INTRODUCTION TO QUANTUM INVARIANTS OF 3-MANIFOLDS, TOPOLOGICAL QUANTUM FIELD THEORIES AND MODULAR CATEGORIES

CHRISTIAN BLANCHET

LMAM, Université de Bretagne-Sud, BP 573, F-56017 Vannes, France,
Christian.Blanchet@univ-ubs.fr

Introduction

In 1988 E. Witten introduced the notion of Topological Quantum Field Theory. This was the starting point of a fascinating interaction between mathematics and theoretical physics, deeply relating many domains such as knot theory, Von Neumann algebras, Hopf algebras, Lie algebras and quantum groups, Chern-Simons theory, conformal field theory. A key point in Witten' ideas is the interpretation, in term of Quantum Field Theory, of a famous knot invariant discovered by V. Jones in 1984. A Topological Quantum Field Theory includes topological invariants of 3-dimensional manifolds. Following Witten' ideas, a rigourous construction of such invariants was first obtained from knot theory and quantum groups by Reshetikin and Turaev. It was then shown by Turaev that the relevant algebraic structure was that of a *modular category*. From a modular category one can derive a Topological Quantum Field Theory and in particular, invariants of links and 3-manifolds.

Our purpose here is to give an introduction to the topological aspect of the subject.

In the first part we focus on the construction of knots and links invariants. We discuss the diagrammatic description of knots and links. We introduce the notion of a *solid* link and use it to describe 3-manifolds; this description is so called *surgery presentation* of 3-manifolds. We also develop a categorical formalism which will be used further and give indications for the construction of specializations of the Homfly polynomial invariant of links, including the famous Jones polynomial.

In the second part we give the axiomatic definition of a Topological Quantum Field Theory. Such a theory associates to a genus g surface a finite dimensional vector space which corresponds to the space of so called *conformal* *blocks* in conformal field theory.

In the third part we show that a Topological Quantum Field Theory can be derived from a modular category, which is a braided category with some additional algebraic features. In the last part we sketch examples. These fundamental examples can be obtained either by *skein* theory (some linearization of links) or by using quantum groups.

In conformal field theory, the dimension of the space of conformal blocks is the Verlinde formula. It depends on a simple simply connected complex Lie group and an integer called the level. The interested reader will be able to recover the combinatorics of the $SU(N)$ Verlinde formula from the TQFTs associated with the Homfly polynomial, and will find in the literature the material for the other classical series.

These are lecture notes of a course given at the Summer School *Geometric and Topological Methods for Quantum Field Theory* in Villa de Leyva, Columbia, July 2001. We thank Sylvie Paycha and Sergio Adarve, the CIMPA, the Universidad de los Andes and ECOS-Nord for organizing and supporting this Summer School.

1 LINKS, 3-MANIFOLDS AND THEIR INVARIANTS

We work in the smooth category. For 3-dimensional manifolds, the topological and smooth classifications coincide. This means that a topological 3-manifold admits a unique C^∞ structure, and that two 3-manifolds are homeomorphic if and only if they are diffeomorphic. It may be useful to consider manifolds equipped with a riemannian metric, for example if we need to consider the bundle of (oriented) orthonormal frames; we point out that whenever we do so, the choice of the metric is irrelevant.

1.1 Knots, links and diagrams

For an elementary introduction to knot theory, see [12, 28, 19, 17]. An oriented knot is an embedding of the circle S^1 in the euclidean space $\mathbb{R}^3$ or in its one point compactification, identified with the sphere S^3. An oriented link with m components is an embedding of a disjoint union of m copies of S^1. We are interested in knots and links up to the equivalence given by ambient isotopy.

Definition 1.1. Two embeddings $k_0, k_1 : S^1 \to S^3$ are ambient isotopic if and only ifthere exists a smooth 1-parameter family of diffeomorphisms

$$\begin{aligned} h : S^3 \times [0,1] &\to S^3 \\ (x,t) \quad &\mapsto h(x,t) = h_t(x) \end{aligned}$$

such that (for each t, h_t is a diffemorphism and)

$$h_0 = Id_{S^3} \text{ and } k_1 = h_1 \circ k_0 \ .$$

This relation includes oriented reparametrization. A knot (resp. link) is an embedding of the circle S^1 (resp. disjoint union of copies of S^1) up to (oriented or not) reparametrization. Two isotopic knots are represented in figure 1, called Reidemeister moves. An immersed closed curve in the plane

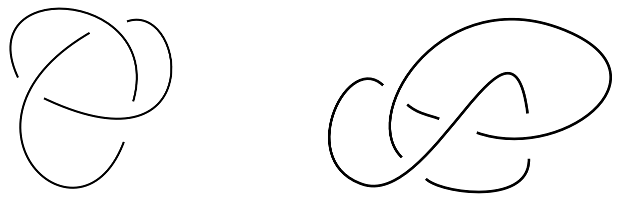


Figure 1. Isotopic knots

$\mathbb{R}^2$ is generic if and only if any multiple point is a double point with non colinear tangent vectors. A regular diagram in the plane is a closed generic immersed curve, and a choice, for each double point, of the arc which passes over. Such a diagram gives a link in $\mathbb{R}^3$ or S^3 well defined up to isotopy. Any link can be obtained this way, and the well known Reidemeister theorem describes the equivalence relation on diagrams induced by isotopy of links.

Theorem 1.2 (Reidemeister theorem). *Two diagrams represent isotopic links if and only if they are related by isotopy in the plane and a finite sequence of the moves in figure 3.*

We will mainly consider here thick knots and links: oriented embeddings of solid tori $D^2 \times S^1$, up to isotopy. (This is equivalent to considering framed knots and links.)

An oriented diagram in the plane gives a thick link by using a blackboard convention. The core of the link is given by the diagram, and the image of

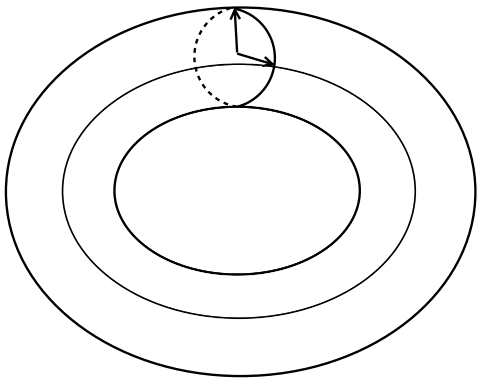
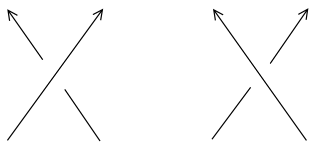


Figure 2. A (trivial) solid knot

the longitude $1 \times S^1$ follows the normal vector pointing outside the blackboard. The Reidemeister theorem for solid knots and links in $\mathbb{R}^3$ or S^3 holds if we replace the first Reidemeister move (R1) by (R1') in figure 4. In the Reidemeister moves one has to consider the various choices of orientations.

Exercise 1.1. We define the weight of a crossing to be $+1$ for the first crossing below (positive crossing), and to be -1 for the second one.



a) Let (K, K') be a 2 components link given by a diagram. Show that the sum of the weights over all mixed crossings is an invariant of the link. Show that it is an even integer. This invariant is twice the *linking number*. For further use, we denote the linking number by $lk(K, K')$.

b) Let K be a solid knot. Show that the sum of the weights over all self crossings is an invariant of the solid knot. This invariant will be called the *framing coefficient* of the solid knot.

(R1)

(R2)

(R3)

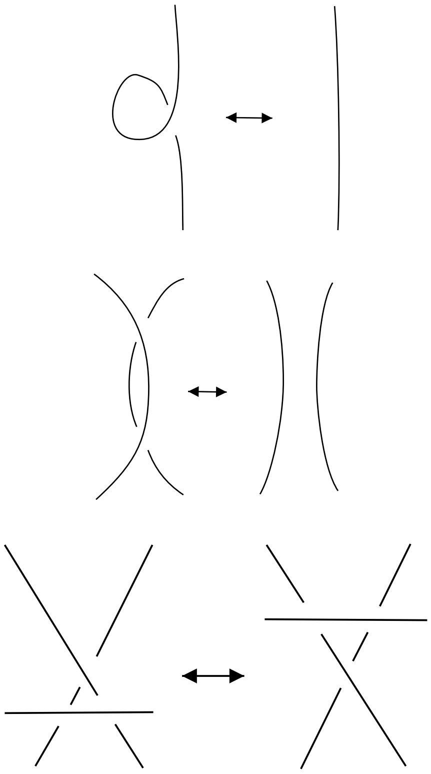


Figure 3. Reidemeister moves

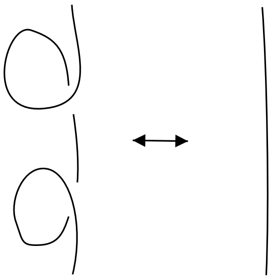


Figure 4. Reidemeister move (R1')

1.2 Surgery on links and Kirby calculus

Let us consider a solid knot in S^3

$$g : D^2 \times S^1 \hookrightarrow S^3 \ .$$

We use it to build a new 3-manifold as follows. We remove the interior of the embedded solid torus, and we glue back another solid torus $-S^1 \times D^2$ using the restriction $g_{|S^1\times S^1}$ (the minus sign here is for reversing orientation). This process is known as surgery. The manifold only depends on the isotopy class K of the solid knot represented by g; we denote it by $S^3(K)$.
Exercise 1.2. a) Show that S^3 is a union of two solid tori with common boundary.
b) Show that $S^2 \times S^1$ is also a union of two solid tori with common boundary.
c) Show that the manifold $S^2 \times S^1$ is obtained by surgery on the solid knot represented by the diagram in figure 5 (0-framed unknot).

