AN EULER SYSTEM OF HEEGNER TYPE

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Abstract. This is about $U(n-1,1) \subset SO(2n-2,2) \subset SO(2n-1,2)$. As a warm up: any algebraic representation of $SO(2n+1)$ contains the trivial representation of $U(n)$ with multiplicity one [35].

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Date: August 24, 2018.

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This is a preliminary version. The introduction should start from the Bloch-Kato conjecture, gradually shift from motives to automorphic representations, and to Shimura varieties and the actual statement of our main result, Theorem 6.4. The Galois representation V considered throughout this paper would ideally be a member of the compatible system of λ-adic Galois representations attached in [15] to an automorphic representation Π of $GL_{2n}(\mathbb{A}_F)$, where F is a totally real number field and Π is regular, algebraic, self-dual, cuspidal, with trivial infinitesimal character. Such V's are symplectic Galois representations by [5]. Our V is realized in the λ-adic middle étale cohomology of a Shimura variety $X = \mathrm{Sh}(\mathbf{G}, \mathcal{X})$ of odd dimension $2n-1$, from which it should be cut out by an admissible representation π_f of $\mathbf{G}(\mathbb{A}_f)$, where $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}} SO(\mathcal{V}, \varphi)$ for some quadratic space $(\mathcal{V}, \varphi)$ of dimension $2n+1$ over F, signature $(2n-1, 2)$ at a fixed place $v \mid \infty$ of F, and anisotropic at the other archimedean places of F. There is a Langlands L-packet $\{\pi_{\infty,1}, \cdots, \pi_{\infty,n}\}$ of discrete representations of $\mathbf{G}(\mathbb{R})$, such that each $\pi^i = \pi_f \otimes \pi^i_\infty$ should be an automorphic representation of $\mathbf{G}(\mathbb{A})$ in the global Vogan L-packet determined by Π. Given E, Π, and a ring class character χ of E satisfying a Heegner assumption, the pair $(\mathbf{G}, (\pi^i)_{i=1}^n)$ should be determined uniquely by a Gan-Gross-Prasad recipe using local signs. It follows from this recipe that $\mathbf{G}$ contains $\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}} U(\mathcal{W}, \psi)$ for some E-hermitian F-subspace $\mathcal{W}$ of $\mathcal{V}$, of E-dimension n, giving rise to a sub Shimura datum $(\mathbf{H}, \mathcal{Y})$ of $(\mathbf{G}, \mathcal{X})$, to the corresponding collection of codimension n special cycles in X, and – using Abel-Jacobi maps – to classes in the Galois cohomology of V. We show that these classes form an Euler System, and use it to establish the analog of Kolyvagin's theorem [33] on Heegner points.

The other shortcomings of this version are in section 6. I have not dealt with the non-proper case (which perhaps requires working with intersection cohomology). I have not provided any kind of automorphic construction of V. In particular, I have not tried to relate my assumptions on V to the many known properties of the Galois representations from [15]. Beyond these assumptions, I also have to assume the truth of a rather strong variant of a conjecture of Blasius and Rogawski (to obtain the "congruence relation" for my Euler system). Much of my work on this project in the last few years has been spent on this conjecture, but I finally decided to leave it here as an assumption. Also in section 6, there is an easy but non-canonical trivialization of my cycles, and I wish I could have done better.

Content. The first part explains the ad-hoc Kolyvagin System argument, following mostly the robust and flexible approach of [49], with a parameter γ as in [52]. Other sources of inspiration include [27, 41], but none of these references was sufficient for our case. Anyone familiar with such arguments could read our definition of Kolyvagin systems in 2.4.3, the statement of Theorem 4.1, and jump to the second part, where we construct the Euler system, the derived Kolyvagin system, and establish our main result, Theorem 6.4. The appendices survey various relations between quadratic and hermitian spaces, and compute the relevant Hecke polynomial.

Part 1. Kolyvagin Systems of Heegner Type

1. Selmer groups

1.1. The Galois representations.

1.1.1. We fix a prime number p and a finite extension Φ of $\mathbb{Q}_p$. We denote by

$$\Phi \supset R \supset \mathfrak{m} \ni \pi, \quad D \stackrel{\text{def}}{=} \Phi/R \quad \text{and} \quad R_r \stackrel{\text{def}}{=} R/\mathfrak{m}^r$$

the ring of integers R in Φ, its maximal ideal $\mathfrak{m}$, a chosen uniformizer π in $\mathfrak{m}$, the dualizing (divisible) R-module D and the quotient ring R_r of length $r \in \mathbb{N}$. For an R-module M, we denote by $M_{\text{tors}} \hookrightarrow M$ and $M_{\text{div}} \hookrightarrow M$ the maximal R-torsion and R-divisible submodules of M, and set $M_{/\text{tors}} = M/M_{\text{tors}}$, $M_{/\text{div}} = M/M_{\text{div}}$. For an R-module M and $x \in M$, we define $\exp(M)$ and $\exp(x)$ in $\mathbb{N} \cup \{\infty\}$ by

$$\exp(M) \stackrel{\text{def}}{=} \inf \{r \in \mathbb{N} : \mathfrak{m}^r \cdot M = 0\} \quad \text{and} \quad \exp(x) \stackrel{\text{def}}{=} \inf \{r \in \mathbb{N} : \mathfrak{m}^r \cdot x = 0\}.$$

Thus $\exp(M) = 0$ if and only if $M = 0$, $\exp(M) < \infty$ if and only if M is an R_r-module for some $r \in \mathbb{N}$ and $\exp(M) = \sup\{\exp(x) : x \in M\}$. For $a \in \mathrm{End}_R(M)$,

$$M^a \stackrel{\text{def}}{=} \ker(a) \quad \text{and} \quad M_a \stackrel{\text{def}}{=} \mathrm{coker}(a).$$

1.1.2. Let K be a field, with algebraic closure $\overline{K}$, separable closure $K^{\text{sep}} \subset \overline{K}$, and Galois group $\mathrm{Gal}_K \stackrel{\text{def}}{=} \mathrm{Gal}(K^{\text{sep}}/K)$. For a collection $\mathcal{X}$ of sets equipped with an action of Gal_K and any subextension L of K^{sep}/K, we denote by $L(\mathcal{X})$ the subextension of K^{sep}/L fixed by the kernel of $\mathrm{Gal}_L \hookrightarrow \mathrm{Gal}_K \to \prod_{x \in \mathcal{X}} \mathrm{Aut}(x)$.

1.1.3. Suppose now that

$$\rho : \mathrm{Gal}_K \to \mathrm{Aut}_\Phi(V)$$

is a finite dimensional continuous Galois representation. For a given Gal_K-stable R-lattice $T \subset V$, we will consider the following dual Galois representations:

$$\begin{array}{ccccc} T & \hookrightarrow & V & \twoheadrightarrow & A \\ \downarrow & & & & \uparrow \\ T_r & & \xrightarrow{\simeq} & & A_r \end{array} \qquad \begin{array}{ccccc} A^*(1) & \twoheadleftarrow & V^*(1) & \hookleftarrow & T^*(1) \\ \uparrow & & & & \downarrow \\ A_r^*(1) & & \xrightarrow{\simeq} & & T_r^*(1) \end{array}$$

Here r is a non-negative integer, $X \mapsto X(1)$ is the Tate twist,

$$A \stackrel{\mathrm{def}}{=} V/T, \qquad T_r \stackrel{\mathrm{def}}{=} T/\mathfrak{m}^r T, \qquad A_r \stackrel{\mathrm{def}}{=} A[\mathfrak{m}^r]$$

and the dual representation of $X \in \{V, T, A, T_r, A_r\}$ is given by

$$\begin{aligned} V^*(1) &\stackrel{\mathrm{def}}{=} \mathrm{Hom}_\Phi\left(V, \Phi(1)\right) \\ A^*(1) &\stackrel{\mathrm{def}}{=} \mathrm{Hom}_R\left(T, D(1)\right) \\ T^*(1) &\stackrel{\mathrm{def}}{=} \mathrm{Hom}_R\left(A, D(1)\right) \\ A_r^*(1) &\stackrel{\mathrm{def}}{=} \mathrm{Hom}_R\left(T_r, D(1)\right) \\ T_r^*(1) &\stackrel{\mathrm{def}}{=} \mathrm{Hom}_R\left(A_r, D(1)\right) \end{aligned}$$

With these definitions, we also have compatible perfect pairings

$$\begin{array}{ccccc} T_r & \times & T_r^*(1) & \longrightarrow & R_r(1) \\ \uparrow & & \uparrow & & \uparrow \\ T & \times & T^*(1) & \longrightarrow & R(1) \\ \downarrow & & \downarrow & & \downarrow \\ V & \times & V^*(1) & \longrightarrow & \Phi(1) \end{array}$$

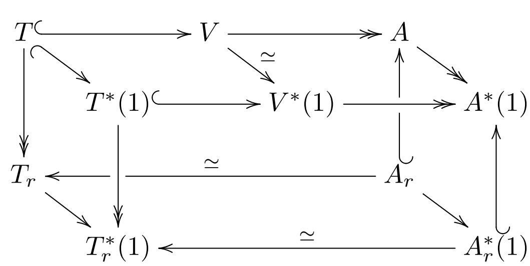
Multiplication by π^r on V or $V^*(1)$ induces Gal_K-equivariant isomorphisms

$$A_r \simeq T_r \quad \text{and} \quad A_r^*(1) \simeq T_r^*(1).$$

1.1.4. In the applications, our Galois representation will be equipped with a Gal_K-equivariant ε-symmetric (for some $\varepsilon \in \{\pm 1\}$) Φ-bilinear perfect pairing

$$\langle -, - \rangle : V \times V \to \Phi(1)$$

Moreover, our Gal_K-stable R-lattice $T \subset V$ will be chosen such that $\langle T, T \rangle \subset R(1)$. The isomorphism $\xi : V \to V^*(1)$ defined by $\xi(v) = \langle v, - \rangle$ then induces compatible linear and Gal_K-equivariant morphisms between the two diagrams of 1.1.3:



1.2. **Local conditions.** Suppose that K is a finite extension of $\mathbb{Q}_\ell$ for some prime number ℓ. If $\ell = p$, we furthermore require V to be a de Rham representation.

1.2.1. A local condition on $X \in \{V, T, A, T_r, A_r\}$ over K is an R or Φ-submodule

$$H^1_{\mathcal{F}}(K, X) \subset H^1(K, X).$$

We refer to the local condition as $\mathcal{F}$. It may also be given by the quotient

$$H^1_{/\mathcal{F}}(K, X) \stackrel{\text{def}}{=} \frac{H^1(K, X)}{H^1_{\mathcal{F}}(K, X)}.$$

Given local conditions $\mathcal{F}$ and $\mathcal{F}'$, we write $\mathcal{F} \leq \mathcal{F}'$ if $H^1_{\mathcal{F}}(K, X) \subset H^1_{\mathcal{F}'}(K, X)$.

1.2.2. A local condition $\mathcal{F}$ on any $X \in \{V, T, A, T_r, A_r\}$ over K induces a local condition $\mathcal{F}^*(1)$ on the dual Y of X as defined above, over K, given by

$$H^1_{\mathcal{F}^*(1)}(K, Y) \stackrel{\text{def}}{=} H^1_{\mathcal{F}}(K, X)^{\perp}$$

where the orthogonal complement is for the perfect pairing of class field theory,

$$\begin{array}{rcll} H^1(K, V) \times H^1(K, V^*(1)) & \to & H^2(K, \Phi(1)) & \simeq \Phi \\ H^1(K, T) \times H^1(K, A^*(1)) & \to & H^2(K, D(1)) & \simeq D \\ H^1(K, A) \times H^1(K, T^*(1)) & \to & H^2(K, D(1)) & \simeq D \\ H^1(K, T_r) \times H^1(K, A_r^*(1)) & \to & H^2(K, D(1)) & \simeq D \\ H^1(K, A_r) \times H^1(K, T_r^*(1)) & \to & H^2(K, D(1)) & \simeq D \end{array}$$

1.2.3. A local condition $\mathcal{F}$ on V over K induces a local condition, also denoted by $\mathcal{F}$, on any $X \in \{T, A, T_r, A_r)$, over K, which are respectively defined by

$$\begin{aligned} H^1_{\mathcal{F}}(K, T) &\stackrel{\text{def}}{=} \ker\left(H^1(K, T) \to H^1_{/\mathcal{F}}(K, V)\right), \\ H^1_{\mathcal{F}}(K, A) &\stackrel{\text{def}}{=} \operatorname{im}\left(H^1_{\mathcal{F}}(K, V) \to H^1(K, A)\right), \\ H^1_{\mathcal{F}}(K, T_r) &\stackrel{\text{def}}{=} \operatorname{im}\left(H^1_{\mathcal{F}}(K, T) \to H^1(K, T_r)\right), \\ H^1_{\mathcal{F}}(K, A_r) &\stackrel{\text{def}}{=} \ker\left(H^1(K, A_r) \to H^1_{/\mathcal{F}}(K, A)\right). \end{aligned}$$

The last two Selmer structures are compatible with the isomorphism $A_r \simeq T_r$.

1.2.4. A local condition $\mathcal{F}$ on $X \in \{V, T, A, T_r, A_r\}$ over a finite extension $L \subset \overline{K}$ of K induces a local condition $\mathrm{res}^*\mathcal{F}$ on X over K, defined by

$$H^1_{\mathrm{res}^*\mathcal{F}}(K, X) \stackrel{\text{def}}{=} \ker\left(H^1(K, X) \to H^1_{/\mathcal{F}}(L, X)\right).$$

Since $\mathrm{cores} \circ \mathrm{res} = [L : K]$ on $H^1(K, X)$, we have

$$[L : K] H^1_{\mathrm{res}^*\mathcal{F}}(K, X) \subset \mathrm{cores}\left(H^1_{\mathcal{F}}(L, X)\right) \subset H^1(K, X).$$

If L/K is Galois and $\mathcal{F}$ is $\mathrm{Gal}(L/K)$-stable, then also

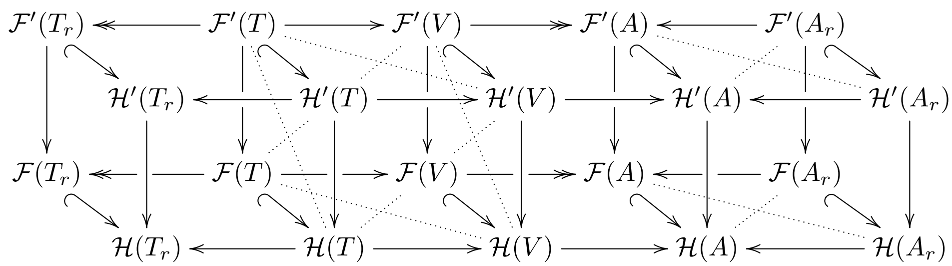
$$\mathrm{cores}\left(H^1_{\mathcal{F}}(L, X)\right) \subset H^1_{\mathrm{res}^*\mathcal{F}}(K, X)$$

since $\mathrm{res} \circ \mathrm{cores} = \mathrm{tr}_{L/K}$ on $H^1(L, X)$. Thus if moreover $p \nmid [L : K]$ or $X = V$, then

$$H^1_{\mathrm{res}^*\mathcal{F}}(K, X) = \mathrm{cores}\left(H^1_{\mathcal{F}}(L, X)\right) \subset H^1(K, X).$$

1.2.5. The first two constructions are related as follows. Starting from a local condition $\mathcal{F}$ on V, the duals of the induced Selmer structures $\mathcal{F}$ on T, A, T_r and A_r, which are Selmer structures on respectively $A^*(1)$, $T^*(1)$, $A_r^*(1)$ and $T_r^*(1)$, are also the Selmer structures which are induced by the dual $\mathcal{F}^*(1)$ of $\mathcal{F}$ on $V^*(1)$.

