Conjugacy problems in braid groups and other Garside groups

Part III

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Problèmes algorithmiques liés aux tresses et à la topologie de basse dimension

GDR Tresses et topologie de basse dimension

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Project to solve CDP & CSP in braid groups

Input: Two braids X and Y.

I. Determine their geometric type.

Open: Is there a polynomial algorithm that finds CRS(X)?

II. pseudo-Anosov braids.

Pass to powers: Can restrict to the **rigid case**.

Open: How many times one must cycle to go from SSS(X) to USS(X)?

Open: Is there a polynomial upper bound for #(USS(X))?

III. Periodic braids.

There is a **polynomial algorithm.**

IV. Reducible braids.

Decompose along CRS(X).

Open: Is there a polynomial algorithm to find the two generators of the centralizer of a pseudo-Anosov braid?

Periodic braids

$$\{\alpha \in B_n : \quad \alpha^k = \Delta^{2m}\}$$

They correspond to finite order elements
in the mapping class group of the punctured disc.

Kérékjartó (1919), **Eilenberg** (1934):

Every periodic automorphism of the disc is conjugate to a rotation.

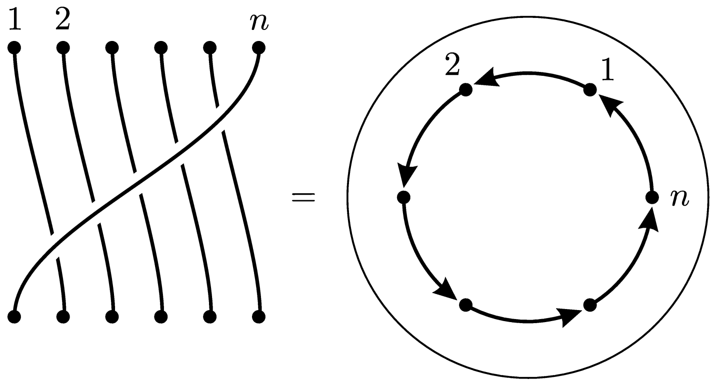
If the automorphism preserves a finite set of points, this set must be:

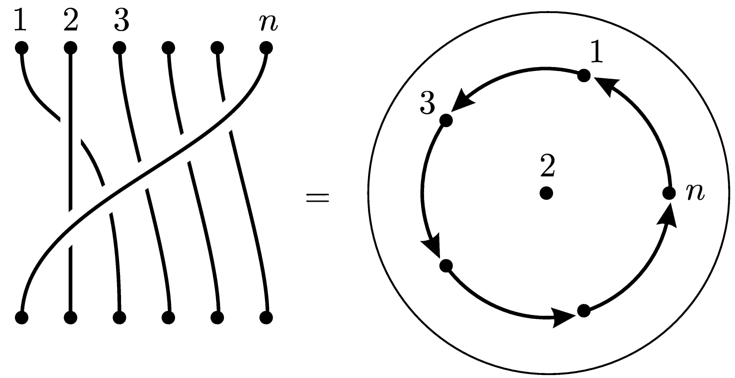
- A union of orbits of k elements each. (angle of rotation $2\pi/k$)
- Possibly one fixed point. (the center of the disc)

Periodic braids

Translation to braids:

Every periodic braid is conjugate to a power of either δ or ε.





$$\delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$$

$$\varepsilon = \sigma_1(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$$

Determining if a braid is periodic

Corollary: X is conjugate to $\delta^k \quad \Leftrightarrow \quad X^n = \Delta^{2k}.$

X is conjugate to $\varepsilon^k \quad \Leftrightarrow \quad X^{n-1} = \Delta^{2k}.$

Proof: $X = c^{-1}\delta^k c \quad \Rightarrow \quad X^n = c^{-1}\delta^{nk}c = c^{-1}\Delta^{2k}c = \Delta^{2k}.$

Same for ε.

One just needs to compute X^{n-1} and X^n in order to know:

- Whether X is periodic.
- The power of δ or ε which is conjugate to X.

This solves the CDP for periodic braids in polynomial time!

Known algorithms for the CSP

Is Gebhardt's algorithm polynomial for periodic braids?

No! It is exponential in *n*.

