Chapter 1

Heegaard-Floer homologyfor knots, grid model

1.1 Grid diagrams

Définition 1.1.1. A grid diagram of size $n \geq 2$ is a couple of sets of n points $\mathbb{X}$ and $\mathbb{O}$ in the $[0, n]^2 \cap (\frac{1}{2} + \mathbb{Z})^2$, such that:

- the $2n$ points are all distinct,
- each *column* and each row contain exactly one point in $\mathbb{X}$ and one point in $\mathbb{O}$.

A grid diagram defines an oriented link diagram by joining vertically $\mathbb{X}$ to $\mathbb{O}$ and horizontally $\mathbb{O}$ to $\mathbb{X}$, with vertical arcs going over at crossings.

Théorème 1.1.2 (Cromwell, see Dynnikov proposition 4)**.** *Any oriented link admits a grid diagram presentation. The relation on grid diagrams corresponding to isotopy is generated by*

- *cyclic permutation of rows or columns,*
- *exchange of non interleaving adjacent rows or columns,*
- *stabilization.*

Remarque 1.1.3. We may consider grid diagrams on the torus rather than on the square, which removes the cyclic permutation move in the preceeding theorem.

Exercice 1.1.4. 1. Give a presentation of the fundamental group of the complement of a link given by a grid diagram, using meridian curves around vertical arcs as generating set.
2. Give a presentation of the Alexander module of a knot given by a grid diagram.
3. Give a formula for Alexander polynomial of a knot given by a grid diagram.

1.2 Combinatorial Heegaard-Floer complex, tilde version

Here we are working with scalar field $\mathbb{F}_2$. Let $G = (\mathbb{X}, \mathbb{O})$ be a grid diagram of size n. We will consider this diagram on the torus. The complex $\widetilde{CK}(G)$ has dimension $n!$ and basis S formed with sets of n integral points in the grid representing permutations (graphs of permutations).

$$S = \{\mathbf{x}_\sigma, \sigma \in \mathcal{S}_n\} \ .$$

For $\mathbf{x}$, $\mathbf{y}$ in S, we consider an associated set of empty rectangles in the torus $\mathrm{Rect}^0(\mathbf{x}, \mathbf{y})$ which is empty unless the two permutations are obtain from each other by composing with a tranposition. In the case where $\mathbf{x} = \mathbf{x}_\sigma$, $\mathbf{y} = \mathbf{y}_\tau$ with $\tau = \sigma \circ (ij)$, then $\mathrm{Rect}^0(\mathbf{x}, \mathbf{y})$ contains the rectangles in the square with

- corners in $\mathbf{x}$, $\mathbf{y}$,
- horizontal edges from $\mathbf{x}$ to $\mathbf{y}$,
- vertical edges from $\mathbf{y}$ to $\mathbf{x}$,
- no points from $\mathbf{x}$ and $\mathbf{y}$ inside.

The formula for the boundary map $\widetilde{\partial}$ is given by

$$\widetilde{\partial}\mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{r \in \mathrm{Rect}^0(\mathbf{x},\mathbf{y}) \\ \mathbb{X}(r) = \mathbb{O}(r) = 0}} \mathbf{y} \ .$$

Here, $\mathbb{X}(r)$ and $\mathbb{O}(r)$ are counting $\mathbb{X}$ or $\mathbb{O}$ points in the rectangle.

Proposition 1.2.1. *One has $\widetilde{\partial} \circ \widetilde{\partial} = 0$.*

Gradings

The M grading is uniquelly defined by

$$M(\mathbf{y}) - M(\mathbf{x}) = 1 - 2\mathbb{O}(r) \text{ ,for all } r \in \mathrm{Rect}^0(\mathbf{x}, \mathbf{y}) \ ,$$

and normalisation $M(\mathbf{x}_0 = 1 - n$, where $\mathbf{x}_0$ is formed with lower left corners of $\mathbb{O}$ points.

The unormalised Alexander grading is defined as the sum of minus winding numbers:

$$a(\mathbf{x}) = \sum_{x \in \mathbf{x}} a(x) \ ,$$

where $-a(x)$ is the linking number between the knot and *the oriented vertical* throw x.

Proposition 1.2.2. *The boundary map $\widetilde{\partial}$ decreases M by 1 and respects a.*

We get a bigraded homology $\widetilde{HK}(G) = \sum_{i,j} \widetilde{HK}_{i,j}(G)$. Its graded Euler characteristic is the polynomial

$$\chi(\widetilde{HK}(G)) = \chi(\widetilde{HK}(G)) = \sum_{i,j} (-1)^i \mathrm{rk}(\widetilde{CK}_{i,j}(G))\, t^j \ .$$

Théorème 1.2.3. *Suppose that G represents a knot K, then the graded Euler characteristic $\chi(\widetilde{HK}(G))$ is related to the Alexander polynomial $\Delta_K(t)$ by the formula*

$$\chi(\widetilde{HK}(G)) \doteq (t-1)^{n-1}\Delta_K(t) \ .$$

Here the $\doteq$ means equality up to sign and power of t. One can normalise the Alexander grading in $\frac{1}{2}\mathbb{Z}$ in order to get an exact formula with *symmetrised Alexander polynomial.*

One can show that $\widetilde{HK}(G)$ is invariant by exchange move, but not by stabilisation. The changing in stabilisation move can be understood, and in fact $\widetilde{HK}(G)$ determines the hat Heegaard-Floer homology $\widehat{HFK}(K)$.

1.3 Combinatorial Heegaard-Floer homology of knots, full package

We consider the case of a knot given by a grid diagram od size n, $G = (\mathbb{X}, \mathbb{O})$. We introduce n indeterminates $U_1, \ldots, U_n$. The complex $CFK(G)$ is feely generated over $\mathbb{F}_2[U_1, \ldots, U_n]$ by the previous set S. The boundary map is defined by

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{r \in \mathrm{Rect}^0(\mathbf{x},\mathbf{y}) \\ \mathbb{X}(r)=0}} U_1^{\mathbb{O}_1(r)} \ldots U_n^{\mathbb{O}_n(r)}\, \mathbf{y} \ .$$

The M grading is extended by setting $M(U_k) = -2$. The unormalised Alexander grading is extended by $a(U_k) = -1$, and further normalised by $A(\mathbf{x}) = a(\mathbf{x}) - A_0$ (see formula for A_0 in reference MOST).

Théorème 1.3.1. *a) $(CFK(G), \partial)$ is a bigraded complex.*
b) Exchange or stabilisation moves produces homotopy equivalent complexes as modules over $\mathbb{F}_2[U_1]$.
c) The bigraded $\mathbb{F}_2[U_1]$-module $HFK(G) = HFK(K)$ is an invariant of the knot.
d) By setting $U_1 = 0$ one get the bigraded homology $\widehat{HFK}(K)$ whose graded Euler characteristic is the symmetrised Alexander polynomial.

The full package is developed in reference MOST for links and produces a complex with filtration whose filtrated homotopy type is an invariant.