Homology of configurations in ribbon graphs

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In honor of the 80th birthday of Louis H. Kauffman

References

- C. Blanchet, Martin Palmer, Awais Shaukat, *Heisenberg homology on surface configurations* arXiv:2109.00515 .
- C. Blanchet, *Heisenberg homology of ribbon graphs*, Essays in Geometry: Dedicated to Norbert A'Campo , ed. A. Papadopoulos, European Mathematical Society Publishing House, Berlin, 2023.

Quantum invariants versus classical

- From algebraic topology: classical invariants, e.g. Euler characteristic, homology, torsion, Alexander polynomial.
- Quantum constructions: quantum invariants, e.g Jones polynomial, colored Jones, Reshetikhin-Turaev invariants, TQFTs, link homologies.
- Lou Kauffman: States models, *Quantum Topology*.
- Quantum Topology from classical algebraic topology ?

Lawrence representations of braid groups

- $\mathbf{B}_m = \pi_1(\mathrm{Conf}_m(D^2)) \simeq \mathrm{Mod}(D^2_m)$.
- Ruth Lawrence (1990): Family of representations

$$L_n : \mathbf{B}_m \to GL(H_n(\widetilde{\mathrm{Conf}}_n(D^2_m)),\ n \geq 2\ .$$

  $\widetilde{\mathrm{Conf}}_n$ is a $\mathbb{Z}^2$-cover of the unordered configuration space $\mathrm{Conf}_n(D^2_m)$ of n distinct points in the m-punctured disc.
- Theorem (Bigelow, Krammer, 2001-2002)
  Braids group are linear: $\mathbf{B}_m \hookrightarrow GL(\mathbb{R}^{\frac{m(m-1)}{2}})$.
- Bigelow proof: L_2 is **faithful**.
- Kohno, Jackson-Kerler: Lawrence (LKB) representations are equivalent to $sl(2)$ **quantum** representations on heighest weight spaces.

Surface configurations

- Goal: LKB type representation for $\mathrm{Mod}(\Sigma)$, from homology groups on a regular cover of the configuration space $\mathrm{Conf}_n(\Sigma)$, for $\Sigma = \Sigma_{g,m}$ an oriented genus g surface with m boundary components, $g, m > 0$.

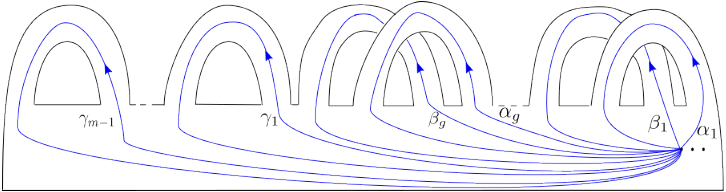


Figure: Model for $\Sigma_{g,m}$.

Paolo Bellingeri presentation (2002) (after G. P. Scott (1970), J. Gonzales-Meneses (2001))

The braid group $\mathbf{B}_n(\Sigma) = \pi_1(\mathrm{Conf}_n(\Sigma))$ has generators $\alpha_1, \dots, \alpha_g$, $\beta_1, \dots, \beta_g$, $\gamma_1, \dots, \gamma_{m-1}$ together with the classical generators $\sigma_1, \dots, \sigma_{n-1}$, and relations:

$$\begin{cases} \textbf{(BR1)} \ [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geq 2, \\ \textbf{(BR2)} \ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ \textbf{(CR1)} \ [\zeta, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } \zeta \text{ among the } \alpha_r, \beta_s, \gamma_t, \\ \textbf{(CR2)} \ [\zeta, \sigma_1\zeta\sigma_1] = 1 & \text{for all } \zeta \text{ among the } \alpha_r, \beta_s, \gamma_t, \\ \textbf{(CR3)} \ [\zeta, \sigma_1^{-1}\eta\sigma_1] = 1 & \text{for all } \zeta \neq \eta \text{ among the } \alpha_r, \beta_s, \gamma_t, \text{ with} \\ & \{\zeta, \eta\} \neq \{\alpha_r, \beta_r\}, \zeta \text{ on the left of } \eta, \\ \textbf{(SCR)} \ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases}$$

Composition of loops is written from right to left.