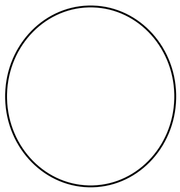


Figure 5. Surgery diagram yielding $S^2 \times S^1$

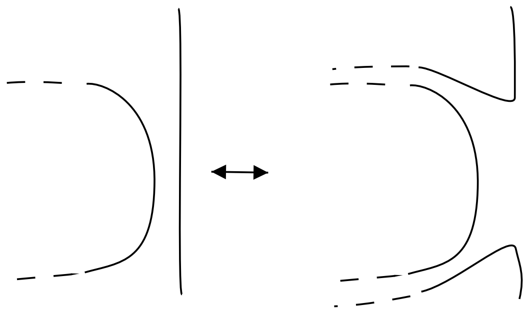
More generally, we will do surgery on a solid link with m components whose isotopy class is denoted by $L = (L_1, \dots, L_m)$; we get a *surgered manifold* denoted by $S^3(L)$.

A theorem of Lickorish and Wallace asserts that any closed oriented 3-manifold is homeomorphic to $S^3(L)$ for some L. As a consequence, we have that any oriented closed 3-manifold can be represented by a diagram in the plane. A famous Kirby theorem describes the equivalence relation on diagrams induced by oriented homeomorphism of 3-manifolds.

Theorem 1.3. *The equivalence relation on diagrams induced by oriented homeomorphism of oriented* 3*-manifolds is generated by isotopy in the plane, the Reidemeister moves (R1'), (R2), (R3), reversing the orientation of one component, and by the two Kirby moves (K1), (K2) described in figure 6.*

The (K1) move is called the handle slide; here the two arcs belong to distinct components of the link. The (K2) move is called stabilization.

(K1)



(K2) Add or remove

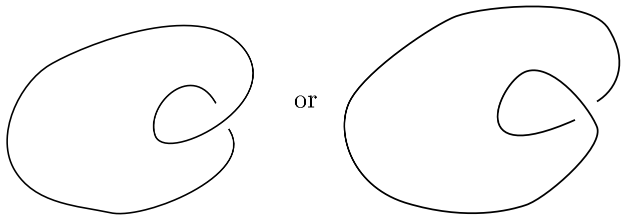


Figure 6. Kirby moves

Exercise 1.3. a) Show that the diagrams in figure 7 represent the projective manifold $\mathbb{R}P^3$, and the same manifold with reversed orientation: $-\mathbb{R}P^3$.
b) Use Kirby calculus to show that $\mathbb{R}P^3$ and $-\mathbb{R}P^3$ are positively homeomorphic oriented manifolds (i.e. there exists an oriented homeomorphism between $\mathbb{R}P^3$ and $-\mathbb{R}P^3$).

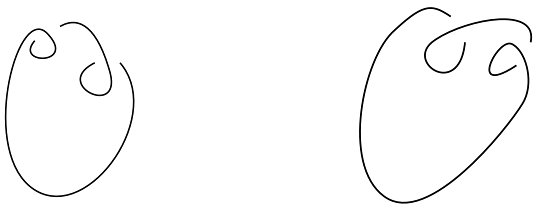


Figure 7. Projective space with positive and negative orientation

1.3 The tangle category

The main idea in the following is to consider invariants of links obtained by associating some algebraic quantities to small pieces of diagram, and algebraic operations to elementary gluings of the pieces. In order to do so, we consider horizontal slices of solid links.

Suppose that we have a solid link in $\mathbb{R}^3$ which intersects the planes $\mathbb{R}^2 \times 0$ and $\mathbb{R}^2 \times 1$ transversally. Then the intersection of the core of the solid link with these planes gives a finite family of framed points in each plane. Here a framing at a point is a trivialization of the tangent bundle to the plane at this point; the orientation of this framing coincides or not with the orientation of the plane, depending if the orientation of the core of the link induced by the embedding goes upwards or downwards at this point.

We define a category* $\mathcal{T}$ in which objects are finite sets of framed points in the disc D^2, and morphisms are pieces of solid links in the cylinder $D^2 \times [0;1]$, with boundary the framed points in the given source and target objects, up to isotopy which is identity on the boundary of the cylinder. Composition is given by stacking the cylinders. Such a morphism is called a tangle and $\mathcal{T}$ is called the tangle category.

People often call tangle category the full sub-category of $\mathcal{T}$ build up with objects whose points have standard positions along the real axis and whose framings are the standard positive or negative one as above. This sub-category is equivalent to $\mathcal{T}$; we will denote it by $\mathcal{T}^s$ and call it the strict tangle category. Morphisms in this category are represented by diagrams in $[-1,1] \times [0;1]$, and the Reidemeister theorem holds. The proof in [12] can be adapted; see [14] for a detailed presentation of the tangle category. An example of tangle is depicted in figure 8.

In such a figure the time parameter goes upwards, so that the morphism gf will be depicted with g lying above f; if the string at a boundary point is upwards, then it is understood that the framing is the standard one in the horizontal plane, and if the string is downwards then the framing is obtained

*A category $\mathcal{C}$ consists of a class of objects $Ob(\mathcal{C})$, for each pair of objects (V,W) a set of morphisms $Hom_{\mathcal{C}}(V,W)$, for each triple of objects (U,V,W) a composition

$$Hom_{\mathcal{C}}(U,V) \times Hom_{\mathcal{C}}(V,W) \to Hom_{\mathcal{C}}(U,W)$$

which is associative and such that any object V admits an identity morphism $\mathbb{1}_V \in End_{\mathcal{C}}(V) = Hom_{\mathcal{C}}(V,V)$.

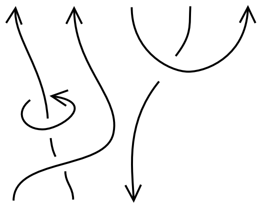


Figure 8. A tangle

from the standard one by applying $[z \mapsto -\overline{z}]$.

We define a tensor product on the category $\mathcal{T}$ by using

$$j = j_{-1} \amalg j_1 : D^2 \amalg D^2 \hookrightarrow D^2 \ ,$$

where, for $\epsilon = \pm 1$, $j_\epsilon : D^2 \hookrightarrow D^2$ is the embedding sending z to $\frac{\epsilon}{2} + \frac{1}{4}z$. Equipped with this tensor product, $\mathcal{T}$ is a monoidal category [20] (the associativity constraints are the obvious ones) and $\mathcal{T}^s$ is a strict[†] monoidal category. If the morphisms f and g are given by diagrams, then $f \otimes g$ is represented by putting these diagrams next to each other , with f on the left and g on the right.

The strategy in the following is to use representations of the strict tangle category. We would like a monoidal functor with source the strict tangle category $\mathcal{T}^s$, and target a category where computations can be done, for example where set of morphisms are finite dimensional vector spaces.

1.4 Yang-Baxter operator

The Yang-Baxter equation was motivated by physical considerations (statistical physics, integrable systems ...) and played a key role in the fundation works on quantum groups. A Yang-Baxter operator is a solution of this famous equation.

Let V be a finite dimensional vector space over the field $\mathbf{k}$. Here any field $\mathbf{k}$ could be used; in most examples one can use complex numbers $\mathbb{C}$. When we introduce a parameter s it may be useful to work with the field of rational functions $\mathbb{Q}(s)$ (generic s). In order to get a 3-manifold invariant we need to

[†]This means that the objects $(m \otimes n) \otimes p$ and $m \otimes (n \otimes p)$ are equal; in a general monoidal category the associativity constraint only gives an isomorphism.

specialized this parameter to a root of unity, and then the convenient scalar field would be a cyclotomic field.

Definition 1.4. A Yang-Baxter operator is an isomorphism $r : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$(r \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes r)(r \otimes \mathbb{1}_V) = (\mathbb{1}_V \otimes r)(r \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes r) \tag{1}$$

We denote by B_n the Artin braid group. We may define B_n as the group generated by the invertible tangles $\sigma_1, \ldots, \sigma_{n-1}$ represented in Figure 9. This

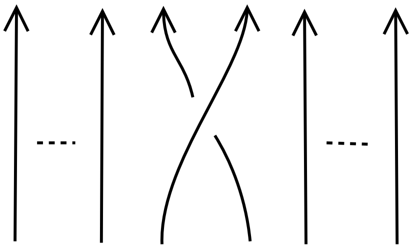


Figure 9. Braid generator σ_i

group has a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$, and relations

$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1 \ldots n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \geq 2 \ . \end{cases}$$

Note that a braid $t \in B_n$ defines a element of the tangle category. In fact we have $B_n \subset End_{\mathcal{T}^s}(n)$.

If r is a Yang-Baxter operator, then we can define a representation $B_n \to GL(V^{\otimes n})$ by sending σ_i to $\mathbb{1}_V^{\otimes i-1} \otimes r \otimes \mathbb{1}_V^{\otimes n-i-1}$. We would like an extension to the strict tangle category, i.e. we want a functor from the tangle category $\mathcal{T}^s$ to the category of vector spaces.

Generators of the tangle category $\mathcal{T}^s$ are depicted in Figure 10. The two first ones are braidings and will be sent to Yang-Baxter operators. The remaining ones should be understood as evaluations and co-evaluations. Our functor will send these generators to *duality morphisms* satisfying certain compatibility conditions.