1.2.6. The last two constructions are related as follows. Starting from a local condition $\mathcal{F}$ on V over $L \supset K$, let $\mathcal{F}' = \mathrm{res}^*\mathcal{F}$ be the induced local condition on V over K. For $X \in \{V, T, A, T_r, A_r\}$, set $\mathcal{F}(X) = H^1_{\mathcal{F}}(L, X)$ inside $\mathcal{H}(X) = H^1(L, X)$ and $\mathcal{F}'(X) = H^1_{\mathcal{F}'}(K, X)$ inside $\mathcal{H}'(X) = H^1(K, X)$. We thus have the following commutative diagram, whose vertical maps are the restriction maps from K to L:



The crossed squares are cartesian, and multiplication by π^r on V identifies the first and last vertical squares. In particular, $\mathcal{F}'(T) = \mathrm{res}^*\mathcal{F}(T)$ while

$$\mathcal{F}'(T_r) \subset \mathrm{res}^*\mathcal{F}(T_r), \quad \mathcal{F}'(A) \subset \mathrm{res}^*\mathcal{F}(A) \quad \text{and} \quad \mathcal{F}'(A_r) \subset \mathrm{res}^*\mathcal{F}(A_r)$$

with more precisely,

$$\frac{\mathrm{res}^*\mathcal{F}(T_r)}{\mathcal{F}'(T_r)} \simeq \frac{\mathrm{res}^*\mathcal{F}(A_r)}{\mathcal{F}'(A_r)} \hookrightarrow \frac{\mathrm{res}^*\mathcal{F}(A)}{\mathcal{F}'(A)}.$$

1.2.7. The first and third constructions are related as follows. Let $L \subset \overline{K}$ be a finite extension of K, $\mathcal{F}$ a local condition on $X \in \{V, T, A, T_r, A_r\}$ over L, and $Y \in \{V^*(1), A^*(1), T^*(1), A_r^*(1), T_r^*(1)\}$ the dual of X, as defined above. Then

$$[L : K] H^1_{\mathrm{res}^*(\mathcal{F}^*(1))}(K, Y) \subset H^1_{(\mathrm{res}^*\mathcal{F})^*(1)}(K, Y) \subset H^1(K, Y).$$

If L/K is Galois and $\mathcal{F}$ is $\mathrm{Gal}(L/K)$-stable, then also

$$H^1_{(\mathrm{res}^*\mathcal{F})^*(1)}(K, Y) \subset H^1_{\mathrm{res}^*(\mathcal{F}^*(1))}(K, Y) \subset H^1(K, Y).$$

Thus if moreover $p \nmid [L : K]$ or $X = V$,

$$H^1_{(\mathrm{res}^*\mathcal{F})^*(1)}(K, Y) = H^1_{\mathrm{res}^*(\mathcal{F}^*(1))}(K, Y).$$

1.2.8. The unramified local condition on $X \in \{V, T, A, T_r, A_r\}$ is defined by

$$H^1_{ur}(K, X) \stackrel{\mathrm{def}}{=} \ker\left(H^1(K, X) \to H^1(K^{\mathrm{ur}}, X)\right)$$

where $K^{\mathrm{ur}} \subset \overline{K}$ is the maximal unramified extension of K. Thus

$$H^1_{ur}(K, X) \simeq H^1\left(\mathrm{Gal}(K^{\mathrm{ur}}/K), X^I\right) \simeq \left(X^I\right)_{\mathrm{Fr}-1}$$

where $I = \mathrm{Gal}(\overline{K}/K^{\mathrm{ur}})$ and Fr is the geometric Frobenius.

1.2.9. The Bloch-Kato local condition $\mathcal{F} = f$ on $X \in \{T, A, T_r, A_r\}$ is the local condition induced by the Bloch-Kato local condition $\mathcal{F} = f$ on V, defined by

$$H^1_f(K, V) \stackrel{\mathrm{def}}{=} \begin{cases} \ker\left(H^1(K, V) \to H^1\left(K^{\mathrm{ur}}, V\right)\right) & \text{if } \ell \neq p, \\ \ker\left(H^1(K, V) \to H^1\left(K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\right)\right) & \text{if } \ell = p \end{cases}$$

where ℓ is the residue characteristic of K and B_{crys} is Fontaine's period ring. The Bloch-Kato local condition is self-dual on $X \in \{V, T, A, T_r, A_r\}$, in the sense that its dual local condition $f^*(1)$ on the dual $Y \in \{V^*(1), A^*(1), T^*(1), A_r^*(1), T_r^*(1)\}$ of X is the Bloch-Kato local condition f on Y. This follows from [7, 3.8] and 1.2.5.

1.2.10. If $\ell \neq p$, these local conditions are related as follows, see [52, 1.3.5]:

$$H^1_f(K,V) = H^1_{ur}(K,V), \quad H^1_f(K,T) \supset H^1_{ur}(K,T), \quad H^1_f(K,A) \subset H^1_{ur}(K,A).$$

More precisely, set $\mathcal{A} = (A^I)_{/\mathrm{div}}$. This is a finite length R-module and

$$\begin{array}{rclcl} \frac{H^1_f(K,T)}{H^1_{ur}(K,T)} & = & H^1_{/ur}(K,T)_{\mathrm{tors}} & = & \mathcal{A}^{\mathrm{Fr}-1}, \\ \frac{H^1_{ur}(K,A)}{H^1_f(K,A)} & = & H^1_{ur}(K,A)_{/\mathrm{div}} & = & \mathcal{A}_{\mathrm{Fr}-1}. \end{array}$$

If V is unramified, then $A^I = A$ is R-divisible, thus $\mathcal{A} = 0$ and

$$H^1_f(K,X) = H^1_{ur}(K,X)$$

for every $X \in \{V, T, A, T_r, A_r\}$.

1.2.11. Let L be a finite unramified extension of K. We wish to compare the Bloch-Kato local condition f on $X \in \{V, T, A, T_r, A_r\}$ over K and the restriction $\mathrm{res}^* f$ of the Bloch-Kato local condition f on X over L. For $X = V$, one checks that $H^1_f(K,V) = H^1_{\mathrm{res}^* f}(K,V)$. Thus by 1.2.6, $H^1_f(K,T) = H^1_{\mathrm{res}^* f}(K,T)$ and

$$\frac{H^1_{\mathrm{res}^* f}(K,T_r)}{H^1_f(K,T_r)} \simeq \frac{H^1_{\mathrm{res}^* f}(K,A_r)}{H^1_f(K,A_r)} \hookrightarrow \frac{H^1_{\mathrm{res}^* f}(K,A)}{H^1_f(K,A)}.$$

1.2.12. If $\ell \neq p$, we have seen that $H^1_f(L,A) \subset H^1_{ur}(L,A)$, thus plainly

$$H^1_{\mathrm{res}^* f}(K,A) \subset H^1_{\mathrm{res}^* ur}(K,A) = H^1_{ur}(K,A)$$

using that L/K is unramified for the equality. Therefore by 1.2.10,

$$\frac{H^1_{\mathrm{res}^* f}(K,A)}{H^1_f(K,A)} \subset \frac{H^1_{ur}(K,A)}{H^1_f(K,A)} \simeq \mathcal{A}_{\mathrm{Fr}-1}.$$

If moreover V is unramified, we find that

$$H^1_{\mathrm{res}^* f}(K,X) = H^1_f(K,X) \quad \big(= H^1_{ur}(K,X) \simeq X_{\mathrm{Fr}-1}\big)$$

for every $X \in \{V, T, A, T_r, A_r\}$. We have shown:

Proposition 1.1. *Let $\mathcal{A} = A^I / A^I_{\mathrm{div}}$ and $c = \exp(\mathcal{A}_{\mathrm{Fr}-1})$. Then for every finite unramified extension L of K as above and any $X \in \{A, T_r, A_r\}$,*

$$\mathfrak{m}^c \cdot H^1_{\mathrm{res}^* f}(K,X) \subset H^1_f(K,X) \subset H^1_{\mathrm{res}^* f}(K,X).$$

If moreover V is unramified, then $c = 0$ and for every $X \in \{V, T, A, T_r, A_r\}$,

$$H^1_f(K,X) = H^1_{\mathrm{res}^* f}(K,X).$$

1.2.13. Suppose that $\ell = p$. Let $D_{cr}(\star) = H^0(K, \star \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})$, with Frobenius φ.

Proposition 1.2. *If $D_{cr}(V^*(1))^{\varphi-1} = 0$, there is a smallest constant $c \in \mathbb{N}$ such that for every finite unramified extension L of K as above and any $X \in \{A, T_r, A_r\}$,*

$$\mathfrak{m}^c \cdot H^1_{\mathrm{res}^* f}(K,X) \subset H^1_f(K,X) \subset H^1_{\mathrm{res}^* f}(K,X).$$

Suppose moreover that K is unramified over $\mathbb{Q}_p$, V is crystalline with Hodge-Tate weights $a_1 \leq \cdots \leq a_n$ ($n = \dim_{\mathbb{Q}_p} V$) such that $\max(a_n, -1) - \min(a_1, -2) < p$. We may then take $c = 0$, and for every $X \in \{V, T, A, T_r, A_r\}$,

$$H^1_f(K,X) = H^1_{\mathrm{res}^* f}(K,X).$$

Proof. By 1.2.11, it is sufficient to prove the proposition for $X = A$. We have to investigate the kernel $H^1_{\mathrm{res}^* f}(K, A)/H^1_f(K, A)$ of the restriction map

$$\mathrm{res} : H^1_{/f}(K, A) \to H^1_{/f}(L, A).$$

It is Pontryagin dual, via local duality, to the cokernel of the corestriction map

$$\mathrm{cores} : H^1_f(L, T^*(1)) \to H^1_f(K, T^*(1)).$$

The first assertion thus follows from [51]. For the second assertion, we have to show that our corestriction map is surjective under the given assumptions. Note that $V^*(1)$ is crystalline with Hodge-Tate weights $-a_n - 1 \le \cdots \le -a_1 - 1$. Thus for $i = \min(0, -a_n - 1)$ and $j = \max(1, -a_1 - 1)$, we have $i \le 0$, $j \ge 1$, $j - i < p$, $\mathrm{Fil}^i D_{dR}(V^*(1)) = D_{dR}(V^*(1))$ and $\mathrm{Fil}^j D_{dR}(V^*(1)) = 0$, which checks the assumptions of Lemma 4.5 of [7]. Let then

$$D \subset D_{\mathrm{cr},K} = H^0\left(K, V^*(1) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\right)$$

be the strongly divisible $\mathcal{O}_K$-lattice corresponding to the Gal_K-stable $\mathbb{Z}_p$-lattice $T^*(1)$ of $V^*(1)$. To properly construct D, one has to combine the covariant twisted version of the Fontaine-Laffaille theory exposed in section 4 of [7] with the essential surjectivity of the original contravariant untwisted version, as established later by Breuil in Proposition 3 of [8]. Let similarly

$$D' \subset D_{\mathrm{cr},L} = H^0\left(L, V^*(1) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\right)$$

be the strongly divisible $\mathcal{O}_L$-lattice corresponding to the Gal_L-stable $\mathbb{Z}_p$-lattice $T^*(1)$ of $V^*(1)$. The compatibility of Fontaine's ($\mathbb{Q}_p$-linear) and Fontaine-Laffaille's ($\mathbb{Z}_p$-linear) functors with unramified base change then implies that

$$D' = D \otimes_{\mathcal{O}_K} \mathcal{O}_L \quad \text{inside} \quad D_{cr,L} = D_{cr,K} \otimes L.$$

Moreover by Lemma 4.5 of [7], there are canonical isomorphisms

$$D'/(\varphi' - 1)\mathrm{Fil}^0 D' \simeq H^1_f\left(L, T^*(1)\right) \quad \text{and} \quad D/(\varphi - 1)\mathrm{Fil}^0 D \simeq H^1_f\left(K, T^*(1)\right)$$

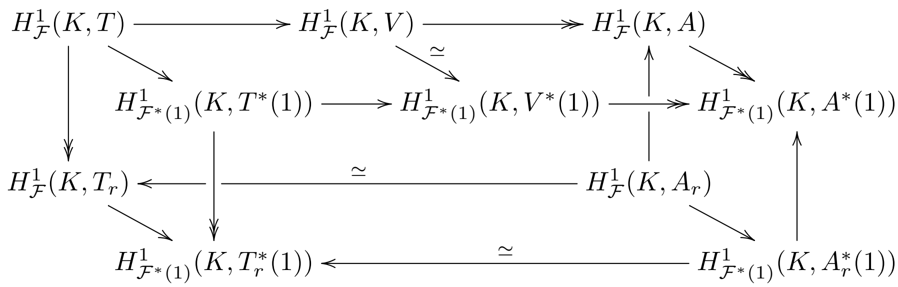
where φ and φ' are the crystalline Frobeniuses on $D_{cr,K}$ and $D_{cr,L}$, while

$$\mathrm{Fil}^0 D = D \cap \mathrm{Fil}^0 D_{cr,K} \quad \text{and} \quad \mathrm{Fil}^0 D' = D' \cap \mathrm{Fil}^0 D_{cr,L}.$$

Under these isomorphisms, the corestriction map is induced by the usual trace map $\mathrm{tr} : D' \to D$, which is indeed surjective since L/K is unramified. □

1.2.14. In the self-dual case of 1.1.4, we say that a local condition $\mathcal{F}$ on V over K is self-dual with respect to the given perfect pairing $\langle -, - \rangle : V \times V \to \Phi(1)$ if the isomorphism $\xi : H^1(K, V) \to H^1(K, V^*(1))$ induced by the Gal_K-equivariant isomorphism $\xi : V \to V^*(1)$ maps $H^1_{\mathcal{F}}(K, V)$ onto $H^1_{\mathcal{F}^*(1)}(K, V^*(1))$. The Bloch-Kato local condition, which is functorial in V and "abstractly" self-dual is also self-dual with respect to any such pairing. Applying $H^1(K, -)$ to the diagram of

1.1.4, we obtain a diagram which is compatible with the induced local conditions:



If L/K is a finite Galois extension and $\mathcal{F}$ is a self-dual, $\mathrm{Gal}(L/K)$-stable local condition on V over L, then $\mathrm{res}^*\mathcal{F}$ is a self-dual local condition on V over K.

1.3. **The finite/singular exact sequence.**

1.3.1. Suppose that $\ell \neq p$, take K to be a finite extension F of $\mathbb{Q}_\ell$, let $F \subset E \subset \overline{F}$ be a quadratic unramified extension of F. We denote by $\mathcal{O}_F \subset \mathcal{O}_E$ the rings of integers in $F \subset E$, by $\mathfrak{m}_F$ and $\mathfrak{m}_E = \mathfrak{m}_F\mathcal{O}_E$ their maximal ideals, by $\mathbb{F} \subset \mathbb{E}$ their residue fields, and we fix a uniformizer $\varpi \in \mathfrak{m}_F$ for F and E. We consider the following diagram of Galois extensions of F in $\overline{F}$, in which all squares represent linearly disjoint extensions and their compositum:

$$\begin{array}{ccccccccc}
 & & & & E^{\mathrm{tr}} & \longrightarrow & E^{\mathrm{tr}}\cdot E^{\mathrm{ab}} & \longrightarrow & \overline{F} \\
 & & & & \uparrow & & \uparrow & & \\
 & & E(\varpi) & \longrightarrow & E^{\mathrm{tr,ab}} & \longrightarrow & E^{\mathrm{ab}} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & E(1) & \longrightarrow & E^{\mathrm{ur}}(1) & \longrightarrow & E^{\circ}(1) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
F & \longrightarrow & E & \longrightarrow & E^{\mathrm{ur}} & \longrightarrow & E^{\circ} & &
\end{array}$$

First, E^{ab}, E^{ur}, E^{tr} and $E^{\mathrm{tr,ab}} = E^{\mathrm{tr}} \cap E^{\mathrm{ab}}$ are respectively the maximal abelian, unramified, tamely ramified and abelian tamely ramified extensions of E. Let

$$\mathrm{rec} : E^\times \hookrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E)$$

be the reciprocity map from local class field theory. The nine extensions between E and E^{ab} correspond to the following cartesian diagram of subgroups of $E^\times$:

$$\begin{array}{ccccc}
\varpi^{\mathbb{Z}}(1+\mathfrak{m}_E) & \longleftarrow & 1+\mathfrak{m}_E & \longleftarrow & 1 \\
\downarrow & & \downarrow & & \downarrow \\
F^\times\mathcal{O}_{E,1}^\times & \longleftarrow & \mathcal{O}_{E,1}^\times & \longleftarrow & [\mathbb{F}^\times] \\
\downarrow & & \downarrow & & \downarrow \\
E^\times & \longleftarrow & \mathcal{O}_E^\times & \longleftarrow & [\mathbb{E}^\times]
\end{array}$$

Here $\mathcal{O}_{E,1} = \mathcal{O}_F + \mathfrak{m}_E$ and $[-] : \mathbb{E}^\times \hookrightarrow \mu(E)$ is the Teichmüller morphism, so that

$$E^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}_E^\times, \quad \mathcal{O}_E^\times = [\mathbb{E}^\times] \times (1+\mathfrak{m}_E), \quad \mathcal{O}_{E,1}^\times = [\mathbb{F}^\times] \times (1+\mathfrak{m}_E).$$

Set $\mathbb{G}=\mathbb{E}^{\times}/\mathbb{F}^{\times}$, a cyclic group of order $|\mathbb{F}|+1$. In the above diagram of extensions, the finite Galois groups are given by the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Gal}(E^{\mathrm{ab}}/E^{\circ}(1)) & \xrightarrow[\simeq]{\mathrm{res}} & \mathrm{Gal}(E^{\mathrm{tr,ab}}/E^{\mathrm{ur}}(1)) & \xrightarrow[\simeq]{\mathrm{res}} & \mathrm{Gal}(E(\varpi)/E(1)) & \xrightarrow[\simeq]{} & \mathbb{F}^{\times} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Gal}(E^{\mathrm{ab}}/E^{\circ}) & \xrightarrow[\simeq]{\mathrm{res}} & \mathrm{Gal}(E^{\mathrm{tr,ab}}/E^{\mathrm{ur}}) & \xrightarrow[\simeq]{\mathrm{res}} & \mathrm{Gal}(E(\varpi)/E) & \xrightarrow[\simeq]{} & \mathbb{E}^{\times} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Gal}(E^{\circ}(1)/E^{\circ}) & \xrightarrow[\simeq]{\mathrm{res}} & \mathrm{Gal}(E^{\mathrm{ur}}(1)/E^{\mathrm{ur}}) & \xrightarrow[\simeq]{\mathrm{res}} & \mathrm{Gal}(E(1)/E) & \xrightarrow[\simeq]{} & \mathbb{G}
\end{array}$$

