UNIVERSITÉ PARIS DIDEROT
Année 2014-2015, Master 2
Topology of low dimensional manifolds.

Exam, 03/03/2015 (duration: 3 hours)

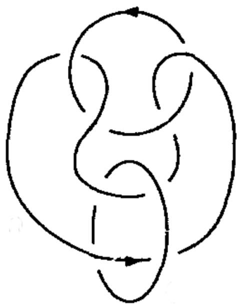
I

We consider the sequence of framed knots K_n, $n \geq 1$, below:

1. Identify knot K_1 (trivial knot, left or right-handed trefoil ?).
2. Write a presentation for the Alexander module of knot K_2.
3. Compute the Alexander polynomial of K_2.
4. Compute the Alexander polynomial of K_n for odd n.

II

1. Compute the homology $H_*(S^3(L))$ of the 3-manifold obtained by surgery on the framed link L depicted below.



2. Let $j = j_{-1} \amalg j_1 : D^2 \times S^0 \to S^2$ be an oriented embedding.
   Give a handle decomposition of the manifold $W = D^3 \times S^1 \cup_g D^2 \times D^1 \times S^1$, where $g : D^2 \times S^0 \times S^1 \to \partial(D^3 \times S^1) = S^2 \times S^1$, is defined by $g(x, -1, y) = (j_{-1}(x), y)$ and $g(x, 1, y) = (j_1(\overline{x}), \overline{y})$.
3. Show that the boundary of W is diffeomorphic to the *mapping tore*

$$T_f = [0,1] \times S^1 \times S^1/(1, x, y) \sim (0, f(x, y)) \ ,$$

   where $f : S^1 \times S^1 \to S^1 \times S^1$ is the map defined by $f(x, y) = (\overline{x}, \overline{y})$.
4. Show that T_f is diffeomorphic to $S^3(L)$.

III

In this exercise we study evaluation of Kauffman bracket at $A = e^{\frac{i\pi}{8}}$. We use the notation $< >$ for this evaluation.

1. Compute evaluation at A of Kauffman bracket a) for the trivial knot with framing k, b) for right-handed trefoil knot with framing k.

2. Show that $< L >$ vanishes for a link L containing a component L_j which has odd total linking number with the rest of the link: $\mathrm{lk}(L_j, L - L_j) = 0$. You may start with the case where L_j is trivial.

   A link L is called a **proper link** if and only if any component has even total linking number with the rest of the link.

3. Show that, for a proper link L , any link L' obtained by connecting two component of L with a band is also a proper link, and

   $$< L >= (-A^2 - A^{-2}) < L' > \quad .$$

   (Observe that in this situation, the link L'' depicted below is not a proper link.)

4. Show that for a knot K with framing k, one has

   $$< K >= (-A^2 - A^{-2})(-A^3)^k \epsilon_K \text{ , where } \epsilon_K = \pm 1.$$

   Remark: $\epsilon_K = (-1)^{Arf(K)}$ gives a definition for the Arf invariant: $Arf(K) \in \{0, 1\}$.