Bestvina (1999): If $Y \in SSS(X)$ is periodic, then $\ell(Y) \leq 1$.

Hence USS(X)=SSS(X)

Let us study the size of USS(δ) and USS(ε).

Characterization of elements in USS(δ)

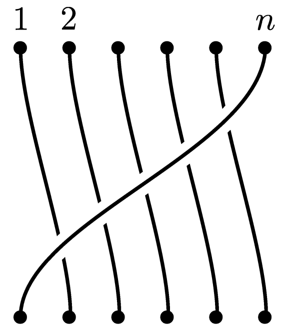
δ is simple. Hence **all elements in USS(δ) are simple.**

They are characterized by their permutation.

We will write permutations as products of disjoint cycles.

$$\delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$$

$$\pi_\delta = (1\ 2\ \cdots\ n)$$



Elements in USS(δ) are conjugates of δ → **They are given by an n-cycle**

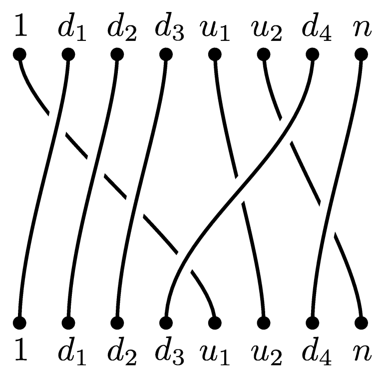
Characterization of elements in USS(δ)

Birman-Gebhardt-GM (2006): $s \in USS(\delta)$ if and only if

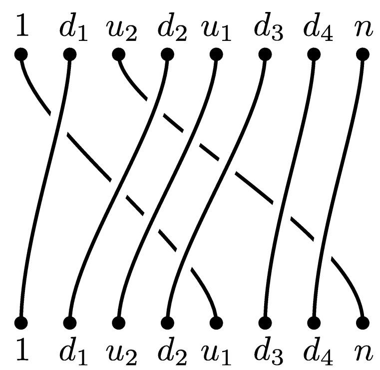
s is simple and $\pi_s = (1\ u_1\ u_2 \cdots\ u_r\ n\ d_t\ d_{t-1} \cdots d_1)$

where $r + t + 2 = n$, $u_1 < u_2 < \cdots < u_r$ and $d_t > d_{t-1} > \cdots > d_1$.

Example:



$s \in USS(\delta)$

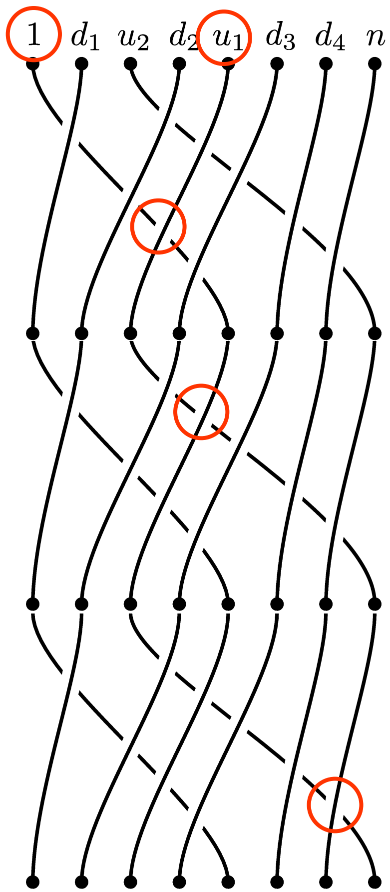


$s \notin USS(\delta)$

Characterization of elements in USS(δ)

Proof: If $s \in USS(\delta)$, then $s^n = \Delta^2$. (Each pair of strands cross **twice**)

If $u_i > u_{i+1}$:



They cross more than twice!

Hence $u_1 < u_2 < \cdots < u_r$.