The vertical exact sequences are equivariant for the actions of $\mathrm{Gal}(E/F)=\{1,\star\}$, with the non-trivial element $\star$ acting by 1 on the subgroup and -1 on the quotient. Inside the procyclic tame inertia group $I^{t}=\mathrm{Gal}(E^{\mathrm{tr}}/E^{\mathrm{ur}})$, we have

$$\mathrm{Gal}(E^{\mathrm{tr}}/E^{\mathrm{tr,ab}})=\left|\mathbb{E}^{\times}\right|I^{t}\quad\text{and}\quad\mathrm{Gal}(E^{\mathrm{tr}}/E^{\mathrm{ur}}(1))=|\mathbb{G}|\,I^{t}.$$

1.3.2. Suppose now that the Gal_F-representation T_r is unramified and the following equivalent conditions hold, where $\mathrm{Fr}\in\mathrm{Gal}(E^{\mathrm{ur}}/F)$ is the geometric Frobenius:

$$\mathrm{Fr}=-1\quad\text{on}\quad R_r(1)\iff\quad|\mathbb{F}|+1\equiv 0\quad\text{in}\quad R_r.$$

We then consider the inflation/restriction exact sequences for T_r and respectively

$$1\to\mathrm{Gal}(\overline{F}/E^{\mathrm{ur}})\to\mathrm{Gal}(\overline{F}/E)\to\mathrm{Gal}(E^{\mathrm{ur}}/E)\to 1$$

$$1\to\mathrm{Gal}(\overline{F}/E(1))\to\mathrm{Gal}(\overline{F}/E)\to\mathrm{Gal}(E(1)/E)\to 1$$

The first one gives the finite/singular exact sequence

$$0\to H^1_f(E,T_r)\to H^1(E,T_r)\to H^1_s(E,T_r)\to 0$$

since $H^2\left(\mathrm{Gal}(E^{\mathrm{ur}}/E),T_r\right)=0$, where

$$\begin{aligned}
H^1_f(E,T_r) &\overset{\mathrm{def}}{=} H^1\left(\mathrm{Gal}(E^{\mathrm{ur}}/E),T_r\right)\\
H^1_s(E,T_r) &\overset{\mathrm{def}}{=} H^1\left(\mathrm{Gal}(\overline{F}/E^{\mathrm{ur}}),T_r\right)^{\mathrm{Gal}(E^{\mathrm{ur}}/E)}\\
&\overset{(1)}{=} \mathrm{Hom}\left(\mathrm{Gal}(\overline{F}/E^{\mathrm{ur}}),T_r\right)^{\mathrm{Fr}^2-1}\\
&\overset{(2)}{=} \mathrm{Hom}\left(\mathrm{Gal}(E^{\mathrm{tr}}/E^{\mathrm{ur}}),T_r\right)^{\mathrm{Fr}^2-1}\\
&\overset{(3)}{=} \mathrm{Hom}\left(\mathrm{Gal}(E^{\mathrm{ur}}(1)/E^{\mathrm{ur}}),T_r\right)^{\mathrm{Fr}^2-1}\\
&\overset{(4)}{=} \mathrm{Hom}\left(\mathrm{Gal}(E^{\mathrm{ur}}(1)/E^{\mathrm{ur}}),T_r^{\mathrm{Fr}^2-1}\right)
\end{aligned}$$

since (1) T_r is unramified, (2) $\ell\neq p$, (3) $|\mathbb{G}|=|\mathbb{F}|+1\equiv 0$ in R_r and (4) $\mathrm{Fr}^2\equiv 1$ on $\mathrm{Gal}(E^{\mathrm{ur}}(1)/E^{\mathrm{ur}})$. The second one gives an exact sequence

$$0\to H^1_t\left(E,T_r\right)\to H^1(E,T_r)\to H^1\left(E(1),T_r\right)^{\mathrm{Gal}(E(1)/E)}$$

where

$$\begin{aligned} H^1_t(E,T_r) &\stackrel{\text{def}}{=} H^1\left(\mathrm{Gal}\,(E(1)/E)\,, T_r^{\mathrm{Gal}\left(\overline{F}/E(1)\right)}\right) \\ &\stackrel{(1)}{=} H^1\left(\mathrm{Gal}\,(E(1)/E)\,, T_r^{\mathrm{Gal}\left(\overline{F}/E\right)}\right) \\ &\stackrel{(2)}{=} \mathrm{Hom}\left(\mathrm{Gal}\,(E(1)/E)\,, T_r^{\mathrm{Fr}^2-1}\right). \end{aligned}$$

since (1) T_r is unramified and $E(1)\cap E^{\mathrm{ur}} = E$, and (2) $\mathrm{Gal}(E(1)/E)$ acts trivially on $T_r^{\mathrm{Gal}(\overline{F}/E)} = T_r^{\mathrm{Fr}^2-1}$. It follows that the composite morphism

$$H^1_t(E,T_r) \hookrightarrow H^1(E,T_r) \twoheadrightarrow H^1_s(E,T_r)$$

is an isomorphism. It yields a splitting of the finite/singular exact sequence:

$$H^1(E,T_r) = H^1_f(E,T_r) \oplus H^1_t(E,T_r).$$

Similarly, the dual representation $T_r^*(1)$ is unramified, and

$$H^1(E,T_r^*(1)) = H^1_f(E,T_r^*(1)) \oplus H^1_t(E,T_r^*(1)).$$

Under the Tate pairing $H^1(E,T_r)\times H^1(E,T_r^*(1)) \to R_r$, we have [41, 1.3.2]

$$H^1_f(E,T_r)^{\perp} = H^1_f(E,T_r^*(1)) \quad \text{and} \quad H^1_t(E,T_r)^{\perp} = H^1_t(E,T_r^*(1)).$$

All of these constructions are plainly equivariant under $\mathrm{Gal}(E/F)$.

1.3.3. Evaluation at $\mathrm{Fr}^2 \in \mathrm{Gal}(E^{\mathrm{ur}}/E)$ yields a Fr-equivariant isomorphism

$$\mathrm{ev}(\mathrm{Fr}^2) : H^1_f(E,T_r) \xrightarrow{\simeq} (T_r)_{\mathrm{Fr}^2-1}.$$

On the other hand, the Fr-equivariant isomorphisms

$$\mathrm{Gal}\,(E^{\mathrm{ur}}(1)/E^{\mathrm{ur}}) \simeq \mathrm{Gal}\,(E(1)/E) \simeq \mathbb{G}$$

induce Fr-equivariant isomorphisms

$$H^1_t(E,T_r)\otimes\mathbb{G} \xrightarrow{\simeq} H^1_s(E,T_r)\otimes\mathbb{G} \xrightarrow{\simeq} T_r^{\mathrm{Fr}^2-1}.$$

The resulting Fr-equivariant morphism

$$H^1(E,T_r)\otimes\mathbb{G} \twoheadrightarrow H^1_s(E,T_r)\otimes\mathbb{G} \xrightarrow{\simeq} T_r^{\mathrm{Fr}^2-1}$$

sends $x\otimes y$ to $c(\mathrm{rec}[\zeta])$ where $c : \mathrm{Gal}(\overline{F}/E) \to T_r$ is any cocycle representing $x \in H^1(E,T_r)$ and $[\zeta]$ is the Teichmüller lift of any $\zeta\in\mathbb{E}^\times$ representing $y\in\mathbb{G}$.

1.3.4. Finally, the local Tate pairing induces a commutative and Fr-equivariant diagram of perfect, bilinear and Fr-equivariant pairings

$$\begin{array}{ccccc}
H^1(E,T_r) & \times & H^1(E,T_r^*(1))\otimes\mathbb{G} & \longrightarrow & R_r\otimes\mathbb{G} \\
\uparrow & & \downarrow & & \| \\
H^1_f(E,T_r) & \times & H^1_s(E,T_r^*(1))\otimes\mathbb{G} & \longrightarrow & R_r\otimes\mathbb{G} \\
\simeq\downarrow & & \simeq\downarrow & & \| \\
(T_r)_{\mathrm{Fr}^2-1} & \times & (T_r^*(1))^{\mathrm{Fr}^2-1} & \longrightarrow & R_r\otimes\mathbb{G} \\
\uparrow & & \downarrow & & \uparrow\simeq \\
T_r & \times & T_r^*(1) & \longrightarrow & R_r(1)
\end{array}$$

where the first line is the twisted Tate pairing with values in $R_r \otimes \mathbb{G}$, the last line is the evaluation pairing between T_r and $T_r^*(1) = \mathrm{Hom}(T_r, R_r(1))$ of 1.1.3, which takes values in $R_r(1) = R_r \otimes \mu(E)$, and the isomorphism $R_r(1) \simeq R_r \otimes \mathbb{G}$ is induced by the reduction map $\mu(E) \twoheadrightarrow \mathbb{E}^\times \twoheadrightarrow \mathbb{G}$.

1.4. **Selmer structures.**

1.4.1. Suppose now that K is a number field with ring of integers $\mathcal{O}_K$. For each finite place v of K, we fix a K-embedding $\overline{K} \hookrightarrow \overline{K}_v$, corresponding to the decomposition group $D(v) = \mathrm{Gal}_{K_v} \subset \mathrm{Gal}_K$, the inertia group $I(v) = \mathrm{Gal}_{K_v^{\mathrm{ur}}} \subset D(v)$ and the geometric Frobenius $\mathrm{Fr}_v \in D(v)/I(v) = \mathrm{Gal}(K_v^{ur}/K_v)$.

1.4.2. We assume that our continuous Galois representation

$$\rho : \mathrm{Gal}_K \to \mathrm{Aut}_\Phi(V)$$

is unramified at all but finitely many places v of F, and de Rham at all $v \mid p$.

1.4.3. A Selmer structure $\mathcal{F}$ on $X \in \{V, T, A, T_r, A_r\}$ (over K) is a collection of local conditions $H^1_{\mathcal{F}}(K_v, X) \subset H^1(K_v, X)$, one for each finite place v of K[1], such that for all but finitely many such v's,

$$H^1_{\mathcal{F}}(K_v, X) = H^1_{ur}(K_v, X) \overset{\mathrm{def}}{=} \ker\left(H^1(K_v, X) \to H^1(K_v^{\mathrm{ur}}, X)\right).$$

The corresponding Selmer module is defined by

$$H^1_{\mathcal{F}}(K, X) \overset{\mathrm{def}}{=} \ker\left(H^1(K, X) \to \prod_v H^1_{/\mathcal{F}}(K_v, X)\right).$$

It is finite dimensional over Φ if $X = V$, a finite R-module if $X = T$, a cofinite R-module if $X = A$, and a finite R_r-module if $X = T_r$ or A_r [52, B.2.7].

1.4.4. The constructions of 1.2.2, 1.2.3, 1.2.4 have analogs for Selmer structures:

- A Selmer structure $\mathcal{F}$ on $X \in \{V, T, A, T_r, A_r\}$ induces a dual Selmer structure $\mathcal{F}^*(1)$ on the dual Y of X.
- A Selmer structure $\mathcal{F}$ on V induces compatible Selmer structures, also denoted by $\mathcal{F}$, on any $X \in \{T, A, T_r, A_r\}$.
- A Selmer structure $\mathcal{F}$ on $X \in \{V, T, A, T_r, A_r\}$ over a finite extension L of K induces a Selmer structure $\mathrm{res}^*\mathcal{F}$ on X over K.

In the first two cases, the local conditions of the induced Selmer structures are the induced local conditions. In the third case, they are given by

$$H^1_{\mathrm{res}^*\mathcal{F}}(K_v, X) \overset{\mathrm{def}}{=} \ker\left(H^1(K_v, X) \to \oplus_{w|v} H^1_{/\mathcal{F}}(L_w, X)\right)$$

where w runs through the places of L above v, and we have commutative diagrams

$$\begin{array}{ccc} H^1_{\mathcal{F}}(L, X) & \xrightarrow{\oplus \mathrm{loc}_w} & \oplus_{w|v} H^1_{\mathcal{F}}(L_w, X) \\ \mathrm{res} \uparrow & & \uparrow \oplus \mathrm{res}_w \\ H^1_{\mathrm{res}^*\mathcal{F}}(K, X) & \xrightarrow{\mathrm{loc}_v} & H^1_{\mathrm{res}^*\mathcal{F}}(K_v, X) \end{array}$$

[1] When $p = 2$, one should also consider local conditions at archimedean real places of K. However for our purposes, the field K will have no real embeddings.

1.4.5. If L is a finite Galois extension of K, a local condition $\mathcal{F}$ on X over L is said to be $\mathrm{Gal}(L/K)$-stable if for every finite place w of L, any $\sigma \in \mathrm{Gal}(L/K)$ induces a commutative diagram of isomorphisms

$$\begin{array}{ccc} H^1_{\mathcal{F}}(L_w, X) & \xrightarrow{\sigma} & H^1_{\mathcal{F}}(L_{\sigma w}, X) \\ \downarrow & & \downarrow \\ H^1(L_w, X) & \xrightarrow{\sigma} & H^1(L_{\sigma w}, X) \end{array}$$

In this case, the induced Selmer structure on X over K is simply given by

$$H^1_{\mathrm{res}^*\mathcal{F}}(K_v, X) = \ker\left(H^1(K_v, X) \to H^1_{/\mathcal{F}}(L_w, X)\right)$$

for any choice of a finite place w of L above v, the corestriction map from L down to K is compatible with the Selmer submodules, and we have commutative diagrams

$$\begin{array}{ccc} H^1_{\mathcal{F}}(L, X) & \xrightarrow{\oplus \mathrm{loc}_w} & \oplus_{w|v} H^1_{\mathcal{F}}(L_w, X) \\ {\scriptstyle \mathrm{cores}}\downarrow & & \downarrow{\scriptstyle \oplus \mathrm{cores}_w} \\ H^1_{\mathrm{res}^*\mathcal{F}}(K, X) & \xrightarrow{\mathrm{loc}_v} & H^1_{\mathrm{res}^*\mathcal{F}}(K_v, X) \end{array}$$

If $p \nmid [L : K]$ or $X = V$, the restriction map yields isomorphisms

$$\begin{array}{ccc} H^1_{\mathrm{res}^*\mathcal{F}}(K, X) & \xrightarrow[\simeq]{\mathrm{res}} & H^1_{\mathcal{F}}(L, X)^{\mathrm{Gal}(L/K)} \\ \downarrow & & \downarrow \\ H^1(K, X) & \xrightarrow[\simeq]{\mathrm{res}} & H^1(L, X)^{\mathrm{Gal}(L/K)} \end{array}$$

1.4.6. If $\mathfrak{a}, \mathfrak{b}$ are relatively prime ideals of $\mathcal{O}_K$ and $\mathcal{F}$ is a Selmer structure on X over K, we define a new Selmer structure $\mathcal{F}_{\mathfrak{a}}^{\mathfrak{b}}$ on X over K as follows:

$$H^1_{\mathcal{F}_{\mathfrak{a}}^{\mathfrak{b}}}(K_v, X) \stackrel{\mathrm{def}}{=} \begin{cases} H^1_{\mathcal{F}}(K_v, X) & \text{if } v \nmid \mathfrak{a}\mathfrak{b}, \\ 0 & \text{if } v \mid \mathfrak{a}, \\ H^1(K_v, X) & \text{if } v \mid \mathfrak{b}. \end{cases}$$

If $\mathfrak{a}$ or $\mathfrak{b}$ equals $\mathcal{O}_K$, we drop it from the notation.

1.4.7. For any ideal $\mathfrak{a}$ of $\mathcal{O}_K$, the Poitou-Tate global duality ([61, 3.1] or [44, 4.10]) gives an exact sequence [41, 2.3.4] which we write as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathcal{F}_{\mathfrak{a}}}(K, T_r) & \longrightarrow & H^1_{\mathcal{F}}(K, T_r) & \xrightarrow{\oplus \mathrm{loc}_v} & \oplus_{v|\mathfrak{a}} H^1_{\mathcal{F}}(K_v, T_r) \\ & & & & & & \updownarrow \\ 0 & \longrightarrow & H^1_{\mathcal{F}^*(1)}(K, T_r^*(1)) & \longrightarrow & H^1_{\mathcal{F}^*(1)^{\mathfrak{a}}}(K, T_r^*(1)) & \xrightarrow{\oplus \mathrm{loc}_v} & \oplus_{v|\mathfrak{a}} H^1_{/\mathcal{F}^*(1)}(K_v, T_r^*(1)) \end{array}$$

In this diagram, the vertical arrow refers to the perfect pairing

$$\oplus_{v|\mathfrak{a}} H^1_{\mathcal{F}}(K_v, T_r) \times \oplus_{v|\mathfrak{a}} H^1_{/\mathcal{F}^*(1)}(K_v, T_r^*(1)) \to R_r$$

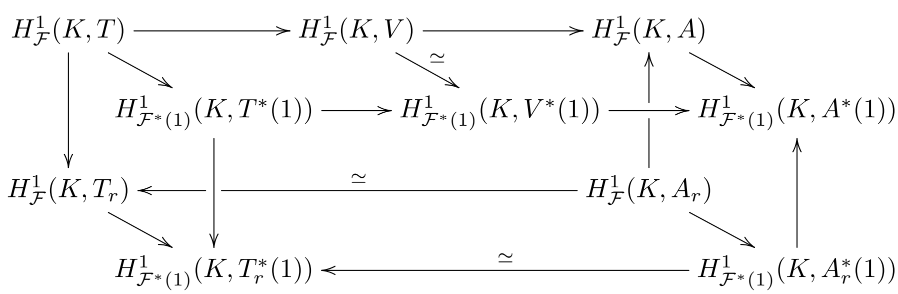
which is the sum of the perfect pairings from Tate's local duality at all $v \mid \mathfrak{a}$, and the exactness of the diagram at this arrow means that under this perfect pairing, the images of both maps $\oplus_{v|\mathfrak{a}} \mathrm{loc}_v$ are orthogonal complements of each other.