In order to do so, we pick $\mu \in GL(V)$, and use the isomorphisms $End(V) \approx Hom(V^* \otimes V, \mathbf{k}) \approx Hom(\mathbf{k}, V^* \otimes V)$ given by the (standard) duality to define the homomorphisms corresponding to μ:

$$e_V : V^* \otimes V \to \mathbf{k} \ ; \ b_{V^*} : \mathbf{k} \to V^* \otimes V \ .$$

Here $e_V(f \otimes x) = f(\mu(x))$ and b_{V^*} is the transposed of e_V. The two homomorphisms below are defined similarly using μ^{-1}.

$$e_{V^*} : V \otimes V^* \to \mathbf{k} \ ; \ b_V : \mathbf{k} \to V \otimes V^* \ .$$

We define $r^{\pm}_{V^*,V} \in Hom(V^* \otimes V, V \otimes V^*)$ and $r^{\pm}_{V,V^*} \in Hom(V \otimes V^*, V^* \otimes V)$ by the formulas below.

$$r^{\pm}_{V^*,V} = (e_V \otimes \mathbb{1}_{V \otimes V^*})(\mathbb{1}_{V^*} \otimes r^{\mp} \otimes \mathbb{1}_{V^*})(\mathbb{1}_{V^* \otimes V} \otimes b_V) \ ,$$

$$r^{\pm}_{V,V^*} = (\mathbb{1}_{V^* \otimes V} \otimes e_{V^*})(\mathbb{1}_{V^*} \otimes r^{\mp} \otimes \mathbb{1}_{V^*})(b_{V^*} \otimes \mathbb{1}_{V \otimes V^*}) \ .$$

Definition 1.5. The triple (V, r, μ) is called an *enhanced Yang-Baxter operator* if and only if
i) $\mu \otimes \mu$ commutes with r;
ii) $(e_V \otimes \mathbb{1}_V)(\mathbb{1}_{V^*} \otimes r)(b_{V^*} \otimes \mathbb{1}_V)(e_V \otimes \mathbb{1}_V)(\mathbb{1}_{V^*} \otimes r^{-1})(b_{V^*} \otimes \mathbb{1}_V) = \mathbb{1}_V$;
iii) $r^{\pm}_{V,V^*} r^{\mp}_{V^*,V} = \mathbb{1}_{V^* \otimes V}$.

This terminology was introduced in [31] with a slightly more restrictive definition. The theorem below is established by using a presentation of the tangle category. As a monoidal category, it is generated by the six morphisms in Figure 10, with the relations described in Figure 11.

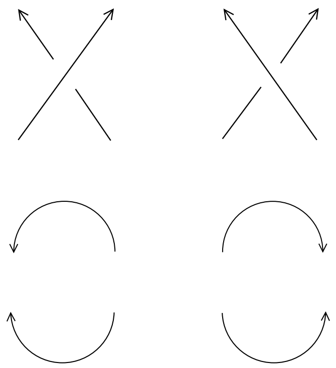


Figure 10. Generators for the strict tangle category $\mathcal{T}^s$

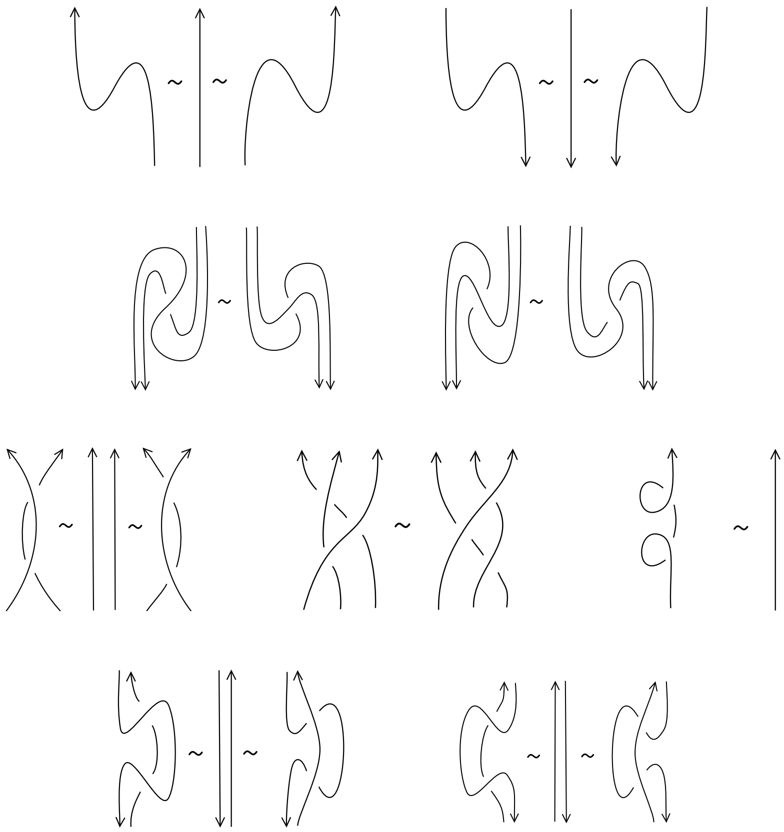


Figure 11. Diagrammatic description of relations in the strict tangle category $\mathcal{T}^s$

For a proof, see for example chapter 12 in [14]. Note that our tangles here involve the framing, so that the relation (2.6) there has to be replaced by the one given by the Reidemeister move (R1').

Theorem 1.6. *If the triple* (V, r, μ) *is an enhanced Yang-Baxter operator then there exists a unique monoidal functor from the strict tangle category* $\mathcal{T}^s$ *to the category of vector spaces whose value on elementary morphisms are given in figure 12.*

Exercise 1.4. (See [31, 30, 24].) Fix an integer $N \geq 2$. Let V be the $\mathbf{k}$-vector space with basis v_i, $i \in \{-N+1, -N+3, \ldots, N-3, N-1\}$. Let s be a non

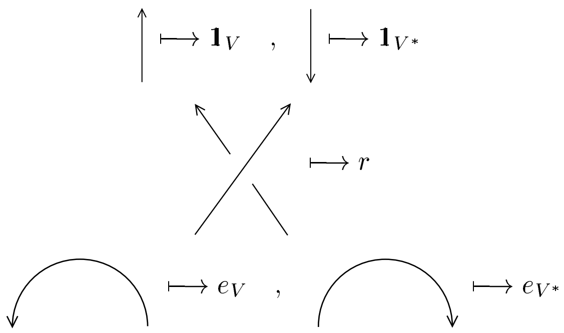


Figure 12. Functor associated with an enhanced Yang-Baxtor operator

zero element in $\mathbf{k}$; denote by $s^{\frac{1}{2}}$ a square root of s. We suppose that $s^{\frac{1}{2}}$ is in $\mathbf{k}$.

Let $E \in End(V \otimes V)$ be defined by

$$E.e_i \otimes e_j = \sum_{k,l} E_{ij}^{kl} e_k \otimes e_l, \text{ with}$$

$$E_{ij}^{kl} = \begin{cases} s & \text{if } i < j, \ k = i \text{ and } l = j; \\ s^{-1} & \text{if } i > j, \ k = i \text{ and } l = j; \\ 1 & \text{if } i \neq j, \ k = j \text{ and } l = i; \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = s\mathbf{1}_{V \otimes V} - E$, and $\mu : V \to V$ be defined by $\mu(v_i) = s^{\frac{i}{2}} v_i$. Note for further use that $r^{-1} = s^{-1}\mathbb{1}_{V \otimes V} - E$, and $r - r^{-1} = (s - s^{-1})\mathbb{1}_{V \otimes V}$.

1. Show that $E^2 = (s + s^{-1})E$.

2. Show that

$$(E \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes E)(E \otimes \mathbb{1}_V) - (E \otimes \mathbb{1}_V) =$$

$$(\mathbf{1}_V \otimes E)(E \otimes \mathbf{1}_V)(\mathbb{1}_V \otimes E) - (\mathbf{1}_V \otimes E) \ .$$

   Deduce that r is a Yang-Baxter operator.

3. Show that (V, r, μ) is an enhanced Yang-Baxter operator.

The invariant of links associated with this enhanced Yang-Baxter operator is the *rank* N specialization of the Homfly polynomial. This terminology will be explained further in relation with quantum groups. The case $N = 2$ gives the famous Jones polynomial.

1.5 Ribbon category

Let $\mathcal{C}$ be a monoidal category with trivial object Θ. A (left) duality on $\mathcal{C}$ associates to any object V an object V^* and morphisms

$$e_V : V^* \otimes V \to \Theta\ ,\ b_V : \Theta \to V \otimes V^*$$

such that

$$(\mathbf{1}_V \otimes e_V)(b_V \otimes \mathbf{1}_V) = \mathbf{1}_V\ , \tag{2}$$

$$(e_V \otimes \mathbf{1}_{V^*})(\mathbf{1}_{V^*} \otimes b_V) = \mathbf{1}_{V^*}\ . \tag{3}$$

A braiding in $\mathcal{C}$ is a natural family of isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V\ \ , V, W \in Obj(\mathcal{C})\ ,$$

such that

$$c_{V,W\otimes W'} = (\mathbf{1}_W \otimes c_{V,W'})(c_{V,W} \otimes \mathbf{1}_{W'})\ , \tag{4}$$

$$c_{V\otimes V',W} = (c_{V,W} \otimes \mathbf{1}_{V'})(\mathbf{1}_V \otimes c_{V',W})\ . \tag{5}$$

A twist in $\mathcal{C}$ compatible with the duality and the braiding is a natural family of isomorphisms

$$\theta_V : V \to V\ \ , V \in Obj(\mathcal{C})\ ,$$

such that

$$(\theta_V \otimes \mathbf{1}_{V^*})b_V = (\mathbf{1}_V \otimes \theta_{V^*})b_V\ , \tag{6}$$

$$\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W)\ . \tag{7}$$

Definition 1.7. A ribbon category is a monoidal category $\mathcal{C}$ with duality, braiding and compatible twist.

The tangle category is a ribbon category. It is universal in the following sense.

Theorem 1.8. *Let $\mathcal{C}$ be a strict ribbon category, and V an object in $\mathcal{C}$, then there exists a unique covariant functor from the strict tangle category $\mathcal{T}^s$ to $\mathcal{C}$, which sends the standard object with one point, denoted by* 1, *to the object V, and preserves tensor product, twist and duality.*

Such a functor is called a ribbon functor. Here strict means that the tensor product is strictly associative. For a proof see Theorem I.2.5 in [32]. Note that a link is an endomorphism of the trivial object, hence we have that this construction provides an invariant of links with values in $End_{\mathcal{C}}(\Theta)$.

