Chapter 3

Symmetric spaces

Let X be a topological space. For $n \geq 1$, we denote by $SP^n(X)$ the quotient of X^n by action of the symmetric group $\mathcal{S}^n$.

3.1 Examples

$SP^n(\mathbb{C}) \cong \mathbb{C}^n$, $SP^n(\mathbb{C}P^1) \cong \mathbb{C}P^n$.
$SP^2(S^1)$ is a Mobius band.

Proposition 3.1.1. *Let Σ be a complex curve, then $SP^n(\Sigma)$ is a smooth complex manifold.*

If X has basepoint $x_0 \in X$, then adding a base point defines an inclusion map $SP^n(X) \in SP^{n+1}(X)$. It will be useful to consider the inductive limite $SP^\infty(X) = \cup_n SP^n(X)$. There is a natural map $SP^n(X) \times SP^m(X) \to SP^{n+m}(X)$. If we fix the base point, then we get a commutative monoid structure on $SP^\infty(X)$. We denote by $*$ the monoid operation.

3.2 Symmetric spaces of complex curve

Let Σ be a genus g complex curve (closed real surface). We denote by $\Sigma^1 = \bigvee_{2g} S^1$ its 1-squeleton for a standard cell decomposition with one 0-cell v_0, $2g$ 1-cells $e_1, \ldots, e_g$, $e_{g+1} = e'_1, \ldots, e_{2g} = e'_g$ and one $2-cell$ denoted d with boundary $\partial d = \Pi_{i=1}^g e_i e'_i \overline{e_i} \overline{e'_i}$.

The space $SP^n(\Sigma^1)$ contains many Mobius bands coming from second symmetric product of 1-cells.

We define $\overline{SP}^n(\Sigma^1)$ (resp. $\overline{SP}^n(\Sigma)$) to be the quotient of $SP^n(\Sigma^1)$ (resp. $SP^n(\Sigma)$) by identification of $x * y$ with $v_0 * xy$ for $x, y \in e_i \cong S^1$, $1 \leq i \leq 2g$.

The space $\overline{SP}^n(\Sigma^1)$ (resp. $\overline{SP}^n(\Sigma)$) is homotopy equivalent to $SP^n(\Sigma^1)$ (resp. $SP^n(\Sigma)$).

The monoid structure on $SP^\infty(\Sigma^1)$ and $SP^\infty(\Sigma)$ induces one on the quotients $\overline{SP}^\infty(\Sigma^1)$ and $\overline{SP}^\infty(\Sigma)$.

Lemma 3.2.1. *$(SP^s(D^2), SP^s(D^2) - SP^s(\mathring{D}^2)$ is a $2s$-cell (i.e. homeomorphic to (D^{2s}, S^{2s-1})).*

We denote by u_s the $2s$-cell $SP^s(d)$.

Using the monoid structure, we get a CW-complex structure on $\overline{SP}^\infty(\Sigma)$ with cells v_0 and

$$e_{i_1} * e_{i_2} * \cdots * e_{i_t} * u_s, \quad i_1 < \cdots < i_t \ , \ s \geq 0 \ .$$

In the cell complex, all boundaries are zero.

Homology and cohomology as modules follow. The monoid structure gives a product on homology:

Theorem 3.2.2. *The algebra $H_*(SP^\infty(\Sigma)) \cong H_*(\overline{SP}^\infty(\Sigma))$ is multiplicatively generated by e_i, $1 \leq i \leq 2g$, u_s, $s \geq 0$ with relations*
$e_i^2 = 0$,
$e_i e_j = -e_j e_i$, *for* $i \neq j$,
$e_i u_s = u_s e_i$,
$u_s u_{s'} = \binom{s+s'}{s} u_{s+s'}$.

Theorem 3.2.3. *The map $\pi^* : H^*(SP^n(\Sigma)) \to H^*(\Sigma^n)$ is an injective homomorphism, with image the subring of invariant classes under action of symmetric group: $H^*(\Sigma^n)^{\mathcal{S}_n}$.*

Theorem 3.2.4. *The ring $H^*(SP^n(\Sigma))$ is generated by the 1-dimensional classes $f_i = e_i^*$, and the 2-dimensional class $b = u_1^*$.*

Exercice 3.2.5. Prove the previous theorem.

Theorem 3.2.6. *For $n \geq 2$, $\pi_1(SP^n(\Sigma)) \cong H_1(\Sigma)$.*

We will need the group $\pi_2'(SP^n(\Sigma))$ which is the quotient of $\pi_2(SP^n(\Sigma))$ by action of $\pi_1(SP^n(\Sigma)$ (the action is trivial for $n \geq 3$).

An hyperelliptic involution on Σ is an involution with $2g+2$ fixed points τ. We then get $\Sigma/\tau \simeq S^2$.

For $n \geq 2$ we defined a map $S : S^2 \to SP^n(\Sigma)$ by quotienting the map

$$Id \times \tau \times const : \Sigma \to \Sigma^n \ .$$

Theorem 3.2.7. *For $n \geq 2$, $\pi_2'(SP^n(\Sigma)) \approx \mathbb{Z}$, generated by S.*

3.3 Symmetric spaces and Heegaard diagrams

ch référence Ozsvath-Szabo, Holomorphic disks and topological invariants for closed three-manifolds, Ann. of Math. (2) 159 (2004), no. 3, 1027–1158, section 2.