From Jones relation to representations of the Mapping Class Groups

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The Jones polynomial

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS

BY VAUGHAN F. R. JONES

For the trace invariant we have

**THEOREM 12.** \(1/tV_{L^-} - tV_{L^+} = (\sqrt{t} - 1/\sqrt{t})V_L.\)

The Jones polynomial invariant of links \(V_L\) is uniquely determined by the above relation and the normalisation value 1 for the unknot.
Kauffman state sum construction


- Kauffman bracket, recursive definition

\[ \langle \chi \rangle = A \langle \rangle + A^{-1} \langle \chi \rangle \]

\[ \langle D \cup O \rangle = (-A^2 - A^{-2}) \langle D \rangle \]

- Global formula

\[ \langle D \rangle = \sum_{s: \{\text{crossings}\} \rightarrow \{\pm 1\}} A^{\sum s(c)} (-A^2 - A^{-2})^{\#D_s} \]

- Jones polynomial is recovered by

\[ V(L) = \left( -A^{-3} \right)^w \frac{\langle D \rangle}{-A^2 - A^{-2}} \bigg|_{A=t^{-\frac{1}{4}}} \]

where \( w \) is the writhe of an oriented diagram \( D \) representing the link \( L \),

\[ w(L_+) = 1, \quad w(L_-) = -1, \quad w(D) = \sum_c w(c) \]
Computation for the right handed trefoil

\[
\langle C \rangle = A^3 (-A^2 - A^{-2})^2 \\
+ 3A (-A^2 - A^{-2}) \\
+ 3A^2 (-A^2 - A^{-2})^2 \\
+ A^{-3} (-A^2 - A^{-2})^3 \\
= (-A^5 - A^{-3} + A^{-7})(-A^2 - A^{-2})
\]

\[V(T) = \left( (-A^{-3})^3 \left\langle T \right\rangle \right)_{A = \zeta^4} = \left( A^{-4} + A^{-12} - A^{-16} \right)_{A = \zeta^4} \]
\[= -\zeta^4 + \zeta^3 + \zeta\]
Witten’s Physical interpretation


Quantum Field Theory and the Jones Polynomial

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Abstract. It is shown that 2 + 1 dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from $S^3$ to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in 1 + 1 dimensions.
Atiyah’s contribution


The Jones polynomial can be profitably studied from many angles and it has been generalized in several ways to produce further knot invariants. Much of this work has involved important ideas from theoretical physics, essentially physics of 2 dimensions. However a major break-through came in 1988 when Witten [10] gave a simple interpretation of the Jones polynomial in terms of 3-dimensional physics. These ideas of Witten are based on a heuristic use of the Feynman integral, but they lead to very explicit results and calculations which can be verified by alternative rigorous methods. A full mathematical treatment of Witten’s theory has yet to appear, so my account has to be somewhat sketchy and incomplete.
Homfly-pt polynomial


**Proposition 6.2.** To each oriented link $L$ (up to isotopy) there is a Laurent polynomial $P_L(t, x)$ in the two variables $t$ and $x$ such that, if $\lambda$ and $q$ satisfy $t = \sqrt{\lambda} \sqrt{q}$, $x = (\sqrt{q} - 1/\sqrt{q})$ then $P_L(t, x) = X_L(q, \lambda)$. Moreover, $P_L(t, x)$ is uniquely defined by the “Skein rule”: If $L_+, L_-$ and $L_0$ are links that have projections identical, except in one crossing where they are as in Figure 6.3:

![Figure 6.3](image)

then $t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}$.
Colored Jones and Statistical models


  Example 1.20. The quantum group formalism of [Ji], [Dr] suggests that there is a vertex model invariant associated with any finite dimensional representation of any complex simple Lie algebra. Indeed, Example 1.18 corresponds to $sl_n$ in its $n$ dimensional identity representation and Example 1.19 embraces the $B_n$, $C_n$ and $D_n$ series in their fundamental representations. In support of this conjecture we give another example, corresponding to the $N$-dimensional irreducible representation of $sl_2$. The matrix $R(0)$ can be deduced from [Dr] and [Ji2]. These examples have apparently also been discovered using braids by Akutsu and Wadati [AW] and Wenzl [W2] although it is difficult to be absolutely sure, as only the first three cases are given in [AW] and only an existence result occurs in [W2] (which also gives

Quantum Group constructions

Construction from skein theory

Jones representation of genus 2 MCG


10. Mapping class groups

The problem of classification of closed 3-manifolds can be reduced via Heegard decompositions to the study of the mapping class groups (= diffeomorphism groups modulo the connected component of the identity) of closed orientable surfaces of arbitrary genus. It would be significant if one could find representations of these groups and an invariant via the Reidemeister–Singer theorem ([36]) as we have done for links via Markov’s theorem. We have not yet succeeded but we would like to describe some progress towards that goal.

