From Subfactors to Quantum Topology - In memory of Vaughan Jones

SwissMAP
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From Jones relation to representations of the Mapping Class Groups

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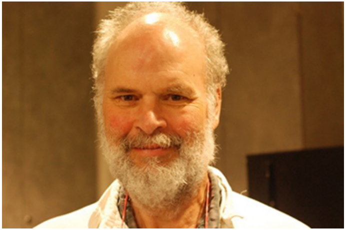


KNOTS
IN HELLAS
2016
MAUI 2014
Subfactor Theory
in Mathematics and Physics


Π5

The Jones polynomial

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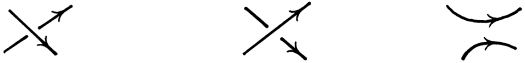
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

BY VAUGHAN F. R. JONES[2]



L^+ $\qquad$ L^- $\qquad$ L

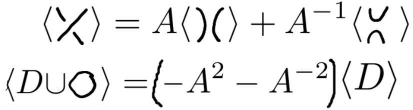
For the trace invariant we have

THEOREM 12. $1/tV_{L^-} - tV_{L^+} = (\sqrt{t} - 1/\sqrt{t})V_L$.

The Jones polynomial invariant of links V_L is uniquely determined by the above relation and the normalisation value 1 for the unknot.

Kauffman state sum construction

- Kauffman Louis, State models and the Jones polynomial. Topology 26 (1987), no. 3, 395–407.
- Kauffman bracket, recursive definition



- Global formula

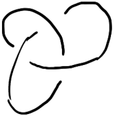
$$\langle D\rangle = \sum_{s:\{\text{crossings}\}\to\{\pm 1\}} A^{\sum s(c)}(-A^2 - A^{-2})^{\sharp D_s}$$

- Jones polynomial is recovered by $V(L) = \left[(-A^{-3})^w \frac{\langle D\rangle}{-A^2 - A^{-2}}\right]_{A=t^{-\frac{1}{4}}}$,

  where w is the writhe of an oriented diagram D representing the link L,

$$w(L_+) = 1\ ,\ w(L_-) = -1\ ,\ w(D) = \sum_c w(c)\ .$$

Computation for the right handed trefoil

$\Big\langle$  $\Big\rangle$

$$= A^3(-A^2-A^{-2})^2 + 3A(-A^2-A^{-2}) + 3A^{-1}(-A^2-A^{-2})^2 + A^{-3}(-A^2-A^{-2})^3$$

$$= (-A^5-A^{-3}+A^{-7})(-A^2-A^{-2})$$

$\| $

T

$$V(T) = \left((-A^{-3})^3 \frac{\langle T\rangle}{-A^2-A^{-2}}\right)\Bigg|_{A=t^{-\frac{1}{4}}} = \left(A^{-4}+A^{-12}-A^{-16}\right)\Big|_{A=t^{-\frac{1}{4}}}$$

$$= -t^4+t^3+t$$

Witten's Physical interpretation

- Witten Edward, Topological quantum field theory. Comm. Math. Phys. 117 (1988), no. 3, 353–386.
- Witten Edward, Quantum field theory and the Jones polynomial. Comm. Math. Phys. 121 (1989), no. 3, 351–399.

Quantum Field Theory and the Jones Polynomial ⋆

Edward Witten ⋆⋆

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

Atiyah's contribution

- Atiyah Michael, Topological quantum field theory. Inst. Hautes Études Sci. Publ. Math. No. 68 (1988), 175–186.
- Atiyah Michael, The Jones-Witten invariants of knots. Séminaire Bourbaki, Vol. 1989/90. Astérisque No. 189-190 (1990), Exp. No. 715, 7–16.

The Jones polynomial can be profitably studied from many angles and it has been generalized in several ways to produce further knot invariants. Much of this work has involved important ideas from theoretical physics, essentially physics of 2 dimensions. However a major break-through came in 1988 when Witten [10] gave a simple interpretation of the Jones polynomial in terms of 3-dimensional physics. These ideas of Witten are based on a heuristic use of the Feynman integral, but they lead to very explicit results and calculations which can be verified by alternative rigorous methods. A full mathematical treatment of Witten's theory has yet to appear, so my account has to be somewhat sketchy and incomplete.

