Heisenberg homology of surface configurations

Christian Blanchet

IMJ-PRG, Université de Paris

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Recent Developments in Link Homology

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Lawrence representations

- Lawrence (1990): Family of representations, $n \geq 2$,

$$L_n : B_m \to GL(H_n(\widetilde{\mathcal{C}}_n(D^2_m))$$

- $\widetilde{\mathcal{C}}_n$ is a $\mathbb{Z}^2$-cover of the unordered configuration space $\mathcal{C}_n(D^2_m)$ of n points in the m-punctured disc.
- Theorem (Bigelow, Krammer, 2001-2002): L_2 is faithful.
- Kohno: Lawrence (LKB) representations are equivalent to $sl(2)$ quantum representations on heighest weight spaces.

Homological representations of MCG

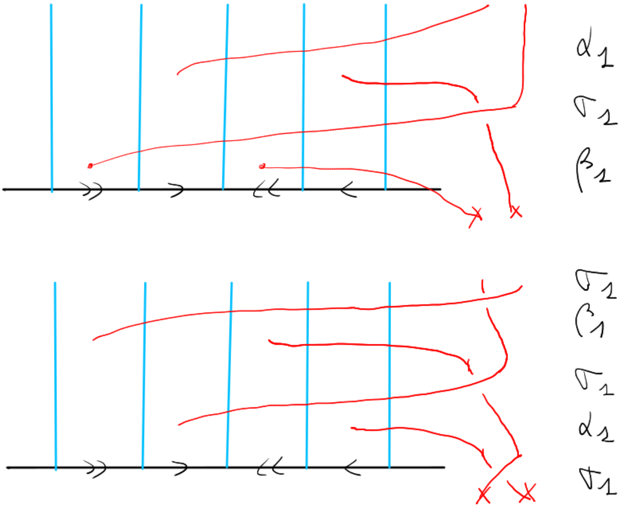
- $B_m = \mathfrak{M}(D^2_m)$ is a mapping class group.
- Goal: LKB type representations for $\mathfrak{M}(\Sigma = \Sigma_{g,1})$, $g \geq 2$.
- A Heisenberg local system on $\mathcal{C}_n(\Sigma)$ is obtained from a representation V of the Heisenberg group $\mathcal{H}(H_1(\Sigma, \mathbb{Z}))$, which will appear as a quotient of $B_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma))$.
- We obtain a twisted action of the MCG on $H_n(\mathcal{C}_n(\Sigma), V)$.
- For the Shrödinger representation ($L^2(\mathbb{R}^g)$ or $L^2(\mathbb{Z}_N^g)$) we obtain linear representations of the *stably universal* central extension of the MCG.
  In the case $L^2(\mathbb{Z}_N^g)$, the dimension is $\binom{2g+n-1}{n} N^g$.

Surface braid groups

- $B_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma))$.
- Bellingeri presentation, revisited by Bellingeri-Godelle: generators $\sigma_1, \ldots, \sigma_{n-1}$, $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and relations:

$$\begin{cases} \textbf{(BR1)}\ [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geq 2, \\ \textbf{(BR2)}\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ \textbf{(CR1)}\ [\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } r, \\ \textbf{(CR2)}\ [\alpha_r, \sigma_1\alpha_r\sigma_1] = [\beta_r, \sigma_1\beta_r\sigma_1] = 1 & \text{for all } r, \\ \textbf{(CR3)}\ [\alpha_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\alpha_r, \sigma_1^{-1}\beta_s\sigma_1] & \\ \qquad = [\beta_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\beta_r, \sigma_1^{-1}\beta_s\sigma_1] = 1 & \text{for all } r < s, \\ \textbf{(SCR)}\ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases}$$

SCR relation $\alpha_1\sigma_1\beta_1 = \sigma_1\beta_1\sigma_1\alpha_1\sigma_1$



Heisenberg group

- The Heisenberg group $\mathcal{H}(H)$ is the central extension of $H = H_1(\Sigma)$ defined with the intersection cocycle.
- $\mathcal{H}(H) = \mathbb{Z} \times H$ with $(k,x)(l,y) = (k+l+x.y, x+y)$.
- Theorem: $B_n(\Sigma)/(\sigma_1$ central) is isomorphic to the Heisenberg group $\mathcal{H}(H)$.
- The above isomorphism is not canonical.