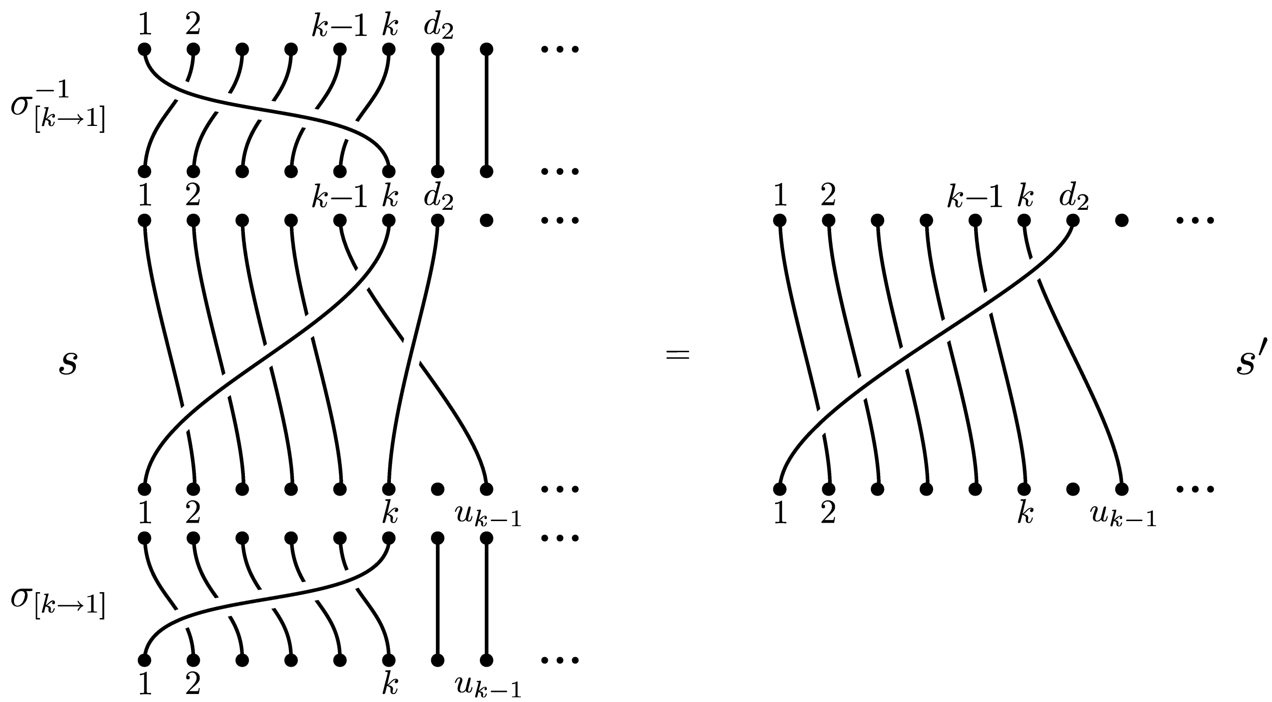
Simlar argument for the d_i's.

Characterization of elements in USS(δ)

Conversely, suppose that $\pi_s = (1\ u_1\ u_2 \cdots\ u_r\ n\ d_t\ d_{t-1} \cdots d_1)$,

where $u_1 < u_2 < \cdots < u_r$ and $d_t > d_{t-1} > \cdots > d_1$.

Let $k = d_1$, and $\sigma_{[k\to 1]} = \sigma_{k-1}\sigma_{k-2}\cdots\sigma_1$.



Characterization of elements in USS(δ)

Conjugating by $\sigma_{[d_1 \to 1]}$ reduces the number of d_i's

$$\alpha = \sigma_{[d_1 \to 1]} \sigma_{[d_2 \to 1]} \cdots \sigma_{[d_t \to 1]}.$$

Conjugating by α removes all d_i's

$$\pi_{\alpha^{-1} s \alpha} = (1\ 2\ \cdots\ n) = \pi_\delta \quad \Rightarrow \quad \boxed{\alpha^{-1} s \alpha = \delta.} \qquad \square$$

This solves the CSP for conjugates of δ.

Characterization of elements in USS(δ)

Corollary: $\#USS(\delta) = 2^{n-2}.$

Proof: Elements in $USS(\delta)$ are characterized by $1 < u_1 < \cdots < u_r < n$.

(possible sequences) = # (subsets of $\{2,3,\ldots,n-1\}$) = 2^{n-2}. □

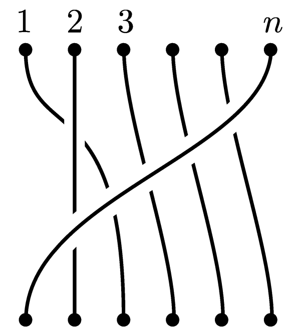
Gebhardt's algorithm is exponential in n for conjugates of δ.