- The genus $g$ surface $\Sigma_g$ is a double covering of the sphere $S^2$ branched over $2g + 2$ points.
  - Birman Hilden homorphism $\mathcal{M}(S^2, 2g + 2) \to \mathcal{M}(\Sigma_g)$ whose image is the so called hyperelliptic MCG.

- In genus 2 the hyperelliptic MCG is equal to $\mathcal{M}(\Sigma_2)$. 
Jones representation of genus 2 MCG

- Vaughan Jones obtained representations of the genus $g$ hyperelliptic MCG from certain representations of the Hecke algebra $H_{2g+2}$.
- The Hecke algebra is the quotient of the braid group algebra by a quadratic relation.
- Irreducible representations of the Hecke algebra are indexed by Young diagrams.

Theorem (V. Jones criterion)

The representation of $H_{2g+2}$ indexed by the Young diagram $Y$ can be renormalised into a representation which extends to the hyperelliptic MCG if and only if $Y$ is rectangular.

Theorem 10.2. Let $Y$ be a Young diagram and let $\pi'_Y$ be the corresponding representation of $B_m$, adjusted as above so that $\pi'_Y(\sigma_1 \ldots \sigma_{m-1})^m = 1$. Then $\pi'_Y$ defines a representation of $M(0, m)$ via $\omega_i \mapsto \pi'_Y(\sigma_i)$ if and only if $Y$ is rectangular.
Jones representation of genus 2 MCG

However, in genus two, the group generated by the $\theta_i$'s is the whole mapping class group so that we do obtain representations of this group $M(2,0)$. Up to symmetry there is only one rectangular tableau on 6 nodes, so in fact there is really only one representation. Here is a choice of matrices corresponding to $\theta_1$ which, when multiplied by $q^{-2/5}$, give a representation

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & q \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & q & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & q
\end{pmatrix}$$

$$\begin{pmatrix}
q & 0 & 0 & 0 & 0 \\
0 & q & 0 & 0 & 0 \\
0 & q & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1
\end{pmatrix}$$

$$\begin{pmatrix}
-1 & 0 & 0 & q & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & q & 0 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}$$

$$\begin{pmatrix}
q & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & q \\
1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & q
\end{pmatrix}$$

$$\begin{pmatrix}
-1 & q & 0 & 0 & 0 \\
0 & q & 0 & 0 & 0 \\
0 & 0 & q & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}$$
Representations of Mapping Class Groups

- 3-dimensional TQFTs contain representations of central extensions of MCG.
- MCG act on skein algebras of surfaces which have interesting connections with character varieties and geometry.
- Using classical topology we construct action of MCG on homologies of surface configurations with local coefficients built from representations of the Heisenberg group.

Lawrence representations

- Lawrence (1990): Family of representations, $n \geq 2$,
  \[ L_n : B_m \rightarrow GL(H_n(\tilde{C}_n(D_m^2))) \]
- $\tilde{C}_n$ is a $\mathbb{Z}^2$-cover of the unordered configuration space $C_n(D_m^2)$ of $n$ points in the $m$-punctured disc.
- Theorem (Bigelow, Krammer, 2001-2002): $L_2$ is faithful.
- Kohno: Lawrence (LKB) representations are equivalent to $sl(2)$ quantum representations on highest weight spaces.
Homological representations of MCG

- $B_m = \mathcal{M}(D_m^2)$ is a mapping class group.
- Goal: LKB type representations for $\mathcal{M}(\Sigma_{g,1}), g \geq 2$.
- A local system on $C_n(\Sigma_{g,1})$ is obtained from a representation $V$ of the Heisenberg group $\mathcal{H}_g$, which will appear as a quotient of $B_n(\Sigma_{g,1}) = \pi_1(C_n(\Sigma_{g,1}))$.
- We obtain a twisted action of the MCG on $H_n(C_n(\Sigma_{g,1}), V)$.
- For the Schrödinger representation ($L^2(\mathbb{R}^g)$ or $L^2(\mathbb{Z}_N^g)$) we obtain representations

$$\mathcal{M}(\Sigma_{g,1}) \to PU(H_n(C_n(\Sigma_{g,1}), V)).$$

In the case $L^2(\mathbb{Z}_N^g)$, the dimension is $\binom{2g+n-1}{n} N^g$.
- For the $2g + 2$ dimensional representation defined from the left regular action we obtain linear representations of the native MCG.
Surface braid groups

- \( B_n(\Sigma_{g,1}) = \pi_1(C_n(\Sigma_{g,1})) \).