Representations

- $\mathcal{H}$ can be realised as a group of matrices, which gives a faithful $(g+2)$-dimensional representation:

$$\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right) \longmapsto \begin{pmatrix} 1 & p & \frac{k+p\cdot q}{2} \\ 0 & I_g & q \\ 0 & 0 & 1 \end{pmatrix},$$

  where $p = (p_i)$ is a row vector and $q = (q_i)$ is a column vector
- Schrödinger representation on the Hilbert space $W \cong L^2(\mathbb{R}^g)$:

$$\left[\rho_W\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\hbar\frac{k-p\cdot q}{2}} e^{i\hbar p\cdot s}\psi(s-q).$$

- Schrödinger representation on the f.d. Hilbert space $W_N \cong L^2(\mathbb{Z}_N^g)$, N even: $\hbar = \frac{2\pi}{N}$

MCG action on Heisenberg group

- For $f = [g] \in \mathfrak{M}(\Sigma)$, the diffeomorphism $\mathcal{C}_n(g)$ induces an automorphism $g_{\mathcal{H}} = f_{\mathcal{H}} \in \mathrm{Aut}^+(\mathcal{H})$ (identity on center).
- $\mathrm{Aut}^+(\mathcal{H}) \simeq Sp(H) \ltimes H^*$ is the affine symplectic group.
- $f_{\mathcal{H}} = (k, x) \mapsto (k + \mathfrak{d}_f(x), f_*(x))$, with $\mathfrak{d}_f \in H^*$.
- $f \mapsto \mathfrak{d}_f$ is a crossed homomorphism, i.e.

$$\mathfrak{d}_{g \circ f}(x) = \mathfrak{d}_f(x) + f^*(\mathfrak{d}_g)(x) \ .$$

- The crossed homomorphism $\mathfrak{d}$ was already defined by Morita and generates $H^1(\mathfrak{M}(\Sigma), H^*) \cong \mathbb{Z}$.

Notation

- H_*^{BM} denotes the Borel-Moore homology,

$$H_n^{BM}(\mathcal{C}_n(\Sigma); V) = \varprojlim_{T} H_n(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma) \setminus T; V),$$

  the inverse limit is taken over all compact subsets $T \subset \mathcal{C}_n(\Sigma)$
- $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ is the properly embedded subspace of $\mathcal{C}_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^-\Sigma \subset \partial\Sigma$.
- Borel-Moore homology is functorial with respect to proper maps and for a proper embedding $B \subset A$, the relative homology $H_*^{BM}(A, B)$ is defined.
- For a representation $\rho : \mathcal{H} \to GL(V)$ and $\tau \in \mathrm{Aut}(\mathcal{H})$, the τ-twisted representation $\rho \circ \tau$ is denoted by ${}_{\tau}V$.

Local system from an Heisenberg group representation

- The (singular or cellular) chain complex of the Heisenberg group cover, denoted by $S_*(\widetilde{\mathcal{C}}_n(\Sigma))$, is a right $\mathbb{Z}[\mathcal{H}]$-module.
- Given a representation $\rho : \mathcal{H} \to GL(V)$, the corresponding local homology is that of the complex $S_*(\mathcal{C}_n(\Sigma), V) := S_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V$.
- For $f = [g] \in \mathfrak{M}(\Sigma)$, the map $\mathcal{C}_n(g)$ lifts to the Heisenberg cover and the lift $\widetilde{\mathcal{C}}_n(g)$ induces a chain map $S_*(\widetilde{\mathcal{C}}_n(g))$ which is twisted linear: $S_*(\widetilde{\mathcal{C}}_n(g))(z.h) = S_*(\widetilde{\mathcal{C}}_n(g))(z).f_{\mathcal{H}}(h)$.
- We get chain maps

$$S_*(\mathcal{C}_n(g), V) : S_*(\mathcal{C}_n(\Sigma), {}_{f_{\mathcal{H}}}V) \to S_*(\mathcal{C}_n(\Sigma), V) \ ,$$

$$S_*(\mathcal{C}_n(g), {}_{\tau}V) : S_*(\mathcal{C}_n(\Sigma), {}_{\tau \circ f_{\mathcal{H}}}V) \to S_*(\mathcal{C}_n(\Sigma), {}_{\tau}V) \ , \tau \in \mathrm{Aut}(\mathcal{H}) \ .$$

Main result

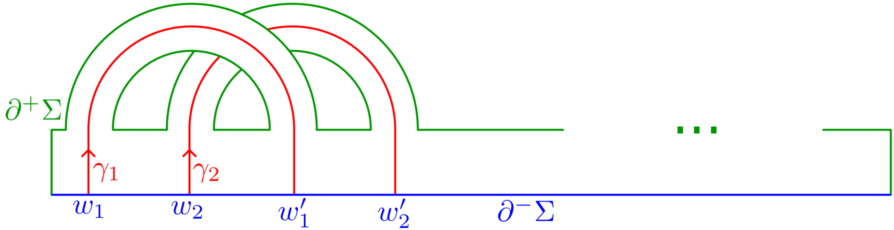
Theorem

Let $n \geq 2$, $g \geq 1$, V a representation of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma = \Sigma_{g,1})$ over a ring R.
a)The module $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$ is isomorphic to the direct sum of $\begin{pmatrix} 2g+n-1 \\ n \end{pmatrix}$ copies of V. Furthermore, it is the only non-vanishing module in $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$.
b) There is a natural twisted representation of the mapping class group $\mathfrak{M}(\Sigma)$ on the R-modules

$$H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_\tau V\right)\ ,\ \tau \in \mathrm{Aut}(\mathcal{H})\ ,$$

where the action of $f \in \mathfrak{M}(\Sigma)$ is $\mathcal{C}_n(f)_*$:
$H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau \circ f_{\mathcal{H}}} V\right) \to H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_\tau V\right)$

