Colored Jones invariants involving $SL_2(\mathbb{C})$-connections

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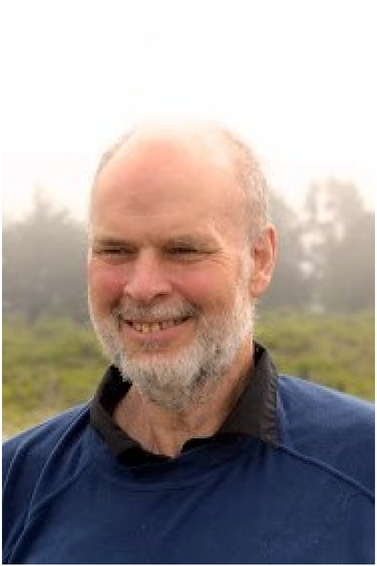
Link invariants, $sl(2)$ family

- V. Jones 1984, Jones polynomial.
- V. Jones 1989, N. Reshetikhin and V. Turaev 1990, Colored Jones polynomial.
  Components are colored with representations of generic quantum $sl(2)$ (integral weights).
  Vector representation recovers Jones polynomial.
- Agutsu-Deguchi-Ohtsuki 1992, J. Murakami, N. Geer and B. Patureau, Renormalised/modified trace invariants from quantum $sl(2)$ at root of 1 (colored Alexander invariants).
  They are based on nilpotent representations with complex highest weights.
  $U(1)$ flat connections are implicite.
  At 4-th root of 1 it recovers Alexander polynomial.



Motivation and context

- The colored Jones polynomial invariant of links uses irreducible representations of quantum $sl(2)$ with integral weights.
- Quantum $sl(2)$ at root of unity has much more irreducible representations.
- At $q^{2p} = 1$, there exists a map

$$Irrep(U_q(sl(2)) \longrightarrow SL_2(\mathbb{C}) ,$$

  which is a p-fold covering on a dense open subset.
- Purpose: extended colored Jones invariants involving $SL_2(\mathbb{C})$ gauge theory.

Main result

Theorem (BGPR)

For each $p \geq 2$, there exists an invariant of gauge classes of generic $SL_2(\mathbb{C})$ flat connections on link complements, enhanced for each component of the link with a root of the degree p equation

$$\mathcal{T}_p(x) = -trace(holonomy\ on\ meridian),$$

where $\mathcal{T}_p(x)$ is the renormalized p-th Chebyshev polynomial determined by $\mathcal{T}_p(2\cos\theta) = 2\cos(p\theta)$.

- This invariant extends the colored Alexander polynomial.
- Up to now this invariant is defined up to a p^2-th root of 1.
- Calvin McPhail-Snyder: For $p = 2$ it recovers $SL_2(\mathbb{C})$ torsion.

References

- Our paper: B-Geer-Patureau-Reshetikhin, *Holonomy braidings, biquandles and quantum invariants of links with SL2(C) flat connections*, Selecta Mathematatica volume 26, (2020) , arXiv:1806.02787.
- Based on: R. Kashaev, N. Reshetikhin - *Braiding for quantum gl2 at roots of unity.* Noncommutative geometry and representation theory in mathematical physics, 183–197, Contemp. Math., 391, Amer. Math. Soc., Providence, RI, 2005.
- Case $p = 2$: C. McPhail-Snyder, *Holonomy invariants of links and non abelian Reidemeister torsion*, arXiv:2005.01133.

Quantum $sl(2)$

- Let $q = e^{\frac{i\pi}{p}}$, $p \geq 2$.
  The $\mathbb{C}$-algebra $\mathcal{U}_q$ is defined by generators E, F, K, K^{-1} and relations $KK^{-1} = K^{-1}K = 1$ and

$$KEK^{-1} = q^2 E \ , \ KFK^{-1} = q^{-2} F \ , \ [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \ .$$

- Hopf algebra structure

$$\begin{aligned}
\Delta(E) &= 1 \otimes E + E \otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\
\Delta(K) &= K \otimes K & \varepsilon(K) &= 1, & S(K) &= K^{-1}, \\
\Delta(K^{-1}) &= K^{-1} \otimes K^{-1} & \varepsilon(K^{-1}) &= 1, & S(K^{-1}) &= K.
\end{aligned}$$

The center

- K^p, E^p, F^p are central.
- They generate a Hopf subalgebra Z_0.
- The center Z is a degree p extension of Z_0, generated by the Casimir element

$$C = (q - q^{-1})^2 FE + Kq + K^{-1}q^{-1},$$

with relation

$$\mathcal{T}_p(C) = (q - q^{-1})^{2p} E^p F^p - (K^p + K^{-p})$$

where $\mathcal{T}_p$ is the renormalized p-th Chebyshev polynomial determined by $\mathcal{T}_p(2\cos\theta) = 2\cos(p\theta)$.

