INTRODUCTION TO QUANTUM INVARIANTS OF KNOTS AND LINKS

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ABSTRACT. These notes provide an introduction to certain quantum invariants of knots and links, and were presented during the first week of the Summer School *Invariants de nœuds et de variétés de dimension* 3 (Grenoble, 1999). Our purpose is to first construct the well known polynomial invariants using elementary methods, then present a version of the colored Homfly invariant and explain its relation with the representation theory of quantum groups.

1. HOMFLY POLYNOMIAL

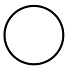
The Homfly polynomial was found by several authors just after the discovery of the Jones polynomial [3, 25]. We will give here a construction using plane graphs [23], which is very closed to the one using Yang-Baxter operators in [28] (see also [12]).

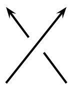
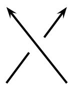
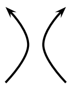
1.1. The main theorem.

Theorem 1.1. *There exists a unique map*

$$L \longmapsto P(L)$$

from isotopy classes of oriented links in the 3*-sphere to* $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ *such that*

$P($  $) = 1$,

$v^{-1}\, P($  $) \;-\; v\, P($  $) \;=\; z\, P($  $)$.

In the last relation, which will be called the Homfly (skein) relation, the links are the same except in a ball where they are as indicated. This theorem is equivalent to the existence of a unique function on diagrams satisfying the Reidemeister relations and the two relations above.

We deduce from the Homfly relation the following property.

$$P(L \cup \bigcirc) = \frac{v^{-1}-v}{z} P(L)$$

Hence the value of P on the m components trivial link must be $(\frac{v^{-1}-v}{z})^{m-1}$. The unicity part of the theorem above is obtained by induction on the number of crossings, using the fact that one can transform any link diagram into that of a trivial link by changing crossings. Note that $P(L)$ belongs to the subring $\Lambda = \mathbb{Z}[v^{\pm 1}][\frac{v^{-1}-v}{z}]$ For existence, we will construct a discrete series of specializations in the sub-section 1.3, and obtain the following proposition.

Proposition 1.2. *For any integer $n \geq 2$, there exists a unique map*

$$L \longmapsto F_n(L)$$

from isotopy classes of oriented links to $\mathbb{Z}[s^{\pm 1}]$ which satisfies the relations in theorem 1.1, with $z = s - s^{-1}$ and $v = s^{-n}$.

Using this proposition, we prove the existence [30]. We consider the Λ-module $\mathcal{H}(S^3)$ freely generated by oriented links in S^3, factored by the Homfly relation. The unicity argument shows that the trivial knot U is a generator. The map F_n extends to a homomorphism of Λ-module

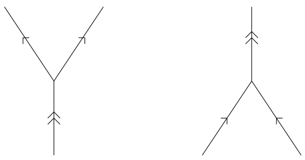
$$\widetilde{F}_n : \mathcal{H}(S^3) \to \mathbb{Z}[s^{\pm 1}]$$

Here, the Λ-module structure of $\mathbb{Z}[s^{\pm 1}]$ is given by the specialization

$$z = s - s^{-1},\ v = s^{-n}\ .$$

Suppose that $R(v, z) \in \Lambda$ is such that $R(v, z)U$ represents zero in $\mathcal{H}(S^3)$, then for any n, we have that $R(s^{-n}, s - s^{-1}) = 0$. This shows that $R(v, s - s^{-1})$ is zero in $\mathbb{Q}(s)[v^{\pm 1}]$, and so $R(v, z)$ is zero. The unknot U represents a basis of $\mathcal{H}(S^3)$. Using this basis, we get an isomorphism $\mathcal{H}(S^3) \simeq \Lambda$ whose value on links give $P(L)$.

1.2. **Invariants of plane graphs.** We will consider here trivalent plane graphs (loops are allowed). If G is such a graph, then we denote by $\mathcal{V}_G$ the set of trivalent vertices in G, and by $\mathcal{E}_G$ the set of connected components of $G \setminus \mathcal{V}_G$ (the edges). A 2-flow for such a graph is an orientation of the edges, together with a map $f : \mathcal{E}_G \to \{1, 2\}$ which defines an integral cycle. We will mark the edges with one or two arrows to indicate the orientation and the value of the 2-flow. The figure below gives the two possibilities at a trivalent vertex.



For $n \geq 2$, we will associate to each graph equipped with a 2-flow a scalar $\langle (G,f) \rangle_n \in \mathbb{Z}[s^{\pm 1}]$, which is invariant under isotopies of the plane. We will further use it to give a global formula for $F_n(L)$.

Let $\mathcal{N} = \{-n+1, -n+3, \ldots, n-1\}$.

Definition. A state of a graph with 2-flow (G,f) is a map

$$\sigma : \mathcal{E}_G \to \mathcal{P}(\mathcal{N})$$

such that for every edge e, one has

$$\sharp\sigma(e) = f(e),$$

and at each vertex, the incoming set is equal to the outgoing one.

To such a state σ, we associate an immersed curve G_σ obtained by replacing each double edge with state $\{i,j\}$ by two simple edges with respective states $\{i\}$ and $\{j\}$, and joining at each trivalent vertex the edges according to their state.

The components C of G_σ have a state $\sigma(C)$, and a rotation number $rot(C)$ (the degree of the Gauss map).

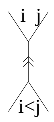
Set

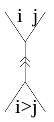
$$\langle G_\sigma \rangle_n = \prod_C s^{\sigma(C) rot(C)} \ . \tag{1}$$

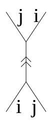
The formula below is invariant under isotopies of the plane.

$$\langle (G,f) \rangle_n = \sum_\sigma \prod_{e \in f^{-1}(2)} w_\sigma(e) \langle G_\sigma \rangle_n \ . \tag{2}$$

The sum is over all states. The product is over the set $f^{-1}(2)$ of double edges. The coefficients $w_\sigma(e)$ are defined by:

$w_\sigma \Bigg($ 

$\Bigg) = s \quad w_\sigma \Bigg($ 

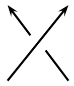
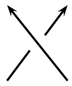
$\Bigg) = s^{-1} \quad w_\sigma \Bigg($ 

$\Bigg) = 1 \ .$

1.3. **The $SU(n)$ specialization of Homfly polynomial.** We consider a link diagram D and we denote by $\mathcal{C}_D$ the set of its crossings. We define $\langle D \rangle_n$ by

$$\langle D \rangle_n = \sum_{X \subset \mathcal{C}_D} s^{\epsilon(X)} (-1)^{\sharp(\mathcal{C}_D \setminus X)} \langle D_X \rangle_n \ . \tag{3}$$

Here, $\epsilon(X) = \sum_{c \in X} \epsilon(c)$, with

$\epsilon($  $) = 1$, $\epsilon($  $) = -1$.