Exercise 1.5. Associate a ribbon category to an enhanced Yang-Baxter operator.

The functor in the above theorem has various extensions; in view of our construction of TQFT in the next sections, we introduce a slightly modified version of the colored ribbon graphs considered in [32], which we call solid graphs. Roughly speaking, a solid graph is a 3-dimensional thickening of a finite graph.

A *solid graph* is a 3-dimensional manifold with boundary, which is decomposed into parametrized copies of cylinders $D^2 \times [0,1]$, and solid tori $D^2 \times S^1$. The cylinders are decomposed in two parts which are called respectively the (solid) edges, and the (solid) vertices. The tori are called the closed components. The bottom and top of each solid vertex have (positively or negatively) framed points and so are objects in the category $\mathcal{T}$; each of these points is glued to the end of a solid edge, using the framing.

A coloring of a solid graph over a ribbon category $\mathcal{C}$, is a labeling of the closed components and the solid edges by objects in $\mathcal{C}$, and of the solid vertices by morphisms in $\mathcal{C}$, which is compatible with the gluings.

We define the category of $\mathcal{C}$-colored solid graphs similarly as we did for tangles; here objects are finite sets of framed points in the disc, labeled with objects of $\mathcal{C}$, and morphisms are embedded colored solid graphs whose free ends of edges are conveniently glued to the source and target points. We denote by $\mathcal{G}_{\mathcal{C}}$ this category, and by $\mathcal{G}^s_{\mathcal{C}}$ the strict sub-category (i.e. the full sub-category built up with objects whose points have standard position along the real axis and have standard positive or negative framing).

Here also, morphisms in this category are represented by diagrams. An example of a diagram representing a solid graph is given in Figure 13. Edges are represented by arcs; solid vertices are represented by boxes, and for each of them top and bottom are given by the diagram. A colored solid vertex will

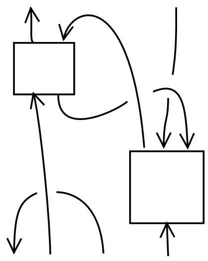


Figure 13. A diagram of a solid graph

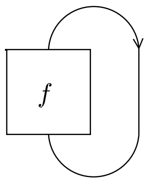
be called a *coupon*.

The following theorem is a reformulation of Theorem I.2.5 in [32].

Theorem 1.9. *Let $\mathcal{C}$ be a strict ribbon category, then there exists a unique monoidal functor from the strict $\mathcal{C}$-colored solid graphs category $\mathcal{G}^s_{\mathcal{C}}$ to $\mathcal{C}$, which respects duality, braiding and twist, and sends a colored positively framed standard point to its label, and a coupon to the morphism given by its label.*

Note that this theorem allows to use the graphical calculus for morphisms in the ribbon category. Whenever we do so, the functor is implicit.

In a ribbon category there is a notion of trace of morphisms and dimension of objects. The trace of a morphism f is denoted by $\langle f \rangle$.

$\langle f \rangle =$ 

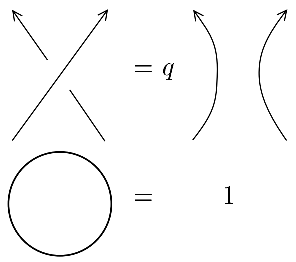
The dimension of an object V is the trace of $\mathbb{1}_V$; we will use the notation $\langle V \rangle$ as well as $\langle \mathbb{1}_V \rangle$. We often say *quantum* trace and dimension to distinguish from the usual trace and dimension in vector spaces.

1.6 The skein method

The tangle category $\mathcal{T}$ is in a natural way a ribbon category. It will be useful to switch to a category in which the objects are the same, but morphisms are linear combinations of tangles with coefficients in some ring $\mathbf{k}$. This category is denoted by $\mathbf{k}[\mathcal{T}]$; note that composition and tensor product are bilinear.

The skein method builds ribbon categories by quotienting the linear tangle category $\mathbf{k}[\mathcal{T}]$ by local relations; the local relations generate a tensor ideal in the monoidal category $\mathbf{k}[\mathcal{T}]$, and the quotient is a ribbon category.

Example 1.1 (Simplest example). The ring is $\mathbb{Z}[q^{\pm 1}]$, the local relation is given below.



Example 1.2 (Homfly skein theory). Here $\mathbf{k}$ is a ring which contains $a^{\pm 1}$, $v^{\pm 1}$, $s^{\pm 1}$, $\frac{v^{-1}-v}{s-s^{-1}}$. The local relations are given in figure 14.

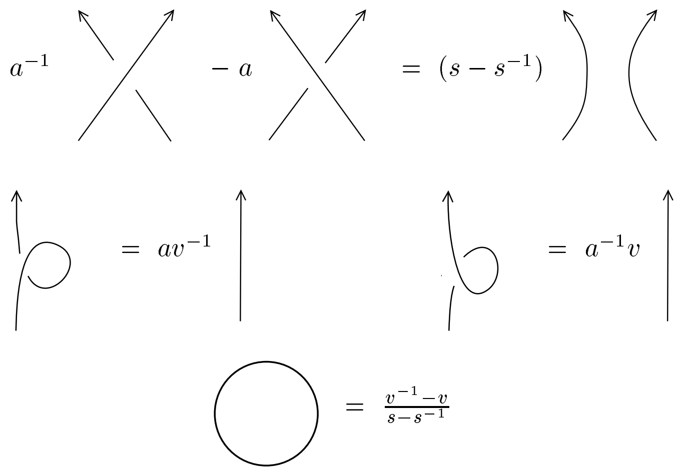


Figure 14. Homfly relations

This example with specific specializations of the parameters will play a key role in our constructions. The ribbon category defined with these relations is called the Hecke category; we denote it by H. The trivial object is denoted by 0, the standard object with n points is denoted by n, the module $End_H(n)$ is denoted by H_n.

Exercise 1.6. We suppose that $\mathbf{k}$ is a field, $N \geq 2$, $v = s^{-N}$, $a = 1$ in $\mathbf{k}$, and that $s^{\frac{1}{2}}$ is in $\mathbf{k}$. Show that the functor in exercise 1.4 induces a functor from the category H to the category of $\mathbf{k}$-vector spaces.

Theorem 1.10. *H_0 is isomorphic to $\mathbf{k}$.*

This theorem gives an invariant of links $\langle L \rangle \in \mathbf{k}$, which is the Homfly polynomial (framed version); here we fix the isomorphism with the normalization $\langle \emptyset \rangle = 1$. The Homfly polynomial was obtained by several authors [13, 26]. A proof based on exercise 1.4 can be found in [31], as well as a proof of the following result.

Theorem 1.11. *The algebra H_n is isomorphic to the quotient of the Artin braid group algebra $\mathbf{k}[B_n]$ by the quadratic relations $a^{-1}\sigma_i - a\sigma_i^{-1} = s - s^{-1}$.*

This algebra is the Hecke algebra; it is a deformation of the symmetric group algebra.

1.7 Hopf algebra and quantum groups

We saw in exercise 1.4 that some $r \in End(V \otimes V)$ satisfies the Yang-Baxter equation. This can be explained by the fact that V is a module over the quantum group $U_q sl(N)$. We give an outline of this approach (see [14] for a basic exposition).

Definition 1.12. A bialgebra over a field $\mathbf{k}$ is an algebra (with unit) A equipped with two algebra morphisms

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to \mathbf{k}$$

such that Δ (the comultiplication) is coassociative, and ϵ (the augmentation) is a counit.

A Hopf algebra[‡] (A, Δ, ϵ, S) is a bialgebra with an antipode S, i.e. an endomorphism S of A such that (here $m : A \otimes A \to A$ denote the multiplication) for any $x \in A$

$$(m \circ (S \otimes Id_A) \circ \Delta)(x) = (m \circ (Id_A \otimes S) \circ \Delta)(x) = \epsilon(x) 1_A \ .$$

[‡]See Section 1 in the lecture notes *Hopf algebras in noncommutative geometry* by Joseph C. Várilly in this book.

As a fundamental example, we have the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$.

Consider the category of finite dimensional representations of a Hopf algebra A, denoted by $Rep(A)$. This category has a tensor product, defined by using the coproduct on A, making $Rep(A)$ into a monoidal category. Moreover, by using the antipode, we can provide $Rep(A)$ with a duality. Denote the coproduct Δ by

$$\Delta(a) = \sum_{(a)} a' \otimes a'' \ ,$$

then the A-module structure on a tensor product and on the dual are defined by

$$a.(v \otimes w) = \sum_{(a)} a'.v \otimes a''.w \ ,$$

$$(a.f)(v) = f(S(a).w) \ .$$

A universal R-matrix on A is an invertible element in $A \otimes A$ such that

$$\forall a \in A \ , \ \Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1} \tag{8}$$

$$(Id_A \otimes \Delta)(R) = R_{13}R_{12} \tag{9}$$

$$(\Delta \otimes Id_A)(R) = R_{13}R_{23} \ . \tag{10}$$

The element v in the center of A is then a universal twist if and only if

$$\Delta(v) = P_A(R)R(v \otimes v) \ , \ \epsilon(v) = 1 \ , \ S(v) = v \ . \tag{11}$$

Here $P_A : A \otimes A \to A \otimes A$ is the flip, $\Delta^{\mathrm{op}}(a) = P_A\Delta(a)$ and

$$R_{12} = R \otimes 1_A \ , \ R_{23} = 1_A \otimes R \ , \ R_{13} = (Id_A \otimes P_A)(R_{12}) \ .$$

Definition 1.13. A ribbon Hopf algebra is a Hopf algebra with universal R-matrix and twist.