Characterization of elements in USS(ε)

Analogous for conjugates of ε.

$$\varepsilon = \sigma_1(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$$

$$\pi_\varepsilon = (2)(1\ 3\ 4\ \cdots\ n)$$



Birman-Gebhardt-GM (2006): $s \in USS(\varepsilon)$ if and only if

s is simple and $\pi_s = (a)(1\ u_1\ u_2 \cdots\ u_r\ n\ d_t\ d_{t-1} \cdots d_1)$

where $r + t + 3 = n$, $u_1 < u_2 < \cdots < u_r$ and $d_t > d_{t-1} > \cdots > d_1$.

Characterization of elements in USS(ε)

Corollary: $\#USS(\varepsilon) = (n-2)2^{n-3}.$

Proof: $n-2$ choices for a.

2^{n-3} choices for $1 < u_1 < \cdots < u_r < n.$ □

Gebhardt's algorithm is exponential in n for conjugates of ε.

These argumenst are not easy to generalize to other periodic elements.

A polynomial algorithm

Birman-Gebhardt-GM (2006) Idea: change of Garside structure!

Case 1: Conjugates of δ^k.

Birman-Ko-Lee (1998): There is a Garside structure of B_n whose Garside element is precisely δ.

With this structure:

$$USS(\delta^k) = \{\delta^k\}$$

Conjugating X to δ^k is very fast!

Conjugates of powers of δ

Algorithm: Input: Two braids X and Y.

1) Check that $X^n = Y^n = \Delta^{2k}$.

2) Translate $X, Y \in B_n^{[Artin]}$ to $X', Y' \in B_n^{[BKL]}$.

3) Conjugate X' and Y' to δ^k by iterated cycling and decycling. This finds a conjugating element $c' \in B_n^{[BKL]}$.

4) Translate c' to $c \in B_n^{[Artin]}$, and return c.

$O(l^3 n^2 \log n)$

Conjugates of powers of ε

$$\begin{array}{ccccc} B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\ \cup & & & & \cup \\ P_{n,2} & \cong & \mathcal{A}(\mathbf{B}_{n-1}) & \cong & Sym_{2n-2} \end{array}$$