Quotients of surface braid groups

- Connected regular covers are associated with quotient maps $\mathbf{B}_n(\Sigma) \twoheadrightarrow G$.
- Trivial quotient: Action of $\mathrm{Mod}(\Sigma)$ on $H_*(\mathrm{Conf}_n(\Sigma), \mathbb{Z}))$ recovers Johnson filtration (Moriyama 2007).
- Abelianisation: $\mathbf{B}_n(\Sigma)^{\mathrm{abelian}} = \mathbb{Z}/2\mathbb{Z} \times H_1(\Sigma, \mathbb{Z})$.
- Heisenberg group quotient:

Theorem (B-Palmer-Shaukat)

For $n \geq 2$,

$$\mathbf{B}_n(\Sigma)/[\sigma_1, \mathbf{B}_n(\Sigma)]^{\mathrm{normal}} \approx \mathcal{H}(\Sigma) = \mathbb{Z} \times H_1(\Sigma, \mathbb{Z})$$

Heisenberg group product $(k, x)(l, y) = (k + l + x \cdot y, x + y)$.

- Bellingeri-Gervais-Ghaschi (2008) identified the second nilpotent quotient for $n \geq 3$.

$$\mathbf{B}_n(\Sigma)/\Gamma_3(\mathbf{B}_n(\mathrm{Conf}_n(\Sigma))) \approx \mathcal{H}(\Sigma)$$

Heisenberg homologies

- From the quotient map $\phi : \mathbf{B}_n(\Sigma) \to \mathcal{H}(\Sigma)$ we get the *Heisenberg cover* $\widetilde{\mathrm{Conf}}_n^{\mathcal{H}}(\Sigma))$.
- Action of a mapping class lifts to the Heisenberg cover, $\Rightarrow$ action on homology.
- From deck action, $H_*(\widetilde{\mathrm{Conf}}_n^{\mathcal{H}}(\Sigma))$ is a right $\mathbb{Z}[\mathcal{H}]$-module and action of a mapping class f is twisted over this action by an automorphism $f_{\mathcal{H}} : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma)$.
- Given a representation $\rho : \mathcal{H}(\Sigma) \to GL(V)$, we get $\rho \circ \phi : \mathbf{B}_n(\Sigma) \to GL(V)$ and we have homology with local coefficient in V, $H_*(\mathrm{Conf}_n(\Sigma), V)$.
- If the representation ρ is good, meaning that $\rho \circ f_{\mathcal{H}}$ is canonically (resp. canonically up to scalar) isomorphic to ρ, then we obtain a (resp. projective) representation of $\mathrm{Mod}(\Sigma)$.

The linearised translation action

- The left regular action $\mathfrak{l}_{(k,x)} : \mathcal{H}(\Sigma) = \mathbb{Z} \times H_1(\Sigma, \mathbb{Z}) \to \mathcal{H}(\Sigma)$,
  $\mathfrak{l}_{(k_0,x_0)} : (k,x) \mapsto (k + k_0 + x_0.x, x + x_0)$ defines

an affine action $\mathfrak{l} : \mathcal{H}(\Sigma) \to GA(\mathcal{H}(\Sigma)) = GA(\mathbb{Z}^{2g+1})$,
a linearised action $\rho_{\mathfrak{l}} : \mathcal{H}(\Sigma) \to GL(L = \mathbb{Z} \oplus \mathcal{H}(\Sigma)) = GL(\mathbb{Z}^{2g+2})$,
$\rho_{\mathfrak{l}}(k_0, x_0) : (\nu, (k,x)) \mapsto (\nu, (k + \nu k_0 + x_0.x, x + \nu x_0))$.