Proposition 1.14. *If A is a ribbon Hopf algebra, then $Rep(A)$ is a ribbon category.*

The universal enveloping algebra $U\mathfrak{g}$ of a finite dimensional simple Lie algebra admits a deformation $U_q\mathfrak{g}$ which is almost a ribbon Hopf algebra. Here the *almost* means that the universal R-matrix lives in a suitable completion. If q is a generic parameter, the representation category (which is a ribbon category) is very similar to the classical case, and (type 1) irreducible representations are deformations of the classical ones. There exists an invariant for a link whose components are colored with irreducible representations. For $\mathfrak{g} = sl(N)$, if the *vector representation* is used (deformation of the standard representation $\mathbb{C}^n$), then one can show [30] that the corresponding invariant is the specialization of the Homfly invariant obtained in Exercise 1.4.

2 TOPOLOGICAL QUANTUM FIELD THEORY

In this section, we introduce the axiomatic definition of a TQFT in dimension 3. This concept first appeared in [35, 4]. For basic references, see [32, 5].

2.1 Axiomatic definition

A TQFT in dimension 3 will provide the following assignments.

$$\Sigma \mapsto \mathbf{V}(\Sigma) \ ,$$

where Σ is a surface (possibly with additional structure) and $\mathbf{V}(\Sigma)$ is a finite dimensional $\mathbf{k}$-vector space;

$$M \mapsto \mathbf{V}(M) \in Hom(\mathbf{V}(\Sigma), \mathbf{V}(\Sigma')) \ ,$$

where M is a 3-dimensional cobordism between Σ and Σ', i.e. a 3-manifold with boundary $-\Sigma \amalg \Sigma'$.

This has to satisfy axioms which are conveniently expressed in the categorical language.

The cobordism category C^2 is defined as follows. Objects are closed oriented surfaces; morphisms are cobordisms up to equivalence.

A morphism between Σ_0 and Σ_1 is a 3-manifold M together with a diffeomorphism j between the boundary $-\Sigma_0 \amalg \Sigma_1$ and ∂M. The appropriate notion of equivalence on the cobordisms is orientation preserving diffeomorphism rel. boundary. The cobordisms (M, j) and (M', j') are equivalent if and only if there exists an oriented diffeomorphism $\phi : M \to M'$ such that $j' = \phi \circ j$.

Disjoint union makes this category into a monoidal category. The surface with opposite orientation is a left and right dual; the duality morphisms are given by the product of the surface with the interval $[0;1]$; the evaluation e_Σ is equal to $\Sigma \times [0;1]$, viewed as a cobordism from $-\Sigma \amalg \Sigma$ to $\emptyset$, and the coevaluation b_Σ is equal to $\Sigma\times[0;1]$, viewed as a cobordism from $\emptyset$ to $\Sigma\amalg -\Sigma$. We get a (symmetric) ribbon category.

Definition 2.1. A TQFT in dimension 3 is a monoidal functor $\mathbf{V}$ from C^2 to the category of finite dimensional vector spaces.

From the definition, we have that $\mathbf{V}(\emptyset)=\mathbf{k}$. Hence for a closed 3-manifold M we have an invariant $\mathbf{V}(M)\in End(\mathbf{V}(\emptyset))=\mathbf{k}$. If $\partial M=\Sigma$, then we have a vector $\mathbf{V}(M)\in Hom(\mathbf{k},\mathbf{V}(\Sigma))\simeq\mathbf{V}(\Sigma)$.

Exercise 2.1. Use the duality cobordisms e_Σ and b_Σ to construct inverse isomorphisms between $\mathbf{V}(-\Sigma)$ and $\mathbf{V}(\Sigma)^*$.

Exercise 2.2. Show that the dimension of $\mathbf{V}(\Sigma)$ is equal to the invariant of $\Sigma\times S^1$.

2.2 Framings

In most cases, we have to consider a cobordism category where surfaces and bordisms are equipped with additional structure. Many kinds of structures may be considered; see [29] for an extensive exposition on cobordism with structure. Note that gluing structures along a surface and changing orientation of the surface have to be well defined. Here are some examples.

Framed manifolds. The surfaces are equipped with a trivialization of the oriented stable tangent bundle; cobordisms are equipped with an extension of the given trivialization on the boundary, up to homotopy rel. boundary. Here one has to fix the identification on the boundary. Our convention is 'outgoing normal vector first' for the oriented boundary. The discussion of existence of trivialization of an oriented bundle, and parameterization up to homotopy, is a piece of obstruction theory (see e.g. [22]). The first homotopy groups of $SO(n)$ are: $\pi_1(SO(n))=\mathbb{Z}/2$ for $n\geq(3)$, $\pi_2(SO(n))=0$ for all n and $\pi_3(SO(n))=\mathbb{Z}$ for $n\geq 5$. In the dimensions we are interested in (up to dimension 4), obstruction theory and parameterization up to homotopy will rest on the Stiefel-Whitney class w_2, and the Pontryagin class p_1. If we do so, we have to consider spin structures [10]; the theory has interesting features, but for a basic exposition it is better to work in a simpler context.

Complex framing. It happens that in most cases, because of a so called *framing anomaly* which will be explained further, some kind of framing should be considered. A weak version of framing was proposed by Atiyah to solve this question; this was called 2-framing. The key point is that obstruction theory in dimension up to 4 rests exclusively on the Pontryagin class p_1, or equivalently on signatures of 4-manifold, by the Hirzebruch formula. An equivalent notion called p_1-structure was considered in [9]. We would like to give here a geometric definition.

The operation of adding a trivial line bundle to a vector bundle is called stabilization. If we do this repeatedly on a real vector bundle, then we will get a $\mathbb{R}^\infty$ vector bundle, where $\mathbb{R}^\infty$ is the inductive limit $\lim\limits_{n\to\infty} \mathbb{R}^n$. If we do so on the tangent bundle of a manifold, then we get the stable tangent bundle. If the manifold is oriented, then the complexification of its stable tangent bundle is a SU bundle.

Definition 2.2. A complex framing of an oriented manifold M is a trivialization of the complexified stable tangent bundle (as a SU bundle).

The first obstruction in the existence problem is the second Chern class of the complex bundle $c_2 = -p_1(M)$. Here $p_1(M)$ is the first Pontryagin class of the manifold M. Milnor and Stasheff book [22] is an excellent reference for what is needed here about characteristic classes.

We deduce the following.

Proposition 2.3. *a) An oriented surface Σ admits a complex framing which is unique up to homotopy.*
b) A closed oriented 3-manifold M admits a complex framing and homotopy classes are affinely isomorphic to $H^3(M,\mathbb{Z})$.
c) A closed oriented 4-manifold W admits a complex framing if and only if its signature is zero and homotopy classes are affinely isomorphic to $H^3(W,\mathbb{Z})$.

We may also consider the extension problem. If we fix a complex framing on the boundary of an oriented manifold, we are interested in extensions of it to the whole manifold up to homotopy rel. boundary.

Proposition 2.4. *a) For an oriented 3-manifold M, a complex framing of the boundary extends, and relative homotopy classes of extensions are affinely isomorphic to $H^3(M,\partial M,\mathbb{Z})$.*
b) For an oriented 4-manifold W, a complex framing α on the boundary $\partial W = M$ extends, if and only if some relative obstruction $c_2(\alpha) = -p_1(\alpha) \in H^4(W,\partial W,\mathbb{Z})$ vanishes; moreover another complex framing α' on M is homotopic to α if and only if $p_1(\alpha) = p_1(\alpha')$.

The lemma below will be the key point for further use.

Lemma 2.5. *Let M be a connected closed oriented 3-manifold. There exists a bijection $\alpha \mapsto \sigma(\alpha)$ between complex framings on M up to homotopy and $\mathbb{Z}$,* *such that for any oriented W with $\partial W = M$ one has*

$$\sigma(\alpha) = 3signature(W) - \langle p_1(\alpha), [W] \rangle \ .$$

The integer $\sigma(\alpha)$ will be called the *sigma invariant* of α.

2.3 Colored solid graphs

Let $\mathcal{C}$ be a ribbon category. We have already considered (Section 1.5) $\mathcal{C}$-colored solid graphs. It will be useful in the construction to extend the cobordism category with embedded $\mathcal{C}$-colored solid graphs.

The cobordism category $\mathrm{Cob}(\mathcal{C})$ is defined as follows. Objects are oriented surfaces with a trivialization of the complexified stable tangent bundle, and a finite set of points inside with trivialized neighborhood labeled with objects in $\mathcal{C}$; in brief such an object will be called a $\mathcal{C}$-marked surface.

Morphisms are cobordisms with embedded $\mathcal{C}$-colored solid graphs, and extension of the given complex framings, up to the equivalence given by homeomorphism rel. boundary, isotopy of the graph rel. boundary and homotopy of the complex framing rel. boundary.

2.4 The universal construction

In this subsection we fix a ribbon category $\mathcal{C}$, and we work in the cobordism category $\mathrm{Cob}(\mathcal{C})$. The whole TQFT can be derived from a convenient invariant of closed 3-manifolds. Our method follows the construction in [9].

Here a closed 3-manifold $M = (M, \alpha, G)$ is an oriented closed 3-manifold M with a complex framing α and a $\mathcal{C}$-colored solid graph G. Recall that the complex framing α can be defined by its sigma invariant $\sigma(\alpha)$; we denote simply by S^3 and $S^2 \times S^1$ the corresponding 3-manifold equipped with empty graph and the complex framing α_0 whose σ-invariant is zero.

Let Z be a multiplicative (for disjoint union) invariant of closed 3-manifolds, with value in a field $\mathbf{k}$. We say that Z satisfies the surgery axioms provided (S0), (S1) and (S2) below are satisfied.