Homfly-pt polynomial

- Freyd, P.; Yetter, D.; Hoste, J.; Lickorish, W. B. R.; Millett, K.; Ocneanu, A. A new polynomial invariant of knots and links. Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 2, 239–246.
- Przytycki Józef H., Traczyk, Paweł, Invariants of links of Conway type. Kobe J. Math. 4 (1988), no. 2, 115–139.
- Jones, V. F. R. Hecke algebra representations of braid groups and link polynomials. Ann. of Math. (2) 126 (1987), no. 2, 335–388.

PROPOSITION 6.2. *To each oriented link L (up to isotopy) there is a Laurent polynomial $P_L(t, x)$ in the two variables t and x such that, if λ and q satisfy $t = \sqrt{\lambda}\sqrt{q}$, $x = (\sqrt{q} - 1/\sqrt{q})$ then $P_L(t, x) = X_L(q, \lambda)$. Moreover, $P_L(t, x)$ is uniquely defined by the "Skein rule": If L_+, L_- and L_0 are links that have projections identical, except in one crossing where they are as in Figure 6.3:*

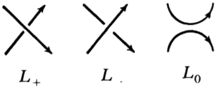


FIGURE 6.3

then $t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}$.

Colored Jones and Statistical models

- Jones Vaughan, On knot invariants related to some statistical mechanical models. Pacific J. Math. 137 (1989), no. 2, 311–334.

EXAMPLE 1.20. The quantum group formalism of **[Ji]**, **[Dr]** suggests that there is a vertex model invariant associated with any finite dimensional representation of any complex simple Lie algebra. Indeed, Example 1.18 corresponds to sl_n in its n dimensional identity representation and Example 1.19 embraces the B_n, C_n and D_n series in their fundamental representations. In support of this conjecture we give another example, corresponding to the N-dimensional irreducible representation of sl_2. The matrix $R(0)$ can be deduced from **[Dr]** and **[Ji2]**. These examples have apparently also been discovered using braids by Akutsu and Wadati **[AW]** and Wenzl **[W2]** although it is difficult to be absolutely sure, as only the first three cases are given in **[AW]** and only an existence result occurs in **[W2]** (which also gives

- Turaev Vladimir, The Yang-Baxter equation and invariants of links. Invent. Math. 92 (1988), no. 3, 527–553.

Quantum Group constructions

- Reshetikhin Nikolai, Turaev Vladimir, Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. 127 (1990), no. 1, 1–26.
- Reshetikhin Nikolai, Turaev Vladimir, Invariants of 3-manifolds via link polynomials and quantum groups. Invent. Math. 103 (1991), no. 3, 547–597.
- Turaev Vladimir, Quantum invariants of knots and 3-manifolds. De Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

Construction from skein theory

- Lickorish, W. B. R. Three-manifolds and the Temperley-Lieb algebra. Math. Ann. 290 (1991), no. 4, 657–670
- Lickorish, W. B. R. Invariants for 3-manifolds from the combinatorics of the Jones polynomial. Pacific J. Math. 149 (1991), no. 2, 337–347.
- Blanchet, C.; Habegger, N.; Masbaum, G.; Vogel, P. Three-manifold invariants derived from the Kauffman bracket. Topology 31 (1992), no. 4, 685–699
- Blanchet, C.; Habegger, N.; Masbaum, G.; Vogel, P. Topological quantum field theories derived from the Kauffman bracket. Topology 34 (1995), no. 4, 883–927.