- Bellingeri presentation, revisited by Bellingeri-Godelle: generators \( \sigma_1, \ldots, \sigma_{n-1}, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \) and relations:

\[
\begin{align*}
\text{(BR1)} & \quad [\sigma_i, \sigma_j] = 1 \quad \text{for } |i - j| \geq 2, \\
\text{(BR2)} & \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1, \\
\text{(CR1)} & \quad [\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1 \quad \text{for } i > 1 \text{ and all } r, \\
\text{(CR2)} & \quad [\alpha_r, \sigma_1 \alpha_r \sigma_1] = [\beta_r, \sigma_1 \beta_r \sigma_1] = 1 \quad \text{for all } r, \\
\text{(CR3)} & \quad [\alpha_r, \sigma_1^{-1} \alpha_s \sigma_1] = [\alpha_r, \sigma_1^{-1} \beta_s \sigma_1] \\
& \quad = [\beta_r, \sigma_1^{-1} \alpha_s \sigma_1] = [\beta_r, \sigma_1^{-1} \beta_s \sigma_1] = 1 \quad \text{for all } r < s, \\
\text{(SCR)} & \quad \sigma_1 \beta_r \sigma_1 \alpha_r \sigma_1 = \alpha_r \sigma_1 \beta_r \quad \text{for all } r.
\end{align*}
\]

Composition is written from right to left.

- First presentation in closed case:
SCR relation $\alpha_1 \sigma_1 \beta_1 = \sigma_1 \beta_1 \sigma_1 \alpha_1 \sigma_1$

\[
\begin{array}{c}
\sigma_1 \\
\beta_1 \\
\alpha_1 \\
\sigma_1 \\
\end{array}
\]
The Heisenberg group $\mathcal{H}_g$ is the central extension of $H = H_1(\Sigma_{g,1})$ defined with the intersection cocycle.

$\mathcal{H}_g = \mathbb{Z} \times H$ with $(k, x)(l, y) = (k + l + x.y, x + y)$.

Theorem: $B_n(\Sigma_{g,1})/(\sigma_1 \text{ central})$ is isomorphic to the Heisenberg group $\mathcal{H}_g$.

We consider the associated regular covering $\tilde{C}_n(\Sigma_{g,1})$ and its homology which support a right action of $\mathcal{H}_g$. 
Model surface for $\Sigma = \Sigma_{g,1}$

$\partial \Sigma$

$\gamma_1$

$\gamma_2$

$w_1$

$w_2$

$w'_1$

$w'_2$

$\partial^- \Sigma$

$\triangleright C_n(\Sigma, \partial^- (\Sigma))$ is the properly embedded subspace of $C_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^- \Sigma \subset \partial \Sigma$.

$\triangleright H_{BM}^*$ denotes the Borel-Moore homology,

$$H_{BM}^n(\tilde{C}_n(\Sigma), \tilde{C}_n(\Sigma, \partial^- (\Sigma)); \mathbb{Z}) = \lim_{\leftarrow T} H_n(\tilde{C}_n(\Sigma), \tilde{C}_n(\Sigma, \partial^- (\Sigma)) \cup (\tilde{C}_n(\Sigma) \setminus \tilde{T}); \mathbb{Z}),$$

the inverse limit is taken over all compact subsets $T \subset C_n(\Sigma)$.

**Theorem**

For $g \geq 1$, $n \geq 2$, the module $H_{BM}^n(\tilde{C}_n(\Sigma), \tilde{C}_n(\Sigma, \partial^- (\Sigma)), \mathbb{Z})$ is a free $\mathbb{Z}[[\mathcal{H}]]$-module of rank $\binom{2g + n - 1}{n}$. Furthermore, it is the only non-vanishing module in $H_{BM}^*(\tilde{C}_n(\Sigma), \tilde{C}_n(\Sigma, \partial^- (\Sigma)), \mathbb{Z})$. 
Let $a_i, b_i \in H_1(\Sigma, \mathbb{Z})$ be the classes of $\alpha_i, \beta_i$, $1 \leq i \leq g$.

- $\mathcal{H}_g$ can be realised as a group of matrices, which gives a faithful $(g + 2)$-dimensional representation:

$$
\begin{pmatrix}
 k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i \\
 1 p \frac{k+p\cdot q}{2} \\
 0 l_g q \\
 0 0 1
\end{pmatrix},
$$

where $p = (p_i)$ is a row vector and $q = (q_i)$ is a column vector.