The graph with 2-flow D_X is obtained by replacing each crossing c by two parallel edges if c is in X, and by a double edge otherwise.

Theorem 1.3. *$\langle D \rangle_n$ defines an invariant of oriented framed links.*

We give a sketchy proof.

Let V be the free Λ-module with basis $(v_i)_{i \in \mathcal{N}}$. We define an endomorphism E of $V \otimes V$ by the following matrix.

$$E_{ij}^{kl} = \begin{cases} s & \text{if } i < j \ , \ k = i \text{ and } l = j; \\ s^{-1} & \text{if } i > j \ , \ k = i \text{ and } l = j; \\ 1 & \text{if } i \neq j \ , \ k = j \text{ and } l = i; \\ 0 & \text{else.} \end{cases}$$

We can show.

Proposition 1.4. *a)* $E^2 = (s + s^{-1})E$.
b) $(E \otimes Id_V) \circ (Id_V \otimes E) \circ (E \otimes Id_V) - (E \otimes Id_V) =$
$(Id_V \otimes E) \circ (E \otimes Id_V) \circ (Id_V \otimes E) - (Id_V \otimes E)$.

Let R be defined by $R = sId_{V \otimes V} - E$.

Proposition 1.5. *a)* $R^{-1} = s^{-1} Id_{V \otimes V} - E$,
b) $R - R^{-1} = (s - s^{-1}) Id_{V \otimes V}$.

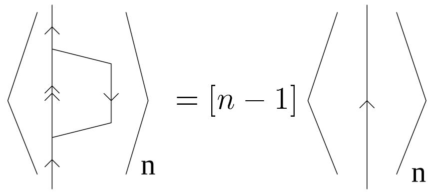
The key point is that R is a Yang-Baxter operator.

Theorem 1.6 (Yang-Baxter equation)**.**

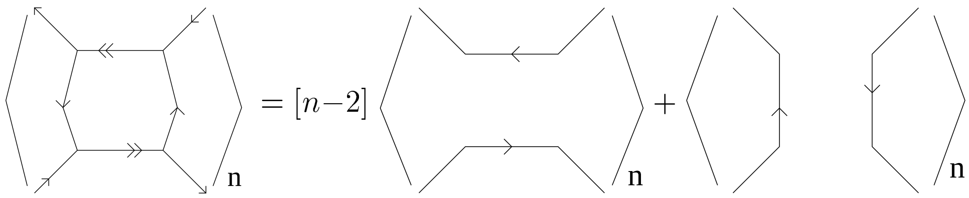
$$(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R) \ .$$

The statement above is the Yang-Baxter equation and prove that $\langle D \rangle_n$ satisfies the Reidemeister relation III (with coherent orientations). For the Reidemeister relation II, we have to consider two cases depending on the orientations of the strings. The first one follows from the proposition 1.5. The second one uses the graphical lemmas below. Here the quantum integer $[m]$ is defined by $[m] = \frac{s^m - s^{-m}}{s - s^{-1}}$.

Lemma 1.7.



Lemma 1.8.



We can show that

Hence we get an invariant $\langle L\rangle_n$ of oriented framed links L. Morever

$$F_n(L) = \frac{s - s^{-1}}{s^n - s^{-n}} s^{-nL.L}\langle L\rangle_n$$

satisfies the conditions in theorem 1.2. Here $L.L$ is the writhe of the framed link L (the sum of the signs of the crossings in a diagram).

The framed Homfly polynomial. Here k is an integral domain containing the invertible elements a, v, s; we suppose moreover that $s - s^{-1}$ is invertible in k. Let $z = s - s^{-1}$. We will use later the following invariant of framed links, and call it the framed Homfly polynomial:

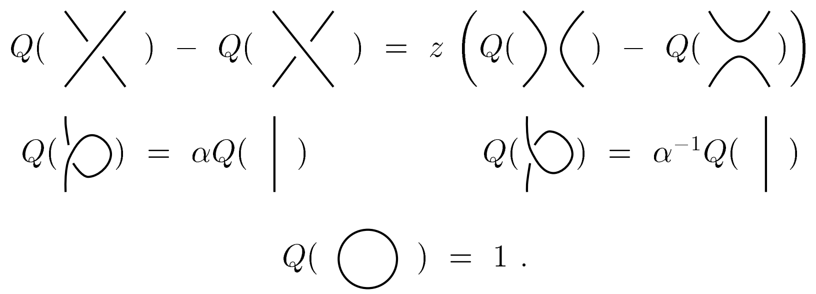
$$\langle L\rangle = \frac{v^{-1} - v}{s - s^{-1}}(av^{-1})^{L.L}P(L) \ .$$

2. The Kauffman polynomial

Theorem 2.1. *There exists a unique map*

$$L \longmapsto Q(L)$$

from isotopy classes of unoriented framed links in the 3*-sphere to* $\mathbb{Z}[\alpha^{\pm 1}, z^{\pm 1}]$ *such that*



We can deduce the following relation.

$$Q(L \cup \bigcirc) = \left(\frac{\alpha-\alpha^{-1}}{z}+1\right) Q(L)$$

If L is oriented, then $\alpha^{-L.L}Q(L)$ is an invariant of oriented links. The unicity part of the theorem above is obtained similarly as in section 1.

The existence can also be proved by constructing a discrete series of specializations, as we did in Section 1. See [22] for a graphical construction. In fact one can construct three series, using the Yang-Baxter operators associated with the deformations of the fundamental representations of the classical simple Lie algebras $so(2n+1)$, $sp(2n)$ and $so(2n)$ [28, 30]. See also the original construction [15], and an unpublished construction of Jaeger using the Homfly polynomial (a sketchy proof of Jaeger's theorem is given in [16, I.14])[1].