1.4.8. The Bloch-Kato Selmer structure $\mathcal{F} = f$ on V over K is defined by

$$H^1_f(K_v, V) \stackrel{\text{def}}{=} \begin{cases} \ker\left(H^1(K_v, V) \to H^1\left(K_v^{\mathrm{ur}}, V\right)\right) & \text{if } v \nmid p, \\ \ker\left(H^1(K_v, V) \to H^1\left(K_v, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\right)\right) & \text{if } v \mid p. \end{cases}$$

It is self-dual in the sense that the dual Selmer structure $f^*(1)$ on $V^*(1)$ is the Bloch-Kato Selmer structure f on $V^*(1)$. The Bloch-Kato Selmer structure f on any $X \in \{T, A, T_r, A_r\}$ is the Selmer structure induced by the Bloch-Kato Selmer structure on V. The dual Selmer structures $f^*(1)$ on the dual $Y \in \{A^*(1), T^*(1), A_r^*(1), T_r^*(1)\}$ of X is therefore also the Bloch-Kato Selmer structure f on Y, induced by the Bloch-Kato Selmer structure $f = f^*(1)$ on $V^*(1)$.

1.4.9. In the self-dual case of 1.1.4, we say that a Selmer structure $\mathcal{F}$ on V over K is self-dual with respect to the given perfect pairing $\langle -, - \rangle : V \times V \to \Phi(1)$ if its local conditions are self-dual with respect to the pairing, in the sense of 1.2.14. For instance, the Bloch-Kato Selmer structure is self-dual in this sense. Applying $H^1(K, -)$ to the diagram of 1.1.4, we obtain a diagram of Selmer submodules:



If L/K is a finite Galois extension and $\mathcal{F}$ is a self-dual, $\mathrm{Gal}(L/K)$-stable Selmer structure on V over L, then $\mathrm{res}^*\mathcal{F}$ is a self-dual Selmer structure on V over K.

2. Kolyvagin Systems

2.1. **The Galois representation.**

2.1.1. Let now $K = F$ be the totally real subfield of a CM field E. Let $E^{\mathrm{ab}} \subset \overline{F}$ be the maximal abelian extension of E and let $E \subset E[\infty] \subset E^{\mathrm{ab}}$ be the subextension which is fixed by the image of the verlagerung morphism $\mathrm{Ver} : \mathrm{Gal}_F^{\mathrm{ab}} \to \mathrm{Gal}_E^{\mathrm{ab}}$.

2.1.2. We denote by τ the complex conjugation on the compositum $E(\mu_\infty) \subset E^{\mathrm{ab}}$ of E and the cyclotomic extension $\mathbb{Q}(\mu_\infty)$ of $\mathbb{Q}$. Note that $E(\mu_\infty) \subset \mathbb{Q}^{\mathrm{cm}}$ where $\mathbb{Q}^{\mathrm{cm}} \subset \overline{F}$ is the union of all CM fields inside $\overline{F}$, thus τ is indeed well-defined on $E(\mu_\infty)$. It fixes F and induces the non-trivial automorphism $\star$ of $\mathrm{Gal}(E/F)$.

2.1.3. We denote by H a given finite subextension of $E[\infty]/E$. Any such extension is Galois over F and its Galois group $\mathrm{Gal}(H/F)$ is a (generalized) dihedral extension of $\mathrm{Gal}(E/F) = \{1, \star\}$ by the abelian group $\mathrm{Gal}(H/E)$.

2.1.4. We assume that our continuous Galois representation

$$\rho : \mathrm{Gal}_F \to \mathrm{Aut}_\Phi(V)$$

is unramified at all but finitely many places v of F, de Rham at all $v \mid p$, and equipped with a Gal_F-equivariant ε-symmetric Φ-bilinear perfect pairing

$$\langle -, - \rangle : V \times V \to \Phi(1).$$

Here $\varepsilon \in \{+1, -1\}$. We require that the Gal_F-stable R-lattice $T \subset V$ satisfies

$$\langle T, T \rangle \subset R(1).$$

2.2. **The assumption $\mathbf{BI}_0$.** We now make the following assumption:

BI_0 There is a $\gamma \in \mathrm{Gal}_F$ with $\gamma = \tau$ on $E(\mu_{p^\infty})$ and $\dim_\Phi V^{\gamma-1} = 1$.

2.2.1. Let $P(t) \overset{\text{def}}{=} \det(\rho(\gamma) - t\,\mathrm{Id} : V)$ be the characteristic polynomial of $\rho(\gamma)$.

Lemma 2.1. *We have $P(t) = (t^2-1)Q(t)$ for some $Q \in \Phi[t]$. Moreover,*

$$Q(\gamma) : V_{\gamma^2-1} \to V^{\gamma^2-1} \quad \textit{and} \quad Q(\gamma)(\gamma \pm 1) : V_{\gamma \mp 1} \to V^{\gamma \mp 1}$$

are isomorphisms, with

$$\begin{array}{rcccl} \dim_\Phi V_{\gamma^2-1} & = & \dim_\Phi V^{\gamma^2-1} & = & 2, \\ \dim_\Phi V_{\gamma\mp1} & = & \dim_\Phi V^{\gamma\mp1} & = & 1. \end{array}$$

Proof. We first show that with $\dim_\Phi V^{\gamma-1} = 1$, also $\dim_\Phi V^{\gamma+1} = 1$. Let $\star$ be the involution of $\mathrm{End}_\Phi(V)$ induced by the perfect pairing $V \times V \to \Phi(1)$. Then

$$\rho(\gamma)^\star = \chi_{\mathrm{cyc}}(\gamma)\rho(\gamma)^{-1} = -\rho(\gamma)^{-1}$$

and thus indeed $\dim_\Phi V^{\gamma+1} = 1$ since

$$V^{\gamma+1} = \ker(\rho(\gamma) + \mathrm{Id}) = (\mathrm{im}(\rho(\gamma) + \mathrm{Id})^\star)^\perp = (\mathrm{im}(\rho(\gamma) - \mathrm{Id}))^\perp.$$

Viewing V as a $\Phi[t]$-module with t acting on V by $\rho(\gamma)$, we may write

$$V \simeq \left(\oplus_{i=1}^{s_+} \Phi[t]/(t-1)^{n_i^+}\right) \oplus \left(\oplus_{j=1}^{s_-} \Phi[t]/(t+1)^{n_j^-}\right) \oplus \left(\oplus_{k=1}^{s} \Phi[t]/Q_k(t)\right)$$

for integers $s_+, s_-, s \geq 0$, positive integers $n_i^\pm \geq 1$ and polynomials $Q_k(t) \in \Phi[t]$ with $Q_k(\pm 1) \neq 0$. Since $s_\mp = \dim_\Phi V^{\gamma\pm1} = 1$, $P(t) = (t^2-1)Q(t)$ with

$$Q(t) \quad = \quad (t-1)^{n_1^+-1} \cdot (t+1)^{n_1^--1} \cdot \textstyle\prod_{k=1}^{s} Q_k(t).$$

The lemma easily follows. $\square$

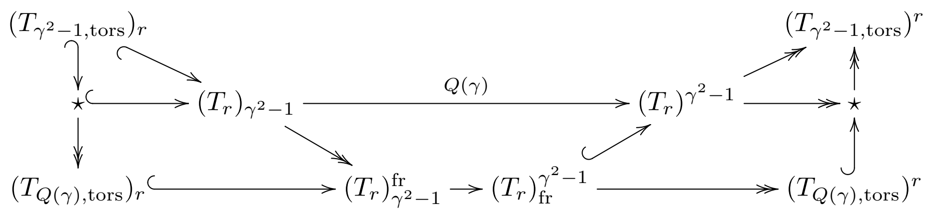
2.2.2. Since $\rho(\gamma)T = T$, $P(t)$ and $Q(t)$ belong to $R[t] \subset \Phi[t]$. Plainly,

$$\begin{array}{rcccl} \mathrm{rank}_R T_{\gamma^2-1} & = & \mathrm{rank}_R T^{\gamma^2-1} & = & 2, \\ \mathrm{rank}_R T_{\gamma\mp1} & = & \mathrm{rank}_R T^{\gamma\mp1} & = & 1. \end{array}$$

Lemma 2.2. *There is an exact sequence*

$$0 \longrightarrow T_{\gamma^2-1,\mathrm{tors}} \longrightarrow T_{\gamma^2-1} \xrightarrow{Q(\gamma)} T^{\gamma^2-1} \longrightarrow T_{Q(\gamma),\mathrm{tors}} \longrightarrow 0$$

and a commutative diagram with exact lines



with $(T_r)^{\mathrm{fr}}_{\gamma^2-1}$ *and* $(T_r)^{\gamma^2-1}_{\mathrm{fr}}$ *free of rank* 2 *over* R_r, *given by*

$$(T_r)^{\mathrm{fr}}_{\gamma^2-1} \stackrel{\mathrm{def}}{=} (T_{\gamma^2-1}/T_{\gamma^2-1,\mathrm{tors}})_r \quad \textit{and} \quad (T_r)^{\gamma^2-1}_{\mathrm{fr}} \stackrel{\mathrm{def}}{=} (T^{\gamma^2-1})_r.$$

Proof. Suppose that $\alpha\beta = 0 = \beta\alpha$ on X. Then α and β induce morphisms

$$\overline{\alpha} : X_\beta \to X^\beta \quad \text{and} \quad \overline{\beta} : X_\alpha \to X^\alpha$$

where as usual $X_\star = \mathrm{coker}(\star)$ and $X^\star = \ker(\star)$, with

$$\ker(\overline{\alpha}) = \frac{\ker(\alpha)}{\mathrm{im}(\beta)} = \mathrm{coker}(\overline{\beta}) \quad \text{and} \quad \mathrm{coker}(\overline{\alpha}) = \frac{\ker(\beta)}{\mathrm{im}(\alpha)} = \ker(\overline{\beta}).$$

We apply this to $\alpha = (\gamma^2 - 1)$ and $\beta = Q(\gamma)$ on $X = V$, T or T_r. For $X = V$, we have seen in Lemma 2.1 that $Q(\gamma) : V_{\gamma^2-1} \to V^{\gamma^2-1}$ is an isomorphism, and so is therefore also $(\gamma^2 - 1) : V_{Q(\gamma)} \to V^{Q(\gamma)}$. It follows that for $X = T$, the kernels of

$$Q(\gamma) : T_{\gamma^2-1} \to T^{\gamma^2-1} \quad \text{and} \quad (\gamma^2 - 1) : T_{Q(\gamma)} \to T^{Q(\gamma)}$$

are the torsion submodules $T_{\gamma^2-1,\mathrm{tors}}$ and $T_{Q(\gamma),\mathrm{tors}}$, which yields the first exact sequence. Finally for $X = T_r$, we consider the factorization

$$(T_r)_{\gamma^2-1} \twoheadrightarrow (T_r)^{\mathrm{fr}}_{\gamma^2-1} \longrightarrow (T_r)^{\gamma^2-1}_{\mathrm{fr}} \hookrightarrow (T_r)^{\gamma^2-1}$$

of $Q(\gamma) : (T_r)_{\gamma^2-1} \to (T_r)^{\gamma^2-1}$ which is given by

$$(T_{\gamma^2-1})_r \twoheadrightarrow (T_{\gamma^2-1}/T_{\gamma^2-1,\mathrm{tors}})_r \xrightarrow{Q(\gamma)} (T^{\gamma^2-1})_r \hookrightarrow (T_r)^{\gamma^2-1}$$

One checks easily that

$$\begin{aligned}
\ker\left((T_r)_{\gamma^2-1} \twoheadrightarrow (T_r)^{\mathrm{fr}}_{\gamma^2-1}\right) &= (T_{\gamma^2-1,\mathrm{tors}})_r \\
\mathrm{coker}\left((T_r)^{\gamma^2-1}_{\mathrm{fr}} \hookrightarrow (T_r)^{\gamma^2-1}\right) &= (T_{\gamma^2-1,\mathrm{tors}})^r \\
\ker\left((T_r)^{\mathrm{fr}}_{\gamma^2-1} \to (T_r)^{\gamma^2-1}_{\mathrm{fr}}\right) &= (T_{Q(\gamma),\mathrm{tors}})^r \\
\mathrm{coker}\left((T_r)^{\mathrm{fr}}_{\gamma^2-1} \to (T_r)^{\gamma^2-1}_{\mathrm{fr}}\right) &= (T_{Q(\gamma),\mathrm{tors}})_r
\end{aligned}$$

which finishes the proof. □

Lemma 2.3. *For* $\epsilon \in \{\pm 1\}$, *there is an exact sequence*

$$0 \longrightarrow T_{\gamma-\epsilon,\mathrm{tors}} \longrightarrow T_{\gamma-\epsilon} \xrightarrow{Q(\gamma)(\gamma+\epsilon)} T^{\gamma-\epsilon} \longrightarrow T_{Q(\gamma)(\gamma+\epsilon),\mathrm{tors}} \longrightarrow 0$$

and a commutative diagram with exact lines

$$\begin{array}{ccccccc}
(T_{\gamma-\epsilon,\mathrm{tors}})_r & & & & & & (T_{\gamma-\epsilon,\mathrm{tors}})^r \\
\downarrow & \searrow & & & & \nearrow & \uparrow \\
\star & \hookrightarrow & (T_r)_{\gamma-\epsilon} & \xrightarrow{Q(\gamma)(\gamma+\epsilon)} & (T_r)^{\gamma-\epsilon} & \twoheadrightarrow & \star \\
\downarrow & & \searrow & & \nearrow & & \uparrow \\
(T_{Q(\gamma)(\gamma+\epsilon),\mathrm{tors}})_r & \hookrightarrow & (T_r)^{\mathrm{fr}}_{\gamma-\epsilon} & \to & (T_r)^{\gamma-\epsilon}_{\mathrm{fr}} & \twoheadrightarrow & (T_{Q(\gamma)(\gamma+\epsilon),\mathrm{tors}})^r
\end{array}$$

with $(T_r)^{\mathrm{fr}}_{\gamma-\epsilon}$ *and* $(T_r)^{\gamma-\epsilon}_{\mathrm{fr}}$ *free of rank* 1 *over* R_r, *given by*

$$(T_r)^{\mathrm{fr}}_{\gamma-\epsilon} \stackrel{\mathrm{def}}{=} (T_{\gamma-\epsilon}/T_{\gamma-\epsilon,\mathrm{tors}})_r \quad \textit{and} \quad (T_r)^{\gamma-\epsilon}_{\mathrm{fr}} \stackrel{\mathrm{def}}{=} (T^{\gamma-\epsilon})_r.$$

Proof. The proof is similar. □

Lemma 2.4. *There is a commutative diagram of split exact sequences*

$$\begin{array}{ccccccccc}
0 & \to & \left(T_{\gamma^2-1,\mathrm{tors}}\right)_r & \to & (T_r)_{\gamma^2-1} & \to & (T_r)^{\mathrm{fr}}_{\gamma^2-1} & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow & & \\
0 & \to & (T_{\gamma-1,\mathrm{tors}})_r \oplus (T_{\gamma+1,\mathrm{tors}})_r & \to & (T_r)_{\gamma-1} \oplus (T_r)_{\gamma+1} & \to & (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1} & \to & 0
\end{array}$$

The kernels and cokernels of all vertical maps are killed by 2.

