Chapter 2

Combinatorial Heegaard-Floer homology

2.1 Combinatorial Heegaard-Floer complex, tilde version

Here we are working with scalar field $\mathbb{F}_2$. Let $G = (\mathbb{X}, \mathbb{O})$ be a grid diagram of size n. We will consider this diagram on the torus. The complex $\widetilde{GC}(G)$ has dimension $n!$ and basis $S(G)$ formed with sets of n integral points in the grid representing permutations (graphs of permutations).

$$S(G) = \{\mathbf{x}_\sigma, \sigma \in \mathcal{S}_n\} \ .$$

For $\mathbf{x}$, $\mathbf{y}$ in S, we consider an associated set of empty rectangles in the torus $\mathrm{Rect}^0(\mathbf{x}, \mathbf{y})$ which is empty unless the two permutations are obtain from each other by composing with a tranposition. In the case where $\mathbf{x} = \mathbf{x}_\sigma$, $\mathbf{y} = \mathbf{y}_\tau$ with $\tau = \sigma \circ (ij)$, then $\mathrm{Rect}^0(\mathbf{x}, \mathbf{y})$ contains the rectangles in the square with

- corners in $\mathbf{x}$, $\mathbf{y}$,
- horizontal edges from $\mathbf{x}$ to $\mathbf{y}$,
- vertical edges from $\mathbf{y}$ to $\mathbf{x}$,
- no points from $\mathbf{x}$ in its interior.

The formula for the boundary map $\widetilde{\partial}$ is given by

$$\widetilde{\partial}\mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{r \in \mathrm{Rect}^0(\mathbf{x}, \mathbf{y}) \\ \mathbb{X}(r) = \mathbb{O}(r) = 0}} \mathbf{y} \ .$$

Here, $\mathbb{X}(r)$ and $\mathbb{O}(r)$ are counting $\mathbb{X}$ or $\mathbb{O}$ points in the rectangle.

Proposition 2.1.1. *One has $\widetilde{\partial} \circ \widetilde{\partial} = 0$.*

Gradings

The M grading is uniquely defined by

$$M(\mathbf{y}) - M(\mathbf{x}) = 1 - 2\mathbb{O}(r) \text{ , for all } r \in \text{Rect}^0(\mathbf{x}, \mathbf{y}) \ ,$$

and normalisation $M(\mathbf{x}^{NW\mathbb{O}}) = 0$, where $\mathbf{x}^{NW\mathbb{O}}$ is formed with upper left corners of $\mathbb{O}$ points.

The Alexander grading is defined by using winding numbers. For a point x with non integral coordinates x, $I(x)$ is the linking number between the knot and *the oriented vertical* line throw x. For marks, $\mathbb{X}_J$, $\mathbb{O}_j$, we define $I(\mathbb{X}_J)$, $I(\mathbb{O}_j)$ as the sum of winding numbers of the fours centers of the neighbouring grid squares. Then the Alexander grading is defined by:

$$A(\mathbf{x}) = -\frac{n-1}{2} + \sum_j I(\mathbf{x}_j) + \frac{1}{8} \sum_j \left(I(\mathbb{X}_j) + I(\mathbb{O}_j) \right) \ .$$

Proposition 2.1.2. *The boundary map $\widetilde{\partial}$ decreases M by 1 and respects a.*

We get a bigraded homology $\widetilde{GH}(G) = \sum_{i,j} \widetilde{GH}_{i,j}(G)$. Its graded Euler characteristic is the polynomial

$$\chi(\widetilde{GH}(G)) = \chi(\widetilde{GH}(G)) = \sum_{i,j} (-1)^i \text{rk}(\widetilde{GC}_{i,j}(G))\, t^j \ .$$

Theorem 2.1.3. *Suppose that G represents a knot K, then the graded Euler characteristic $\chi(\widetilde{GH}(G))$ is related to the Alexander polynomial $\Delta_K(t)$ by the formula*

$$\chi(\widetilde{GH}(G)) \doteq (t-1)^{n-1} \Delta_K(t) \ .$$

Here the $\doteq$ means equality up to sign and power of t. The refined formula using Alexander-Conway polynomial also holds for link.

Theorem 2.1.4. *If L is the link represented by the grid diagram G. Then the graded Euler characteristic $\chi(\widetilde{GH}(G))$ is related to the Alexander-Conway polynomial by the formula*

$$\chi(\widetilde{GH}(G)) = (1-t)^{n-1} \nabla_L(\sqrt{t}) \ .$$

One can show that $\widetilde{GH}(G)$ is invariant by exchange move, but not by stabilisation. The changing in stabilisation move can be understood, and in fact $\widetilde{GH}(G)$ determines the hat version of Heegaard-Floer homology $\widehat{GH}(K)$, which is an isotopy invariant.

2.2 Combinatorial Heegaard-Floer homology of knots, full package

We consider the case of a knot given by a grid diagram of size n, $G = (\mathbb{X}, \mathbb{O})$. We introduce n indeterminates $V_1, \ldots, V_n$. The complex $GC^-(G)$ is freely generated over $\mathbb{F}_2[V_1, \ldots, V_n]$ by the previous set $S(G)$. The graded boundary map is defined by

$$\partial_{\mathbb{X}}^- \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{r \in \mathrm{Rect}^0(\mathbf{x},\mathbf{y}) \\ \mathbb{X}(r)=0}} V_1^{\mathbb{O}_1(r)} \ldots V_n^{\mathbb{O}_n(r)}\, \mathbf{y} \ .$$

The M grading is extended by setting $M(V_k) = -2$. The unormalised Alexander grading is extended by $A(V_k) = -1$.

Theorem 2.2.1. *a) $(GC^-(G), \partial_{\mathbb{X}}^-)$ is a bigraded complex.*
b) Exchange move produces an homotopy equivalent complex.
c) Stabisation move produces a complex with isomorphic homology.

Theorem 2.2.2. *Let G be a grid diagram of a knot K.*
a) The bigraded $\mathbb{F}_2[V_1]$-module $GH^-(G) = H(GC^-(G), \partial_{\mathbb{X}}^-)$ is an invariant of the knot K.
b) By setting $V_1 = 0$ one get the bigraded homology $\widehat{GH}(K)$, which is an isotopy invariant of the knot K whose graded Euler characteristic is the Alexander-Conway polynomial.

The filtrated boundary map is defined by

$$\partial^- \mathbf{x} = \sum_{\mathbf{y}} \sum_{r \in \mathrm{Rect}^0(\mathbf{x},\mathbf{y})} V_1^{\mathbb{O}_1(r)} \ldots V_n^{\mathbb{O}_n(r)}\, \mathbf{y} \ .$$

The full package is developped in reference MOST for links and produces a complex with filtration whose filtrated homotopy type is an invariant.