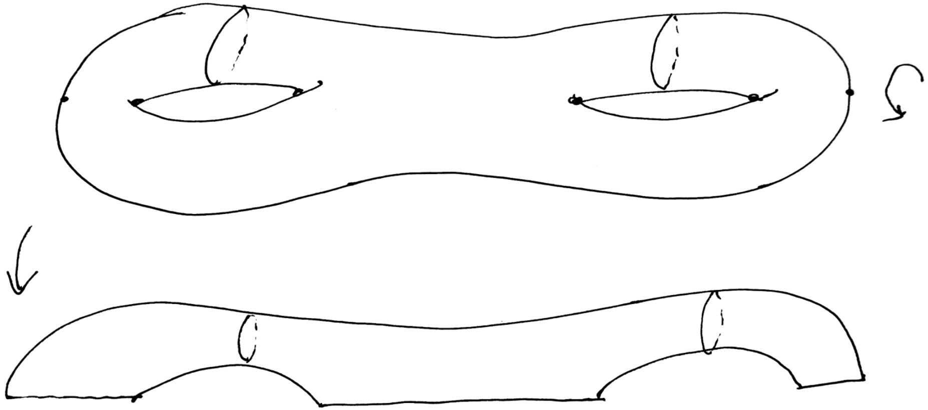
Jones representation of genus 2 MCG

- Jones, V. F. R. Hecke algebra representations of braid groups and link polynomials. Ann. of Math. (2) 126 (1987), no. 2, 335–388.

10. Mapping class groups

The problem of classification of closed 3-manifolds can be reduced via Heegard decompositions to the study of the mapping class groups (= diffeomorphism groups modulo the connected component of the identity) of closed orientable surfaces of arbitrary genus. It would be significant if one could find representations of these groups and an invariant via the Reidemeister–Singer theorem ([36]) as we have done for links via Markov's theorem. We have not yet succeeded but we would like to describe some progress towards that goal.

- The genus g surface Σ_g is a double covering of the sphere S^2 branched over $2g+2$ points.
  Birman Hilden homorphism $\mathcal{M}(S^2, 2g+2) \to \mathcal{M}(\Sigma_g)$ whose image is the so called hyperelliptic MCG.
- In genus 2 the hyperelliptic MCG is equal to $\mathcal{M}(\Sigma_2)$.



Jones representation of genus 2 MCG

- Vaughan Jones obtained representations of the genus g hyperelliptic MCG from certain representations of the Hecke algebra H_{2g+2}.
- The Hecke algebra is the quotient of the braid group algebra by a quadratic relation.
- Irreducible representations of the Hecke algebra are indexed by Young diagrams.

Theorem (V. Jones criterion)

The representation of H_{2g+2} indexed by the Young diagram Y can be renormalised into a representation which extends to the hyperelliptic MCG if and only if Y is rectangular.

Theorem 10.2. *Let Y be a Young diagram and let π'_Y be the corresponding representation of B_m, adjusted as above so that $\pi'_Y(\sigma_1 \ldots \sigma_{m-1})^m = 1$. Then π'_Y defines a representation of $M(0, m)$ via $\omega_i \mapsto \pi'_Y(\sigma_i)$ if and only if Y is rectangular.*

Jones representation of genus 2 MCG

However, in genus two, the group generated by the θ_i's *is* the whole mapping class group so that we do obtain representations of this group $M(2,0)$. Up to symmetry there is only one rectangular tableau on 6 nodes, so in fact there is really only one representation. Here is a choice of matrices corresponding to $\begin{smallmatrix}\square&\square\\ \square&\square\\ \square&\square\end{smallmatrix}$ which, when multiplied by $q^{-2/5}$, give a representation

$$\theta_1:\begin{pmatrix} -1 & 0 & 0 & 0 & q \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix} \qquad \theta_2:\begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & q & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\theta_3:\begin{pmatrix} -1 & 0 & 0 & q & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \qquad \theta_4:\begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & q \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}$$

$$\theta_5:\begin{pmatrix} -1 & q & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Representations of Mapping Class Groups

- 3-dimensional TQFTs contain representations of central extensions of MCG.
- MCG act on skein algebras of surfaces which have interesting connections with character varieties and geometry.
- Using classical topology we construct action of MCG on homologies of surface configurations with local coefficents built from representations of the Heisenberg group.
  CB, Martin Palmer, Awais Shaukat, Heisenberg homology on surface configurations. arXiv:2109.00515 .

