

$$< \Omega_{2r}^{(\epsilon_1(L'))}, \ldots, \Omega_{2r}^{(\epsilon_l(L'))}, z, \ldots, z >_{L \cup K} = << L, L', K >>$$

If L' is empty, there is nothing to show. If not, let L_i be a component of L. By induction, we may assume the formula is proved for $L' - L_i$, so that we have

$$<< L, L' - L_i, K >> = A^{r^2((L'-L_i)\cdot(L'-L_i)+2(L'-L_i)\cdot K)} << L, \emptyset, K >>$$

Assume first that L_i is unknotted. Then we can view $(L - L_i) \cup K$ as lying in the complementary solid torus, so that after cabling with the Ω_{2r}^{ϵ}'s it gives rise to an element $x \in \mathcal{B}$. Applying lemma 2.7, it is clear that x is homogeneous with respect to the $\mathbf{Z}/2$-grading of $\mathcal{B}$, and its parity is precisely

$$L_i \cdot K + \sum_{j \neq i} \epsilon_j(L') L_i \cdot L' = L_i \cdot (L' - L_i) + L_i \cdot K$$

modulo 2. Hence

$$\begin{aligned} << L, L', K >> &= < t^{L_i \cdot L_i} \Omega_{2r}^{(1)}, x > \\ &= A^{r^2(L_i \cdot L_i + 2(L_i \cdot (L' - L_i) + L_i \cdot K))} < t^{L_i \cdot L_i} \Omega_{2r}^{(0)}, x > \\ &= A^{r^2(L_i \cdot L_i + 2(L_i \cdot (L' - L_i) + L_i \cdot K))} \\ &\quad << L, L' - L_i, K >> \\ &= A^{r^2(L' \cdot L' + 2L' \cdot K)} << L, \emptyset, K >> \end{aligned}$$

where we have applied lemma 2.8 and the induction hypothesis. Thus we have proved the proposition in the case where L_i is unknotted.

Finally, if L_i is knotted, we may unknot it in the standard way at the expense of introducing new components cabled by Ω_{2r} (compare the proof of prop. 5.4 of [BHMV1].) But $L_i \cdot L_i$ mod 4 remains unchanged, and the new components have even linking number with L_i. Since A^{r^2} is a fourth root of unity, this reduces the case where

L_i is knotted to the special case treated above. This completes the proof of prop. 2.3.

Remark 2.9. If r is even, then the decomposition

$$\Omega_{2r} = \Omega_{2r}^{(0)} + \Omega_{2r}^{(1)}$$

yields a "splitting" of the invariant $\theta_{2r}(M,K)$ as a sum of "structured" invariants. This result was found by Kirby -Melvin [KM2] and Turaev [T2] (in terms of their invariant τ_r), and also independently by Blanchet in his thesis. Here, we will just state the result. A proof using our approach can be found in [B].

Define an s-structure on M to be a spin structure if $r/2$ is even, and an element of $H^1(M, \mathbf{Z}/2)$ if $r/2$ is odd. If K is a banded link in M, an s-structure on (M,K) is an s-structure on $M - K$ which does not extend over any component of K. If $M = M_L$ and $K \subset S^3 - L$ as above, then there is a natural $1-1$-correspondence between s-structures on (M_L, K) and sublinks $L' \subset L$ such that

$$(L' + K) \cdot L_i = \begin{cases} L_i \cdot L_i \mod 2 & \text{if } r/2 \text{ is even} \\ 0 \mod 2 & \text{if } r/2 \text{ is odd} \end{cases}$$

where L_i are the components of L. (If $K = \emptyset$, then a sublink L' of L satisfying the above condition for $r/2$ even is usually called a characteristic sublink.)

Theorem 2.10 [B] [KM2] [T2] *If r is even, then the expression*

$$\frac{< \Omega_{2r}^{(\epsilon_1(L'))}, \ldots, \Omega_{2r}^{(\epsilon_l(L'))}, z, \ldots, z >_{L \cup K}}{< t(\Omega_{2r}) >^{b_+(L)} < t^{-1}(\Omega_{2r}) >^{b_-(L)}}$$

is zero unless L' satisfies the condition above, and in that case it is an invariant, denoted by $\theta_{2r}(M,K,s)$, of the s-structure s on (M,K) determined by L'. In particular, we have a "splitting"

$$\theta_{2r}(M,K) = \sum_s \theta_{2r}(M,K,s)$$

Remark 2.11. The Jones-Kauffman module $\mathcal{B} = K(S^1 \times \mathbf{I} \times \mathbf{I})$ is related to the representation ring $RSU(2)$ as follows. Its basis $\{e_i : i \geq 0\}$ satisfies