In (S1) and (S2), the surgery gives a four dimensional cobordism between the two closed 3-manifolds M and M'. We fix the (homotopy class of the) complex framing on M' from the given one on M, by extending to this cobordism.

(S0) $Z(S^3) \neq 0$.

(S1) There exists a scalar $\mathcal{D}$ such that, if M' is obtained from M by an index one surgery, then $Z(M') = \mathcal{D}Z(M)$.

(S2) There exists a linear combination of objects of $\mathcal{C}$, ω, such that, if M' is obtained from M by an index two surgery on a solid knot K, then $Z(M') = Z(M, K(\omega))$.

Here the component K is colored by a linear combination of objects in the ribbon category $\mathcal{C}$; the corresponding invariant $Z(M, K(\omega))$ is obtained by expanding linearly.

Exercise 2.3. Suppose that Z satisfies the surgery axioms, then show that $\mathcal{D}Z(S^3) = 1$ and $Z(S^2 \times S^1) = 1$.

Given an invariant Z which satisfies the surgery axioms, then we associate to each object Σ of the category $\mathrm{Cob}(\mathcal{C})$ vector spaces $\mathcal{V}(\Sigma)$ and $\mathbf{V}(\Sigma)$ as follows. The vector space $\mathcal{V}(\Sigma)$ is freely generated by morphisms with boundary Σ, i.e. by $Hom_{\mathrm{Cob}(\mathcal{C})}(\emptyset, \Sigma)$. The vector space $\mathbf{V}(\Sigma)$ is the quotient of $\mathcal{V}(\Sigma)$ by the right kernel of the natural bilinear map

$$\mathcal{V}(-\Sigma) \otimes \mathcal{V}(\Sigma) \to \mathbf{k} \ ,$$

defined on the basis by evaluating the invariant Z on the glued manifold.

From the construction we have that $\mathbf{V}$ is a functor from the cobordism category $\mathrm{Cob}(\mathcal{C})$ to the category of vector spaces. However, we will need additional hypothesis in order to have finite dimension and multiplicativity. The role of the surgery axioms is given by the following important lemma ([9], Proposition 1.9).

Lemma 2.6 (Connexion lemma). *Let N (resp. N') be a connected oriented manifold with boundary the surface underlying the object Σ (resp. $-\Sigma$), then*

a) $\mathbf{V}(\Sigma)$ is generated by bordisms M with underlying manifold equal to N.

b) A linear combination $\sum \lambda_i M_i$ of bordisms is zero in $\mathbf{V}(\Sigma)$ if and only if, for any bordism M' with underlying manifold N', one has $\sum \lambda_i Z(M_i \cup_\Sigma M') = 0$.

3 MODULAR CATEGORIES

3.1 Pre-modular and modular categories

Let $\mathbf{k}$ be a field. A monoidal category $\mathcal{C}$, with trivial object Θ, is said to be $\mathbf{k}$-additive if the Hom sets are $\mathbf{k}$-vector spaces, composition and tensor product are bilinear, and $End(\Theta) = \mathbf{k}$. An object X of a $\mathbf{k}$-additive monoidal category is said to be *simple* if the map $u \mapsto u\mathbf{1}_X$ from $\mathbf{k} = End(\Theta)$ to $End(X)$ is an isomorphism. Note that this simplicity condition is stronger that the usual one in module theory.

Definition 3.1. A *modular category* [32], over the field $\mathbf{k}$, is a $\mathbf{k}$-additive ribbon category $\mathcal{C}$ in which there exists a finite family Γ of simple objects λ satisfying the axioms below.

Domination axiom For any object X of the category there exists a finite decomposition $\mathbb{1}_X = \sum_i f_i \mathbf{1}_{\lambda_i} g_i$, with $\lambda_i \in \Gamma$ for every i.

Non-degeneracy axiom The following matrix is invertible.

$$S = (S_{\lambda\mu})_{\lambda,\mu\in\Gamma} \ ,$$

where $S_{\lambda\mu} \in \mathbf{k}$ is the endomorphism of the trivial object associated with the (λ, μ)-colored, 0-framed Hopf link with linking $+1$.

It follows that Γ is a representative set of isomorphism classes of simple objects; note that the trivial object Θ is simple, so that we may suppose that Θ is in Γ. If we remove the last axiom, we get the definition of a *pre-modular* category [11].

If the category $\mathcal{C}$ has direct sums and a zero object, then it is pre-modular if and only if it has finitely many isomorphism classes of simple objects, and any object splits as a direct sum of simple objects.

Note. Direct sums may be added formally. By using the domination property, one can show that any pre-modular category with direct sums is abelian.

Definition 3.2. An object ρ of a pre-modular category A is called *transparent*, if for any object V in A the equality in Figure 15 holds.

Such an object is also called a central object. It is enough to have the above equality for any V in a representative set of simple objects. Note that a category containing a non trivial transparent simple object cannot be modular, simply because the row in the S-matrix corresponding to this transparent object is collinear to the row of the trivial one. In the next subsection we show

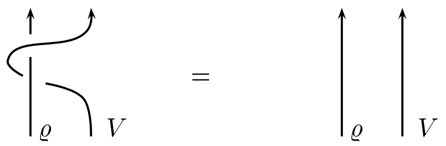


Figure 15. A transparent object ρ

that the absence of non trivial transparent simple objects implies (under a mild assumption) that the category is modular.

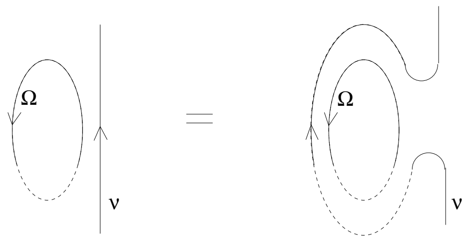
3.2 Properties of pre-modular categories

We will first give some general facts about pre-modular categories. Let $\mathcal{C}$ be a pre-modular category over the field $\mathbf{k}$ and let $\Gamma(\mathcal{C})$ be a representative set of isomorphism classes of its simple objects.

A morphism $f \in Hom_{\mathcal{C}}(X,Y)$ is called negligible if for any $g \in Hom_{\mathcal{C}}(Y,X)$ one has $\langle fg \rangle = 0$. We suppose that $\mathcal{C}$ has no non trivial negligible morphism (we quotient out by negligible morphisms if necessary).

We denote by Ω the *Kirby color*, i.e. $\Omega = \sum_{\lambda \in \Gamma(\mathcal{C})} \langle \lambda \rangle \lambda$. Here and above $\langle \mu \rangle$ is the dimension of μ.

Proposition 3.3. *(Sliding property) Let ν be an object in the pre-modular category $\mathcal{C}$, then the two colored solid graphs depicted below define the same morphism in $End_{\mathcal{C}}(\nu)$.*

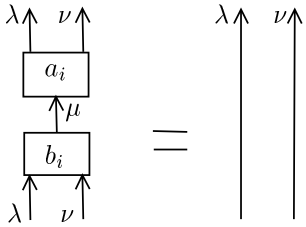


Here the dashed line represents a part of the closed component colored by Ω. This part can be knotted or linked with other components of a colored solid graph representing the morphism. Note that the morphism is unchanged if we reverse the orientation of this closed component.

Proof. For $\lambda, \mu, \nu \in \Gamma(\mathcal{C})$ we have a canonical isomorphism $Hom_{\mathcal{C}}(\lambda \otimes \nu, \mu) \simeq Hom_{\mathcal{C}}(\mu, \lambda \otimes \nu)^*$. (Note that the obvious pairing is non singular, because we have no negligible morphisms.) We denote by a_i, $i \in I_\mu^{\lambda\nu}$, a basis in $Hom_{\mathcal{C}}(\mu, \lambda\otimes\nu)$, and by b_i the dual basis in $Hom_{\mathcal{C}}(\lambda\otimes\nu, \mu)$. By the domination property, we get that a basis in $Hom_{\mathcal{C}}(\lambda \otimes \nu, \lambda \otimes \nu)$, is given by the $b_j a_i$, $i, j \in I_\mu^{\lambda\nu}$, $\mu \in \Gamma(\mathcal{C})$.

By writing the identity of $\lambda \otimes \nu$ in this basis, we get the following decomposition formula (*fusion formula*)

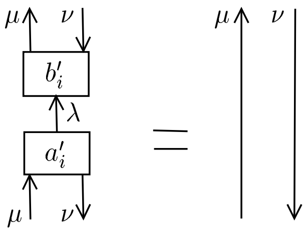
$$\sum_{\mu} \sum_{i \in I_\mu^{\lambda\nu}} \langle \mu \rangle$$



(12)

We denote by $a'_i \in Hom_{\mathcal{C}}(\mu \otimes \nu^*, \lambda)$ (resp. $b'_i \in Hom_{\mathcal{C}}(\lambda, \mu \otimes \nu^*)$) the element corresponding to a_i (resp. to b_i) via the isomorphism given by duality. We also have a fusion formula.

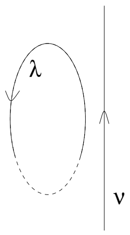
$$\sum_{\mu} \sum_{i \in I_\mu^{\lambda\nu}} \langle \mu \rangle$$



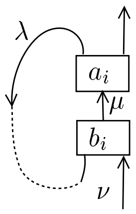
(13)

The calculations below establish the sliding property.

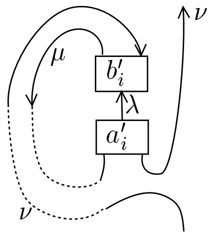
$$\sum_{\lambda \in \Gamma(\mathcal{C})} \langle \lambda \rangle$$



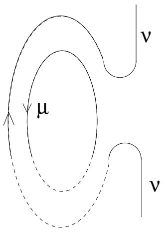
$$= \sum_{\lambda, \mu} \sum_{i \in I_\mu^{\lambda\nu}} \langle \lambda \rangle \langle \mu \rangle$$



$$= \sum_{\lambda,\mu} \sum_{i \in I_{\mu}^{\lambda\nu}} \langle\lambda\rangle\langle\mu\rangle$$



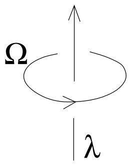
$$= \sum_{\mu \in \Gamma(\mathcal{C})} \langle\mu\rangle$$



In the first and third equalities we use the fusion formula, the second equality holds by isotopy.