Proof. The middle vertical map is given by $x \mapsto (x \bmod \gamma-1, x \bmod \gamma-1)$ on $(T_r)_{\gamma^2-1}$. Its restriction to $\left(T_{\gamma^2-1,\mathrm{tors}}\right)_r$ is induced by the restriction of the analogous map $T_{\gamma^2-1} \to T_{\gamma-1} \oplus T_{\gamma+1}$ to the torsion submodules, which yields the first vertical map. The commutativity of the first square yields the third vertical map. The first two maps have kernels and cokernels killed by 2, thus so does the third map. Both exact sequences are split since their last terms are free. □

2.2.3. The same discussion applies equally well to the action of γ on $V^*(1)$.

Lemma 2.5. *There is a commutative diagram of perfect γ-equivariant pairings*

$$\begin{array}{ccccc}
T_r & \times & T_r^*(1) & \longrightarrow & R_r(1) \\
\downarrow & & \uparrow & & \| \\
(T_r)_{\gamma-\epsilon} & \times & T_r^*(1)^{\gamma+\epsilon} & \longrightarrow & R_r(1) \\
\downarrow & & \uparrow & & \| \\
(T_r)^{\mathrm{fr}}_{\gamma-\epsilon} & \times & T_r^*(1)^{\gamma+\epsilon}_{\mathrm{fr}} & \longrightarrow & R_r(1)
\end{array}$$

Proof. Since the first perfect pairing is γ-equivariant, the adjoint of γ acting on T_r equals $-\gamma^{-1}$ acting on $T_r^*(1)$. The orthogonal complement of the image of $\gamma-\epsilon$ on T_r thus equals the kernel of $\gamma+\epsilon$ acting on $T_r^*(1)$, which gives the second perfect pairing. For the third line, we have to show that this second pairing is trivial on

$$\mathrm{im}\left((T_{\gamma-\epsilon,\mathrm{tors}})_r \hookrightarrow (T_r)_{\gamma-\epsilon}\right) \times \mathrm{im}\left((T^*(1)^{\gamma+\epsilon})_r \hookrightarrow (T_r^*(1))^{\gamma+\epsilon}\right).$$

But the corresponding pairing on $(T_{\gamma-\epsilon,\mathrm{tors}})_r \times (T^*(1)^{\gamma+\epsilon})_r$ is induced by the restriction to $T_{\gamma-\epsilon,\mathrm{tors}} \times T^*(1)^{\gamma+\epsilon}$ of the pairing $T_{\gamma-\epsilon} \times T^*(1)^{\gamma+\epsilon} \to R(1)$ which is itself induced by $T \times T^*(1) \to R(1)$: it is trivial since $R(1)$ is torsion free. □

Lemma 2.6. *There is a commutative diagram of perfect γ-equivariant pairings*

$$\begin{array}{ccccc} T_r & \times & T_r^*(1) & \longrightarrow & R_r(1) \\ \downarrow & & \uparrow & & \| \\ (T_r)_{\gamma^2-1} & \times & T_r^*(1)^{\gamma^2-1} & \longrightarrow & R_r(1) \\ \downarrow & & \uparrow & & \| \\ (T_r)^{\mathrm{fr}}_{\gamma^2-1} & \times & T_r^*(1)^{\gamma^2-1}_{\mathrm{fr}} & \longrightarrow & R_r(1) \end{array}$$

Proof. The proof is similar. □

2.2.4. The Gal_F-equivariant isomorphism $\xi : V \to V^*(1)$ given by $\xi(x) = \langle x, -\rangle$ induces embeddings $T \hookrightarrow T^*(1)$, $T^{\gamma^2-1} \hookrightarrow T^*(1)^{\gamma^2-1}$ and $T^{\gamma-\epsilon} \hookrightarrow T^*(1)^{\gamma-\epsilon}$ with finite cokernels Ξ', $\Xi \subset (\Xi')^{\gamma^2-1}$ and $\Xi(\epsilon) \subset \Xi^{\gamma-\epsilon}$. We thus obtain exact sequences

$$0 \to \Xi^r \to (T_r)^{\gamma^2-1}_{\mathrm{fr}} \xrightarrow{\xi} (T_r^*(1))^{\gamma^2-1}_{\mathrm{fr}} \to \Xi_r \to 0$$

$$0 \to \Xi(\epsilon)^r \to (T_r)^{\gamma-\epsilon}_{\mathrm{fr}} \xrightarrow{\xi} (T_r^*(1))^{\gamma-\epsilon}_{\mathrm{fr}} \to \Xi(\epsilon)_r \to 0$$

2.2.5. We define the following constants:

$$\begin{aligned} e(\epsilon) &\stackrel{\mathrm{def}}{=} \exp\left(T_{\gamma-\epsilon,\mathrm{tors}}\right) \\ e(Q,\epsilon) &\stackrel{\mathrm{def}}{=} \exp\left(T_{Q(\gamma)(\gamma+\epsilon),\mathrm{tors}}\right) \\ e(\Xi,\epsilon) &\stackrel{\mathrm{def}}{=} \exp\left(\Xi(\epsilon)\right) \\ e(\Xi) &\stackrel{\mathrm{def}}{=} \exp(\Xi) \\ e(\gamma) &\stackrel{\mathrm{def}}{=} \exp\left(T_{\gamma^2-1,\mathrm{tors}}\right) \\ e(Q) &\stackrel{\mathrm{def}}{=} \exp\left(T_{Q(\gamma),\mathrm{tors}}\right) \\ r_0 &\stackrel{\mathrm{def}}{=} \exp\left(\frac{R(1)}{\langle T, T\rangle}\right) \end{aligned}$$

For any $\sigma \in \mathrm{Gal}_F$ and $r \geq 1$,

$$\sigma \equiv \gamma \text{ on } T_{r+r_0} \Longrightarrow \sigma \equiv \tau \text{ on } \pi^{r_0} R_{r+r_0}(1) \simeq R_r(1) \Longrightarrow \chi_{\mathrm{cyc}}(\sigma) + 1 \equiv 0 \text{ in } R_r.$$

For $a \in (T_r)^{\mathrm{fr}}_{\gamma+\epsilon}$ and $b \in (T_r)^{\gamma-\epsilon}_{\mathrm{fr}}$ with $\langle a, b\rangle = 0$ in $R_r(1)$,

$$\exp(a) + \exp(b) \leq r + e(\Xi, \epsilon)$$

For $b \in (T_r)^{\gamma^2-1}_{\mathrm{fr}}$ with orthogonal complement $b^\perp \subset (T_r)^{\mathrm{fr}}_{\gamma^2-1}$,

$$b^\perp \simeq R_r \oplus R_s \quad \text{with} \quad s \leq r - \exp(b) + e(\Xi).$$

It follows that for every $a, a' \in b^\perp$,

$$\pi^{2r+e(\Xi)-\exp(a)-\exp(b)} a' \in R_r a.$$

2.3. Kolyvagin primes for T and γ.

2.3.1. A Kolyvagin prime of level r (for T and γ) is a prime $\ell \nmid p$ of F unramified in $E'_r \stackrel{\mathrm{def}}{=} E(T_r, R_r(1))$, whose geometric Frobenius conjugacy class in $\mathrm{Gal}(E'_r/F)$ contains $\gamma|_{E_r}$. We denote by $\mathcal{P}_r = \mathcal{P}_r(T, \gamma)$ the set of Kolyvagin primes of level r. It is infinite by Chebotarev's density theorem.

2.3.2. Let ℓ be a Kolyvagin prime of level r which is unramified in H. Then ℓ is unramified in the compositum $H'_r \stackrel{\text{def}}{=} H(T_r, R_r(1))$ of H and E'_r, inert in E/F, and the unique prime $\ell\mathcal{O}_E$ of $\mathcal{O}_E$ above ℓ splits completely in H. Let $\mathcal{L} \mid \lambda \mid \ell$ be primes of H'_r and H above ℓ, so that $E \hookrightarrow H$ induces an isomorphism $E_\ell \simeq H_\lambda$. Let $\mathbb{F}(\ell) \subset \mathbb{E}(\ell) = \mathbb{E}(\lambda)$ be the residue fields of ℓ, $\ell\mathcal{O}_E$ and λ. Fix an embedding $\overline{F} \hookrightarrow \overline{F}_\ell$ inducing $\mathcal{L}$ on H'_r, which thus maps H'_r into $F_\ell^{\mathrm{ur}} = E_\ell^{\mathrm{ur}} = H_\lambda^{\mathrm{ur}}$. With notations as in 1.3, let $\mathrm{Fr}_\ell \in \mathrm{Gal}(F_\ell^{\mathrm{ur}}/F_\ell)$ and $\mathrm{Fr}_\lambda = \mathrm{Fr}_\ell^2 \in \mathrm{Gal}(E_\ell^{\mathrm{ur}}/E_\ell) = \mathrm{Gal}(H_\lambda^{\mathrm{ur}}/H_\lambda)$ be the Frobenius elements, so that $\mathrm{Fr}_{\mathcal{L}} = \mathrm{Fr}_\ell|_{H'_r}$ is the Frobenius of $\mathcal{L}$ in $\mathrm{Gal}(H'_r/F)$. Since $\mathrm{Fr}_\ell|_{E'_r}$ is conjugate to $\gamma|_{E'_r}$, $\mathrm{Fr}_\ell \equiv \gamma \equiv \tau$ on $R_r(1)$, thus $|\mathbb{F}(\ell)| + 1 \equiv 0$ in R_r and we may apply 1.3.2. In particular, we have $\mathrm{Fr}_{\mathcal{L}}$-equivariant isomorphisms

$$\mathrm{ev}(\mathrm{Fr}_{\mathcal{L}}^2) : H^1_f(H_\lambda, T_r) \xrightarrow{\simeq} (T_r)_{\mathrm{Fr}_{\mathcal{L}}^2 - 1} \quad \text{and} \quad H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) \xrightarrow{\simeq} (T_r)^{\mathrm{Fr}_{\mathcal{L}}^2 - 1}$$

where $\mathbb{G}(\ell) \stackrel{\text{def}}{=} \mathbb{E}(\ell)^\times/\mathbb{F}(\ell)^\times$. Since again $\mathrm{Fr}_\ell|_{E'_r}$ is conjugate to $\gamma|_{E'_r}$, the characteristic polynomials $P_\ell(t)$ and $P(t) = (t^2 - 1)Q(t)$ of Fr_ℓ and γ acting on T_r are equal, thus $(\mathrm{Fr}_{\mathcal{L}}^2 - 1)Q(\mathrm{Fr}_{\mathcal{L}}) = 0$ on T_r and we obtain a $\mathrm{Fr}_{\mathcal{L}}$-equivariant map

$$Q(\mathrm{Fr}_{\mathcal{L}}) : (T_r)_{\mathrm{Fr}_{\mathcal{L}}^2 - 1} \to (T_r)^{\mathrm{Fr}_{\mathcal{L}}^2 - 1}$$

2.3.3. The finite/singular morphism is the composite map

$$\begin{array}{ccc} H^1_f(H_\lambda, T_r) & \xrightarrow{\Phi_{\mathcal{L}}} & H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) \\ \simeq \downarrow & & \uparrow \simeq \\ (T_r)_{\mathrm{Fr}_{\mathcal{L}}^2 - 1} & \xrightarrow{Q(\mathrm{Fr}_{\mathcal{L}})} & (T_r)^{\mathrm{Fr}_{\mathcal{L}}^2 - 1} \end{array}$$

It is Fr_ℓ-equivariant and independent of the choice of $\mathcal{L}$ above λ. More generally if $\mathcal{L}' \mid \lambda' \mid \ell$ is any other pair of primes of H'_r and H above ℓ, there is a σ in $\mathrm{Gal}(H'_r/E)$ mapping $\mathcal{L}$ to $\sigma\mathcal{L} = \mathcal{L}'$, so that $\sigma \mathrm{Fr}_{\mathcal{L}} \sigma^{-1} = \mathrm{Fr}_{\mathcal{L}'}$. Extending σ to $\overline{F}_\ell$, we obtain compatible isomorphisms which do not depend upon this extension,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_f(H_\lambda, T_r) & \longrightarrow & H^1(H_\lambda, T_r) & \longrightarrow & H^1_s(H_\lambda, T_r) & \longrightarrow & 0 \\ & & \simeq \downarrow \sigma & & \simeq \downarrow \sigma & & \simeq \downarrow \sigma & & \\ 0 & \longrightarrow & H^1_f(H_{\lambda'}, T_r) & \longrightarrow & H^1(H_{\lambda'}, T_r) & \longrightarrow & H^1_s(H_{\lambda'}, T_r) & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccc} H^1_f(H_\lambda, T_r) & \xrightarrow{\simeq} & (T_r)_{\mathrm{Fr}_{\mathcal{L}}^2 - 1} \\ \simeq \downarrow \sigma & & \simeq \downarrow \sigma \\ H^1_f(H_{\lambda'}, T_r) & \xrightarrow{\simeq} & (T_r)_{\mathrm{Fr}_{\mathcal{L}'}^2 - 1} \end{array} \qquad \begin{array}{ccc} H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) & \xrightarrow{\simeq} & (T_r)^{\mathrm{Fr}_{\mathcal{L}}^2 - 1} \\ \simeq \downarrow \sigma \otimes 1 & & \simeq \downarrow \sigma \\ H^1_s(H_{\lambda'}, T_r) \otimes \mathbb{G}(\ell) & \xrightarrow{\simeq} & (T_r)^{\mathrm{Fr}_{\mathcal{L}'}^2 - 1} \end{array}$$

Since also $\sigma Q(\mathrm{Fr}_{\mathcal{L}}) = Q(\mathrm{Fr}_{\mathcal{L}'})\sigma$ on T_r, we obtain a commutative diagram

$$\begin{array}{ccc} H^1_f(H_\lambda, T_r) & \xrightarrow{\Phi_{\mathcal{L}}} & H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) \\ \simeq \downarrow \sigma & & \simeq \downarrow \sigma \otimes 1 \\ H^1_f(H_{\lambda'}, T_r) & \xrightarrow{\Phi_{\mathcal{L}'}} & H^1_s(H_{\lambda'}, T_r) \otimes \mathbb{G}(\ell) \end{array}$$

If $\lambda = \lambda'$, then $\sigma|_H = 1$ and both vertical maps are the identity, thus $\Phi_\lambda \stackrel{\text{def}}{=} \Phi_{\mathcal{L}}$ does not depend upon the chosen $\mathcal{L}$ above λ and for every $\sigma \in \mathrm{Gal}(H/E)$, the diagram

$$\begin{CD}
H^1_f(H_\lambda, T_r) @>{\Phi_\lambda}>> H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) \\
@V{\simeq}V{\sigma}V @V{\simeq}V{\sigma\otimes 1}V \\
H^1_f(H_{\sigma\lambda}, T_r) @>{\Phi_{\sigma\lambda}}>> H^1_s(H_{\sigma\lambda}, T_r) \otimes \mathbb{G}(\ell)
\end{CD}$$

is commutative. Since Φ_λ is itself Fr_ℓ-equivariant, the diagram

$$\begin{CD}
H^1_f(H_\lambda, T_r) @>{\Phi_\lambda}>> H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) \\
@V{\simeq}V{\sigma}V @V{\simeq}V{\sigma\otimes \sigma}V \\
H^1_f(H_{\sigma\lambda}, T_r) @>{\Phi_{\sigma\lambda}}>> H^1_s(H_{\sigma\lambda}, T_r) \otimes \mathbb{G}(\ell)
\end{CD}$$

is commutative for all $\sigma \in \mathrm{Gal}(H/F)$.

2.3.4. A Kolyvagin prime for H'_r is a prime $\mathcal{L} \nmid p$ of H'_r which is unramified over F with Frobenius $\mathrm{Fr}_{\mathcal{L}} = \gamma|_{H'_r}$ in $\mathrm{Gal}(H'_r/F)$. Then $\mathcal{L}$ sits over a Kolyvagin prime $\ell \in \mathcal{P}_r$ which is unramified in H. Conversely for any $\ell \in \mathcal{P}_r$ which is unramified in H, there is a Kolyvagin prime $\mathcal{L}$ for H'_r above ℓ. If λ is the prime of H induced by $\mathcal{L}$, we define morphisms $\mathrm{loc}^{\mathrm{fr}}_\lambda$ and $\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}$ by the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & (T_r)^{\mathrm{fr}}_{\gamma^2-1} \\
 & & & & & \nearrow & \downarrow \\
\mathrm{loc}_\lambda^{-1}\left(H^1_f(H_\lambda, T_r)\right) & \xrightarrow{\mathrm{loc}_\lambda} & H^1_f(H_\lambda, T_r) & \xrightarrow[\simeq]{\mathrm{ev}(\gamma)} & (T_r)_{\gamma^2-1} & & \downarrow \\
 & & & & & \searrow & \\
 & & & & & & (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}
\end{array}$$

with $\mathrm{loc}^{\mathrm{fr}}_\lambda : \mathrm{loc}_\lambda^{-1}\left(H^1_f(H_\lambda, T_r)\right) \to (T_r)^{\mathrm{fr}}_{\gamma^2-1}$ and $\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon} : \mathrm{loc}_\lambda^{-1}\left(H^1_f(H_\lambda, T_r)\right) \to (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}$.

Note also that by Lemma 2.2, there is a factorization

$$\begin{CD}
H^1_f(H_\lambda, T_r) @>{\simeq}>> (T_r)_{\gamma^2-1} @>>> (T_r)^{\mathrm{fr}}_{\gamma^2-1} \\
@V{\Phi_\lambda}VV @VV{Q(\gamma)}V @VVV \\
H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell) @<{\simeq}<< (T_r)^{\gamma^2-1} @<<< (T_r)^{\gamma^2-1}_{\mathrm{fr}}
\end{CD}$$

and similarly by Lemma 2.3 for $\epsilon \in \{\pm 1\}$, there is a factorization

$$\begin{CD}
H^1_f(H_\lambda, T_r)_{\mathrm{Fr}_\ell-\epsilon} @>{\simeq}>> (T_r)_{\gamma-\epsilon} @>>> (T_r)^{\mathrm{fr}}_{\gamma-\epsilon} \\
@V{(\mathrm{Fr}_\ell+\epsilon)\Phi_\lambda}VV @VV{(\gamma+\epsilon)Q(\gamma)}V @VVV \\
\left(H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\ell)\right)^{\mathrm{Fr}_\ell-\epsilon} @<{\simeq}<< (T_r)^{\gamma-\epsilon} @<<< (T_r)^{\gamma-\epsilon}_{\mathrm{fr}}
\end{CD}$$

2.4. **Kolyvagin systems for T and γ.**

2.4.1. We fix a $\mathrm{Gal}(H/F)$-stable self-dual Selmer structure $\mathcal{F}$ on V over H and still denote by $\mathcal{F}$ the induced Selmer structures on T and T_r. We fix a subset $\mathcal{P}_{\mathcal{F}}$ of $\mathcal{P} = \mathcal{P}_1(T,\gamma)$ such that $\mathcal{P} \setminus \mathcal{P}_{\mathcal{F}}$ is finite and contains all the bad ℓ's: those where V or H ramifies, and those where $H^1_{\mathcal{F}}(H_\lambda, V) \neq H^1_{ur}(H_\lambda, V)$ for some $\lambda \mid \ell$. Then for any $\lambda \mid \ell \in \mathcal{P}_{\mathcal{F}}$, ℓ is unramified in H'_r and $H^1_{\mathcal{F}}(H_\lambda, T_r) = H^1_f(H_\lambda, T_r)$. We set $\mathcal{P}_{\mathcal{F},r} = \mathcal{P}_{\mathcal{F}} \cap \mathcal{P}_r$ and let $\mathcal{N}_{\mathcal{F},r}$ be the set of square-free products of elements of $\mathcal{P}_{\mathcal{F},r}$.