Irreducible representations

- An irreducible representation induces a character Ξ on Z, and its restriction χ on Z_0.
- A character χ on Z_0 is generic if and only if the Casimir equation below has simple roots.

$$\mathcal{T}_p(c) = (q - q^{-1})^{2p}\chi(E^p F^p) - \chi(K^p + K^{-p}).$$

  (Critical values of $\mathcal{T}_p(x)$ are ± 2.)
- A generic character χ on Z_0 is realized by p non isomorphic irreducible representations which are p-dimensional: $V_{(\chi,c)}$, with c a solution of the Casimir equation.

Irreducible representations

- Nilpotent case: $\chi(E^p) = \chi(F^p) = 0$. An irreducible representation is generated by a highest weight vector v_λ, $Ev_\lambda = 0$, $Kv_\lambda = q^\lambda v_\lambda = e^{\frac{i\pi\lambda}{p}} v_\lambda$.
- Case $\chi(E^p F^p) \neq 0$: cyclic representations, basis composed with weight vectors, E and F rotate with appropriate coefficients.

Subalgebra Z_0^+ generated by K^p, E^p

- Hopf algebra structure

$$\Delta(E^p) = 1 \otimes E^p + E^p \otimes K^p, \quad \varepsilon(E^p) = 0, \quad S(E^p) = -E^p K^{-p},$$
$$\Delta(K^p) = K^p \otimes K^p \qquad \varepsilon(K^p) = 1, \quad S(K^p) = K^{-p}$$

- Z_0^+ is isomorphic to the coordinate Hopf algebra on the group

$$G^u = \left\{ \begin{pmatrix} 1 & \epsilon \\ 0 & \kappa \end{pmatrix}, \kappa \in \mathbb{C}^*, \epsilon \in \mathbb{C} \right\} \subset GL_2(\mathbb{C})$$

- $\begin{pmatrix} 1 & \epsilon \\ 0 & \kappa \end{pmatrix} \begin{pmatrix} 1 & \epsilon \\ 0 & \kappa' \end{pmatrix} = \begin{pmatrix} 1 & \epsilon' + \epsilon\kappa' \\ 0 & \kappa\kappa' \end{pmatrix}$, hence coproduct δ:

$$\delta(\kappa) = \kappa \otimes \kappa \ , \ \delta(\epsilon) = 1 \otimes \epsilon + \epsilon \otimes \kappa \ .$$

Hopf subalgebra Z_0

- The Hopf algebra Z_0 is isomorphic to the coordinate Hopf algebra on the group (subgroup of $GL_2(\mathbb{C})^2$)

$$G^* = \left\{ M(\kappa, \epsilon, \phi) = \left(\begin{pmatrix} \kappa & 0 \\ \phi & 1 \end{pmatrix}, \begin{pmatrix} 1 & \epsilon \\ 0 & \kappa \end{pmatrix} \right) : \epsilon, \phi \in \mathbb{C}, \kappa \in \mathbb{C}^* \right\}$$

- We identify G^* with characters on Z_0 by

$$M(\kappa, \epsilon, \phi)(K^p) = \kappa, \quad M(\kappa, \epsilon, \phi)(E^p) = (q - q^{-1})^{-p}\epsilon,$$

$$M(\kappa, \epsilon, \phi)(F^p) = (q - q^{-1})^{-p}\phi\kappa^{-1}.$$

- $G^* = SL_2(\mathbb{C})^*$ is the Poisson-Lie dual of $SL_2(\mathbb{C})$.