3. Skein modules

The notion of a skein module was introduced in [24]. We present here a framed version of Homfly skein theory.

Let M be an oriented 3-manifold. We denote by $\mathcal{H}(M)$ the k-module freely generated by isotopy classes of (oriented) framed links (the empty link is allowed) in M, factored by the Homfly relations given in figure 1.

Here k is an integral domain containing the invertible elements a, v, s; we suppose moreover that $s - s^{-1}$ is invertible in k. (For $L \neq \emptyset$, the third equality is a consequence of the others.)

Note about the framing. Here a framing is a trivialization of the normal bundle up to homotopy. This is equivalent to an orientation of the link together with a non-singular normal vector field up to homotopy. In the figures, we use the blackboard framing.

[1]We plan to write in an annexe a version of Jaeger's construction.

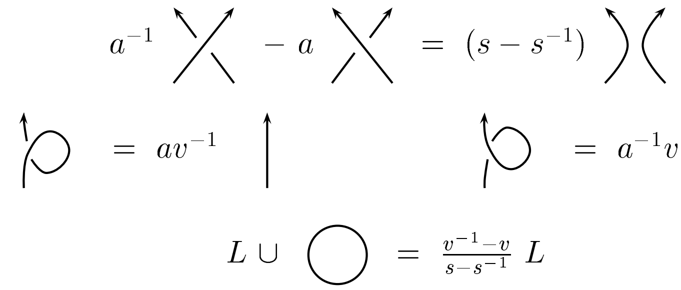


FIGURE 1. Homfly relations

The Homfly polynomial defined at the end of Section 1 gives an isomorphism

$$\begin{array}{rccl} \langle \dots \rangle : & \mathcal{H}(S^3) & \to & k \\ & L & \mapsto & \langle L \rangle = \frac{a^{-1}-a}{s-s^{-1}} a^{L.L} P(L) \end{array}$$

normalized by $\langle \emptyset \rangle = 1$. Another useful example is the solid torus $D^2 \times S^1$. Links in this solid torus are represented by diagrams in the annulus $[-1, 1] \times S^1$. There exist a natural product on $\mathcal{H}(D^2 \times S^1)$, which is commutative, and can be described on diagrams by gluing the two copies of $[-1, 1] \times S^1$.

Theorem 3.1. *$\mathcal{H}(D^2 \times S^1)$ is a polynomial algebra with free generators z_j, $j \in \mathbb{Z} \setminus \{0\}$.*

The generators z_j and z_{-j}, $j > 0$ are represented by the closure of the *positive braid* corresponding to the cycle $(1, 2, \dots, j)$ (figure below), with distinct orientations. For a proof of this theorem, see [29] or [4].

We can use the skein module of the solid torus to obtain many invariants of framed links, via an operation named cabling. which we describe in the following.

The Homfly functor. An embedding $j : M \hookrightarrow N$ gives a well defined operator $\mathcal{H}(j) : \mathcal{H}(M) \to \mathcal{H}(N)$. This makes $\mathcal{H}$ into a functor from the category of 3-manifolds whose morphisms are isotopy classes of embeddings, to the category of k-modules.

To an oriented embedding of a disjoint union of solid tori

$$g = \coprod_{i=1}^{m} g_i : \coprod_{i=1}^{m} D_i^2 \times S_i^1 \to S^3 ,$$

is associated a multilinear map

$$\mathcal{H}(g) : \mathcal{H}(D^2 \times S^1)^{\otimes m} \to \mathcal{H}(S^3) \approx k .$$

This map only depends on the isotopy class of the framed link $L = (L_1, \dots, L_m)$ underlying g. The image of $x_1 \otimes \cdots \otimes x_m$ under this map is said to be obtained by cabling the components L_i with the skein elements x_i, and is denoted by $\langle L_1(x_1), \dots, L_m(x_m)\rangle$ or $\langle L(x_1, \dots, x_m)\rangle$.

4. The Hecke category and the Homfly invariant of colored links

4.1. **Relative Homfly skein module.** In the case where M has non-empty boundary, we may fix a finite set of points, l, in the boundary of M, equipped with a trivialization of the tangent bundle of ∂M at each point, and define the relative skein module $\mathcal{H}(M, l)$. Here generators are links $L \subset M$ with boundary l, equipped with relative framing (extension to L of the trivialization fixed on l, up to relative homotopy). Note that a point in l will be incoming or outgoing, depending on the orientation of the trivialization.

4.2. **The Hecke category.** The *k-linear Hecke category* H is defined as follows. An object in this category is a disc D^2 equipped with a set of points with trivialization as above. If $\alpha = (D^2, l_0)$ and $\beta = (D^2, l_1)$ are two objects, the module $Hom_H(\alpha, \beta)$ is $\mathcal{H}(D^2 \times [0;1], l_0 \times 0 \amalg l_1 \times 1)$. The notation $H(\alpha, \beta)$ and H_α will be used respectively for $Hom_H(\alpha, \beta)$ and $End_H(\alpha)$. When we draw a figure to describe a morphism, the *time* parameter goes upwards, so that the morphism gf is depicted with g lying above f.

Note. The terminology *Hecke category* was introduced by Turaev in [30]. His definition gives a category equivalent to an unframed version of ours.

We will use the notation $\hat{f}$ for the closure in $\mathcal{H}(D^2 \times S^1)$ of the morphism $f \in H_\alpha$. Let U_0 be a 0-framed unknot in S^3. We use the notation $\langle f \rangle$ for $\langle U_0(\hat{f}) \rangle$ (the quantum trace).

We will simply denote by n the object formed by the n points $-1 + \frac{2j-1}{n}$, $j = 1, \ldots, n$, equipped with the standard trivialization.

For $\epsilon = \pm 1$ let $j_\epsilon : D^2 \hookrightarrow D^2$ be the embedding which sends z to $\frac{\epsilon}{2} + \frac{1}{4}z$. By using $j = j_{-1} \amalg j_1 : D^2 \amalg D^2 \hookrightarrow D^2$, we make H into a monoidal category[2].