$P_{n,2} = \{\text{Braids fixing the second puncture}\}.$

$\wr\|$

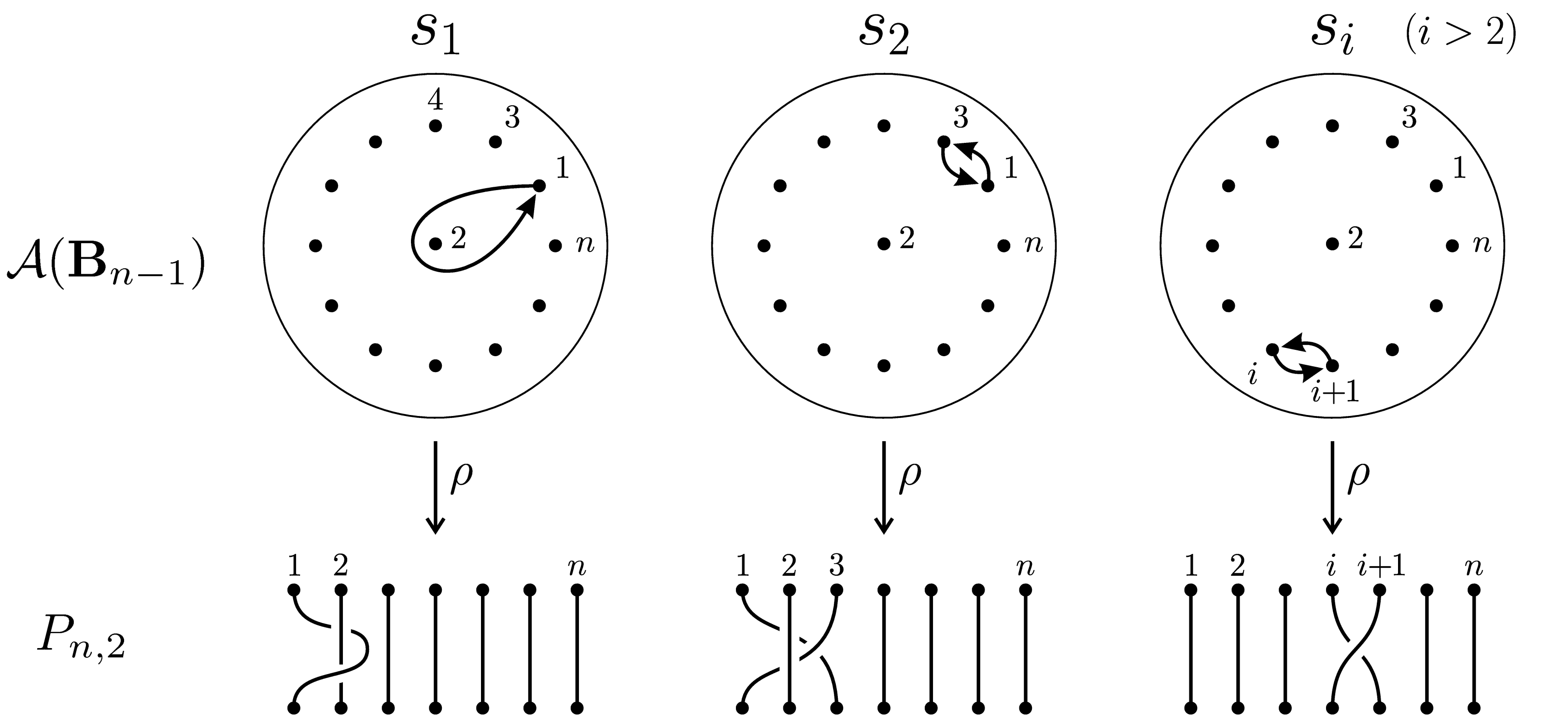
Braid group of the annulus, with *n-1* strands.

$\wr\|$

Artin-Tits group of type $\boldsymbol{B}_{n-1}$.

Conjugates of powers of ε

$$\begin{array}{ccccc} B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\ \cup & & & & \cup \\ P_{n,2} & \cong & \mathcal{A}(\mathbf{B}_{n-1}) & \cong & Sym_{2n-2} \end{array}$$



Conjugates of powers of ε

$$
\begin{array}{ccccccc}
B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\
\cup & & & & \cup \\
P_{n,2} & \stackrel{\rho}{\longleftarrow} & \mathcal{A}(\mathbf{B}_{n-1}) & \cong & Sym_{2n-2}
\end{array}
$$