- The left regular action of the Heisenberg group $\mathcal{H}_g$ is affine on $\mathcal{H}_g \cong \mathbb{Z}^{2g+1}$. Its linearisation gives a $2g + 2$ dimensional representation $L$. 
Unitary representations

- Schrödinger representation on the Hilbert space $\mathcal{W} \cong L^2(\mathbb{R}^g)$:

$$\rho_{\mathcal{W}} \left( k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i \right) \psi(s) = e^{i\hbar \frac{k-p \cdot q}{2}} e^{i\hbar p \cdot s} \psi(s - q).$$

- Schrödinger representation on the f.d. Hilbert space $\mathcal{W}_N \cong L^2(\mathbb{Z}_N^g)$. For $N$ even, $\hbar = \frac{2\pi}{N}$. 
MCG action on Heisenberg group

- For $[f] \in M(\Sigma)$, the diffeomorphism $C_n(f)$ induces an automorphism $f_H \in Aut^+(\mathcal{H})$ (identity on center).
- For a representation $\rho : \mathcal{H}_g \to GL(V)$ and $\tau \in Aut(\mathcal{H})$, the $\tau$-twisted representation $\rho \circ \tau$ is denoted by $\tau V$. 
Local system from an Heisenberg group representation

- The (singular or cellular) chain complex of the Heisenberg group cover, denoted by $S_*(\tilde{C}_n(\Sigma))$, is a right $\mathbb{Z}[\mathcal{H}_g]$-module.

- Given a representation $\rho : \mathcal{H}_g \to GL(V)$, the corresponding local homology is that of the complex $S_*(C_n(\Sigma), V) := S_*(\tilde{C}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}_g]} V$.

- For $[f] \in \mathcal{M}(\Sigma)$, the map $C_n(f)$ lifts to the Heisenberg cover and the lift $\tilde{C}_n(f)$ induces a chain map $S_*(\tilde{C}_n(f))$ which is twisted linear over $\mathbb{Z}[\mathcal{H}_g]$,

$$S_*(\tilde{C}_n(f))(z.h) = S_*(\tilde{C}_n(f))(z).f_{\mathcal{H}}(h).$$

- We get chain maps

$$S_*(C_n(f), V) : S_*(C_n(\Sigma), f_{\mathcal{H}}V) \to S_*(C_n(\Sigma), V),$$

$$S_*(C_n(f), \tau V) : S_*(C_n(\Sigma), \tau \circ f_{\mathcal{H}}V) \to S_*(C_n(\Sigma), \tau V), \tau \in Aut(\mathcal{H}_g).$$
Notation

- $H^{BM}_*$ denotes the Borel-Moore homology,

$$H^{BM}_n(C_n(\Sigma); V) = \lim_{\leftarrow T} H_n(C_n(\Sigma), C_n(\Sigma) \setminus T; V),$$

the inverse limit is taken over all compact subsets $T \subset C_n(\Sigma)$.

- $C_n(\Sigma, \partial^- (\Sigma))$ is the properly embedded subspace of $C_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^- \Sigma \subset \partial \Sigma$. 
Twisted representation

**Theorem**

Let $n \geq 2$, $g \geq 1$, $V$ a representation of the discrete Heisenberg group $\mathcal{H}_g$ over a ring $R$.

a) The module $H_n^{BM}(C_n(\Sigma), C_n(\Sigma, \partial^-(\Sigma)); V)$ is isomorphic to the direct sum of
$$2g + n - 1 \choose n$$ copies of $V$. Furthermore, it is the only non-vanishing module in $H_n^{BM}(C_n(\Sigma), C_n(\Sigma, \partial^-(\Sigma)); V)$.

b) There is a natural twisted representation of the mapping class group $\mathcal{M}(\Sigma)$ on the modules

$$H_n^{BM}(C_n(\Sigma), C_n(\Sigma, \partial^-(\Sigma)); \tau V), \quad \tau \in \text{Aut}(\mathcal{H}),$$

where the action of $f \in \mathcal{M}(\Sigma)$ is $C_n(f)_* :$

$$H_n^{BM}(C_n(\Sigma), C_n(\Sigma, \partial^-(\Sigma)); \tau_\circ f \mathcal{H} V) \rightarrow H_n^{BM}(C_n(\Sigma), C_n(\Sigma, \partial^-(\Sigma)); \tau V)$$
Schrödinger representation