4.3. **Idempotents in Hecke algebras.** Let $\sigma_i \in H_n$, $i = 1, \ldots, n-1$, be represented by the standard generators of the braid group B_n (the strand numbered i crosses over the strand numbered $i+1$).

Recall that the group B_n is generated by $\sigma_1, \ldots, \sigma_n$, with the relations

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ i = 1, \ldots, n-2 \end{cases}$$

There exists an obvious homomorphism from the braid group algebra $k[B_n]$ to the algebra H_n.

Theorem 4.1. *The algebra H_n is isomorphic to the quotient of the braid group algebra $k[B_n]$ by the Homfly relations*

$$a^{-1}\sigma_i - a\sigma_i^{-1} = (s - s^{-1})\mathbb{1}_n \ i = 1, \ldots, n-1 \ .$$

A basis for this algebra is represented by any family of braids b_π, $\pi \in \mathcal{S}_n$, such that the braid b_π induces the permutation π.

As an example of basis of H_n, we have the positive braids. The positive braid w_π associated with a permutation π, is the braid inducing π that can be represented by a diagram with the minimal number of crossings, all of them having positive sign. This positive braid w_π is obtained by replacing the transposition $(i, i+1)$ by w_i in the standard decomposition of π,

$$\pi = \ldots (n-2, n-1) \ldots (i_{n-1}, i_{n-1}+1)(n-1, n) \ldots (i_n+1), i_n+2)(i_n, i_n+1)$$

H_n is called a Hecke algebra. It is a deformation of the algebra of the symmetric group. The above theorem was proved independently by Morton-Traczyk [21] and Turaev [30].

[2] The associativity isomorphisms will be omitted. They connect the corresponding points in the obvious way. Note that the object $1^{\otimes n}$ is defined up to these associativity isomorphisms, and is canonically isomorphic to the object n.

Symmetrizers. For $n \geq 0$, the quantum integer $[n]$, and the quantum factorial $[n]!$ are defined by $[n] = \frac{s^n - s^{-n}}{s - s^{-1}}$ and $[n]! = \prod_{j=1}^{n}[j]$.
For a Young diagram λ, we will use the notation $[hl(\lambda)]$, for the product over all cells of the quantum hook-lengths.

$$[hl(\lambda)] = \prod_{\text{cells}} [hl(c)]$$

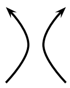
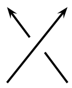
Proposition 4.2. *If $[n]!$ is invertible in k, then there exists a unique idempotent $f_n \in H_n$ such that $\forall i$ $\sigma_i f_n = as f_n = f_n \sigma_i$, and a unique idempotent $g_n \in H_n$ such that $\forall i$ $\sigma_i g_n = -as^{-1} g_n = g_n \sigma_i$.*

Proof. It is shown in [20, 2] that the deformation f_n of the Young symmetrizer, given below, satisfies the required condition.

$$f_n = \frac{1}{[n]!} s^{-\frac{n(n-1)}{2}} \sum_{\pi \in \mathcal{S}_n} (as^{-1})^{-l(\pi)} w_\pi$$

Here w_π is the positive braid associated with the permutation π, and $l(\pi)$ is the length of π.

One can also construct f_n recursively using the formulas below [35].

$[2] f_2 = s^{-1}$  $+ a^{-1}$ 

$$[n+1] f_{n+1} = -[n-1] f_n \otimes 1_1 + [2][n](f_n \otimes 1_1)(1_{n-1} \otimes f_2)(f_n \otimes 1_1)$$

(Here, $1_p \in H_p$ is the identity.)

Suppose now that f'_n satisfies the condition of the theorem. Then we have that $f_n f'_n = f'_n = \zeta f_n$. By considering $f_n {f'}^2_n$, we get $\zeta = 1$.

For g_n we can proceed similarly, either with the deformed antisymmetrizer

$$g_n = \frac{1}{[n]!} s^{\frac{n(n-1)}{2}} \sum_{\pi \in \mathcal{S}_n} (-as)^{-l(\pi)} w_\pi \ ,$$

or with the recursive formulas

$$g_1 = 1_1 \ ,$$

$$g_{n+1} = 1_1 \otimes g_n - \frac{[2][n]}{[n+1]} (1_1 \otimes g_n)(f_2 \otimes 1_{n-1})(1_1 \otimes g_n) \ .$$

$\square$

Note that $1_2 = f_2 + g_2$; this can be used to obtain more symmetric recursive formulas for f_n and g_n. Our choice immediately gives the following lemma [35], which is useful in Yokota's skein computations.

Lemma 4.3. *For any two integers p, q such that $[p+1]!$ and $[q+1]!$ are invertible, one has*

$$\begin{aligned}[p+q]f_p \otimes g_q &= [p+1][q](1_p \otimes g_q)(f_{p+1} \otimes 1_{q-1})(1_p \otimes g_q)\\ &\quad +[p][q+1](f_p \otimes 1_q)(1_{p-1} \otimes g_{q+1})(f_p \otimes 1_q)\ .\end{aligned}$$

The skein computations frequently use the absorbing property below whose proof follows from the definition of the symmetrizers.

Lemma 4.4. *Suppose that $[n]!$ is invertible. For any $m < n$, one has*
a) $(f_m \otimes 1_{n-m})f_n = f_n(f_m \otimes 1_{n-m}) = f_n$;
b) $(g_m \otimes 1_{n-m})g_n = g_n(g_m \otimes 1_{n-m}) = g_n$.

Aiston-Morton description of Young symmetrizers. For a Young diagram λ of size n, we denote by $\square_\lambda$ the object of the category H formed with one point for each cell c of λ, equipped with the standard trivialization; if c has index (i,j) (i-th row, and j-th column), then the corresponding point in D^2 is $\frac{j+i\sqrt{-1}}{n+1}$. Suppose that $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 1)$, and that $\lambda^\vee = (\lambda_1^\vee \geq \cdots \geq \lambda_q^\vee \geq 1)$ is the transposed Young diagram.
Let F_λ (resp. G_λ) be the element in $H_{\square_\lambda}$ formed with one copy of $[\lambda_i]!f_{\lambda_i}$ along the row i, for $i = 1, \ldots, p$ (resp. one copy of $[\lambda_j^\vee]!g_{\lambda_j^\vee}$ along the column j, for $j = 1, \ldots, q$). Note that these expressions have no denominators.