Model surface



Schrödinger representation

- The left action of $\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$ on the Hilbert space $W \cong L^2(\mathbb{R}^g)$, parametrised by the non zero real number $\hbar$.

$$\left[\rho_W\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\hbar\frac{k-p\cdot q}{2}} e^{i\hbar p\cdot s}\psi(s-q)\,.$$

Theorem (Stone-von Neumann)

W is an irreducible representation of $\mathcal{H}_{\mathbb{R}}$ and up to isomorphism is the unique irreducible representation whose character on the center is $(k,0) \mapsto e^{i\hbar\frac{k}{2}}$.

Finite dimensional Schrödinger representation

- For $N \geq 2$ even, $\mathcal{H}$ acts on the f.d. Hilbert space $W_N \cong L^2(\mathbb{Z}_N^g)$:

$$\left[\rho_{W,N}\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\pi\frac{k-p\cdot q}{N}} e^{i\frac{2\pi}{N}p\cdot s}\psi(s-q)\,.$$

Theorem (Stone-von Neumann)

W_N is an irreducible representation of $\mathcal{H}$ and up to unitary isomorphism is the unique irreducible unitary representation whose character on the center is $(k,0) \mapsto e^{i\pi\frac{k}{N}}$.

Untwisted representation of MCG

- For $\tau \in \mathrm{Aut}(\mathcal{H})$, The Stone-von Neumann theorem provides a unitary isomorphism ${}_\tau W \cong W$ (resp. ${}_\tau W_N \cong W_N$) defined up to S^1.
- We deduce projective actions

$$\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_n) \ , \ \mathcal{V}_n = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W)$$

$$\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_{N,n}) \ , \ \mathcal{V}_{N,n} = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W_N)$$

Stable universal central extension

- For $g \geq 4$, $\mathfrak{M}(\Sigma_{g,1})$ has a universal central extension

$$\mathbb{Z} \hookrightarrow \widetilde{\mathfrak{M}}(\Sigma_{g,1}) \twoheadrightarrow \mathfrak{M}(\Sigma_{g,1}) \ .$$

  which is compatible with the inclusion $\mathfrak{M}(\Sigma_{g,1}) \subset \mathfrak{M}(\Sigma_{g+1,1})$.
- By pulling back to $\Sigma_{g,1}$, $g < 4$, we have a *stable* universal extension for every g.
- The previous projective actions lift to unitary representations

$$\widetilde{\mathfrak{M}}(\Sigma_{g,1}) \to U(\mathcal{V}_n) \ , \ \widetilde{\mathfrak{M}}(\Sigma_{g,1}) \to U(\mathcal{V}_{N,n}) \ ,$$

  $\mathcal{V}_n = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W\big)$,
  $\mathcal{V}_{N,n} = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W_N\big)$.

Subgroups of MCG

- Action on Heisenberg group

$$\mathfrak{M}(\Sigma) \to \mathrm{Aut}(\mathcal{H}) \cong Sp(H) \ltimes H^* \ , \ f \mapsto f_{\mathcal{H}} = (f_*, \mathfrak{d}_f) \ .$$

- $f_{\mathcal{H}}$ is inner iff f is in the Torelli subgroup $\mathcal{T}(\Sigma)$; $f_{\mathcal{H}} = Id$ iff f is in the Chillingworth subgroup $\mathrm{Chill}(\Sigma) \subset \mathcal{T}(\Sigma)$.
- For any representation V of the Heisenberg group, the homology action gives a representation of $\mathrm{Chill}(\Sigma)$ which extends (via untwisting) as a linear representation on $\mathcal{T}(\Sigma)$.
- For the Schrödinger representations the homological action can be untwisted as a linear representation on the subgroup $\mathrm{Mor}(\Sigma) = \ker(\mathfrak{d})$ (Morita subgroup).

Questions

- Kernel ?
  From Moriyama, the Johnson filtration is recovered from the trivial representation.
- Does the Hilbert representation have almost invariant vectors?
- Classical or Quantum ?
- Relation with categorification of infinite generators Heisenberg algebra ?