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Author(s): Robert Connelly

Source: *Proceedings of the American Mathematical Society*, Jan., 1971, Vol. 27, No. 1 (Jan., 1971), pp. 180-182

Published by: American Mathematical Society

Stable URL: https://www.jstor.org/stable/2037284



PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, January 1971

# A NEW PROOF OF BROWN'S COLLARING THEOREM

ROBERT CONNELLY

The aim of this note is to give a new proof that if a subspace $B$, compact for convenience, is locally collared in a space $X$, then it is collared. The idea of the proof is simply to add a collar $B\times I$ to $X$ to get $X^+$ and then to construct a homeomorphism of $X$ with $X^+$ by pushing $B$ down on one collared open set at a time.

The theorem, of course, is essentially that of [1]. However, the proof easily works in the piecewise linear (PL) category (i.e. all maps are PL and spaces are polyhedra), and when $B$, the boundary, is a pair or flag, cf. [3]. At the end of the paper we shall note briefly how the noncompact case and the PL case can be handled by our techniques.

A closed subspace $B\subset X$ is *locally collared* if $B$ is covered by sets $U$, open in $B$, such that for each $U$ there is a closed embedding $h:\overline{U}\times[0, 1]\to X$ such that $h^{-1}(B)=\overline{U}\times\{0\}$, $h(x, 0)=x$ for $x\in\overline{U}$, and $h(U\times[0, 1))$ is open in $X$. For metric spaces this is equivalent to the definition in [1]. $B$ is said to be *collared* if one $U$ can be taken to be all of $B$.

THEOREM. *If $B\subset X$ is compact and locally collared in $X$, which is Hausdorff, then $B$ is collared in $X$.*

PROOF. Let $U_1, U_2, \cdots, U_n$ be an open cover of $B$ such that each $U_i$ is as in the definition. By the normality of $B$, shrink the cover to find another cover $V_1, \cdots, V_n$ such that $\overline{V}_i\subset U_i$, $i=1, \cdots, n$. Let $X^+=X\cup B\times[-1, 0]$ where $(x, 0)$ is identified to $x$, and let $h_i:\overline{U}_i\times[0, 1]\to X$ be the embeddings given by the local collars. Let $H_i:\overline{U}_i\times[-1, 1]\to X^+$, $i=1, \cdots, n$, be the embedding defined by

$$\begin{aligned} H_i(x) &= h_i(x) \quad \text{for } x\in\overline{U}_i\times[0, 1], \\ &= x \quad\quad \text{for } x\in\overline{U}_i\times[-1, 0]. \end{aligned}$$

Inductively we shall define maps $f_i:B\to[-1, 0]$ and embeddings $g_i:X\to X^+$, $i=0, 1, \cdots, n$, such that

Received by the editors May 22, 1969.
*AMS subject classifications.* Primary 5701.

(a) $f_i(x) = -1$ if $x \in \bigcup_{j \leq i} \overline{V}_j$,
(b) $g_i(x) = (x, f_i(x))$ if $x \in B$, and
(c) $g_i(X) = X \cup \{(x, t) \mid t \geq f_i(x)\}$.

Note that since the $V_i$'s cover $B$, $g_n(X) = X^+$ and thus $g_n^{-1}$ will give the required collar.

Define $g_0 = 1$, and inductively suppose $g_{i-1}$ has been defined. Let $\phi_i : H_i^{-1} g_{i-1}(X) \to \overline{U}_i \times [-1, 1]$ be an embedding that pushes down along fibers such that $\phi_i H_i^{-1} g_{i-1}(\overline{V}_i) = \overline{V}_i \times \{-1\}$ and $\phi_i \mid (\overline{U}_i - U_i) \times [-1, 1] \cup \overline{U}_i \times \{1\} = 1$. Such a $\phi_i$ can be defined as follows: Let $\lambda_i : \overline{U}_i \to [0, 1]$ be a Urysohn function which is 0 on $\overline{U}_i - U_i$ and 1 on $\overline{V}_i$. Let $s_x : [f_{i-1}(x), 1] \to [(1 - \lambda_i(x)) f_{i-1}(x) + \lambda_i(x)(-1), 1]$ be the unique order preserving simplicial homeomorphism given by $s_x(t) = ((b-1)/(a-1))(t-1) + 1$ where $a = f_{i-1}(x)$ and $b = (1 - \lambda_i(x)) f_{i-1}(x) + \lambda_i(x)(-1)$. Now define $\phi_i(x, t) = (x, s_x(t))$. Clearly $\phi_i$ is continuous. Then define $\Phi_i : g_{i-1}(X) \to X^+$ by:

$$\begin{aligned} \Phi_i(x) &= H_i \phi_i H_i^{-1}(x) && \text{for } x \in g_{i-1}(X) \cap H_i(\overline{U} \times [-1, 1]), \\ &= x && \text{otherwise,} \end{aligned}$$

and $g_i = \Phi_i g_{i-1}$. Clearly $\Phi_i$ and thus $g_i$ is well defined and an embedding since $\phi_i \mid (\overline{U}_i - U_i) \times [-1, 1] \cup \overline{U}_i \times \{1\} = 1$, $\phi_i$ is an embedding (since each $s_x$ is), and $g_{i-1}(X) \cap H_i(\overline{U}_i \times [-1, 1]) = H_i(\overline{U}_i \times [0, 1]) \cup \{(x, t) \mid t \geq f_{i-1}(x)$ and $x \in \overline{U}_i\}$ by (c) for $g_{i-1}$. Note that (b) now defines $f_i(x)$, and (a) and (c) are satisfied by construction.

REMARK 1. *The noncompact case.* The method of proof used above can be extended to certain cases when $X$ is not compact. For instance, the proof works if we assume that $X$ has a slightly stronger property than paracompactness, namely if every open cover has a star finite refinement (cf. [2] for definitions). In this case it is possible to order the $U_i$'s, although infinite, so that every point in $X^+$ has an open neighborhood which moves only finitely often.

REMARK 2. *The* PL *case.* The theorem is still true if all spaces and maps mentioned (including the definition of local collaring) are polyhedra and PL respectively. The same proof goes through except that the particular definition of $\phi_i$ must be altered slightly. Namely to make $\phi_i$ PL it is easiest to triangulate $\overline{U}_i \times [-1, 1]$ so that $\overline{V}_i \times [-1, 1]$ and $H_i^{-1} g_{i-1}(X)$ are subcomplexes and projection $\pi : \overline{U}_i \times [-1, 1] \to \overline{U}_i$ is simplicial. Then it is easy to define a simplicial map $\phi_i$ so that it has the desired properties.

The author wishes to thank the referee for many useful comments and suggestions.

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Cornell University, Ithaca, New York 14850

University of Michigan, Ann Arbor, Michigan 48104