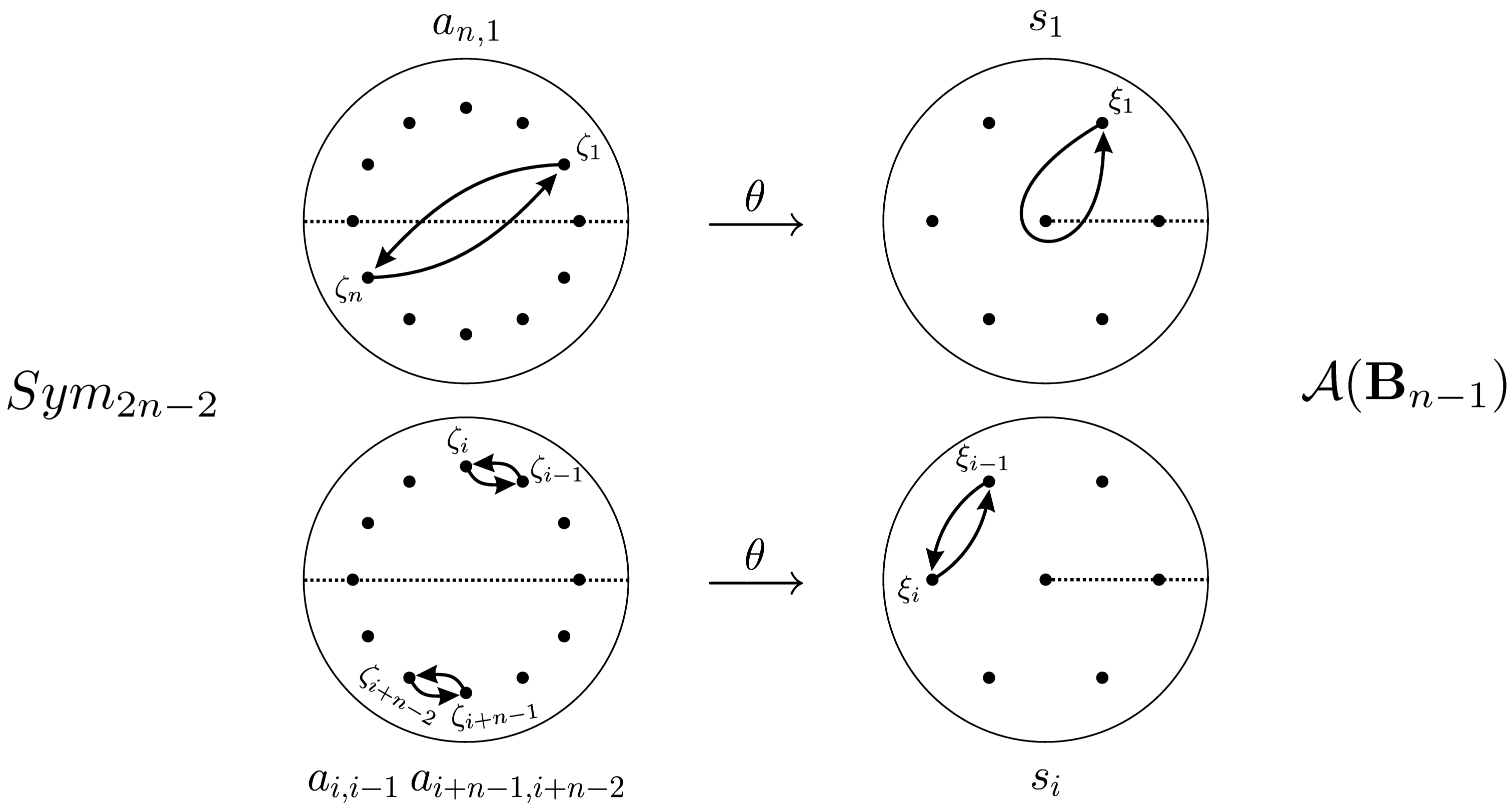
Bessis-Digne-Michel (2002):

$\mathcal{A}(\mathbf{B}_{n-1}) \cong Sym_{2n-2}$ = {Braids on *2n-2* strands, invariant under rot(180º)}

Isomorphism induced by a two-sheeted covering of the annulus:

Conjugates of powers of ε

$$\begin{array}{ccccc} B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\ \cup & & & & \cup \\ P_{n,2} & \stackrel{\rho}{\longleftarrow} & \mathcal{A}(\mathbf{B}_{n-1}) & \cong & Sym_{2n-2} \end{array}$$



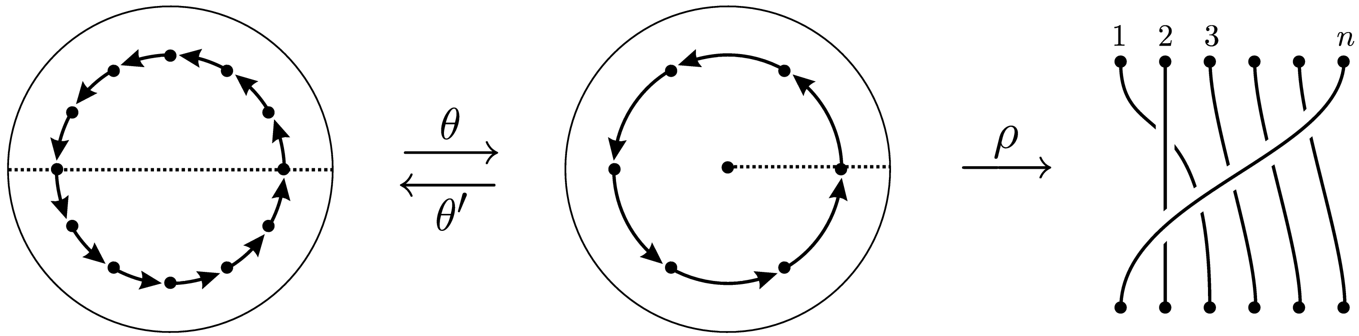
Conjugates of powers of ε

$$\begin{array}{ccccc} B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\ \cup & & & & \cup \\ P_{n,2} & \stackrel{\rho}{\longleftarrow} & \mathcal{A}(\mathbf{B}_{n-1}) & \stackrel{\theta}{\longrightarrow} & Sym_{2n-2} \end{array}$$