In the proposition below, $<$ denotes the lexicographic ordering, and λ and μ are Young diagrams with the same number of cells.

Proposition 4.5. *a) If $\mu < \lambda$, then $F_\lambda H(\square_\mu, \square_\lambda)G_\mu = 0$.*
b) If $\lambda < \mu$, then $G_\lambda H(\square_\mu, \square_\lambda)F_\mu = 0$.
c) One has $F_\lambda H_{\square_\lambda} G_\lambda = kF_\lambda G_\lambda$.
d) One has $G_\lambda H_{\square_\lambda} F_\lambda = kG_\lambda F_\lambda$.

This proposition is proved in [2], section 4. We outline here the proof given there.

Proof. a) Every family of $|\lambda|!$ braids in $H(\square_\mu, \square_\lambda)$, that induces all bijections between the cells of λ and the cells of μ, is a basis of $H(\square_\mu, \square_\lambda)$. The hypothesis $\mu < \lambda$ implies that any bijection between the cells of λ and the cells of μ carries at least two cells in some row of λ, to cells in the same column of μ. So we can find a basis of $H(\square_\mu, \square_\lambda)$ represented by braids which connect in a separate cylinder two points in some row of $\square_\lambda$ with two points in some column of $\square_\mu$. One can deduce that $F_\lambda H(\square_\mu,\square_\lambda)G_\mu=0$. Assertion b) is shown similarly.

c) We say that a permutation π between the cells of λ does not separate, if some pair of cells in the same row is mapped to some pair of cells in the same column; other permutations are said to separate. Using that any separating permutation π of the cells can be written $\pi=\pi_R\pi_C$ (permutations act on the left), where π_R (resp. π_C) preserves the rows (resp. the columns), we see that we can find a basis represented by braids b_π indexed by permutations such that

$$\begin{cases} F_\lambda b_\pi G_\mu = 0 & \text{if } \pi \text{ does not separate,}\\ F_\lambda b_\pi G_\mu = \zeta_\pi F_\lambda G_\mu & \text{if } \pi \text{ separates.}\end{cases}$$

Statement c) follows. Assertion d) is shown similarly. $\square$

The argument used in the proof of a) above gives.

Lemma 4.6. *a) If $p>\lambda_1$, and for some object α containing $|\lambda|$ points with positively oriented trivialization, $x=x_1\otimes f_p\otimes x_2\in H_\alpha$, then one has*

$$xH(\square_\lambda,\alpha)G_\lambda=0,\ \textit{and}\ \ G_\lambda H(\alpha,\square_\lambda)x=0\ .$$

b) If $p>\lambda_1^\vee$, and for some object α containing $|\lambda|$ points with positively oriented trivialization, $x=x_1\otimes g_p\otimes x_2\in H_\alpha$, then one has

$$F_\lambda H(\alpha,\square_\lambda)x=0,\ \textit{and}\ \ xH(\square_\lambda,\alpha)F_\lambda=0\ .$$

Let $\tilde{y}_\lambda=F_\lambda G_\lambda$. A consequence of proposition 4.5 is that $\tilde{y}_\lambda$ is a quasi-idempotent.

Proposition 4.7. *One has $\tilde{y}_\lambda^2=[hl(\lambda)]\tilde{y}_\lambda$.*

We note here that Yokota's idempotents in [35] are based on $G_\lambda F_\lambda G_\lambda$ rather than on $F_\lambda G_\lambda$. This makes essentially no difference in the computation of the normalizing coefficient. The proposition above follows from the lemma below, which is a version of [35, lemma 2.3] (see also [1]). Here $1^n=(1,\dots,1)$ denotes the Young diagram with one column containing n cells.

Lemma 4.8 (Yokota's lemma)**.** *If $\lambda=1^n+\mu$ is a Young diagram with n rows, then one has*
a) $F_\lambda G_\lambda F_\lambda G_\lambda=\prod_{i=1}^n[\lambda_i+n-i]\ F_\lambda(1_{\square_{1^n}}\otimes G_\mu)(1_{\square_{1^n}}\otimes F_\mu)G_\lambda$;
b) $G_\lambda F_\lambda G_\lambda F_\lambda=\prod_{i=1}^n[\lambda_i+n-i]\ G_\lambda(1_{\square_{1^n}}\otimes F_\mu)(1_{\square_{1^n}}\otimes G_\mu)F_\lambda$.

Here the obvious isomorphism between $\square_\lambda$ and $\square_{1^n} \otimes \square_\mu$ is omitted.

Proof. We will show the formula with generic parameters. In the following, the cells in the superscript indicate which way the corresponding symmetrizer or antisymmetrizer is inserted.

We denote by V_i the skein element obtained from F_λ by replacing, for $j = i+1, \ldots, n$, the copy of $[\lambda_j]! f_{\lambda_j}$ by $[\lambda_j - 1]! f_{\lambda_j - 1}^{(j,2),\ldots,(j,\lambda_j)}$. Note that $V_n = F_\lambda$, and $V_0 = 1_{\square_{1^n}} \otimes F_\mu$. We will show the following lemma, which proves the required formulas.

Lemma. *For $i = 1, \ldots, n$, one has*
a) $[n-i+1] F_\lambda G_\lambda V_i G_\lambda = [\lambda_i + n - i] F_\lambda G_\lambda V_{i-1} G_\lambda$;
b) $[n-i+1] G_\lambda V_i G_\lambda F_\lambda = [\lambda_i + n - i] G_\lambda V_{i-1} G_\lambda F_\lambda$.