Lawrence representations

- Lawrence (1990): Family of representations, $n \geq 2$,

$$L_n : B_m \to GL(H_n(\widetilde{\mathcal{C}}_n(D^2_m))$$

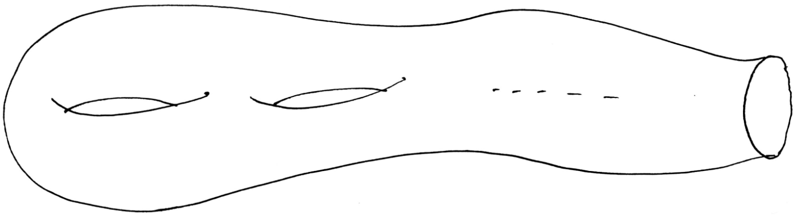
- $\widetilde{\mathcal{C}}_n$ is a $\mathbb{Z}^2$-cover of the unordered configuration space $\mathcal{C}_n(D^2_m)$ of n points in the m-punctured disc.
- Theorem (Bigelow, Krammer, 2001-2002): L_2 is faithful.
- Kohno: Lawrence (LKB) representations are equivalent to $sl(2)$ quantum representations on heighest weight spaces.

Homological representations of MCG

- $B_m = \mathfrak{M}(D^2_m)$ is a mapping class group.
- Goal: LKB type representations for $\mathfrak{M}(\Sigma_{g,1})$, $g \geq 2$.
- A local system on $\mathcal{C}_n(\Sigma_{g,1})$ is obtained from a representation V of the Heisenberg group $\mathcal{H}_g$, which will appear as a quotient of $B_n(\Sigma_{g,1}) = \pi_1(\mathcal{C}_n(\Sigma_{g,1}))$.
- We obtain a twisted action of the MCG on $H_n(\mathcal{C}_n(\Sigma_{g,1}), V)$.
- For the Shrödinger representation ($L^2(\mathbb{R}^g)$ or $L^2(\mathbb{Z}_N^g)$) we obtain representations

$$\mathcal{M}(\Sigma_{g,1}) \to PU(H_n(\mathcal{C}_n(\Sigma_{g,1}), V)) \ .$$

  In the case $L^2(\mathbb{Z}_N^g)$, the dimension is $\binom{2g+n-1}{n} N^g$.
- For the $2g+2$ dimensional representation defined from the left regular action we obtain linear representations of the native MCG.



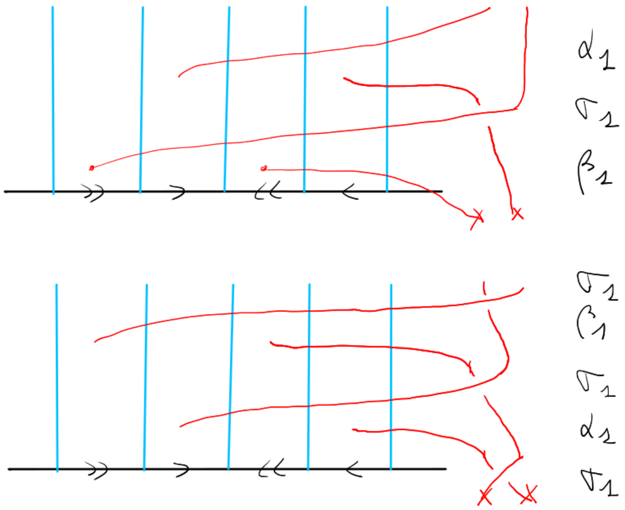
Surface braid groups

- $B_n(\Sigma_{g,1}) = \pi_1(\mathcal{C}_n(\Sigma_{g,1}))$.
- Bellingeri presentation, revisited by Bellingeri-Godelle: generators $\sigma_1, \ldots, \sigma_{n-1}$, $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and relations:

$$\begin{cases} \text{(BR1)} \ [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geq 2, \\ \text{(BR2)} \ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ \text{(CR1)} \ [\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } r, \\ \text{(CR2)} \ [\alpha_r, \sigma_1\alpha_r\sigma_1] = [\beta_r, \sigma_1\beta_r\sigma_1] = 1 & \text{for all } r, \\ \text{(CR3)} \ [\alpha_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\alpha_r, \sigma_1^{-1}\beta_s\sigma_1] & \\ \qquad = [\beta_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\beta_r, \sigma_1^{-1}\beta_s\sigma_1] = 1 & \text{for all } r < s, \\ \text{(SCR)} \ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases}$$