	Garside structure	**Garside element**
	$B_{2n-2}^{[BKL]}$	δ_{2n-2}
Restricts to:	Sym_{2n-2}	δ_{2n-2}
Maps to:	$\mathcal{A}(\mathbf{B}_{n-1})$	α
Maps to:	$P_{n,2}$	ε_n

Conjugates of powers of ε

$$\begin{array}{ccccc} B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\ \cup & & & & \cup \\ P_{n,2} & \stackrel{\rho}{\longleftarrow} & \mathcal{A}(\mathbf{B}_{n-1}) & \stackrel{\theta}{\longrightarrow} & Sym_{2n-2} \end{array}$$



δ_{2n-2} α ε_n

Conjugating in a subgroup

Start with X, conjugate of ε^k.

If $\pi_{\varepsilon^k} = 1, \quad \text{then } \varepsilon^k = \Delta^{2m} = X.$ Trivial case.

Otherwise, **the only puncture fixed by X (and by ε^k) is the second one.**

Hence, **every conjugating element must fix the second puncture** as well.

$$c^{-1}\, X\, c = \varepsilon^k \quad \Rightarrow \quad c \in P_{n,2}.$$

Algorithm for conjugates of ε^k

$$
\begin{array}{ccccc}
B_n^{[Artin]} & & & & B_{2n-2}^{[BKL]} \\
\cup & & & & \cup \\
P_{n,2} & \stackrel{\rho}{\longleftarrow} & \mathcal{A}(\mathbf{B}_{n-1}) & \stackrel{\theta}{\longrightarrow} & Sym_{2n-2}
\end{array}
$$

Input: Two braids $X, Y \in B_n^{[Artin]}$.

1) Check whether $X^{n-1} = Y^{n-1} = \Delta^{2k}$.

2) Translate X and Y to generators of $P_{n,2}$. (Reidmeister-Schreier)

3) Send them to Sym_{2n-2}. (They are conjugate to δ here)

4) By iterated cycling and decycling in BKL generators, find c.

5) Translate c to Artin generators.

$O(l^3 n^2 \log n)$

Comparison with USS algorithm

k	1						2					
n	5	7	10	15	20	50	5	7	10	15	20	50
U[δ]	0.16	0.40	2.05	85.20	—	—	0.15	0.65	20.42	—	—	—
B	0.16	0.32	0.67	1.21	1.83	8.24	0.16	0.33	0.66	1.22	1.76	6.79
U[ε]	0.16	0.49	4.32	—	—	—	0.14	0.42	59.76	—	—	—
C	0.19	0.40	0.83	1.51	2.37	10.75	0.14	0.38	0.86	1.57	2.42	10.69

k	3					4				6		
n	7	10	15	20	50	10	15	20	50	15	20	50
U[δ]	0.33	56.14	—	—	—	4.36	—	—	—	—	—	—
B	0.31	0.69	1.14	1.81	6.86	0.65	1.22	1.66	6.26	1.11	1.76	6.36
U[ε]	0.31	9.64	—	—	—	0.99	—	—	—	—	—	—
C	0.29	0.83	1.59	2.47	11.06	0.85	1.60	2.55	10.85	1.57	2.52	11.19

k	7			8		9		10		11	12
n	15	20	50	20	50	20	50	20	50	50	50
U[δ]	7.72	—	—	—	—	—	—	1.44	—	—	—
B	1.15	1.89	6.79	1.62	6.37	1.70	6.80	1.41	5.23	7.00	6.53
U[ε]	1.04	—	—	—	—	162.83	—	90.88	—	—	—
C	0.99	2.50	11.57	2.49	11.47	2.43	11.55	2.21	11.70	12.51	11.93