$$< e_i >= (-1)^i \frac{A^{2i+2} - A^{-2i-2}}{A^2 - A^{-2}}$$

Hence if we set $q = A^4$, then $< e_i >$ is the character of an $i+1$-dimensional irreducible representation of $SU(2)$ (with respect to the maximal torus $diag(q^{1/2}, q^{-1/2}) \subset SU(2)$.) It follows that V_{2r} is isomorphic to a quotient of $RSU(2)$. (Strictly speaking, the representation ring here has to be taken with coefficients in Λ_{2r}.) If r is odd, then we have seen that V_r can be obtained as a quotient of the subring of $\mathcal{B}$ generated by the e_{2i}, which geometrically means it is generated by banded links that meet a meridinal disk of $S^1 \times \mathbf{I} \times \mathbf{I}$ an even number of times. This subring corresponds to the inclusion $RSO(3) \subset RSU(2)$. Hence, for odd r, V_r is isomorphic to a quotient of the representation ring $RSO(3)$ (with coefficients in Λ_r.)

Proof of prop. 2.2. Since (for $p \geq 3$) $\Omega_p = \sum_{i=0}^{n(p)-1} < e_i > e_i$, one may write

$$\theta_p(M_L) = \frac{\sum_{0 \leq i_\nu \leq n(p)-1} \left(\prod_{\nu=1}^{k} < e_{i_\nu} >\right) < e_{i_1}, \ldots, e_{i_k} >_L}{< t(\Omega_p) >^{b_+(L)} < t^{-1}(\Omega_p) >^{b_-(L)}}$$

This resembles Kirby-Melvin's definition of their invariant, the meta-bracket
$< e_{i_1}, \ldots, e_{i_k} >_L$ playing the role of what Kirby and Melvin call the "colored Jones polynomial" of the banded link $L = L_1 \cup \ldots \cup L_k$ colored by $i_1 + 1, \ldots, i_k + 1$. We will not make this explicit here, but rather quote [Li4]. Let us call $L_r(M)$ the invariant of [Li4] (Lickorish didn't give it a name.) Lickorish (prop. 8 of [Li4]) has shown that

$$L_r(M)\big|_{A=-e^{2\pi i/4r}} = e^{i\pi(6-3r)b_1(M)/4r} \tau_r(M)$$

On the other hand [BHMV1], it is clear from the definition that

$$\theta_{2r}(M) =< t\Omega_{2r} >^{b_1(M)} L_r(M)$$

Using lemmas 6.6 and 6.7 of [BHMV1], it is not hard to see that

$$< t\Omega_{2r} > |_{A=-e^{2\pi i/4r}} = \frac{\sqrt{2r}}{2\sin\frac{\pi}{r}} e^{i\pi(3r-6)/4r}$$

This implies the formula we gave for $\tau_r(M)$. Presumably, the formula for $\tau'_r(M)$ can be obtained similarly (we haven't checked this.) However, if $\theta_2(M)$ is non-zero, the formula for $\tau'_r(M)$ follows from theorem 2.1 evaluated at $A = -e^{2\pi i/4r}$, together with Kirby-Melvin's formula relating $\tau_3(M)$, $\tau_r(M)$, and $\tau'_r(M)$, as given in the remark following prop. 2.2.

3. Wenzl's theorem

Using special trace functionals on the infinite ribbon braid group, Wenzl [We2] has defined invariants of a 3-manifold $M = M_L$ as a "thermodynamic limit" of polynomial invariants of cablings of L. In this section, we will give a simple proof that the invariant θ_p can be obtained in a similar way. Here is the result.

Theorem 3.1. (compare [We2]) *Let $p \geq 5$. Embed Λ_p into* **C** *by setting*

$$A = \begin{cases} e^{2\pi i/2p} & \text{if } p \text{ is even} \\ e^{2\pi i(1+p^2)/4p} & \text{if } p \text{ is odd} \end{cases}$$

Define $u_N \in V_p \underset{\Lambda_p}{\otimes} \mathbf{C}$ by

$$u_N = \begin{cases} \frac{1}{2}\left(1+\frac{z}{\delta}\right)\left(\frac{z}{\delta}\right)^{2N} & \text{if } p \text{ is even} \\ \left(\frac{z}{\delta}\right)^{N} & \text{if } p \text{ is odd} \end{cases}$$

Then

$$\lim_{N\to\infty} u_N = \frac{\Omega_p}{<\Omega_p>}$$

Corollary 3.2. *Under the hypotheses of theorem 3.1*

$$\theta_p(M_L) = \frac{<\Omega_p>^{\sharp L}}{<t(\Omega_p)>^{b_+(L)}<t^{-1}(\Omega_p)>^{b_-(L)}}$$

$$\lim_{N\to\infty} J < u_N, \ldots, u_N >_L$$

This follows immediately from theorem 1.1.

Remark. For $p \leq 4$ there is no need to take a limit, since $\Omega_p = 1$ for $p \in \{1, 3, 4\}$, and $\Omega_2 = 1 + \frac{z}{2}$.