- $\rho_{\mathfrak{l}}$ is good, i. e. we obtain a native (untwisted) homological action

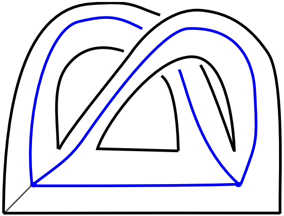
$$\mathrm{Mod}(\Sigma) \to GL(H_*(\mathrm{Conf}_n(\Sigma), L)) \ .$$

Recovering quantum representations

- For q a p-th root of 1, p odd, the modulo p Heisenberg group $\mathcal{H}_p = \mathbb{Z}/p \times H_1(\Sigma, \mathbb{Z}/p\mathbb{Z})$ has an irreducible unitary representation W_q where $u = (1,0)$ acts by q, which is unique up to scalar.
- W_q is a finite dimensional version of the famous Schrödinger representation.
- Using W_q for $\Sigma_{g,1}$, Marco de Renzi and Jules Martel obtained
  - an action of quantum $sl(2)$ on a sum of homologies,
  - inside this sum a subrepresentaion isomorphic to $\left(\mathcal{U}_q(sl2)^{\mathrm{adjoint}}\right)^{\otimes g}$ with a projective mapping class group action isomorphic to Kerler-Lyubashenko non semisimple TQFT representation.

Ribbon graphs

- A ribbon structure on a finite graph Γ is a cyclic ordering of the adjacent edges at each vertex.
- Ribbon graph Σ_Γ: compact surface containing Γ as a strong deformation retract, obtained by gluing bands $[-1,1] \times e$ according to the ribbon structure.



Configurations

- For $n \geq 2$,

$$\mathrm{Conf}_n(\Gamma) \subset \mathrm{Conf}_n(\Sigma_\Gamma)$$

$$\mathbf{B}_n(\Gamma) = \pi_1(\mathrm{Conf}_n(\Gamma)) \to \mathbf{B}_n(\Sigma_\Gamma)$$

- Goal: compute $H_*(\mathrm{Conf}_n(\Sigma_\Gamma), V)$ from the graph Γ + ribbon structure.
- A local system on $\mathrm{Conf}_n(\Sigma_\Gamma)$ induces a local system on the subspace $\mathrm{Conf}_n(\Gamma)$.

The compression trick

- Borel-Moore homology + cohomology with compact support.

$$H_n^{BM}(\mathrm{Conf}_n(\Sigma); V) = \varprojlim_T H_n(\mathrm{Conf}_n(\Sigma), \mathrm{Conf}_n(\Sigma) \setminus T; V),$$

$$H_c^*(\mathrm{Conf}_n(\Sigma); V) = \varinjlim_T H^*(\mathrm{Conf}_n(\Sigma), \mathrm{Conf}_n(\Sigma) \setminus T; V),$$

  the limit is taken over all compact subsets $T \subset \mathrm{Conf}_n(\Sigma)$.
- Following a so called *compression trick* first used by Stephen Bigelow (2004) we can prove the following.

Theorem

The inclusion $\mathrm{Conf}_n(\Gamma) \subset \mathrm{Conf}_n(\Sigma_\Gamma)$ induces for a representation $\rho : \mathcal{H}(\Sigma_\Gamma) \to GL(V)$,
a) an isomorphism on Borel-Moore homology

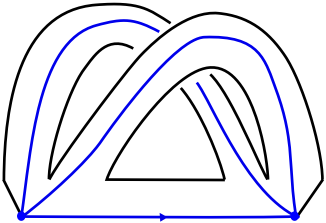
$$H_*^{BM}(\mathrm{Conf}_n(\Gamma); V) \cong H_*^{BM}(\mathrm{Conf}_n(\Sigma_\Gamma); V)\ ,$$

b) an isomorphism on cohomology with compact support

$$H_c^*(\mathrm{Conf}_n(\Sigma_\Gamma); V) \cong H_c^*(\mathrm{Conf}_n(\Gamma); V)\ .$$

Relative ribbon graphs

- Relative ribbon structure:
  a finite graph Γ,
  a subgraph whose components A_i are oriented curves,
  a compatible ordering of the adjacent edges at each vertex: outgoing A_i is first, ingoing A_i is last.
- There is a regular thickening $\Sigma_{(\Gamma,A)}$ of a graph with relative ribbon structure. For an edge e in A we take a *half band* $[0,1] \times e$.