Factorization map

- Given a character χ on Z_0 ($\chi \in G^*$), let

$$\phi_+(\chi) = \begin{pmatrix} \kappa & 0 \\ \phi & 1 \end{pmatrix} \text{ and } \phi_-(\chi) = \begin{pmatrix} 1 & \epsilon \\ 0 & \kappa \end{pmatrix}$$

- Let $\psi : G^* \to SL_2(\mathbb{C})$

$$\psi(\chi) = \phi_+(\chi)\left(\phi_-(\chi)\right)^{-1} = \begin{pmatrix} \kappa & -\epsilon \\ \phi & \dfrac{1}{\kappa} - \dfrac{\epsilon\phi}{\kappa} \end{pmatrix}.$$

- The map ψ is a bijection from G^* to the set of matrices $M = (m_{ij}) \in SL_2(\mathbb{C})$ such that $m_{11} \neq 0$.

Outer automorphism

- Kashaev-Reshetikhin: Outer algebra automorphism $\mathcal{R}$ of the division ring $\mathcal{Q}(U_q^{\otimes 2})$ (comes from conjugation by R-matrix on h-adic quantum $sl(2)$).
- The map $\mathcal{R}$ is given on $Z_0 \otimes Z_0$ by

$$\mathcal{R}(K^p \otimes 1) = (K^p \otimes 1)W, \quad \mathcal{R}(1 \otimes K^p) = (1 \otimes K^p)W^{-1},$$
$$\mathcal{R}(E^p \otimes 1) = E^p \otimes K^p, \quad \mathcal{R}(1 \otimes F^p) = K^{-p} \otimes F^p,$$
$$\mathcal{R}(1 \otimes E^p) = K^p \otimes E^p + (E^p \otimes 1)(1 - (1 \otimes K^{2p})W^{-1}),$$
$$\mathcal{R}(F^p \otimes 1) = F^p \otimes K^{-p} + (1 \otimes F^p)(1 - (K^{-2p} \otimes 1)W^{-1}).$$

where

$$W = 1 + (q - q^{-1})^{2p} K^{-p} E^p \otimes F^p K^p.$$

Outer automorphism

- Yang-Baxter relation: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.
- More relations

$$
\begin{aligned}
(\Delta \otimes 1)\mathcal{R}(u \otimes v) &= \mathcal{R}_{13}\mathcal{R}_{23}(\Delta(u) \otimes v),\\
(1 \otimes \Delta)\mathcal{R}(u \otimes v) &= \mathcal{R}_{13}\mathcal{R}_{12}(u \otimes \Delta(v)),\\
(\epsilon \otimes 1)\mathcal{R}(u \otimes v) &= \epsilon(u)v,\\
(1 \otimes \epsilon)\mathcal{R}(u \otimes v) &= \epsilon(v)u,\\
\mathcal{R}(\Delta(a)) &= (\tau \circ \Delta)(a)\ .
\end{aligned}
$$

Partial biquandle map

- From $\mathcal{R}$ we deduce a (partial) map on (generic) pairs of characters on Z_0:

$$B(\chi_1, \chi_2) = (\chi_4, \chi_3),$$

  so that $(\chi_3 \otimes \chi_4) \circ \mathcal{R} = \chi_1 \otimes \chi_2$.
- B is the transpose of the map

$$\mathcal{R}^{-1} \circ \tau : Z_0 \otimes Z_0 \to Z_0 \otimes Z_0 \ .$$

- B satisfies the set braiding relation (Yang-Baxter equation).
- (G^*, B) is a (generically defined) biquandle.

Definition

A *biquandle* is a set X with a bijective map $B = (B_1, B_2) : X \times X \to X \times X$ such that:

1. The map B satisfies the set Yang-Baxter equation

$$(\mathrm{id} \times B) \circ (B \times \mathrm{id}) \circ (\mathrm{id} \times B) = (B \times \mathrm{id}) \circ (\mathrm{id} \times B) \circ (B \times \mathrm{id}).$$

2. There exists a unique bijective map $S : X \times X \to X \times X$ such that

$$S(B_1(x,y),x) = (B_2(x,y),y)$$

for all $x, y \in X$.

3. The map S induces a bijection $\alpha : X \to X$ on the diagonal:

$$S(x,x) = (\alpha(x), \alpha(x))$$

for all $x \in X$.

Quandle

- A *quandle* is a set Q with a binary operation $(a, b) \to a \triangleright b$ such that
  1. for all $a, b, c \in Q$, $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$,
  2. for all $a, b \in Q$ there is a unique $c \in Q$ such that $a = b \triangleright c$,
  3. for any $a \in Q$, $a \triangleright a = a$.
- A group G with $a \triangleright b = a^{-1}ba$ is a quandle, any union of conjugacy classes is a subquandle.
- A biquandle (X, B) with $B_2(x_1, x_2) = x_1$ is a quandle with $a \triangleright b = B_1(a, b)$.