  Composition is written from right to left.
- First presentation in closed case:
  Scott, G. P. Braid groups and the group of homeomorphisms of a surface. Proc. Cambridge Philos. Soc. 68 (1970), 605–617.

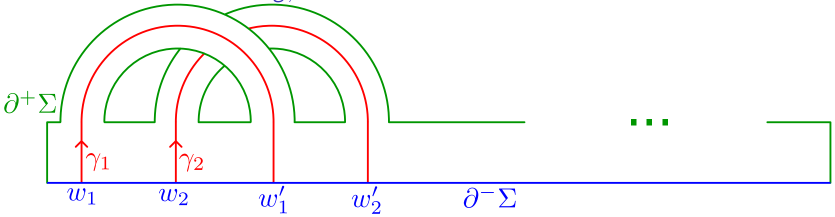
SCR relation $\alpha_1\sigma_1\beta_1 = \sigma_1\beta_1\sigma_1\alpha_1\sigma_1$



Heisenberg group

- The Heisenberg group $\mathcal{H}_g$ is the central extension of $H = H_1(\Sigma_{g,1})$ defined with the intersection cocycle.
- $\mathcal{H}_g = \mathbb{Z} \times H$ with $(k, x)(l, y) = (k + l + x.y, x + y)$.
- Theorem: $B_n(\Sigma_{g,1})/(\sigma_1$ central) is isomorphic to the Heisenberg group $\mathcal{H}_g$.
- We consider the associated regular covering $\widetilde{\mathcal{C}}_n(\Sigma_{g,1})$ and its homology which support a right action of $\mathcal{H}_g$.

Model surface for $\Sigma = \Sigma_{g,1}$



- $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ is the properly embedded subspace of $\mathcal{C}_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^-\Sigma \subset \partial\Sigma$.
- H_*^{BM} denotes the Borel-Moore homology,

$$H_n^{BM}(\widetilde{\mathcal{C}}_n(\Sigma), \widetilde{\mathcal{C}}_n(\Sigma, \partial^-(\Sigma)); \mathbb{Z}) = \varprojlim_T H_n(\widetilde{\mathcal{C}}_n(\Sigma), \widetilde{\mathcal{C}}_n(\Sigma, \partial^-(\Sigma)) \cup (\widetilde{\mathcal{C}}_n(\Sigma) \setminus \widetilde{T}); \mathbb{Z}),$$

the inverse limit is taken over all compact subsets $T \subset \mathcal{C}_n(\Sigma)$.

Theorem

For $g \geq 1$, $n \geq 2$, the module $H_n^{BM}(\widetilde{\mathcal{C}}_n(\Sigma), \widetilde{\mathcal{C}}_n(\Sigma, \partial^-(\Sigma)), \mathbb{Z})$ is a free $\mathbb{Z}[\mathcal{H}]$-module of rank $\binom{2g+n-1}{n}$. Furthermore, it is the only non-vanishing module in $H_*^{BM}(\widetilde{\mathcal{C}}_n(\Sigma), \widetilde{\mathcal{C}}_n(\Sigma, \partial^-(\Sigma)), \mathbb{Z})$.

Representations

Let $a_i, b_i \in H_1(\Sigma, \mathbb{Z})$ be the classes of α_i, β_i, $1 \leq i \leq g$.

- $\mathcal{H}_g$ can be realised as a group of matrices, which gives a faithful $(g+2)$-dimensional representation:

$$\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right) \longmapsto \left(\begin{array}{ccc} 1 & p & \frac{k+p\cdot q}{2} \\ 0 & I_g & q \\ 0 & 0 & 1 \end{array} \right),$$

where $p = (p_i)$ is a row vector and $q = (q_i)$ is a column vector.