We first expand $G_\lambda V_i G_\lambda$, using the lemmas 4.3 and 4.4.

$$\begin{aligned} G_\lambda V_i G_\lambda &= [\lambda_i] G_\lambda g_{n-i+1}^{(i,1),\ldots,(n,1)} f_{\lambda_i}^{(i,1),\ldots,(i,\lambda_i)} g_{n-i+1}^{(i,1),\ldots,(n,1)} V_{i-1} G_\lambda \\ &= \frac{[\lambda_i + n - i]}{[n-i+1]} G_\lambda V_{i-1} G_\lambda \\ &\quad + \mathrm{coeff} \times G_\lambda f_{\lambda_i - 1}^{(i,2),\ldots,(i,\lambda_i)} g_{n-i+2}^{(i,2),(i,1),\ldots,(n,1)} V_{i-1} G_\lambda \end{aligned}$$

We have to show that the second term is zero. We expand the copy of $f_{\lambda_i - 1}$ as a linear sum of braids, so that we have only to consider

$$\eta_i = G_\lambda \sigma^{(i,2),\ldots,(i,\lambda_i)} g_{n-i+2}^{(i,2),(i,1),\ldots,(n,1)} V_{i-1} G_\lambda \ ,$$

where $\sigma^{(i,2),\ldots,(i,\lambda_i)}$ is represented by a braiding of the strings corresponding to the superscript.

For $\nu = 1, \ldots, i$, let $\eta_{i\nu}$ be defined by

$$\eta_{i\nu} = G_\lambda \sigma^{(i,2),\ldots,(i,\lambda_i)} g_{n-\nu+2}^{(i,2)(\nu,1),\ldots,(n,1)} V_{i-1} G_\lambda \ .$$

We have, for $\nu = 1, \ldots, i-1$,

$$\eta_{i,\nu+1} = G_\lambda \sigma^{(i,2),\ldots,(i,\lambda_i)} g_{n-\nu+1}^{(\nu,1),\ldots,(n,1)} g_{n-\nu+1}^{(i,2)(\nu+1,1),\ldots,(n,1)} V_{i-1} G_\lambda \ .$$

The recursive formula for the antisymmetrizers shows that we can write $\eta_{i,\nu+1}$ as a linear combination of $\eta_{i,\nu}$ and $\xi_{i,\nu}$, where

$$\xi_{i\nu} = G_\lambda \sigma^{(i,2),\ldots,(i,\lambda_i)} \tau^{(\nu,1),(i,2)} V_{i-1} G_\lambda \ ,$$

and $\tau^{(\nu,1),(i,2)}$ is represented by a braiding of the two strings indicated by the superscript, inserted between the two first columns. Morever we can choose $\tau^{(\nu,1),(i,2)}$ so that in $\xi_{i\nu}$ the copy of the symmetrizer f_{λ_ν} is directly joined to $g_{\lambda_2^\vee} \otimes \cdots \otimes g_{\lambda_{\lambda_\nu}^\vee}$, hence $\xi_{i\nu}$ is zero. (Lemma 4.6 can be applied.)

This is the key argument in Yokota's proof (a moment of thought and some drawings may be useful). Note that we have $\eta_{ii} = \eta_i$. The conclusion follows from the lemma below.

Lemma. *For $\nu = 1, \ldots, i$, one has $F_\lambda \eta_{i\nu} = \eta_{i\nu} F_\lambda = 0$.*

This lemma is proved recursively. The case $\nu = 1$ comes from lemma 4.6. □

If $[hl(\lambda)]$ is invertible, we define the idempotent y_λ by

$$y_\lambda = [hl(\lambda)]^{-1} \tilde{y}_\lambda \ .$$

Suppose that λ and μ are Young diagrams with the same number of cells, such that $[hl(\lambda)]$ and $[hl(\mu)]$ are invertible. From proposition 4.5, we get.

Proposition 4.9. *a) If $\mu \neq \lambda$, then $y_\lambda H(\square_\mu, \square_\lambda) y_\mu = 0$.*
b) One has that $y_\lambda H_{\square_\lambda} y_\lambda = k y_\lambda$.

For two Young diagrams $\lambda \subset \mu$, the complement of λ in μ is called a skew Young diagram and is denoted by μ/λ. As above we can define the object $\square_{\mu/\lambda}$ in the category H with one point for each cell. The following is proven in the same way as proposition 4.7.

Lemma 4.10. *Let $\lambda \subset \mu$ be two Young diagrams, and let $x \in H_{\square_{\mu/\lambda}}$. One has that*

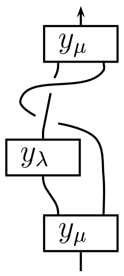
$$G_\mu(\tilde{y}_\lambda \otimes x) F_\mu = [hl(\lambda)] G_\mu(1_{\square_\lambda} \otimes x) F_\mu \ .$$

Corollary 4.11 (Absorbing property)**.** *Let $\lambda \subset \mu$ be two Young diagrams, and suppose that $[hl(\mu)]$ is invertible. Then*

$$y_\mu (y_\lambda \otimes 1_{\square_{\mu/\lambda}}) y_\mu = y_\mu \ .$$

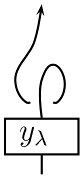
We will need the following formulas.

Proposition 4.12. *a) Suppose that $\lambda \subset \mu$ are two Young diagrams such that the skew diagram μ/λ has only one cell c ($|\mu| = |\lambda| + 1$).*



$$= a^{2|\lambda|} s^{2cn(c)} y_\mu$$

b) (Framing coefficient)



$$= a^{|\lambda|^2} v^{-|\lambda|} s^{2\sum_{cells} cn(c)} y_\lambda$$

Statements a) and b) can be deduced from each other. Statement b) is theorem 5.5 of [2]; the proof given there is an elementary skein calculation which contains statement a). The framing coefficients b) were first computed by Wenzl [34, lemma 3.2.1].

4.4. **Structure of the generic Hecke algebra and path idempotents.** Here we suppose that n is fixed, and that, in the domain k, $[j]$ is invertible for every $j \leq n$, so that all the idempotents y_λ exist in H_n.

A standard tableau t with shape a Young diagram $\lambda = \lambda(t)$ is a labelling of the cells with the integers 1 to n, that is increasing along rows and columns. We denote by t' the tableau obtained by removing the cell numbered by n. We define $\alpha_t \in H(\square_\lambda, n)$ and $\beta_t \in H(n, \square_\lambda)$ by

$$\alpha_1 = \beta_1 = 1_1 \ ,$$

$$\alpha_t = (\alpha_{t'} \otimes 1_1)\varrho_t y_\lambda \ ,$$

$$\beta_t = y_\lambda \varrho_t^{-1}(\beta_{t'} \otimes 1_1) \ .$$

Here $\varrho_t \in H(\square_\lambda, \square_{\lambda(t')} \otimes 1)$ is composed of two isomorphisms, using the intermediate object obtained from $\square_{\lambda(t')} \otimes 1$ by moving the added point to its place in $\square_\lambda$; these two isomorphisms are obtained by connecting the corresponding points in the obvious way.

