Khovanov Homology I

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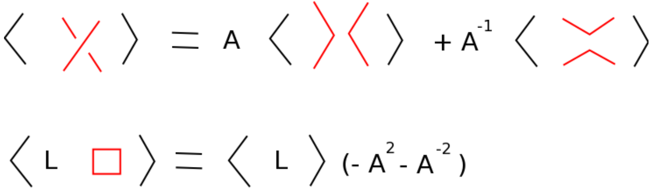
Introduction

Categorification of quantum invariants

- Given a link invariant $\langle L \rangle \in \mathbb{Z}[q^{\pm}1]$, does there exist a bigraded homology (resp. cohomology) theory $H_{i,j}(L)$ such that:
$$\langle L \rangle = \sum_{i,j} (-1)^i \dim(H_{i,j}(L) \otimes \mathbb{Q})\; q^j \;?$$
- Khovanov obtained such a theory for the Jones polynomial. The construction rests on the Kauffman bracket model.

Kauffman bracket

- There exists a unique invariant $\langle L\rangle \in \mathbb{Z}[A, A^{-1}]$ of isotopy classes of ribbon links (unoriented and equipped with an homotopy class of section of the normal bundle), whose value is 1 for the empty link, and such that:



- State sum formula.

$$\langle L\rangle = \sum_s A^{\sum_c s(c)}\langle D_s\rangle = \sum_s A^{\sum_c s(c)}(-A^2 - A^{-2})^{\sharp D_s} .$$

The sum is over states $s : \{\text{crossings } c\} \to \{-1, 1\}$.

Khovanov model

- Based on Kauffman bracket.
- For a diagram, each state s is enhanced with a sign on each component of D_s. Enhanced states generate a bigraded complex.
- The boundary is defined by a rule giving incidence numbers for states which differ in exactly one crossing.
- For each Reidemeister move: homotopy equivalence.

An oriented model

- We will present a slightly different construction based on a state model for the Jones polynomial using planar graphs.
- Tool: TQFT for curves and surfaces equipped with vice-orientation (Viro terminology).
- Khovanov construction extends to cobordisms, and is functorial up to sign. A version of the oriented construction is strictly functorial and the cohomology isomorphism associated with a Reidemeister move is canonical (Scott Morrison, Kevin Walker).

Basic bibliography

- Mikhail Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (2000), no. 3, 359–426.
- Dror Bar Natan, On Khovanov's categorification of the Jones polynomial, Algebraic and Geometric Topology 2 (2002) 337-370.
- Oleg Viro, Remarks on definition of Khovanov homology, math.GT/0202199.

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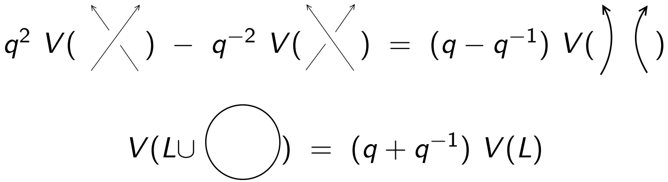
Jones polynomial



$$V(\emptyset) = 1$$

(Our q is $-t^{-\frac{1}{2}}$ in the original definition.)

State sum for trivalent planar graphs

- To each elementary planar graph we associate a linear map.
- E: basis (e_-, e_+), $U = q^{\frac{1}{2}} e_+^+ + q^{-\frac{1}{2}} e_-^- \in \mathrm{End}(E)$.

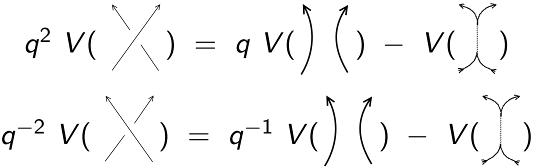
 $\mapsto qe_{-+}^{-+} + q^{-1}e_{+-}^{+-} + e_{-+}^{+-} + e_{+-}^{-+}$

 $\mapsto \mathrm{eval}_E \circ (\mathrm{Id} \otimes U)$,  $\mapsto \mathrm{eval}_{E^*} \circ (U^{-1} \otimes \mathrm{Id})$

 $\mapsto (\mathrm{Id} \otimes U^{-1})\mathrm{coeval}_E$,  $\mapsto (U \otimes \mathrm{Id}) \circ \mathrm{coeval}_{E^*}$

- Use composition and tensor product (monoidal functor).
- Global formula (state sum): planar isotopy invariant $V(G)$ ($sl(2)$ invariant of planar graphs).

State sum for Jones polynomial



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TQFT for oriented surfaces

- oriented curve $\mapsto$ module;
  surface considered as a cobordism $\mapsto$ linear map;
  Functoriality;
  Multiplicativity: disjoint union is sent to tensor product.
- The module associated with the circle is a commutative Frobenius algebra.
- TQFTs in dimension 2 $\longleftrightarrow$ commutative Frobenius algebras.

TQFT: an example

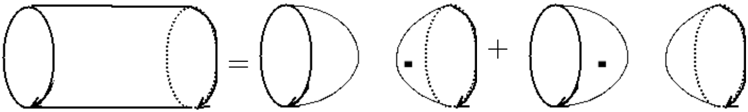
- Example of Frobenius algebra: $\mathbf{A} = \mathbb{Z}[X]/X^2 \approx H^*(\mathbb{C}P^1)$.
- The corresponding invariant of closed surfaces:
  2 for the torus,
  0 for a genus $g \neq 1$ surface.
- Extension to oriented surfaces with points: a disc containing a point is sent to the element X in the algebra associated with the oriented boundary circle.
  For a closed surface with point, the invariant is 0 unless the Euler characteristic of the punctured surface (surface minus the points) is zero.
  For a sphere with one point, the invariant is 1.