The left action of $\mathcal{H}_g^\mathbb{R} \supset \mathcal{H}_g$ on the Hilbert space $W \cong L^2(\mathbb{R}^g)$, parametrised by the non zero real number $\hbar$.

$$
\left[ \rho_W \left( k, x = \sum_{i=1}^g p_i a_i + q_i b_i \right) \psi \right] (s) = e^{i\hbar \frac{k-p \cdot q}{2}} e^{i\hbar p \cdot s} \psi(s-q).
$$

Theorem (Stone-von Neumann)

$W$ is an irreducible representation of $\mathcal{H}_g^\mathbb{R}$ and up to isomorphism is the unique irreducible representation whose character on the center is $(k,0) \mapsto e^{i\hbar \frac{k}{2}}$. 
Finite dimensional Schrödinger representation

- For $N \geq 2$ even, $\mathcal{H}$ acts on the f.d. Hilbert space $W_N \cong L^2(\mathbb{Z}_N^g)$:

$$\rho_{W,N} \left( k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i \right) \psi(s) = e^{i \pi k - p \cdot q \over N} e^{i \frac{2 \pi p \cdot s}{N}} \psi(s - q).$$

**Theorem (Stone-von Neumann)**

$W_N$ is an irreducible representation of $\mathcal{H}$ and up to unitary isomorphism is the unique irreducible unitary representation whose character on the center is $(k,0) \mapsto e^{i \pi k \over N}$. 
Untwisted representation of MCG

- For $\tau \in \text{Aut}(\mathcal{H})$, The Stone-von Neumann theorem provides a unitary isomorphism $\tau W \cong W$ (resp. $\tau W_N \cong W_N$) defined up to $S^1$.

- We deduce projective actions

  $$\mathcal{M}(\Sigma) \to PU(\mathcal{V}_n), \mathcal{V}_n = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^- (\Sigma)); W)$$

  $$\mathcal{M}(\Sigma) \to PU(\mathcal{V}_{N,n}), \mathcal{V}_{N,n} = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^- (\Sigma)); W_N)$$
The left regular action of \((k_0, x_0) \in \mathcal{H}_g\) is an automorphism of 
\(\mathcal{H}_g \equiv \mathbb{Z}^{2g+1}\). We decompose \(x_0 = p_0 + q_0, p_0 \in \Lambda_a = \text{Span}(a_i, 1 \leq i \leq g),\)
\(q_0 \in \Lambda_b = \text{Span}(b_i, 1 \leq i \leq g)\), then the action is written
\[
\begin{align*}
  k' &= k + k_0 + p_0 \cdot q - q_0 \cdot p \\
  p' &= p + p_0 \\
  q' &= q + q_0
\end{align*}
\]

We consider the linearisation \(\rho_L\) of this affine action on \(L = \mathcal{H}_g \oplus \mathbb{Z}\). The linear action of \(\rho_L(k_0, x_0)\) is as follows.
\[
\begin{align*}
  k' &= k + tk_0 + p_0 \cdot q - q_0 \cdot p \\
  p' &= p + tp_0 \\
  q' &= q + tq_0 \\
  t' &= t
\end{align*}
\]

For \(\tau \in \text{Aut}^+(\mathcal{H}_g)\), the linear map \(\tau \times \text{id} : L \mapsto \tau L\) gives an isomorphism
of \(\mathbb{Z}[\mathcal{H}_g]\)-module.
**Theorem**

There is a representation

\[
\mathcal{M}(\Sigma = \Sigma_{g,1}) \rightarrow \text{Aut}(H_n(C_n(\Sigma), C_n(\Sigma, \partial^{-}(\Sigma)); L),
\]

which associates to \( f \in \mathcal{M}(\Sigma_{g,1}) \) the composition of the coefficient isomorphism induced by \( f_{\mathcal{H}} \),

\[
H_n(C_n(\Sigma), C_n(\Sigma, \partial^{-}(\Sigma)); L) \rightarrow H_n(C_n(\Sigma), C_n(\Sigma, \partial^{-}(\Sigma)); f_{\mathcal{H}} L),
\]

with the functorial homology isomorphism

\[
H_n(C_n(\Sigma), C_n(\Sigma, \partial^{-}(\Sigma)); f_{\mathcal{H}} L) \rightarrow H_n(C_n(\Sigma), C_n(\Sigma, \partial^{-}(\Sigma)); L),
\]
Going further

- Is this classical or quantum?
- Action of cobordisms?
- Faithfulness?
- In unitary case, can we find almost invariant vectors?
- What about closed surfaces? Is there a Jones type criterion for extending subrepresentations to closed surfaces?