2.4.2. For $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$, we define a new $\mathrm{Gal}(H/F)$-stable Selmer structure $\mathcal{F}(\mathfrak{n})$ on T_r as follows: for any finite place λ of H,

$$H^1_{\mathcal{F}(\mathfrak{n})}(H_\lambda, T_r) = \begin{cases} H^1_{\mathcal{F}}(H_\lambda, T_r) & \text{if } \lambda \nmid \mathfrak{n}, \\ H^1_t(H_\lambda, T_r) = H^1_t(E_\ell, T_r) & \text{if } \lambda \mid \ell \mid \mathfrak{n}. \end{cases}$$

For $\mathfrak{n}\ell \in \mathcal{N}_{\mathcal{F},r}$ and any $\lambda \mid \ell$, we have a diagram

$$\begin{array}{ccccc} H^1_{\mathcal{F}(\mathfrak{n})}(H, T_r) \otimes \mathbb{G}(\mathfrak{n}) & & H^1_{\mathcal{F}^{\mathfrak{n}\ell}}(H, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell) & \hookleftarrow & H^1_{\mathcal{F}(\mathfrak{n}\ell)}(H, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell) \\ \downarrow{\scriptstyle \mathrm{loc}_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n})}} & & \downarrow{\scriptstyle \mathrm{loc}^s_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n}\ell)}} & & \downarrow{\scriptstyle \mathrm{loc}_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n}\ell)}} \\ H^1_f(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}) & \xrightarrow{\Phi_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n})}} & H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell) & \xleftarrow{\simeq} & H^1_t(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell) \end{array}$$

where $\mathbb{G}(\mathfrak{n}) = \otimes_{\mathfrak{q} \mid \mathfrak{n}} \mathbb{G}(\mathfrak{q})$. We simply write loc_λ, loc^s_λ and Φ_λ for the above maps.

Definition 2.7. A strong (resp. weak) Kolyvagin system of level r if a collection

$$\kappa_r(\mathfrak{n}) \in H^1_{\mathcal{F}(\mathfrak{n})}(H, T_r) \otimes \mathbb{G}(\mathfrak{n}) \ \left(\text{resp. } H^1_{\mathcal{F}^{\mathfrak{n}}}(H, T_r) \otimes \mathbb{G}(\mathfrak{n})\right), \qquad \mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$$

such that for every $\mathfrak{n}\ell \in \mathcal{N}_{\mathcal{F},r}$ and every place $\lambda \mid \ell$ of H,

$$(\Phi_\lambda \circ \mathrm{loc}_\lambda)(\kappa_r(\mathfrak{n})) = \mathrm{loc}^s_\lambda(\kappa_r(\mathfrak{n}\ell)) \quad \text{in} \quad H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell).$$

Plainly, any strong Kolyvagin system of level r induces a weak one.

Remark 2.8. These Kolyvagin system relations imply that our classes have localizations landing into free submodules of the singular cohomology. Indeed, suppose that $\mathcal{L}$ is a Kolyvagin prime for H'_r above $\lambda \mid \ell$, which implies that $\gamma\lambda = \lambda$. Then

$$\begin{array}{c} \mathrm{loc}^s_\lambda(\kappa_r(\mathfrak{n}\ell)) \text{ belongs to } (T_r)^{\gamma^2-1}_{\mathrm{fr}} \otimes \mathbb{G}(\mathfrak{n}) \\ \text{inside } (T_r)^{\gamma^2-1} \otimes \mathbb{G}(\mathfrak{n}) \simeq H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell). \end{array}$$

by Lemma 2.2. Moreover, since then also Φ_λ and loc^s_λ are γ-equivariant, we have

$$\mathrm{loc}^s_\lambda((\gamma+\epsilon)\kappa_r(\mathfrak{n}\ell)) = (\gamma+\epsilon)\Phi_\lambda \circ \mathrm{loc}_\lambda(\kappa_r(\mathfrak{n})) = \Phi_\lambda \circ \mathrm{loc}_\lambda((\gamma+\epsilon)\kappa_r(\mathfrak{n}))$$

and Lemma 2.3 similarly implies that

$$\begin{array}{c} \mathrm{loc}^s_\lambda((\gamma+\epsilon)\kappa_r(\mathfrak{n}\ell)) \text{ belongs to } (T_r)^{\gamma-\epsilon}_{\mathrm{fr}} \otimes \mathbb{G}(\mathfrak{n}) \\ \text{inside } (T_r)^{\gamma^2-1} \otimes \mathbb{G}(\mathfrak{n}) \simeq H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell). \end{array}$$

2.4.3. So far, r was a fixed positive integer. We now allow it to vary as follows. Let $\mathcal{N}_{\mathcal{F}}$ be the set of all square-free products of elements of $\mathcal{P}_{\mathcal{F}}$. For $\mathfrak{n} \in \mathcal{N}_{\mathcal{F}}$, set

$$r(\mathfrak{n}) = \min\{r(\ell) : \ell \mid \mathfrak{n}\} \quad \text{with} \quad r(\ell) = \max\{r : \ell \in \mathcal{P}_{\mathcal{F},r}\}.$$

Since $\ell \in \mathcal{P}_{\mathcal{F},r} \subset \mathcal{P}_r$ implies $|\mathbb{F}(\ell)| + 1 \equiv 0$ in R_r, it follows that $1 \leq r(\ell) < \infty$ for every $\ell \in \mathcal{P}_{\mathcal{F}}$ and $1 \leq r(\mathfrak{n}) < \infty$ for every $\mathfrak{n} \neq 1 \in \mathcal{N}_{\mathcal{F}}$. For $\mathfrak{n}\ell \in \mathcal{N}_{\mathcal{F}}$ and any place

$\lambda \mid \ell$ of H, we now have a diagram

$$\begin{array}{ccccc}
H^1_{\mathcal{F}(\mathfrak{n})}\left(H, T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n}) & & H^1_{\mathcal{F}^{\mathfrak{n}\ell}}\left(H, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}\ell) & \hookleftarrow & H^1_{\mathcal{F}(\mathfrak{n}\ell)}\left(H, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}\ell) \\
\downarrow{\scriptstyle (1)} & & & & \\
H^1_{\mathcal{F}(\mathfrak{n})}\left(H, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}) & & \downarrow{\scriptstyle \mathrm{loc}^s_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n}\ell)}} & & \downarrow{\scriptstyle \mathrm{loc}_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n}\ell)}} \\
\downarrow{\scriptstyle \mathrm{loc}_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n})}} & & & & \\
H^1_f\left(H_\lambda, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}) & \xrightarrow{\Phi_\lambda \otimes 1_{\mathbb{G}(\mathfrak{n})}} & H^1_s\left(H_\lambda, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}\ell) & \xleftarrow{\simeq} & H^1_t\left(H_\lambda, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}\ell)
\end{array}$$

where the new map (1) is induced by the projection $T_{r(\mathfrak{n})} \twoheadrightarrow T_{r(\mathfrak{n}\ell)}$. By convention $r(1) = \infty$, $T_\infty = T$ and $\mathbb{G}(1) = \mathbb{Z}$, thus for $\mathfrak{n} = 1$, the previous diagram becomes

$$\begin{array}{ccccc}
H^1_{\mathcal{F}}(H, T) & & H^1_{\mathcal{F}^{\ell}}\left(H, T_{r(\ell)}\right) \otimes \mathbb{G}(\ell) & \hookleftarrow & H^1_{\mathcal{F}(\ell)}\left(H, T_{r(\ell)}\right) \otimes \mathbb{G}(\ell) \\
\downarrow & & & & \\
H^1_{\mathcal{F}}\left(H, T_{r(\ell)}\right) & & \downarrow{\scriptstyle \mathrm{loc}^s_\lambda \otimes 1_{\mathbb{G}(\ell)}} & & \downarrow{\scriptstyle \mathrm{loc}_\lambda \otimes 1_{\mathbb{G}(\ell)}} \\
\downarrow{\scriptstyle \mathrm{loc}_\lambda} & & & & \\
H^1_f\left(H_\lambda, T_{r(\ell)}\right) & \xrightarrow{\Phi_\lambda} & H^1_s\left(H_\lambda, T_{r(\ell)}\right) \otimes \mathbb{G}(\ell) & \xleftarrow{\simeq} & H^1_t\left(H_\lambda, T_{r(\ell)}\right) \otimes \mathbb{G}(\ell)
\end{array}$$

Again, we simply write loc_λ and Φ_λ for the above maps.

Definition 2.9. A strong (resp. weak) Kolyvagin system is a collection of elements

$$\kappa(\mathfrak{n}) \in H^1_{\mathcal{F}(\mathfrak{n})}\left(H, T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n}) \ \left(\text{resp. } H^1_{\mathcal{F}^{\mathfrak{n}}}\left(H, T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n})\right), \qquad \mathfrak{n} \in \mathcal{N}_{\mathcal{F}}$$

such that for every $\mathfrak{n}\ell \in \mathcal{N}_{\mathcal{F}}$ and every place $\lambda \mid \ell$ of H,

$$\left(\Phi_\lambda \circ \mathrm{loc}_\lambda\right)(\kappa(\mathfrak{n})) = \mathrm{loc}^s_\lambda(\kappa(\mathfrak{n}\ell)) \quad \text{in} \quad H^1_s\left(H_\lambda, T_{r(\mathfrak{n}\ell)}\right) \otimes \mathbb{G}(\mathfrak{n}\ell).$$

A strong Kolyvagin system induces a weak one, and a strong (resp. weak) Kolyvagin system $(\kappa(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_{\mathcal{F}}}$ induces a strong (resp. weak) Kolyvagin system $(\kappa_r(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_{\mathcal{F},r}}$ of level r as follows: for $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$, $\kappa_r(\mathfrak{n})$ is the image of $\kappa(\mathfrak{n})$ under the morphism

$$H^1_{\mathcal{F}(\mathfrak{n}) \text{ or } \mathcal{F}^{\mathfrak{n}}}\left(H, T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n}) \to H^1_{\mathcal{F}(\mathfrak{n}) \text{ or } \mathcal{F}^{\mathfrak{n}}}\left(H, T_r\right) \otimes \mathbb{G}(\mathfrak{n})$$

induced by the projection $T_{r(\mathfrak{n})} \twoheadrightarrow T_r$; note that $r(\mathfrak{n}) \geq r$ since $\mathfrak{n}$ belongs to $\mathcal{N}_{\mathcal{F},r}$.

2.4.4. We denote by $\mathbf{KS} \subset \mathbf{KS}^w$ (resp. $\mathbf{KS}_r \subset \mathbf{KS}^w_r$) the set of all strong and weak Kolyvagin systems (resp. of level r), and write $\mathbf{KS}^\star_\star = \mathbf{KS}^\star_\star(T, \gamma, H, \mathcal{F})$ when we want to emphasize their dependence on T, γ , H or $\mathcal{F}$. These are left $R[\mathrm{Gal}(H/F)]$-modules (resp. left $R_r[\mathrm{Gal}(H/F)]$-modules. Indeed for every $\mathfrak{n} \in \mathcal{N}_{\mathcal{F}}$ and $r \leq r(\mathfrak{n})$,

$$H^1_{\mathcal{F}(\mathfrak{n})}(H, T_r) \otimes \mathbb{G}(\mathfrak{n}) \quad \text{and} \quad H^1_{\mathcal{F}^{\mathfrak{n}}}(H, T_r) \otimes \mathbb{G}(\mathfrak{n})$$

are $R_r[\mathrm{Gal}(H/F)]$-stable submodules of $H^1(H, T_r) \otimes \mathbb{G}(\mathfrak{n})$, and we may thus set $x\kappa = (x\kappa(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_{\mathcal{F}}}$ (resp. $x\kappa_r = (x\kappa_r(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_{\mathcal{F},r}}$) for any $x \in R[\mathrm{Gal}(H/F)]$. These classes still satisfy the required Kolyvagin system relations. Indeed when x is a group element $\sigma \in \mathrm{Gal}(H/F)$, for $\mathfrak{n}\ell \in \mathcal{N}_{\mathcal{F}}$, $r \leq r(\mathfrak{n}\ell)$ and any $\lambda \mid \ell$, we have

$$\begin{aligned}
\Phi_\lambda \circ \mathrm{loc}_\lambda(\sigma\kappa_r(\mathfrak{n})) &= \Phi_\lambda\left(\sigma \cdot \mathrm{loc}_{\sigma^{-1}\lambda}(\kappa_r(\mathfrak{n}))\right) \\
&= \sigma \cdot \left(\Phi_{\sigma^{-1}\lambda} \circ \mathrm{loc}_{\sigma^{-1}\lambda}(\kappa_r(\mathfrak{n}))\right) \\
&= \sigma \cdot \left(\mathrm{loc}^s_{\sigma^{-1}\lambda}(\kappa_r(\mathfrak{n}\ell))\right) \\
&= \mathrm{loc}^s_\lambda(\sigma\kappa_r(\mathfrak{n}\ell))
\end{aligned}$$

using 2.3.3 for the second equality.

2.4.5. Let $\kappa = (\kappa(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_F}$ be a weak or strong Kolyvagin system (for T over H). Let H' be a subextension of H/E. Let $\mathcal{F}'$ be the $\mathrm{Gal}(H'/F)$-stable self-dual Selmer structure on V over H' induced by $\mathcal{F}$, and still denote by $\mathcal{F}'$ the induced $\mathrm{Gal}(H'/F)$-stable Selmer structures on T or T_r over H'. For $\mathfrak{n} \in \mathcal{N}_\mathcal{F}$ and $1 \leq r \leq \infty$, the corestriction $H^1(H, T_r) \to H^1(H', T_r)$ then maps $H^1_{\mathcal{F}^\mathfrak{n}}(H, T_r)$ to $H^1_{\mathcal{F}'^\mathfrak{n}}(H', T_r)$ and for $r \leq r(\mathfrak{n})$, it also maps $H^1_{\mathcal{F}(\mathfrak{n})}(H, T_r)$ to $H^1_{\mathcal{F}'(\mathfrak{n})}(H', T_r)$. We may thus define

$$\kappa'(\mathfrak{n}) = \left(\mathrm{cores} \otimes 1_{\mathbb{G}(\mathfrak{n})}\right)(\kappa(\mathfrak{n})) \in \begin{cases} H^1_{\mathcal{F}'^\mathfrak{n}}\left(H', T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n}) \\ H^1_{\mathcal{F}'(\mathfrak{n})}\left(H', T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n}) \end{cases}$$

For $\mathfrak{n}\ell \in \mathcal{N}_\mathcal{F}$ and any $\lambda' \mid \ell$ of H', we still have the Kolyvagin system relation

$$\Phi_{\lambda'} \circ \mathrm{loc}_{\lambda'}\left(\kappa'(\mathfrak{n})\right) = \mathrm{loc}^s_{\lambda'}\left(\kappa'(\mathfrak{n}\ell)\right) \quad \text{in} \quad H^1_s(H'_{\lambda'}, T_{r(\mathfrak{n}\ell)}) \otimes \mathbb{G}(\mathfrak{n}\ell).$$

This follows from the commutativity of the diagram in 1.4.5 along with that of

$$\begin{array}{ccc} H^1_f(H_\lambda, T_{r(\mathfrak{n}\ell)}) & \xrightarrow{\Phi_\lambda} & H^1_s(H_\lambda, T_{r(\mathfrak{n}\ell)}) \otimes \mathbb{G}(\ell) \\ {\scriptstyle \mathrm{cores}_\lambda}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \mathrm{cores}_\lambda \otimes 1} \\ H^1_f(H'_{\lambda'}, T_{r(\mathfrak{n}\ell)}) & \xrightarrow{\Phi_{\lambda'}} & H^1_s(H_{\lambda'}, T_{r(\mathfrak{n}\ell)}) \otimes \mathbb{G}(\ell) \end{array}$$

for any place $\lambda \mid \lambda'$ of H, which is obvious since $E_\ell \simeq H'_{\lambda'} \simeq H_\lambda$. We thus obtain a weak or strong Kolyvagin system $\kappa' = \mathrm{cores}(\kappa) = (\kappa'(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_F}$ for T over H'. The same construction also applies to (weak or strong) Kolyvagin systems of level r, and we thus obtain a commutative diagram of left $R[\mathrm{Gal}(H/F)]$-modules

$$\begin{array}{ccccccc} \mathbf{KS}(H) & & \longrightarrow & & \mathbf{KS}(H') & & \\ & \searrow & & & & \searrow & \\ \downarrow & & \mathbf{KS}_r(H) & \longrightarrow & \downarrow & & \mathbf{KS}_r(H') \\ & & & & & & \\ \mathbf{KS}^w(H) & & \longrightarrow & & \mathbf{KS}^w(H') & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & \mathbf{KS}^w_r(H) & \longrightarrow & & & \mathbf{KS}^w_r(H') \end{array}$$

2.4.6. Let us choose for every $\ell \in \mathcal{P}_\mathcal{F}$ a generator ζ_ℓ of $\mathbb{G}(\ell)$. In the above definitions, we may then get rid of the $\mathbb{G}(\mathfrak{n})$-factors by replacing each Φ_λ with the unique morphism $\Phi^\sim_\lambda : H^1_f(H_\lambda, T_r) \to H^1_s(H_\lambda, T_r)$ such that $\Phi_\lambda = \Phi^\sim_\lambda \otimes \zeta_\ell$, where ℓ is the prime of F below λ. The relevant systems will be called ζ-Kolyvagin systems, and denoted with a tilda. We pass back and forth from a Kolyvagin system κ (of any sort) to the corresponding ζ-Kolyvagin system $\tilde{\kappa}$ by the formula

$$\kappa(\mathfrak{n}) = \tilde{\kappa}(\mathfrak{n}) \otimes (\otimes_{\ell\mid\mathfrak{n}}\zeta_\ell).$$

The cumbersome Kolyvagin systems are more canonical than their ζ-variants, a key difference being that while Φ_λ is Fr_ℓ-equivariant, we have $\mathrm{Fr}_\ell \circ \Phi^\sim_\lambda = -\Phi^\sim_\lambda \circ \mathrm{Fr}_\ell$.