Note that $\beta_\tau \alpha_t = 0$ if $\tau \neq t$, and $\beta_t \alpha_t = y_{\lambda(t)}$.

Theorem 4.13. *a) The family $\alpha_t \beta_\tau$ for all standard tableaux t, τ such that $\lambda(t) = \lambda(\tau)$ forms a basis for H_n.*
b) There exists an algebra isomorphism

$$\bigoplus_{|\lambda|=n} \mathcal{M}_{d_\lambda}(k) \approx H_n \ ,$$

where d_λ is the number of standard tableaux with shape λ, and $\mathcal{M}_{d_\lambda}(k)$ is the algebra of $d_\lambda \times d_\lambda$ matrices with coefficients in k.

Proof. We have (here δ is the Kronecker delta)

$$\alpha_t\beta_\tau\alpha_s\beta_\sigma = \delta_{\tau s}\alpha_t\beta_\sigma \ .$$

This shows the independence. Moreover the number of vectors is equal to the dimension. This proves the proposition over the field of quotients of the domain k. The above formula gives the coordinate forms and shows that the result a) is valid over k.

We can index the entries of the matrices in $\mathcal{M}_{d_\lambda}(k)$ with the tableaux of shape λ. The isomorphism in b) is then obtained by sending the elementary matrix $E_{t\tau}$ to $\alpha_t\beta_\tau$. $\square$

The $\alpha_t\beta_\tau$ are called *matrix units* [27]. The isomorphism in b) gives the semi-simple decomposition of H_n. The simple components are indexed by Young diagrams. The component indexed by λ is the two-sided ideal generated by y_λ; its rank is d_λ^2. The minimal central idempotents are the $z_\lambda = \sum_{\lambda(t)=\lambda} \alpha_t\beta_t$.

The minimal idempotents y_λ and the path idempotents p_t satisfy the following branching formula.

Theorem 4.14 (Branching formula)**.**

$$p_t \otimes \mathbb{1}_1 = \sum_{\tau'=t} p_\tau \ ,$$

$$y_\lambda \otimes \mathbb{1}_1 = \sum_{\substack{\lambda\subset\mu \\ |\mu|=|\lambda|+1}} (y_\lambda \otimes \mathbb{1}_1) y_\mu (y_\lambda \otimes \mathbb{1}_1) \ ,$$

We have omitted in these formulas the standard isomorphisms respectively between $\square_\lambda \otimes 1$ and $\square_\mu$, and between $(n-1)\otimes 1$ and n.

Proof. The first formula is deduced from Theorem 4.13.
Let $\zeta \in H(|\lambda|, \square_\lambda)$ be an isomorphism. We have

$$\begin{aligned} y_\lambda \otimes 1_1 &= \sum_\mu \left((y_\lambda\zeta)\otimes 1_1\right) z_\mu \left((\zeta^{-1}y_\lambda)\otimes 1_1\right) \\ &= \sum_\mu \sum_{\lambda(t)=\mu} \left((y_\lambda\zeta)\otimes 1_1\right) p_t \left((\zeta^{-1}y_\lambda)\otimes 1_1\right) \\ &= \sum_\mu \sum_{\lambda(t)=\mu} \left((y_\lambda \dots y_{\lambda(t')})\otimes 1_1\right) y_\mu \left((y_{\lambda(t')}\dots y_\lambda)\otimes 1_1\right) \end{aligned}$$

In the above, only those tableaux t with $\lambda(t') = \lambda$ contribute, and all these contributions are proportional to $(y_\lambda \otimes 1_1)y_\mu(y_\lambda \otimes 1_1)$. Hence we obtain coefficients ξ_μ such that

$$y_\lambda \otimes 1_1 = \sum_{\substack{\lambda\subset\mu \\ |\mu|=|\lambda|+1}} \xi_\mu (y_\lambda \otimes 1_1) y_\mu (y_\lambda \otimes 1_1) \ .$$

For each diagram ν such that $\lambda \subset \nu$ and $|\nu| = |\lambda| + 1$, we get $\xi_\nu = 1$ by writing $y_\nu(y_\lambda \otimes 1_1)$. □

Proposition 4.15 (Quantum dimension)**.**

$$\langle y_\lambda \rangle = \prod_{\text{cells}} \frac{v^{-1} s^{cn(c)} - v s^{-cn(c)}}{s^{hl(c)} - s^{-hl(c)}}$$

Recall that $\langle y_\lambda \rangle$ is the Homflypt polynomial of a 0-framed unknot cabled with the closure $\hat{y}_\lambda$ of y_λ in $D^2 \times S^1$.
We will denote $\langle y_\lambda \rangle$ simply by $\langle \lambda \rangle$, and call it the quantum dimension of λ.

Here is a sketchy proof. We first check the formula for columns 1^n, by using the recursive formula for the antisymetrizers y_{1^n}. We note [18] that the right hand side in theorem 4.15 is the Schur polynomial in the $\langle y_{1^n} \rangle$. We then proceed recursively on the number of cells. If λ contains two distinct sub-diagrams μ and ν with $|\mu| = |\nu| = |\lambda| - 1$, then we get the result by considering $(y_\mu \otimes \mathbb{1}_1)(y_\nu \otimes \mathbb{1}_1)$ (considered as a composition of morphisms in $H_{\square_\lambda}$). We obtain the remaining cases by using the branching formula.

4.5. **Invariants of colored links.** Suppose $\mathcal{C}$ is a set of Young diagrams λ, such that the Young idempotents y_λ exist (in the generic case, $\mathcal{C}$ may contain all Young diagrams). Denote by $\widehat{y}_\lambda$ the closure of y_λ in the skein module of the solid torus $D^2 \times S^1$. We can use these closures of idempotents for cabling. We define the invariant of the link $L_1, \ldots, L_m$ colored with the Young diagrams $\lambda^{(1)}, \ldots, \lambda^{(m)}$ using this cabling procedure:

$$\langle (L_1, \lambda^{(1)}), \ldots, (L_m, \lambda^{(m)}) \rangle = \langle L_1(\widehat{y}_{\lambda^{(1)}}), \ldots, L_m(\widehat{y}_{\lambda^{(m)}}) \rangle \ .$$

We obtain in this way the Jones polynomial of the link $L_1, \ldots, L_m$, whose components are colored with the irreducible representations of the quantum group $U_q sl(N)$ of highest weights $\lambda^{(1)}, \ldots, \lambda^{(m)}$. This will be briefly explained in the next section.

