Chapter 1

Grid diagrams

1.1 Grid diagram for knots and links

Definition 1.1.1. A grid diagram of size $n \geq 2$ is a couple of sets of n points $\mathbb{X}$ and $\mathbb{O}$ in $[0,n]^2 \cap (\frac{1}{2} + \mathbb{Z})^2$, such that:

- the $2n$ points are all distinct,
- each *column* and each row contain exactly one point in $\mathbb{X}$ and one point in $\mathbb{O}$.

A grid diagram defines an oriented link diagram by joining vertically $\mathbb{X}$ to $\mathbb{O}$ and horizontally $\mathbb{O}$ to $\mathbb{X}$, with vertical arcs going over at crossings. In order to obtain a smooth link corners should be smoothed. A link in $\mathbb{R}^3$ whose orthogonal projection in xy plane is a diagram associated with a grid one will be called a grid diagram link.

Definition 1.1.2. An oriented link in $\mathbb{R}^3$ is in pre-grid position if its orthogonal projection on horizontal plane $\mathbb{R}^2 \times 0$ is a generic immersion such that at every crossing the overcrossing tangent line is non collinear to the first coordinate vector $(1,0,0)$.

By transversality, any link is arbitrary closed, and hence isotopic, to a link in pre-grid position. If we are given the projection in xy plane of a link in pre-grid position, then we may find a *staircase* approximation of this projection representing a link which is isotopic to the starting one throw an isotopy among links in pre-grid position. If necessary we use an extra isotopy in order to get a diagram in which any two segments are not collinear. We then obtain (up to vertical and horizontal isotopy) is a grid diagram link.

1.2 Grid moves

Definition of exchange and stabilisation moves.

Theorem 1.2.1 (Cromwell, see Dynnikov proposition 4 or Appendix in OSS book)**.** *Any oriented link is isotopic to a grid diagram link. The relation on grid diagrams corresponding to isotopy is generated by*

- *exchange of non interleaving adjacent rows or columns,*
- *stabilisation.*

We gave above a procedure for producing a grid representative. For proving the theorem we have to show that the following moves can be realised by a finite sequence of exchange/stabilisation moves.

- An isotopy among pre-grid position links.
- An isotopy going throw an overcrossing arc parallel to x coordinate.
- Reidemeister moves.

Remark 1.2.2. Applying a cyclic permutation of rows or colomn gives an isotopic link. Even if this move can be deduced from the previous ones it can be useful. In particular there are two type of exchange moves depending wether the two segments overlap or not. Each type can be deduce from the other togetnher with cyclic permutations.

Exercice 1.2.3. 1. Give a presentation of the fundamental group of the complement of a link given by a grid diagram, using meridian curves around vertical arcs as generating set.
2. Give a presentation of the Alexander module of a knot given by a grid diagram.
3. Give a formula for Alexander polynomial of a knot given by a grid diagram.

1.3 Grid diagrams and Alexander polynomial

Theorem 1.3.1. *Let K be a knot associated with a grid diagram of size n, then a presentation of the fundamental group $\pi_1(S^3 - K, \infty)$ is given with one generator x_j, $1 \leq j \leq n$, for each vertical arc of the diagram, represented by a meridian around the arc, and one relation r_i, $1 \leq i \leq n-1$, corresponding to horizontal grid segments written as words in generators.*

A presentation of the Alexander module can be obtained similarly, or deduced using Fox calculus.

Theorem 1.3.2. *A presentation over $\mathbb{Z}[t, t^{-1}$ for the Alexander module $H_1(\widetilde{X}_K)$ is given with $n-1$ generators $a_j = [\widetilde{x}_j - \lambda_j \widetilde{x}_n]$, $1 \leq j \leq n-1$, and the $n-1$ relations $\widetilde{r}_i$, $1 \leq j \leq n-1$, obtained by lifting the loops in the cyclic covering.*

The Fox derivative $\frac{\partial}{\partial x_j}$ is defined on words in $x_i^{\pm 1}$ with value in the algebra of the free group by the rules

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \ , \ \frac{\partial (vw)}{\partial x_j} = \frac{\partial v}{\partial x_j} + v \frac{\partial w}{\partial x_j} \ .$$

Then a presentation matrix for the Alexander module is given by applying the abelianisation morphism to the Fox matrix

$$\left(\frac{\partial r_i}{\partial x_j}\right)_{1\leq i,j\leq n-1} .$$

Here the relations are read horizontally. The Alexander polynomial is represented, up to $\pm t^k$ by the previous presentation matrix.

We will now extract the Alexander-Conway polynomial of links, $\nabla_L(x)$. Recall that it is uniquely determined by the value 1 on unknot and the skein relation

$$\nabla_{L_+} - \nabla_{L_-} = (x - x^{-1})\nabla_{L_0} .$$

It is a normalised refinement of the Alexander polynomial, which is recovered by substitution $x = \sqrt{t}$.

Definition 1.3.3. Let G be a grid diagram of size n. The grid matrix $M(G)$ is the $n \times n$ matrix whose (i, j) entry is equal to $t^{-I(j-1,n-i)}$, where $I(j-1, n-i)$ is the index (linking number with the link projection) of the $(j-1, n-i)$ lattice point.

Theorem 1.3.4. *The Alexander-Conway polynomial of the grid link L is given by the formula*

$$\nabla_L(\sqrt{t}) = \epsilon(G)\det(M(G))(t^{-\frac{1}{2}} - t^{\frac{1}{2}})^{1-n}t^{a(G)} .$$

Here $a(G) = \frac{1}{8}\sum_j I(\mathbb{X}_j) + I(\mathbb{O}_j)$*, where the index of a point in a square is the sum of indices of the four neighbouring lattice points; and* $\epsilon(G)$ *is the signature of the permutation* $\sigma_{\mathbb{O}} \circ (n, n-1, \ldots, 1)$*, where* $\sigma_{\mathbb{O}}$ *is the permutation whose graph is given by the upper right corner of points in* $\mathbb{O}$*.*

To prove the theorem, show that the given formula is invariant by grid move and satisfy the skein relation. See OSS book, Th. 3.3.6.