The universal construction

- Reconstruction of the TQFT from the invariant of closed surfaces with points.
  The module associated with a curve is

$$\frac{\text{free module generated by surfaces with boundary the given curve}}{\text{kernel of the bilinear form defined by gluing}}$$

- Key lemma: surgery formula.



Graded TQFT

- Grading on $\mathbf{A} = \mathbb{Z}[X]/X^2$: $\deg(\mathbf{1}) = -1$, $\deg(X) = 1$.
- The q-dimension of the module associated with a k components curve is: $(q + q^{-1})^k$.
- The TQFT functor is graded: for a cobordism Σ between C and C', the linear map $\mathbf{V}(\Sigma) : \mathbf{V}(C) \to \mathbf{V}(C')$ has degree $-\chi(\Sigma) + 2\sharp\mathrm{pts}$.

2-*surfaces*

- We consider orientable surfaces with a singular locus which is an oriented curve called the binding; the complement of the singular locus is oriented and induces the given orientation on the singular curve: the singuler locus is a binding with 2 pages; for each component of the binding, the 2 adjacent pages are ordered. Such a surface will be called a 2-surface, or vice-oriented surface.
- If we cut along the binding a closed 2-surface Σ with a k components binding, we obtain an oriented surface $\dot{\Sigma}$ whose boundary is decomposed in k pairs.
- $$\mathbf{V}(\dot{\Sigma}) \in (\mathbf{A} \otimes \mathbf{A})^{\otimes k}$$

  We evaluate this vector in $\mathbb{Z}$ using the scalar product on $\mathbf{A} \otimes \mathbf{A}$.

Remarks

- (Technical extension) It will be useful to consider bindings with singular points where the order of the pages changes. The invariant will be zero if some binding has an odd number of singular points and is computed as before otherwise. This will be used only for Reidemeister moves II- and III.
- A variant construction (Morrison-Walker) can be done by a slight modification of the evaluation.

Vice-oriented TQFT

- A generic curve on a 2-surface has singular points. The complement of the singular points is oriented, and at each singular point an order of the 2 adjacent arcs is given.
- The universal construction extends the TQFT to the *vice-oriented* category.

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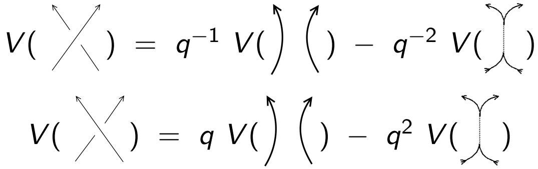
Categorification of the $sl(2)$ *invariant of planar graphs*

- A trivalent planar graph defines a 2-curve; we use a planar convention for the ordering namely right-left relatively to the oriented double edge (we use a dotted line in the pictures).
- Remark: One can show that the q-dimension of the TQFT module associated with a planar graph G is equal to the $sl(2)$ invariant $V(G)$ (in fact $(q+q^{-1})^{\sharp G_1}$, where G_1 is the underlying vice-oriented curve).

Reminding state sum for Jones polynomial



Khovanov complex, 1^{rst} tentative

- Let D be an oriented link diagram. Here a state s assign to a positive (resp. negative) crossing either 0 or 1 (resp. either 0 or -1); the associated planar graph D_s is obtained by smoothing the crossing c if $s(c) = 0$, and by inserting an edge with trivalent vertices if $|s(c)| = 1$.
- $\tilde{K}(D) = \bigoplus_s \mathbf{V}(D_s)$.
- If s' is obtained from s, by changing the value of c from 0 to 1 (resp. from -1 to 0), the elementary cobordism (which is identity except around c where there is a saddle with a singular arc) induces a linear map $\mathbf{V}(D_s) \to \mathbf{V}(D_{s'})$ whose degree is 1.
- We obtain this way a commutative cube whose vertices correspond with states.
- We will obtain a complex by introducing signs, in order to get an anticommutative cube.

Khovanov complex

- For a state s as above, $d_s = \sum |s(c)|$, and Δ_s denotes abelian group freely generated by crossing c with $|s(c)| = 1$;
$\epsilon(c) = \pm 1$ is the sign of the crossing.
-
$$K(D) = \bigoplus_s \mathbf{V}(D_s)\{-\sum_c (\epsilon(c) + s(c))\} \otimes \Lambda^{d_s} \Delta_s$$
is a bigraded belian group.
The homological degree is $-\sum_c s(c)$.
The TQFT grading is shifted as indicated in the bracket, so that the boundary map is homogenous
$(\deg_q(G\{i\}) = q^i \deg_q(G))$.

Boundary map

- If s' is obtaining from s, by switching the positive crossing c from 0 to 1, then

$$\delta : \mathbf{V}(D_s) \otimes \Lambda^{d_s} \Delta_s \to \mathbf{V}(D_{s'}) \otimes \Lambda^{d_{s'}} \Delta_{s'}$$

  is defined as the tensor product of the TQFT operator with $\bullet \wedge c$.

- If s' is obtaining from s, by switching the negative crossing c from -1 to 0, then

$$\delta : \mathbf{V}(D_s) \otimes \Lambda^{d_s} \Delta_s \to \mathbf{V}(D_{s'}) \otimes \Lambda^{d_{s'}} \Delta_{s'}$$

  is defined as the tensor product of the TQFT operator with the contraction $< \bullet, c >$.

- In all other cases, the boundary map δ is zero.

Khovanov categorification result

Theorem: a) $(K(D), \delta)$ is a bigraded complex.
b) The graded Euler characteristic is equal to the Jones polynomial.
c) The homotopy type of $(K(D), \delta)$ is a link invariant.