5. Relation with representation theory of quantum groups

We saw in section 1 that $R \in End(V \otimes V)$ satisfies the Yang-Baxter equation. This can be explained by the fact that V is a module over the quantum group $U_q sl(n)$. We give an outline of this approach. We refer mainly to [14]. See also [13, 31, 9, 26].

5.1. **Hopf algebras.** We suppose here that k is a field, and A is an algebra over k, with unit 1_A.

Definition. a) A bialgebra (A, Δ, ϵ) is an algebra A equipped with two algebra morphisms

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to k$$

such that Δ (the comultiplication) is coassociative, and ϵ (the augmentation) is a counit.
b) A Hopf algebra (A, Δ, ϵ, S) is a bialgebra with an antipode S, i.e. an endomorphism S of A such that

$$\forall x \in A \quad (m \circ (S \otimes Id_A) \circ \Delta)(x) = (m \circ (Id_A \otimes S) \circ \Delta)(x) = \epsilon(x) 1_A \ .$$

As a fundamental example, we have the universal enveloping algebra $U_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$.

Consider the category of finite dimensional representations of a Hopf algebra, denoted by $Rep(A)$. This category has a tensor product, defined by using the coproduct on A, making $Rep(A)$ into a linear monoidal category. Morever, by using the antipode, we can provide $Rep(A)$ with a duality. In order to construct a functor from the tangle category to the category $Rep(A)$, we need, on the category $Rep(A)$ of finite dimensional representations of A, a braiding and a twist, compatible with the duality, we will then obtain a *ribbon category*.

Definition. A ribbon Hopf algebra is a Hopf algebra A, with a universal R-matrix and a universal twist v.

This means that R is an invertible element in $A \otimes A$, and v is an invertible element in the center of A, such that

$$\forall a \in A \ \Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1} \tag{4}$$
$$(Id_A \otimes \Delta)(R) = R_{13}R_{12} \tag{5}$$
$$(\Delta \otimes Id_A)(R) = R_{13}R_{23} \tag{6}$$
$$\Delta(v) = P_A(R)R(v \otimes v) \tag{7}$$

Here $P_A : A \otimes A \to A \otimes A$ is the flip $a \otimes b \mapsto b \otimes a$, and $\Delta^{\mathrm{op}}(a) = P_A\Delta(a)$;

$$R_{12} = R \otimes \mathbb{1}_A,\ R_{23} = \mathbb{1}_A \otimes R,\ R_{13} = (Id_A \otimes P_A)(R_{12})$$

Definition. A k-linear ribbon category is a monoidal category $\mathcal{M}$ with duality $(V, b_V, e_V)_{V \in Obj(\mathcal{M})}$, braiding and twist compatible.

Here monoidal means that $\mathcal{M}$ is equipped with a tensor product $\otimes$, i.e. a functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, associative and with a unit object $\mathbf{T}$.
The associativity and the definition of a unit object may use (coherent) isomorphisms (see [19]). These isomorphisms are often omitted. If these isomorphisms are equalities, then we have a *strict* monoidal category.
k-linear means that the Hom sets are k-module, composition and tensor product are bilinear.
The duality associate to each object V a dual object V^*, a coevaluation b_V from $\mathbb{T}$ to $V \otimes V^*$, and an evaluation e_V from $V^* \otimes V$ to $\mathbb{T}$ such that

$$(\mathbb{1}_V \otimes e_V)(b_V \otimes \mathbb{1}_V) = \mathbb{1}_V , \tag{8}$$
$$(e_V \otimes \mathbb{1}_{V^*})(\mathbb{1}_{V^*} \otimes b_V) = \mathbb{1}_{V^*} . \tag{9}$$

A braiding is a natural family of isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

such that for any three objects U, V, W, we have

$$c_{U,V\otimes W} = (\mathbb{1}_V \otimes c_{U,W})(c_{U,V} \otimes \mathbb{1}_W) , \tag{10}$$
$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathbb{1}_V)(\mathbb{1}_U \otimes c_{V,W}) . \tag{11}$$

A twist is a natural family of isomorphisms

$$\theta_V : V \to V$$

such that

$$\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W) \tag{12}$$

There is one compatibility condition with the duality.

$$(\theta_V \otimes \mathbb{1}_{V^*})b_V \quad = \quad (\mathbb{1}_V \otimes \theta_{V^*})b_V \tag{13}$$

Theorem 5.1. *If A is a ribbon Hopf algebra, then $Rep(A)$ is a ribbon category.*

If we are given a ribbon category $\mathcal{M}$, then there exists an invariant of links colored with objects of $\mathcal{M}$.

5.2. **The quantum groups $U_q sl(N)$.** The universal envelopping algebra $Usl(N)$ admits a deformation $U_q sl(n)$ which is almost a ribbon Hopf algebra. Here the *almost* means that the universal R operator lives in a suitable completion. This is enought to defines invariants of links colored with representations of the quantum group.

If q is a generic parameter, the representation theory of $U_q sl(n)$ is very closed to the classical case, and irreducible representations are indexed by highest weights and correspond bijectively with pairs (λ, ϵ), where λ is a Young diagram, and $\epsilon \in (\mathbb{Z}/2)^N$.

The invariant of a link whose components are colored with the deformation of the fundamental representation (highest weight $(1,0)$), is equal to the specialization $\langle\ \rangle_N$ of the Homfly polynomial. It can be shown that the invariant obtained by coloring the link $L_1, \dots, L_m$ with the irreducible representations of highest weights $(\lambda^{(1)}, O), \dots, (\lambda^{(m)}, 0)$, is equal to our colored invariant

$$\langle (L_1, \lambda^{(1)}), \dots, (L_m, \lambda^{(m)}) \rangle \ .$$

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