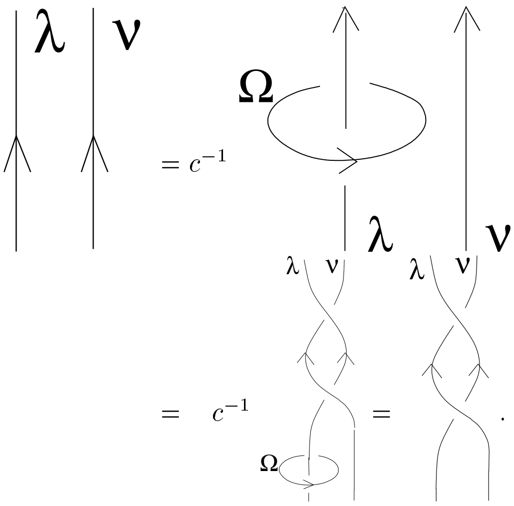
□

Lemma 3.4. *(Killing property) Suppose that $\langle\Omega\rangle$ is nonzero. Let $\lambda \in \Gamma(\mathcal{C})$, then the following morphism is nonzero in $\mathcal{C}$ if and only if λ is transparent.*



Proof. If λ is transparent, then this morphism is equal to $\langle\Omega\rangle\mathbf{1}_\lambda$, which is nonzero. Conversely, if this morphism is nonzero, it is equal to $c\mathbf{1}_\lambda$ for some

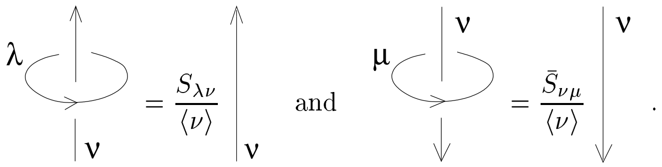
$0 \neq c \in \mathbf{k}$. Then, for any $\nu \in \Gamma(\mathcal{C})$, we have



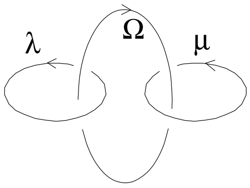
The second equality holds by the sliding lemma. □

Proposition 3.5. *A pre-modular category $\mathcal{C}$ with $\langle \Omega \rangle \neq 0$ which has no non-trivial transparent simple object is modular.*

Proof. We have to check the non-degeneracy axiom. Let us denote by $\bar{S}$ the matrix whose (λ, μ) entry is equal to the value of the 0-framed Hopf link with linking -1 and coloring of the components λ, μ. Then we have that



We deduce that the (λ, μ) entry of the matrix $S\bar{S}$ is equal to the invariant of the colored link depicted below.



By using (12) and the killing property we obtain the formula

$$S\bar{S} = \langle\Omega\rangle I \ ,$$

where I is the identity matrix, which proves the invertibility of the S matrix.
$\square$

3.3 ***A*** 3-***manifold invariant***

If $L = (L_1, \ldots, L_m)$ is a solid link in S^3, let $B_L = (b_{ij})$ be its linking matrix; for $i \neq j$, b_{ij} is the linking number $lk(L_i, L_j)$ (see Exercise 1.1) and b_{ii} is the framing coefficient of L_i.

We denote by b_+ (resp. b_-) the number of positive (resp. negative) eigenvalues of B_L. The signature $sign(L) = b_+ - b_-$ is the signature of the 4-manifold given by surgery on L, and whose boundary is $S^3(L)$.

For $\epsilon = \pm 1$, we denote by U_ϵ the unknot with framing epsilon. From the sliding property we get the following theorem (here Ω is the Kirby color).

Theorem 3.6. *Let $\mathcal{C}$ be a pre-modular category over a field* $\mathbf{k}$ *such that $\langle U_1(\Omega)\rangle\langle U_{-1}(\Omega)\rangle$ is non zero. Then there exists an invariant $\theta_{\mathcal{C}}(M)$ of closed oriented* 3*-manifolds such that for any link $L = (L_1, \ldots, L_m)$ in S^3, one has*

$$\theta_{\mathcal{C}}(S^3(L)) = \frac{\langle L_1(\Omega), \ldots, L_m(\Omega)\rangle}{\langle U_1(\Omega)\rangle^{b_+}\langle U_{-1}(\Omega)\rangle^{b_-}}$$

This invariant can be extended to closed 3-manifolds with a $\mathcal{C}$-colored graph K inside by the formula

$$\theta_{\mathcal{C}}(S^3(L), K) = \frac{\langle L_1(\Omega), \ldots, L_m(\Omega), K\rangle}{\langle U_1(\Omega)\rangle^{b_+}\langle U_{-1}(\Omega)\rangle^{b_-}} \tag{14}$$

In view of the index 2 surgery axiom (S2), we note that a surgery on U_ϵ, $\epsilon = \pm 1$ does not change the 3-manifold. We would like to represent this surgery by inserting some $U_\epsilon(\omega)$. A renormalization $\omega = \zeta\Omega$ should be a good candidate to represents this surgery, but will contribute by a multiplicative coefficient $U_\epsilon(\omega)$. In most cases we cannot choose ζ in such a way that the coefficient is 1 for both values of ϵ. This problem is so called the *framing anomaly*. We will not have the required surgery formula unless we consider an additional structure.

Suppose from now on that $\mathcal{C}$ is a modular category.

Exercise 3.1. Show that $\langle U_1(\Omega)\rangle\langle U_{-1}(\Omega)\rangle = \langle\Omega\rangle \neq 0$.

We extend $\mathbf{k}$ if necessary, and fix κ such that $\kappa^6 = \frac{\langle U_1(\Omega)\rangle}{\langle U_{-1}(\Omega)\rangle}$. Let $\mathcal{D} = \kappa^{-3}\langle U_1(\Omega)\rangle = \kappa^3\langle U_{-1}(\Omega)\rangle$; note that $\mathcal{D}^2 = \langle\Omega\rangle$.

We define the normalized invariant of connected closed 3-manifolds M equipped with complex framing α and colored graph K by

$$Z_{\mathcal{C}}(M,\alpha,K) = \mathcal{D}^{-1}\kappa^{\sigma(\alpha)}\theta_{\mathcal{C}}(M,K)\ . \tag{15}$$

We extend multiplicatively $Z_{\mathcal{C}}$ to non connected closed 3-manifolds.

Proposition 3.7. *The invariant $Z_{\mathcal{C}}$ satisfies the surgery axioms (S0), (S1) and (S2) with $\omega = \mathcal{D}^{-1}\Omega$.*

3.4 TQFT associated with a modular category

As explained in Section 2.4, we use the invariant $Z_{\mathcal{C}}$ to construct a functor $\mathbf{V}_{\mathcal{C}}$ from the category $\mathrm{Cob}(\mathcal{C})$ to the category of vector spaces.

Exercise 3.2. Show that:

1. $\mathbf{V}_{\mathcal{C}}(S^2) = \mathbf{k}$.

2. If λ is a non trivial simple object, then $\mathbf{V}(S^2,\lambda) = 0$.

3. If λ,ν are simple objects, then $\mathbf{V}_{\mathcal{C}}(S^2,\lambda,\nu^*)$ is equal to $\mathbf{k}$ if λ and ν are isomorphic and zero else.

4. If λ,μ,ν are simple objects, then $\mathbf{V}_{\mathcal{C}}(S^2,\lambda,\mu,\nu^*) = Hom_{\mathcal{C}}(\lambda\otimes\mu,\nu) = \mathbf{V}_{\mathcal{C}}(S^2,\lambda,\mu,\nu^*)$ and $\mathbf{V}_{\mathcal{C}}(S^2,\lambda,\mu^*,\nu^*) = Hom_{\mathcal{C}}(\lambda,\mu\otimes\nu)\mathbf{V}_{\mathcal{C}}(S^2,\lambda,\mu^*,\nu^*)$.

Theorem 3.8. *If $\mathcal{C}$ is a modular category, then the functor $\mathbf{V}_{\mathcal{C}}$ is multiplicative and for any $\mathcal{C}$-marked surface Σ the vector space $\mathbf{V}_{\mathcal{C}}(\Sigma)$ has finite dimension.*

Corollary 3.9. *The functor $\mathbf{V}_{\mathcal{C}}$ defines a Topological Quantum Field Theory.*

In the theorem above, multiplicativity is shown by using the surgery axioms. The description of the vector space $\mathbf{V}_{\mathcal{C}}(\Sigma)$ reduces to those in exercise 3.2 by using the splitting formula below.

Lemma 3.10 (Splitting formula). *Let γ be a simple closed curve on a connected surface Σ such that Σ cut along γ has two connected components Σ' and Σ''. Then one has an isomorphism*

$$\sum_{\lambda\in\Gamma}\mathbf{V}_{\mathcal{C}}(\Sigma'_\lambda)\otimes\mathbf{V}_{\mathcal{C}}({}_\lambda\Sigma'')\approx\mathbf{V}_{\mathcal{C}}(\Sigma)\ ,$$

where Σ'_λ is obtained from Σ' by closing with a standard disc whose origin is colored by λ, and ${}_\lambda\Sigma''$ is obtained by closing Σ'' with a copy of the same disc with reversed orientation.