- The left regular action of the Heisenberg group $\mathcal{H}_g$ is affine on $\mathcal{H}_g \cong \mathbb{Z}^{2g+1}$. Its linearisation gives a $2g+2$ dimensional representation L.

Unitary representations

- Schrödinger representation on the Hilbert space $W \cong L^2(\mathbb{R}^g)$:

$$\left[\rho_W\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\hbar\frac{k-p\cdot q}{2}} e^{i\hbar p\cdot s}\psi(s-q).$$

- Schrödinger representation on the f.d. Hilbert space $W_N \cong L^2(\mathbb{Z}_N^g)$. For N even, $\hbar = \frac{2\pi}{N}$.

MCG action on Heisenberg group

- For $[f] \in \mathfrak{M}(\Sigma)$, the diffeomorphism $\mathcal{C}_n(f)$ induces an automorphism $f_{\mathcal{H}} \in \mathrm{Aut}^+(\mathcal{H})$ (identity on center).
- For a representation $\rho : \mathcal{H}_g \to GL(V)$ and $\tau \in \mathrm{Aut}(\mathcal{H})$, the τ-twisted representation $\rho \circ \tau$ is denoted by ${}_\tau V$.

Local system from an Heisenberg group representation

- The (singular or cellular) chain complex of the Heisenberg group cover, denoted by $S_*(\widetilde{\mathcal{C}}_n(\Sigma))$, is a right $\mathbb{Z}[\mathcal{H}_g]$-module.
- Given a representation $\rho : \mathcal{H}_g \to GL(V)$, the corresponding local homology is that of the complex $S_*(\mathcal{C}_n(\Sigma), V) := S_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}_g]} V$.
- For $[f] \in \mathfrak{M}(\Sigma)$, the map $\mathcal{C}_n(f)$ lifts to the Heisenberg cover and the lift $\widetilde{\mathcal{C}}_n(f)$ induces a chain map $S_*(\widetilde{\mathcal{C}}_n(f))$ which is twisted linear over $\mathbb{Z}[\mathcal{H}_g]$,

$$S_*(\widetilde{\mathcal{C}}_n(f))(z.h) = S_*(\widetilde{\mathcal{C}}_n(f))(z).f_{\mathcal{H}}(h) \ .$$

- We get chain maps

$$S_*(\mathcal{C}_n(f), V) : S_*(\mathcal{C}_n(\Sigma), {}_{f_{\mathcal{H}}}V) \to S_*(\mathcal{C}_n(\Sigma), V) \ ,$$

$$S_*(\mathcal{C}_n(f), {}_{\tau}V) : S_*(\mathcal{C}_n(\Sigma), {}_{\tau\circ f_{\mathcal{H}}}V) \to S_*(\mathcal{C}_n(\Sigma), {}_{\tau}V) \ , \tau \in \mathrm{Aut}(\mathcal{H}_g) \ .$$

Notation

- H_*^{BM} denotes the Borel-Moore homology,

$$H_n^{BM}(\mathcal{C}_n(\Sigma); V) = \varprojlim_T H_n(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma) \setminus T; V),$$

  the inverse limit is taken over all compact subsets $T \subset \mathcal{C}_n(\Sigma)$
- $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ is the properly embedded subspace of $\mathcal{C}_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^-\Sigma \subset \partial\Sigma$.

Twisted representation

Theorem

Let $n \geq 2$, $g \geq 1$, V a representation of the discrete Heisenberg group $\mathcal{H}_g$ over a ring R.

a)The module $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$ is isomorphic to the direct sum of $\begin{pmatrix} 2g+n-1 \\ n \end{pmatrix}$ copies of V. Furthermore, it is the only non-vanishing module in $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$.

b) There is a natural twisted representation of the mapping class group $\mathfrak{M}(\Sigma)$ on the modules

$$H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_\tau V\right) \ , \ \tau \in \mathrm{Aut}(\mathcal{H}) \ ,$$

where the action of $f \in \mathfrak{M}(\Sigma)$ is $\mathcal{C}_n(f)_*$:

$H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau \circ f_{\mathcal{H}}} V\right) \to H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_\tau V\right)$

Schrödinger representation

- The left action of $\mathcal{H}_g^{\mathbb{R}} \supset \mathcal{H}_g$ on the Hilbert space $W \cong L^2(\mathbb{R}^g)$, parametrised by the non zero real number $\hbar$.

$$\left[\rho_W\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\hbar\frac{k-p\cdot q}{2}} e^{i\hbar p\cdot s}\psi(s-q) \ .$$

Theorem (Stone-von Neumann)

W is an irreducible representation of $\mathcal{H}_g^{\mathbb{R}}$ and up to isomorphism is the unique irreducible representation whose character on the center is $(k,0) \mapsto e^{i\hbar\frac{k}{2}}$.

Finite dimensional Schrödinger representation

- For $N \geq 2$ even, $\mathcal{H}$ acts on the f.d. Hilbert space $W_N \cong L^2(\mathbb{Z}_N^g)$:

$$\left[\rho_{W,N}\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\pi\frac{k - p\cdot q}{N}} e^{i\frac{2\pi}{N}p\cdot s}\psi(s-q)\ .$$

Theorem (Stone-von Neumann)

W_N is an irreducible representation of $\mathcal{H}$ and up to unitary isomorphism is the unique irreducible unitary representation whose character on the center is $(k, 0) \mapsto e^{i\pi\frac{k}{N}}$.

Untwisted representation of MCG

- For $\tau \in \mathrm{Aut}(\mathcal{H})$, The Stone-von Neumann theorem provides a unitary isomorphism ${}_\tau W \cong W$ (resp. ${}_\tau W_N \cong W_N$) defined up to S^1.
- We deduce projective actions

$$\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_n) \ , \ \mathcal{V}_n = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W\big)$$

$$\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_{N,n}) \ , \ \mathcal{V}_{N,n} = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W_N\big)$$

- The left regular action of $(k_0, x_0) \in \mathcal{H}_g$ is an automorphism of $\mathcal{H}_g \equiv \mathbb{Z}^{2g+1}$. We decompose $x_0 = p_0 + q_0$, $p_0 \in \Lambda_a = \mathrm{Span}(a_i, 1 \leq i \leq g)$, $q_0 \in \Lambda_b = \mathrm{Span}(b_i, 1 \leq i \leq g)$, then the action is written

$$\begin{cases} k' = k + k_0 + p_0.q - q_0.p \\ p' = p + p_0 \\ q' = q + q_0 \end{cases}$$

- We consider the linearisation ρ_L of this affine action on $L = \mathcal{H}_g \oplus \mathbb{Z}$. The linear action of $\rho_L(k_0, x_0)$ is as follows.

$$\begin{cases} k' = k + tk_0 + p_0.q - q_0.p \\ p' = p + tp_0 \\ q' = q + tq_0 \\ t' = t \end{cases}$$

- For $\tau \in \mathrm{Aut}^+(\mathcal{H}_g)$, the linear map $\tau \times \mathrm{id} : L \mapsto {}_\tau L$ gives an isomorphism of $\mathbb{Z}[\mathcal{H}_g]$-module.

Theorem

There is a representation

$$\mathfrak{M}(\Sigma = \Sigma_{g,1}) \to \mathrm{Aut}(H_n\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); L\big) ,$$

which associates to $f \in \mathfrak{M}(\Sigma_{g,1})$ the composition of the coefficient isomorphism induced by $f_{\mathcal{H}}$,

$$H_n\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); L\big) \to H_n\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{f_{\mathcal{H}}}L\big) ,$$

with the functorial homology isomorphism

$$H_n\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{f_{\mathcal{H}}}L\big) \to H_n\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); L\big) ,$$

Going further

- Is this classical or quantum ?
- Action of cobordisms ?
- Faithfulness ?
- In unitary case, can we find almost invariant vectors ?
- What about closed surfaces ? Is there a Jones type criterion for extending subrepresentations to closed surfaces ?