3. An application of Chebotarev's density theorem

3.1. The assumptions $\mathbf{BI}_1$ and $\mathbf{BI}_2$.

3.1.1. We consider the following "big image" assumptions on (V, H):

$\mathbf{BI}_1$ The image of $\rho : \mathrm{Gal}_H \to \mathrm{Aut}_\Phi(V)$ contains a non-trivial homothety.

$\mathbf{BI}_2$ The representation V of Gal_H is absolutely irreducible.

3.1.2. Consider the following diagram, defining the rings in its 3rd and 4th columns:

$$\begin{array}{ccccccccc}
\Phi[\mathrm{Gal}_H] & \twoheadrightarrow & \Phi[\rho(\mathrm{Gal}_H)] & \twoheadrightarrow & S(\Phi) & \hookrightarrow & S^{\max}(\Phi) & = & \mathrm{End}_\Phi(V) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
R[\mathrm{Gal}_H] & \twoheadrightarrow & R[\rho(\mathrm{Gal}_H)] & \twoheadrightarrow & S & \hookrightarrow & S^{\max} & = & \mathrm{End}_R(T) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
R_r[\mathrm{Gal}_H] & \twoheadrightarrow & R_r[\rho_r(\mathrm{Gal}_H)] & \twoheadrightarrow & S_r^\circ & \hookrightarrow & S_r^{\max} & = & \mathrm{End}_{R_r}(T_r)
\end{array}$$

We set $H_r = H(T_r)$, a Galois extension of F with Galois group over H given by

$$G_r = \rho_r(\mathrm{Gal}_H) \subset S_r^{\circ,\times} \subset S_r^\circ \subset S_r^{\max} = \mathrm{End}_{R_r}(T_r).$$

3.1.3. Since $\rho(\mathrm{Gal}_H) \cap R^\times = \rho(\mathrm{Gal}_H) \cap \Phi^\times \neq \{1\}$ by $\mathbf{BI}_1$,

$$z(V,H) \stackrel{\mathrm{def}}{=} \exp\left(R/\left\langle 1-u : u \in \rho(\mathrm{Gal}_H) \cap R^\times \right\rangle\right) < \infty$$

We have the exact inflation/restriction sequence

$$0 \longrightarrow H^1(G_r, T_r) \xrightarrow{\mathrm{inf}} H^1(H, T_r) \xrightarrow{\mathrm{res}} H^1(H_r, T_r)^{G_r}$$

and by Sah's lemma [4, A.2] (or [53, 2.7.b] for the original version),

$$\exp\left(H^1(G_r, T_r)\right) \leq z(V,H).$$

3.1.4. Since $S(\Phi) = S^{\max}(\Phi)$ by $\mathbf{BI}_2$, S is an R-order in $S^{\max}$ and

$$c(T,H) \stackrel{\mathrm{def}}{=} \exp\left(S^{\max}/S\right) < \infty.$$

Since $S_r^{\max}/S_r^\circ$ is a quotient of $S^{\max}/S$, we have

$$\exp\left(S_r^{\max}/S_r^\circ\right) \leq c(T,H).$$

3.1.5. Fix a finite R_r-submodule C' of $H^1(H, T_r)$ with image $C = \mathrm{res}(C')$ in

$$H^1(H_r, T_r)^{G_r} = \mathrm{Hom}_{G_r}\left(\mathrm{Gal}_{H_r}^{\mathrm{ab}}, T_r\right).$$

We may then consider the diagram

$$\mathrm{Gal}_{H_r}^{\mathrm{ab}} \twoheadrightarrow X_r(C) \hookrightarrow Y_r(C) \hookrightarrow Z_r(C) \hookrightarrow \mathrm{Hom}_{R_r}(C, T_r)$$

where $H_r(C)$ is the fixed field of $\ker\left(\mathrm{Gal}_{H_r}^{\mathrm{ab}} \to \mathrm{Hom}_{R_r}(C, T_r)\right)$,

$$X_r(C) \stackrel{\mathrm{def}}{=} \mathrm{Gal}(H_r(C)/H_r)$$

is a finite $\mathbb{Z}_p[G_r]$-submodule of $\mathrm{Hom}_{R_r}(C, T_r)$, and $Y_r(C)$, $Z_r(C)$ are the submodules of $\mathrm{Hom}_{R_r}(C, T_r)$ spanned by $X_r(C)$ over respectively S_r° and $S_r^{\max}$. Since $X_r(C)$ is already $\mathbb{Z}_p[G_r]$-stable, $Y_r(C)$ is also spanned by $X_r(C)$ over R_r. We have

$$\exp\left(Z_r(C)/Y_r(C)\right) \leq c(T,H).$$

On the other hand, we have the left and right non-degenerate pairings

$$\begin{array}{ccc}
C \times X_r(C) & \searrow & \\
\downarrow & & \\
C \times Y_r(C) & \longrightarrow & T_r \\
\downarrow & & \\
C \times Z_r(C) & \nearrow &
\end{array}$$

The last one induces $S_r^{\max}$ and R_r-linear embeddings of finite length modules

$$C \hookrightarrow \mathrm{Hom}_{S_r^{\max}}\left(Z_r(C), T_r\right) \quad \text{and} \quad Z_r(C) \hookrightarrow \mathrm{Hom}_{R_r}\left(C, T_r\right).$$

Since T_r induces a length-preserving Morita equivalence between $S_r^{\max}$ and R_r-modules, it follows that these two maps are isomorphisms:

$$C = \mathrm{Hom}_{S_r^{\max}}\left(Z_r(C), T_r\right) \quad \text{and} \quad Z_r(C) = \mathrm{Hom}_{R_r}\left(C, T_r\right).$$

3.1.6. Since $\gamma \neq 1$ in $\mathrm{Gal}(E/F)$, $\gamma^2 = 1$ in $\mathrm{Gal}(H/F)$, i.e. $\gamma^2 \in G_r = \mathrm{Gal}(H_r/H)$ inside $\mathrm{Gal}(H_r/F)$. We may then consider the following commutative diagram:

$$\begin{array}{ccc}
X_r(C)_{\gamma^2-1} & & \\
\downarrow{\scriptstyle (1)} & & \\
Y_r(C)_{\gamma^2-1} & \xrightarrow{(2)} & \mathrm{Hom}_{R_r}\left(C, T_r\right)_{\gamma^2-1} \\
\downarrow & & \downarrow{\scriptstyle (3)} \\
\mathrm{Hom}_{R_r}\left(C', (T_r)_{\gamma^2-1}\right) & \hookleftarrow & \mathrm{Hom}_{R_r}\left(C, (T_r)_{\gamma^2-1}\right) \\
\downarrow & & \downarrow{\scriptstyle (4)} \\
\mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma^2-1}\right) & \hookleftarrow & \mathrm{Hom}_{R_r}\left(C, (T_r)^{\mathrm{fr}}_{\gamma^2-1}\right) \\
\downarrow & & \downarrow{\scriptstyle (5)} \\
\mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right) & \overset{(6)}{\hookleftarrow} & \mathrm{Hom}_{R_r}\left(C, (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)
\end{array}$$

The image of (1) spans $Y_r(C)_{\gamma^2-1}$ over R_r, the cokernel of (2) has exponent $\leq c(T,H)$, (4) $\circ$ (3) is surjective (since $T_r \twoheadrightarrow (T_r)^{\mathrm{fr}}_{\gamma^2-1}$ has a section), the cokernel of (5) is killed by 2 (since $T_r \twoheadrightarrow (T_r)^{\mathrm{fr}}_{\gamma\pm1}$ both have sections), and finally (6) is injective with cokernel $\mathrm{Hom}_{R_r}(C' \cap H^1(G_r, T_r), \oplus_\epsilon (T_r)^{\mathrm{fr}}_{\gamma-\epsilon})$ of exponent $\leq z(V,H)$. Thus

$$\exp\left(\mathrm{coker}\left(Y_r(C)_{\gamma^2-1} \to \mathrm{Hom}_{R_r}\left(C', \oplus_\epsilon (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}\right)\right)\right) \leq c(T,H) + z(V,H) + v_\Phi(2).$$

Here $v_\Phi : \Phi^\times \twoheadrightarrow \mathbb{Z}$ is the normalized valuation of Φ.

3.1.7. Suppose that $C' \subset H^1(H, T_r)$ is γ-stable. Then so is $C \subset H^1(H_r, T_r)$, thus $H_r(C)$ is a Galois extension of the fixed field H^γ of γ in H, γ acts on $X_r(C)$ and $Y_r(C)$, and the above diagram is γ-equivariant with $\gamma^2 = 1$ everywhere. From its first column, we obtain yet another commutative diagram

$$\begin{array}{ccc}
X_r(C)_{\gamma-1} & \longrightarrow & \left(X_r(C)_{\gamma^2-1}\right)^{\gamma-1} \\
\downarrow & & \downarrow \\
Y_r(C)_{\gamma-1} & \longrightarrow & \left(Y_r(C)_{\gamma^2-1}\right)^{\gamma-1} \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{R_r}\left(C', (T_r)_{\gamma^2-1}\right)_{\gamma-1} & \longrightarrow & \mathrm{Hom}_{R_r}\left(C', (T_r)_{\gamma^2-1}\right)^{\gamma-1} \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma^2-1}\right)_{\gamma-1} & \longrightarrow & \mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma^2-1}\right)^{\gamma-1} \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)_{\gamma-1} & \longrightarrow & \mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1}
\end{array}$$

in which the horizontal maps are induced by $(\gamma+1)$, with kernels and cokernels killed by 2. Note that the last term is given by

$$\mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1} = \oplus_{\epsilon\in\{\pm1\}} \mathrm{Hom}_{R_r}\left(C'_{\gamma-\epsilon}, (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}\right)$$

We denote by Ψ and $\Psi^{\mathrm{fr}}_\bullet = (\Psi^{\mathrm{fr}}_+, \Psi^{\mathrm{fr}}_-)$ the induced morphisms

$$\begin{array}{ccc}
 & \overset{\Psi}{\nearrow} & \mathrm{Hom}_{R_r}\left(C', (T_r)_{\gamma^2-1}\right)^{\gamma-1} \\
 & & \downarrow \mathrm{pr} \\
X_r(C)_{\gamma-1} & \xrightarrow{\Psi^{\mathrm{fr}}_\bullet} & \mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1} \\
 & & \downarrow \mathrm{pr}_\epsilon \\
 & \underset{\Psi^{\mathrm{fr}}_\epsilon}{\searrow} & \mathrm{Hom}_{R_r}\left(C'_{\gamma-\epsilon}, (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}\right)
\end{array}$$

We obtain:

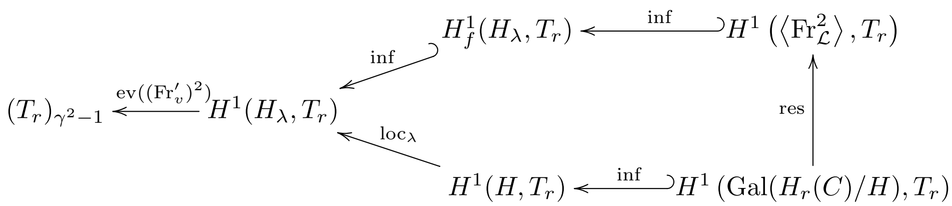
$$\exp\left(\mathrm{coker}\left(\Psi^{\mathrm{fr}}_\bullet \otimes R_r\right)\right) \leq c(T,H) + z(V,H) + 2v_\Phi(2). \tag{3.1}$$

3.2. **The application of Chebotarev's theorem.** We provide here the required generalisations of the material from section 6 of [49], see also [41, §3.6].

3.2.1. *Localizations.* Fix $g \in X_r(C)$. Pick a prime $\mathcal{L}$ of $H_r(C)$ unramified over H^γ with Frobenius $\mathrm{Fr}_\mathcal{L} = g\gamma$ in $\mathrm{Gal}(H_r(C)/H^\gamma)$ and $\mathrm{Fr}^2_\mathcal{L} = g^{1+\gamma}\gamma^2$ in $\mathrm{Gal}(H_r(C)/H)$. Let λ and ℓ be the primes of H and F below $\mathcal{L}$, and fix a place v of $\overline{F}$ over $\mathcal{L}$, with inertia and decomposition groups $I(v) \subset D(v)$ in $\mathrm{Gal}(\overline{F}/H^\gamma)$ and $I(v) \subset D_H(v)$ in $\mathrm{Gal}(\overline{F}/H)$. Fix a lift $\mathrm{Fr}'_v \in D(v)$ of the Frobenius $\mathrm{Fr}_v \in D(v)/I(v)$, so that $(\mathrm{Fr}'_v)^2$ lifts the Frobenius $\mathrm{Fr}^2_v \in D_H(v)/I(v)$. The Fr'_v-equivariant commutative diagram

$$\begin{array}{ccccc}
 & & D_H(v)/I(v) & \twoheadrightarrow & \langle \mathrm{Fr}^2_\mathcal{L} \rangle \\
 & \nearrow & & & \downarrow \\
\mathbb{Z} \xrightarrow{(\mathrm{Fr}'_v)^2} D_H(v) & & & & \\
 & \searrow & & & \\
 & & \mathrm{Gal}_H & \twoheadrightarrow & \mathrm{Gal}(H_r(C)/H)
\end{array}$$

gives a γ-equivariant diagram (where $H^1_f(H_\lambda, T_r) = H^1(D_H(v)/I(v), T_r)$)



By definition of $H_r(C)$, $C' \subset H^1(H, T_r)$ is contained in the image of the bottom inflation map. Thus $\mathrm{loc}_\lambda(C') \subset H^1_f(H_\lambda, T_r)$ and the γ-equivariant morphism

$$\mathrm{loc}_g \stackrel{\mathrm{def}}{=} \mathrm{ev}((\mathrm{Fr}'_v)^2) \circ \mathrm{loc}_\lambda, \qquad \mathrm{loc}_g : C' \to T_{r,\gamma^2-1}$$

only depends upon $\mathrm{Fr}^2_{\mathcal{L}} = g^{1+\gamma} \cdot \gamma^2 \in \mathrm{Gal}(H_r(C)/H)$, i.e. it only depends upon $g \in X_r(C)$ (and *not* upon the auxiliary choices of $\mathcal{L}$, v, and Fr'_v). In fact, this morphism even only depends upon the image of g in $X_r(C)_{\gamma-1}$. More precisely:

Lemma 3.1. *For every* $g \in X_r(C)$,

$$\mathrm{loc}_g = \Psi(g) + \mathrm{loc}_1 \quad \textit{in} \quad \mathrm{Hom}_{R_r}\left(C', T_{r,\gamma^2-1}\right)^{\gamma-1}.$$

Proof. Pick a cocycle c representing some element of $C' \subset H^1(H, T_r)$. Then

$$\begin{array}{rcll} \mathrm{loc}_g(c) & = & c(g^{1+\gamma}\gamma^2) & \text{since } \mathrm{Fr}^2_{\mathcal{L}} = g^{1+\gamma}\gamma^2, \\ & = & g^{1+\gamma}c(\gamma^2) + c(g^{1+\gamma}) & \text{since } c \text{ is a cocycle,} \\ & = & c(\gamma^2) + c(g^{1+\gamma}) & \text{since } g = \mathrm{Id} \text{ on } T_r, \\ & = & \mathrm{loc}_1(c) + \Psi(g)(c) & \text{by definition,} \end{array}$$

thus indeed $\mathrm{loc}_g = \Psi(g) + \mathrm{loc}_1$ in $\mathrm{Hom}_{R_r}\left(C', T_{r,\gamma^2-1}\right)^{\gamma-1}$. □

Set $\mathrm{loc}^{\mathrm{fr}}_{g,\bullet} = \mathrm{pr} \circ \mathrm{loc}_g = \left(\mathrm{loc}^{\mathrm{fr}}_{g,+}, \mathrm{loc}^{\mathrm{fr}}_{g,-}\right)$. Then for every $g \in X_r(C)$,

$$\mathrm{loc}^{\mathrm{fr}}_{g,\bullet} = \Psi^{\mathrm{fr}}_\bullet(g) + \mathrm{loc}^{\mathrm{fr}}_{1,\bullet} \in \mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1}, \tag{3.2}$$

$$\mathrm{loc}^{\mathrm{fr}}_{g,\epsilon} = \Psi^{\mathrm{fr}}_\epsilon(g) + \mathrm{loc}^{\mathrm{fr}}_{1,\epsilon} \in \mathrm{Hom}_{R_r}\left(C'_{\gamma-\epsilon}, (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}\right). \tag{3.3}$$

3.2.2. *Linear algebra.* Let $\mathcal{Z}, \mathcal{Z}', \mathcal{Z}''$ be finite R_r-modules, let $\alpha, \beta : \mathcal{Z} \to \mathcal{Z}'$ be linear maps, let $B : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}''$ be a bilinear map. Define $Q : \mathcal{Z} \to \mathcal{Z}''$ and $B^s : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}''$ by $Q(z) = B(z, z)$ and $B^s(z_1, z_2) = B(z_1, z_2) + B(z_2, z_1)$. Let $\mathcal{X}$ be a subgroup of $\mathcal{Z}$ and let $\mathcal{Y}$ be the R_r-submodule of $\mathcal{Z}$ generated by $\mathcal{X}$, so that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$. Fix $z \in \mathcal{Z}$. Define $a, b : \mathcal{X} \to \mathcal{Z}'$ and $q : \mathcal{X} \to \mathcal{Z}''$ by

$$a(x) = \alpha(x + z), \quad b(x) = \beta(x + z), \quad q(x) = Q(x + z).$$

Let e be the exponent of $\mathcal{Z}/\mathcal{Y}$.