Exercise 3.3. Let γ be a simple closed curve on a connected surface Σ such that Σ cut along γ is a connected surface Σ'. Show that there exists an isomorphism

$$\sum_{\lambda\in\Gamma} \mathbf{V}_{\mathcal{C}}({}_\lambda\Sigma'_\lambda) \approx \mathbf{V}_{\mathcal{C}}(\Sigma) \ ,$$

where ${}_\lambda\Sigma'_\lambda$ is obtained by closing as above.

3.5 Verlinde formula

The Verlinde formula gives the dimension of the TQFT modules.

Proposition 3.11. *Let $\mathcal{C}$ be a modular category with Γ as a representative set of simple objects, then for a closed genus g surface Σ_g without marked point, the dimension of the TQFT module $\mathbf{V}_{\mathcal{C}}(\Sigma_g)$ is*

$$d_{\mathcal{C}}(g) = \mathcal{D}^{2g-2}\sum_{\lambda\in\Gamma}\langle\lambda\rangle^{2-2g} \ .$$

This formula is proved in [32], Corollary 12.1.2. It follows from the computation of the trace of the identity, which is equal to $Z_{\mathcal{C}}(\Sigma_g\times S^1)$.

Exercise 3.4. The 3-manifold $\Sigma_g\times S^1$ is obtained by surgery on the borromean link with $2g+1$ components represented in figure 16. Show 3.11. (Hint: use the fusion formula 12 and the killing lemma 3.4.)

4 EXAMPLES

4.1 The skein method

We give a sketchy description of the whole procedure. This was first applied to the Kauffman bracket skein theory. It was the possible to achieve the construction for the Homfly skein theory [36],[8], and for the skein theory associated with the 2 variables Kauffman polynomial [33, 7].

The relation between these constructions and quantum groups is as follows.

The Kauffman bracket theory corresponds to $U_qsl(2)$.

The Homfly skein theory corresponds to $U_qsl(N)$, $n\geq 2$.

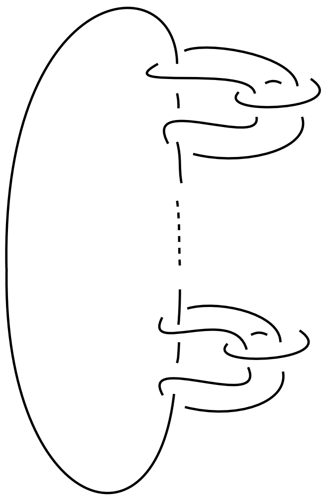


Figure 16. Borromean link: surgery diagram yielding $\Sigma_g \times S^1$

The Kauffman skein theory corresponds to $U_q so(2n+1)$, $U_q sp(2n+1)$ and $U_q so(2n+2)$, $n \geq 1$.

Step 1 Consider a multiplicative (for disjoint union) invariant of (framed) links in the sphere with values in some field $\mathbf{k}$.

Step 2 Define a $\mathbf{k}$-linear ribbon category by considering linear combinations of framed tangles, and quotienting out by negligible morphisms rel. to the invariant given in step 1. Show that the set of morphisms are finite dimensional vector spaces.

Step 3 Add the idempotents of the category as objects and show the (strong) semi-simplicity.

Step 4 Study simple objects. If isomorphism classes of simple objects form a finite set, and there is no non trivial transparent object, then the category is modular. Otherwise, by considering transparent simple objects apply a modularization criterion and procedure developed by Bruguières [11]; see also [25].

Exercise 4.1. Construct a TQFT based on the simplest skein theory in Example 1.1 with q a N-th root of unity if N is odd and a $2N$-th root of unity if N is even (cf [23]).

Exercise 4.2. We consider the Homfly skein theory as defined in Example 1.2. Recall that in this theory H_n denotes the algebra of endomorphisms of n positively oriented points (Hecke algebra).

1. Suppose that the parameters are generic and $\mathbf{k}$ is a field. Show that there exists a unique idempotent $f_n \in H_n$ such that for $i = 1, \dots, n-1$, $\sigma_i f_n = f_n \sigma_i = as f_n$, and a unique idempotent $g_n \in H_n$ such that $i = 1, \dots, n-1$, $\sigma_i g_n = g_n \sigma_i = -as^{-1} g_n$.

2. Construct a TQFT based on Homfly skein theory with parameters $s = a^{-2}$, $v = s^{-2}$, with a a $4l$-th root of unity.

3. Construct a TQFT based on Homfly skein theory with parameters $s = a^{-2}$, $v = s^{-2}$, with a a $2l$-th root of unity, l odd.

4.2 The quantum group machinery

Let $\mathfrak{g}$ be a simple Lie algebra, then its universal enveloping algebra admits a deformation $U_q\mathfrak{g}$ which is called a quantum group. It is a Hopf algebra, and we are interested in the category of its finite dimensional modules. In the generic case, this category is a semi-simple ribbon category, but has infinitely many simple objects, indexed as in the classical case by the fundamental weights. If q is a root of unity, the representation category is much more complicated, however one can obtain the following facts, for an appropriate version of the quantum group (see [32, 16, 5]).

1. It is a ribbon category.

2. There exists a finite set of simple objects (indexed by the affine Weyl alcove) which, after quotienting by negligible morphisms, is closed under decomposition of tensor products and generates a modular category.

The required decomposition of tensor products is established in [2, 3]. The construction was first done in [27]. See [33, 21, 15, 18] for further developments.

References

1. A.K. Aiston, H.R. Morton, *Idempotents of Hecke algebras of type A*, Journal of Knot Theory and Ram. 7 (1998), 463–487.
2. H. Andersen, *Tensor product of quantized tilting modules*, Comm. Math. Physics 149 (1991), 149–159.
3. H. Andersen, Paradowski J. *Fusion categories arising from semi-simple Lie algebras*, Comm. Math. Physics 169 (1995), 563–588.
4. M. Atiyah, *Topological quantum field theories*, Publ. Math. IHES 68 (1989),175–186.
5. B. Bakalov, A. Kirillov, *Lectures on Tensor Categories and Modular Functors*, Univ. Lect. Series Vol. 21, AMS.
6. A. Beliakova, C. Blanchet, *Skein construction of idempotents in the Birman-Murakami-Wenzl algebras*, Math. Ann. 321 (2001) 2, 347–373.
7. A. Beliakova, C. Blanchet, *Modular categories of types B,C and D*, Comment. Math. Helv. 76 (2001) 467–500.
8. C. Blanchet, *Hecke algebras, modular categories and* 3*-manifolds quantum invariants*, Topology, Vol. 39 (2000), 193–223.
9. C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological Quantum Field Theories derived from the Kauffman bracket*, Topology 34, No 4 (1995), 883–927.
10. C. Blanchet, G. Masbaum, *Topological Quantum Field Theories for Surfaces with Spin Structure*, Duke Math. Journal, Vol. 82, No. 2, 229–267.
11. A. Bruguières, *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*, Math. Ann., Vol. 316 (2000), No. 2, 215–236.
12. G. Burde, H. Zieschang, *Knots*, De Gruyter Studies in Math. 5, 1985.
13. P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millet, A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. AMS 12 (1985), 239–246.
14. C. Kassel, Quantum groups, Graduate Texts in Math. 155, Springer-Verlag.
15. A. Kirillov, *On inner product in modular tensor categories*, Journal AMS 9 (1996), 1135–1170.
16. C. Kassel, M. Rosso, V. Turaev, *Quantum groups and knots invariants*, Panoramas et Synthèses No 5, SMF (1997).
17. L. H. Kauffman, Knots and Physics, World Scientific (1991).

18. T. Le, *Quantum invariants of* 3*-manifolds: integrality, splitting, and perturbative expansion*, math.QA/0004099 .
19. W. B. R. Lickorish, *An introduction to Knot Theory*, Springer-Verlag (1997).
20. S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag (1971).
21. G. Masbaum, H. Wenzl, *Integral modular categories and integrality of quantum invariants at root of prime order*, Journal fur die Reine Ang. Math. 505 (1998) 209–235.
22. J. Milnor, J. Stasheff, *Characteristic classes*, Annals of Math. Studies 76, Princeton Univ. Press.
23. H. Murakami, T. Othsuki, and M. Okada, *Invariants of* 3*-manifolds derived from linking matrices of framed links*, Osaka J. Math. 29 (1992), 545-572.
24. H. Murakami, T. Ohtsuki, S. Yamada *Homfly polynomial via an invariant of colored planar graphs*, Ens. Math. 44 (1998), 325–360.
25. M. Müger, *Galois theory for braided tensor categories and the modular closure*, Advances in Math. 150 (2000), 151–201.
26. J. Przytycki, P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math. 4 (1987), 115–139.
27. N. Reshetikhin, V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Inv. Math. 103 (1991), 547–597.
28. D. Rolfsen, *Knots and Links*, Publish Perish (1976).
29. R. Stong, *Notes on cobordism theory*, Princeton Univ. Press (1968).
30. V. Turaev, *The Yang-Baxter equation and invariants of links.*, Invent. Math. 92 (1988), 527–533.
31. V. Turaev, *Operator invariants of tangles and* R*-matrices*, Math. USSR Izvestia Vol. 35 (1990), 411–443
32. V. Turaev, *Quantum invariants of knots and* 3*-manifolds*, De Gruyter Studies in Math. 18, 1994.
33. V. Turaev, H. Wenzl, *Quantum invariants of* 3*-manifolds associated with classical simple Lie algebras*, Int. J. of Math., Vol. 4, No. 2 (1993) 323–358.
34. V. Turaev, H. Wenzl, *Semi-Simple and modular categories from link invariants*, Math. Ann., Vol. 309 (1997), 411–461.
35. E. Witten, *Topological Quantum Field Theory*, Comm. Math. Phys. 117

(1988), 353–386.

36. Y. Yokota, *Skeins and quantum* $SU(N)$ *invariants of* 3*-manifolds*, Math. Ann. 307 (1997), 109–138.