

# (Locally) Garside categories

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- David Bessis, “Garside categories, periodic loops and cyclic sets” [math.GR/0610778](#)

## An example

Let  $(W, S)$  be a Coxeter system.

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Here  $\mathbf{W}$  are the *simples* and  $\mathbf{S}$  the *atoms* of a locally Garside monoid structure on  $B^+$  (Garside monoid if  $W$  is finite).

The braid group, denoted  $B$ , is the group with the same presentation. We will consider it only when  $\mathbf{W}$  is finite.

## Left divisibility, nice posets

In a monoid we write  $x \preceq y$  if there exists  $z$  such that  $xz = y$ . We write  $x \prec y$  if in addition  $x \neq y$ .

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### Definition

A preordered set is *nice* if there does not exist a bounded infinite strictly increasing sequence.

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For the monoid, one can do almost as much which locally Garside as with Garside (normal forms, fixed points of automorphisms, etc. . . ).

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Conjugacy in a monoid or category:  $\forall w' = w'' \Leftrightarrow vw' = w''v$  and  $w'^v = w'' \Leftrightarrow w'v = vw''$

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We will consider the elements  $\mathbf{w} \in B^+$  such that  $\mathbf{I}^{\mathbf{w}} \subset \mathbf{S}$  and  $\alpha_1(\mathbf{w}) = 1$ . This makes sense in any (possibly non-spherical) Artin monoid.

## The category $B^+(\mathcal{I})$

Let  $\mathbf{I} \subset \mathbf{S}$  and let  $\mathcal{I}$  the set of conjugates of  $\mathbf{I}$  in  $B^+$  (they are subsets of  $\mathbf{S}$ ).

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### Definition

We define  $B^+(\mathcal{I})$  as the category whose set of objects is  $\mathcal{I}$  and such that the morphisms between  $\mathbf{I}$  and  $\mathbf{J}$  is the set of elements  $\mathbf{w} \in B^+$  such that  $\mathbf{I}^{\mathbf{w}} = \mathbf{J}$  and  $\alpha_{\mathbf{I}}(\mathbf{w}) = 1$ .

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In a Garside category, there is a  $\Delta$  starting from each object. The  $\Delta$  from  $\mathbf{I}$  is  $\omega_{\mathbf{I}}(\Delta) = \Delta_{\mathbf{I}}^{-1} \Delta_{\mathbf{S}}$ .

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In a category, we have the notion of left divisibility, left gcd, right lcm etc. . . . for morphisms with given source. We write  $xy$  if  $A \xrightarrow{x} B \xrightarrow{y} C$ .

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A left locally Garside monoid is the monoid of endomorphisms of a one-object left locally Garside category.

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We can show that a Garside category is also right locally Garside, so there is no notion of “left Garside”.

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A germ  $(P, \mathcal{O})$  is a pair consisting of a set  $\mathcal{O}$  of objects, and a set  $P$  of morphisms (which have a source and a target, which are objects), with a partially defined “composition” map  $m : P \times P \rightarrow P$

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A germ  $(P, \mathcal{O})$  is a pair consisting of a set  $\mathcal{O}$  of objects, and a set  $P$  of morphisms (which have a source and a target, which are objects), with a partially defined “composition” map  $m : P \times P \rightarrow P$  (for  $a, b \in P$  we will write “ $ab \in P$ ” to mean that  $m(a, b)$  is defined).

If we denote by  $P_{A \rightarrow B}$  the set of morphisms in  $\mathcal{O}$  of source  $A$  and target  $B$ , we require the following axioms:

- 1 For all  $A \in \mathcal{O}$ , there exists  $1_A \in P_{A \rightarrow A}$  such that for any  $a \in P(B, A)$  (resp. any  $a' \in P_{A \rightarrow B}$ )  $a = a.1_A \in P$  (resp.  $a' = 1_A.a' \in P$ ).
- 2 For  $a, b, c \in P$ , we have  $ab, (ab)c \in P$  if and only if  $bc, a(bc) \in P$  and in this case  $a(bc) = (ab)c$ .

## Category generated by a germ

A *path* is a sequence of morphisms  $(p_1, \dots, p_n)$  of  $P$  such that the target of  $p_i$  is the source of  $p_{i+1}$ .

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We denote by  $C(P)$  the generated category. We can show that  $P$  identifies to a subset of  $C(P)$  stable by taking right and left factors. In particular, it makes sense to talk about a nice germ, *i.e.* when left divisibility induces a nice poset.

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However, in practice we can check it if there is an injective map compatible with multiplication from  $P$  into a category with the left cancellation property (e.g. a groupoid).

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## Theorem

*A left locally Garside category  $C$  is the category  $C(P)$  of any germ  $(P, \mathcal{O})$  where  $\mathcal{O}$  is the set of objects of  $C$  and where  $P$  is a set of morphisms of  $C$  which generate  $C$ , stable by taking left factors and stable by taking right lcm when they exist.*

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We show that the number of terms in the normal form is the minimum for all decompositions of  $x$  as a product of elements of  $P$ .

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We call *sub-germ* of a germ  $(P, \mathcal{O})$  a germ  $(P_1, \mathcal{O}_1)$  obtained by taking a part  $\mathcal{O}_1$  of the objects and a part  $P_1$  of the morphisms between objects in  $\mathcal{O}_1$  which is stable by the partial multiplication in  $P$ .

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## Proposition

Let  $C$  be a locally Garside category and let  $\sigma$  be an autoequivalence of  $C$  stabilizing a germ  $P$  such that  $C = C(P)$ ; then  $P^\sigma$  is a left locally Garside sub-germ and  $C(P)^\sigma = C(P^\sigma)$ .

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The category is atomic; if  $W$  is finite the atoms are of the form  $w_J^{-1}w_{J \cup \{a\}}$  where  $w_I$  is the longest element of  $W_I$  and  $a \in \Pi - J$ .

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# The conjugacy category

Let  $B$  be a [locally] Garside monoid  $B$  (or even a [locally] Garside category), and let  $w \in B$ .

We can form a category  $C$  whose objects are the conjugates of  $w$  and such that  $\text{Hom}_C(w', w'') = \{x \in B \mid w'x = xw''\}$ .

## Proposition

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We get a Garside category with simples the simples in some  $\text{Hom}_C(w, ?)$  and atoms the “minimal simples” of Franco and Gonzalez-Meneses.

## Further examples

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**Question:** What can we say about a weak Garside monoid?

Assume  $B$  is such that such that some power  $\Delta^a$  is central.

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He has used that the centralizer of a periodic braid in the dual monoid of a complex reflection group is a Garside category to solve the  $K(\Pi, 1)$  problem for complex reflection groups.

## A Deligne-like result

There is a natural poset structure on the set of decompositions of a morphism into simples:

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n) > (g_1, \dots, g_{i-1}, a, b, g_{i+1}, \dots, g_n)$$

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it means that there does exist a well-defined Deligne-Lusztig variety attached to a morphism since to sequence of canonical maps between different decompositions will coincide.