R-matrix over biquandle

Suppose that $B(\chi_1, \chi_2) = (\chi_4, \chi_3)$. Let c_1, c_2 be solutions of the Casimir equations associated with χ_1, χ_2.

- Then c_1, c_2 are also compatibles with χ_3, χ_4.
- There exists an isomorphism

$$R_{(\chi_1,c_1),(\chi_2,c_2)} : V_{(\chi_1,c_1)} \otimes V_{(\chi_2,c_2)} \to V_{(\chi_3,c_1)} \otimes V_{(\chi_4,c_2)}$$

  which is defined up to a p^2 root of 1.
- The isomorphisms

$$\tau \circ R_{(\chi_1,c_1),(\chi_2,c_2)} : V_{(\chi_1,c_1)} \otimes V_{(\chi_2,c_2)} \to V_{(\chi_4,c_2)} \otimes V_{(\chi_3,c_1)}$$

  are invariant (intertwinner) and satisfies the colored braid relation modulo a p^2-th root of 1.

Colored braid relation

Denote $\tau \circ R_{(\chi_1,c_1),(\chi_2,c_2)}$ by $\check{R}_{(\chi_1,c_1),(\chi_2,c_2)}$.
For any $\Xi_i = (\chi_i, c_i)$, $i = 1, 2, 3$, we have an equality up to a p^2-th root of 1 of isomorphisms

$$(\check{R}_{\diamond,\diamond} \otimes \mathrm{id}_\diamond) \circ (\mathrm{id}_\diamond \otimes \check{R}_{\diamond,\diamond}) \circ (\check{R}_{\Xi_1,\Xi_2} \otimes \mathrm{id}_{V_{\Xi_3}}),$$

$$(\mathrm{id}_\diamond \otimes c_{\diamond,\diamond}) \circ (c_{\diamond,\diamond} \otimes \mathrm{id}_\diamond) \circ (\mathrm{id}_{V_{\Xi_1}} \otimes \check{R}_{\Xi_2,\Xi_3}),$$

where the $\diamond$ objects are completed with the biquandle structure B.

Link diagrams

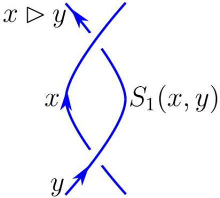
- Reshetikhin-Turaev type functor on tangles colored with irreducible representations.
- It vanishes on link diagrams, but a modified trace gives a non trivial evaluation.
- Evaluation is invariant by colored Reidemeister moves.
- Topological interpretation for the biquandle coloring ?

From biquandle to quandle

- From V. Lebed and L. Vendramin (2017).

  Let (X, B) be a biquandle. For $x, y \in X$ the operation $\triangleright$ given by

  $$x \triangleright y = B_1(x, S_1(x, y)) :$$



  defines a quandle structure on X.

- There is a dictionary between biquandle colorings and associated quandle colorings.

Recovering flat connections

- Recall that we have the factorization map $\psi : G^* \to SL_2(\mathbb{C})$, and a (partial) biquandle structure B on G^*.
- On G^*, the quandle associated with B is the pullback of the conjugacy quandle on $SL_2(\mathbb{C})$.
- G^* colorings $\longleftrightarrow$ generic representations $\pi_1(S^3 - L) \to SL_2(\mathbb{C})$.
- Casimir equation becomes $\mathcal{T}_p(x) = -\mathrm{trace}(\mathrm{holonomy})$.

Conclusion

Theorem (BGPR)

For each $p \geq 2$, there exists an invariant of gauge classes of $SL_2(\mathbb{C})$ flat connections on link complements with trace of holonomies on meridians $\neq \pm 2$, enhanced for each component of the link with a root of the degree p equation

$$\mathcal{T}_p(x) = -trace(holonomy\ on\ meridian),$$

where $\mathcal{T}_p(x)$ is the renormalized p-th Chebyshev polynomial determined by $\mathcal{T}_p(2\cos\theta) = 2\cos(p\theta)$. This invariant is a complex number up to p^2-th root of 1.