Compression trick, relative case

- $\mathrm{Conf}_n(X, Y) \subset \mathrm{Conf}_n(X)$ denotes the space of configurations in X with at least one point in Y.

Theorem

$$H_*^{BM}(\mathrm{Conf}_n(\Gamma), \mathrm{Conf}_n(\Gamma, A); V) \cong H_*^{BM}(\mathrm{Conf}_n(\Sigma_{(\Gamma,A)}), \mathrm{Conf}_n(\Sigma_{(\Gamma,A)}, A); V) .$$

$$H_c^*(\mathrm{Conf}_n(\Sigma_{(\Gamma,A)}), \mathrm{Conf}_n(\Sigma_{(\Gamma,A)}, A); V) \cong H_c^*(\mathrm{Conf}_n(\Gamma), \mathrm{Conf}_n(\Gamma, A), V) .$$

About computation

- $H_*(\mathrm{Conf}_n(\Gamma), V)$ can be obtain from a combinatorial model if Γ is enough subdivided (A. D. Abrams, 2000).
- For Borel-Moore homology, we may work with a finite open cell decomposition which does not confer a CW-complex structure, but nevertherless gives a filtration which works similarly for computing Borel-Moore homology.
- Case $n = 2$, the 0-cells are pairs of vertices (we assume at least 2 vertices), the 1-cells are $v \times e$, v a vertex, e an edge, the 2-cells are either squares $e \times f$, $e \neq f$, or simplices $\mathrm{Conf}_2(e)$.
- The cells are attached along the faces which are not diagonal. Each possibly non compact closed k-cell is properly embedded in $\mathrm{Conf}_n^k(\Gamma)$ and will contribute to a V summand in the homology $H_k^{BM}(\mathrm{Conf}_n^k(\Gamma), \mathrm{Conf}_n^{k-1}(\Gamma), V)$.
- Relative case as well.

Easy case

Theorem

Let $\Sigma = \Sigma_{g,m}$ be a genus g surface with $m > 0$ boundary components, and $A \subset \partial\Sigma_{g,m}$ be an oriented interval. Let $n \geq 2$ and let V be a representation of the surface braid group $\mathbf{B}_n(\Sigma)$ over a ring R.

The Borel-Moore homology module $H_n^{BM}(\mathrm{Conf}_n(\Sigma_{g,m})), \mathrm{Conf}_n(\Sigma_{g,m}, A); V)$ is isomorphic to the direct sum of $\begin{pmatrix} 2g+m+n-2 \\ n \end{pmatrix}$ copies of V. Furthermore, $H_*^{BM}(\mathrm{Conf}_n(\Sigma_{g,m}), \mathrm{Conf}_n(\Sigma_{g,m}, A); V)$ vanishes for $* \neq n$.

Poincaré duality

We suppose that the boundary of $\Sigma = \Sigma_{g,1}$ is decomposed as the union of two intervals, $\partial\Sigma = \partial^- \cup \partial^+$.

- We have isomorphisms

$$H_n^{BM}(\mathrm{Conf}(\Sigma), \mathrm{Conf}(\Sigma, \partial^-); V) \cong H^n(\mathrm{Conf}(\Sigma), \mathrm{Conf}(\Sigma, \partial^+); V)$$

$$H_n(\mathrm{Conf}(\Sigma), \mathrm{Conf}(\Sigma, \partial^+); V) \cong H_c^n(\mathrm{Conf}(\Sigma), \mathrm{Conf}(\Sigma, \partial^-); V)$$

- Suppose that the R-module V is equipped with a non degenerate hermitian pairing, then there exists a sesquilinear pairing
  $H_n^{BM}(\mathrm{Conf}(\Sigma), \mathrm{Conf}(\Sigma, \partial^-); V) \otimes H_n(\mathrm{Conf}(\Sigma), \mathrm{Conf}(\Sigma, \partial^+); V) \to R$,
  which is non degenerate.

Problems

1. More computations.
2. Unitarity.
3. TQFT.
4. Kernel of MCG action.