We now give a proof of theorem 3.1.

Lemma 3.3. *The self-map of V_p induced by multiplication by z has eigenvalues $\lambda_0, \ldots, \lambda_{n(p)-1}$.*

(Recall $n(p) = [(p-1)/2]$ if $p \geq 3$, and $n(p) = p$ if p is 1 or 2.)

Proof. Recall that $\lambda_k = -A^{2k+2} - A^{-2k-2}$ are the eigenvalues of the self-map c on $\mathcal{B}$ which is adjoint to multiplication by z, i.e.

$$< c(u), v >=< u, zv >$$

for all $u, v \in \mathcal{B}$. Hence the lemma follows from the fact that the symmetric bilinear form $< \, , \, >$ on $\mathcal{B}$ induces a nondegenerate symmetric bilinear form on V_p [BHMV1].

Recall that $< z >= \delta = \lambda_0 = -A^2 - A^{-2}$.

Lemma 3.4. $z \, \Omega_p = \delta \, \Omega_p$

Proof. If $p \geq 3$, then

$$\begin{aligned} z\Omega_p &= \frac{1}{4}\sum_{i=1}^{2p} < e_i > ze_i = \frac{1}{4}\sum_{i=1}^{2p} < e_i > (e_{i-1} + e_{i+1}) \\ &= \frac{1}{4}\sum_{i=1}^{2p}(< e_{i+1} > + < e_{i-1} >)\, e_i = \frac{1}{4}\sum_{i=1}^{2p} < ze_i > e_i \\ &=< z > \Omega_p = \delta\Omega_p \end{aligned}$$

The case $p = 1$ is trivial, and if $p = 2$, the result follows from $z^2 - 4 = 0$ in V_2.

Lemma 3.5. *Let $p \geq 3$ be odd. Embed Λ_p into $\mathbf{C}$ by setting $A = e^{2\pi i(1+p^2)/4p}$. Then for all $x \in V_p \otimes \mathbf{C}$ such that $< x > \neq 0$, one has*

$$\lim_{N\to\infty} \left(\frac{z}{\delta}\right)^N \frac{x}{< x >} = \frac{\Omega_p}{< \Omega_p >}$$

(Notice that $\phi_{2p}(e^{2\pi i(1+p^2)/4p}) = 0$ since $gcd(1+p^2, 4p) = 2$.)

Proof. For the chosen value of A, we have $\lambda_k = 2\cos(\pi(k+1)/p)$. Hence $0 < \lambda_k < \lambda_0 = \delta$ for $0 < k \leq n(p)-1$. By lemma 3.3, it follows that $\lim_{N\to\infty} \left(\frac{z}{\delta}\right)^N$ is a projector onto the eigenspace corresponding to the eigenvalue δ. The result now follows from lemma 3.4.

Lemma 3.6. *Let $p \geq 6$ be even. Embed Λ_p into $\mathbf{C}$ by setting $A = e^{2\pi i/2p}$. Let $\epsilon \in \{0,1\}$. Then for all $x \in V_p^{(\epsilon)} \otimes \mathbf{C}$ such that $< x > \neq 0$, one has*

$$\lim_{N\to\infty} \left(\frac{z}{\delta}\right)^{2N} \frac{x}{< x >} = \frac{\Omega_p^{(\epsilon)}}{< \Omega_p^{(\epsilon)} >}$$

Proof. By lemma 3.3, the self-map of V_p induced by multiplication by z^2 has eigenvalues λ_k^2 where $k = 0, \ldots, n(p)-1$. Since λ_k^2 takes $[(n(p)+1)/2]$ different values, it is easy to see that the eigenspaces of z^2 are two-dimensional, except for the eigenvalue zero which occurs if $p \equiv 0 \bmod 4$, and whose eigenspace is one-dimensional. Moreover, each of the two-dimensional eigenspaces splits into even and odd parts which are exchanged by multiplication by z. Hence for both $\epsilon = 0$ and $\epsilon = 1$, multiplication by z^2 has distinct non-negative eigenvalues on $V_p^{(\epsilon)}$, and for the chosen value of A, the strictly largest eigenvalue is $\lambda_0^2 = \delta^2$. But lemma 3.4 implies $z\Omega_p^{(0)} = \delta\Omega_p^{(1)}$ and $z\Omega_p^{(1)} = \delta\Omega_p^{(0)}$, hence $z^2\Omega_p^{(\epsilon)} = \delta^2\Omega_p^{(\epsilon)}$. The result follows as in the proof of lemma 3.5.

Proof of theorem 3.1. The result follows from lemmas 3.5 and 3.6. (If p is even, this uses the fact that $< \Omega_p^{(0)} > = < \Omega_p^{(1)} >$.)

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