Proposition 3.2. *We have the following implications:*

$$B = 0 \Rightarrow Q = 0 \Rightarrow B^s = 0 \Rightarrow 2Q = 0$$

$$a = 0 \Rightarrow \pi^e\alpha = 0, \quad b = 0 \Rightarrow \pi^e\beta = 0, \quad q = 0 \Rightarrow \pi^{2e}B^s = 0$$

Let also $\mathcal{H}$ *be a subgroup of* $\mathcal{X}$ *and* $x_\circ \in \mathcal{X}$. *Then*

$$|\mathcal{X}/\mathcal{H}| \geq 3 \quad \textit{and} \quad b|_{\mathcal{X} \setminus x_\circ + \mathcal{H}} = 0 \Rightarrow b = 0.$$

$$|\mathcal{X}/\mathcal{H}| \geq 4 \quad \textit{and} \quad q|_{\mathcal{X} \setminus x_\circ + \mathcal{H}} = 0 \Rightarrow 2q = 0.$$

Proof. We establish the hardest two implications and leave the others to the reader. See also [41, 3.6.2] and [49, 6.6]. For every $x, y \in \mathcal{Z}$, we have

$$Q(x+y) - Q(x) - Q(y) \overset{(1)}{=} B^s(x,y)$$

thus for any $x \in \mathcal{X}$,

$$q(x) \quad \overset{(2)}{=} \quad Q(z) + B^s(x,z) + Q(x),$$

and for any $x, y \in \mathcal{X}$,

$$2\left(q(x)+q(y)\right) - \left(q(x+y)+q(x-y)\right) \overset{(3)}{=} 2Q(z) + 2B^s(y,z).$$

Suppose that $q \equiv 0$ on $\mathcal{X}$. Then $Q(z) = 0$ and $Q(x) = -B^s(x,z)$ for all $x \in \mathcal{X}$ by (2), therefore $B^s(x,y) = 0$ for all $x, y \in \mathcal{X}$ by (1), thus $B^s \equiv 0$ on $\mathcal{Y}$ by linearity and $\pi^{2e}B^s \equiv 0$ on $\mathcal{X}$. Suppose now that $q \equiv 0$ on $\mathcal{X}' = \mathcal{X} \setminus x_\circ + \mathcal{H}$ with $|\mathcal{X}/\mathcal{H}| \geq 4$. Then for every $y \in \mathcal{X}'$, there is an $x \in \mathcal{X}'$ with also $x \pm y \in \mathcal{X}'$ thus by (3),

$$2Q(z) + 2B^s(y,z) \overset{(4)}{=} 0.$$

In particular, $2B^s(-,z)$ is constant on $\mathcal{X}'$, which implies that $2B^s(-,z) \equiv 0$ on $\mathcal{X}$ since $|\mathcal{X}/\mathcal{H}| \geq 3$. Then $2Q(z) = 0$ by (4), and the right hand side of (3) is trivial. For $x \notin \mathcal{X}'$ pick $y \in \mathcal{X}'$ with also $x \pm y \in \mathcal{X}'$, which exists since $|\mathcal{X}/\mathcal{H}| \geq 3$. Then $2q(x) = 0$ by (3) and therefore indeed $2q \equiv 0$ on $\mathcal{X}$. □

Corollary 3.3. *Let $\delta_S : \mathbb{N} \to \{0,1\}$ be the Kronecker symbol of $S \subset \mathbb{N}$.*

(1) *There is an $x \in \mathcal{X}$ such that*

$$\exp\left(a(x)\right) \geq \exp\left(\alpha\right) - e - \delta_{\{2\}}(p) \quad \text{and} \quad \exp\left(b(x)\right) \geq \exp(\beta) - e.$$

(2) *There is an $x \in \mathcal{X}$ such that*

$$\exp\left(a(x)\right) \geq \exp\left(\alpha\right) - e - \delta_{\{2,3\}}(p) \quad \text{and} \quad \exp\left(q(x)\right) \geq \exp(Q) - 2e - 2v_\Phi(2).$$

Proof. Again, we only prove the second assertion. Set $\delta = \delta_{\{2,3\}}(p)$ and suppose first that $\exp(\alpha) - e - \delta \leq 0$, so that the first condition is empty. For every $i \geq 0$,

$$\pi^i q = 0 \Rightarrow \pi^{i+2e} B^s = 0 \Rightarrow 2\pi^{i+2e} Q = 0.$$

It follows that $\exp(q) \geq \exp(Q) - 2e - v_\Phi(2)$, which proves our claim in this case. Otherwise, let us write $\exp(\alpha) = e + \delta + f$ with $f \geq 1$, so that $\pi^{\delta+f-1}a \neq 0$ and

$$\mathcal{X}' = \left\{x \in \mathcal{X} : \pi^{f-1}a(x) \neq 0\right\} = \left\{x \in \mathcal{X} : \exp\left(a(x)\right) \geq \exp(\alpha) - e - \delta\right\}$$

is a non-empty subset of $\mathcal{X}$. If $\mathcal{X}' = \mathcal{X}$, then again for every $i \geq 0$,

$$\pi^i q|_{\mathcal{X}'} = 0 \Leftrightarrow \pi^i q = 0 \Rightarrow \pi^{i+2e} B^s = 0 \Rightarrow 2\pi^{i+2e} Q = 0.$$

Otherwise $\mathcal{X} \setminus \mathcal{X}' = x_\circ + \mathcal{H}$ with $\mathcal{H} = \ker\left(\pi^{f-1}\alpha|_{\mathcal{X}}\right)$ for some $x_\circ \in \mathcal{X}$. Then

$$x_\circ + \ker(\pi^{\delta+f-1}\alpha|_{\mathcal{X}}) = \left\{x \in \mathcal{X} : \pi^{\delta+f-1}a = 0\right\} \neq \mathcal{X}$$

thus $\pi^{f-1}\alpha(\mathcal{X}) \not\subset \mathcal{Z}'[\pi^\delta]$ and $|\mathcal{X}/\mathcal{H}| = \left|\pi^{f-1}\alpha(\mathcal{X})\right| \geq 4$. For every $i \geq 0$,

$$\pi^i q|_{\mathcal{X}'} = 0 \Rightarrow 2\pi^i q = 0 \Rightarrow 2\pi^{i+2e} B^s = 0 \Rightarrow 4\pi^{i+2e} Q = 0.$$

In both cases, there is an $x \in \mathcal{X}'$ (so that $\exp(a(x) \geq \exp(\alpha) - e - \delta)$ with

$$\exp\left(q(x)\right) \geq \exp(Q) - 2e - 2v_\Phi(2),$$

which proves the corollary. □

3.2.3. We shall apply the above formalism to

$$\begin{aligned} \mathcal{Z} &= \mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1}, \\ \mathcal{X} &= \text{image of } \Psi^{\mathrm{fr}}_{\bullet} : X_r(G)_{\gamma-1} \to \mathcal{Z}, \\ \mathcal{Y} &= \text{image of } \Psi^{\mathrm{fr}}_{\bullet} \otimes R_r : X_r(G)_{\gamma-1} \otimes R_r \to \mathcal{Z}. \end{aligned}$$

Fix $c_1, c_2, c_3 \in C'$ with images $\overline{c}_i = \overline{c}_{i,+} \oplus \overline{c}_{i,-}$ in $C'_{\gamma-1} \oplus C'_{\gamma+1}$. Set

$$\mathcal{Z}' = (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}$$

and define $\alpha_i : \mathcal{Z} \to \mathcal{Z}'$ by $\alpha_i(f) = f(c_i)$ for $f \in \mathcal{Z}$, so that for any $g \in X_r(C)$,

$$\mathrm{loc}^{\mathrm{fr}}_{g,\bullet}(c_i) = \alpha_i(x+z) = a_i(x) \quad \text{with} \quad \begin{cases} x = \Psi^{\mathrm{fr}}_{\bullet}(g) & \text{in } \mathcal{X}, \\ z = \mathrm{loc}^{\mathrm{fr}}_{1,\bullet} & \text{in } \mathcal{Z}, \end{cases} \tag{3.4}$$

by (3.2). For any $f = (f_+, f_-)$ in $\mathcal{Z} = \oplus_\epsilon \mathrm{Hom}_{R_r}\left(C'_{\gamma-\epsilon}, (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}\right)$,

$$f(c_i) = \left(f_+(\overline{c}_{i,+}), f_-(\overline{c}_{i,-})\right)$$

in $\mathcal{Z}' = (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}$, therefore

$$\exp\left(\alpha_i(f)\right) = \exp\left(f(c_i)\right) = \max\left\{\exp\left(f_\pm(\overline{c}_{i,\pm})\right)\right\} \le \max\left\{\exp\left(\overline{c}_{i,\pm}\right)\right\} = \exp\left(\overline{c}_i\right)$$

and plainly since $(T_r)^{\mathrm{fr}}_{\gamma\pm1}$ is free over R_r,

$$\exp(\alpha_i) = \max\left\{\exp\left(\overline{c}_{i,\pm}\right)\right\} = \exp\left(\overline{c}_i\right) \ge \exp(c_i) - v_\Phi(2). \tag{3.5}$$

Set $\mathcal{Z}'' = \Lambda^2\left(\mathcal{Z}'[\pi^d]\right) \simeq R_d$ where

$$d \overset{\mathrm{def}}{=} \max\left\{\exp(\overline{c}_1), \exp(\overline{c}_2)\right\} \le \exp\left(\langle c_1, c_2\rangle\right).$$

Thus $\alpha_1, \alpha_2 : \mathcal{Z} \to \mathcal{Z}'[\pi^d]$ and we may define $B : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}''$ by

$$B(f_1, f_2) = \alpha_1(f_1) \wedge \alpha_2(f_2).$$

Then $Q(f) = f(c_1) \wedge f(c_2)$ and for any $i \ge 0$,

$$\pi^i Q(f) = 0 \iff i + d \ge \mathrm{length} f\left(\langle c_1, c_2\rangle\right).$$

It follows that $\exp(Q(f)) + d = \max\left\{d, \mathrm{length} f\left(\langle c_1, c_2\rangle\right)\right\}$ and

$$\exp(Q) + d = \max\left\{\mathrm{length} f\left(\langle c_1, c_2\rangle\right) : f \in \mathcal{Z}\right\} \tag{3.6}$$

since for $i \in \{1,2\}$ and $\epsilon_i \in \{\pm1\}$ with $\exp(\overline{c}_i) = \exp(\overline{c}_{i,\epsilon_i})$, there is an $f \in \mathcal{Z}$ with

$$\exp(\overline{c}_i) = \exp(\overline{c}_{i,\epsilon_i}) = \exp(f_{\epsilon_i}(\overline{c}_{i,\epsilon_i})) = \exp(f(c_i)) \le \mathrm{length} f\left(\langle c_1, c_2\rangle\right).$$

Proposition 3.4. *For every c_1, c_2 and c_3 in C' as above,*

(1) *There is a $g \in X_r(C)$ such that*

$$\begin{aligned} \exp\left(\mathrm{loc}^{\mathrm{fr}}_{g,\bullet} c_1\right) &\ge \exp(c_1) - \mathcal{C}^\circ_1 \\ \textit{and} \quad \exp\left(\mathrm{loc}^{\mathrm{fr}}_{g,\bullet} c_2\right) &\ge \exp(c_2) - \mathcal{C}^\circ_2 \end{aligned}$$

with

$$\begin{aligned} \mathcal{C}^\circ_1 &\overset{\mathrm{def}}{=} c(T,H) + z(V,H) + 3v_\Phi(2) \\ \mathcal{C}^\circ_2 &\overset{\mathrm{def}}{=} \mathcal{C}^\circ_1 + \delta_{\{2\}}(p). \end{aligned}$$

(2) *There is a* $g \in X_r(C)$ *such that*

$$\begin{aligned} \max\left\{d(c_1,c_2), \mathrm{length}\left(\mathrm{loc}^{\mathrm{fr}}_{g,\bullet}\left(\langle c_1,c_2\rangle\right)\right)\right\} &\geq \mathcal{C}(c_1,c_2) - 2\mathcal{C}^\circ_1, \\ \textit{and} \quad \exp(\mathrm{loc}^{\mathrm{fr}}_{g,\bullet} c_3) &\geq \exp(c_3) - \mathcal{C}^\circ_3. \end{aligned}$$

with $\mathcal{C}^\circ_1$ *as above,* $\mathcal{C}^\circ_3 \stackrel{\mathrm{def}}{=} \mathcal{C}^\circ_1 + \delta_{2,3}\{p\}$ *and*

$$\begin{aligned} d(c_1,c_2) &\stackrel{\mathrm{def}}{=} \max\left\{\exp(\bar{c}_1), \exp(\bar{c}_2)\right\} \leq \exp\left(\langle c_1,c_2\rangle\right), \\ \mathcal{C}(c_1,c_2) &\stackrel{\mathrm{def}}{=} \max\left\{\mathrm{length} f\left(\langle c_1,c_2\rangle\right) : f \in \mathcal{Z}\right\}. \end{aligned}$$

Proof. By (3.4), this follows from corollary 3.3 along with (3.1), (3.5) and (3.6). □

3.2.4. Suppose now that $C' \subset H^1(H,T_r)$ is $\mathrm{Gal}(H/F)$-stable.

Proposition 3.5. *For every* c_1, c_2 *and* c_3 *in* C' *as above,*

(1) *There are infinitely many Kolyvagin primes* $\mathcal{L}$ *for* H'_r *such that*

$$\exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\bullet} c_1\right) \geq \exp\left(c_1\right) - \mathcal{C}_1 \quad \textit{and} \quad \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\bullet} c_2\right) \geq \exp\left(c_2\right) - \mathcal{C}_2.$$

for the prime λ *of* H *below* $\mathcal{L}$*, with*

$$\mathcal{C}_1 \stackrel{\mathrm{def}}{=} \mathcal{C}^\circ_1 + r_0 \quad \textit{and} \quad \mathcal{C}_2 \stackrel{\mathrm{def}}{=} \mathcal{C}^\circ_2 + r_0.$$

(2) *There are infinitely many Kolyvagin primes* $\mathcal{L}$ *for* H'_r *such that*

$$\begin{aligned} \max\left\{d(c_1,c_2), \mathrm{length}\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda}\left(\langle c_1,c_2\rangle\right)\right)\right\} &\geq \mathcal{C}(c_1,c_2) - 2\mathcal{C}_1, \\ \textit{and} \quad \exp(\mathrm{loc}^{\mathrm{fr}}_{\lambda} c_3) &\geq \exp(c_3) - \mathcal{C}_3. \end{aligned}$$

for the prime λ *of* H *below* $\mathcal{L}$*, with* $\mathcal{C}_1$ *as above and* $\mathcal{C}_3 \stackrel{\mathrm{def}}{=} \mathcal{C}^\circ_3 + r_0$.

Proof. The exact sequence $0 \to T_r \to T_{r+r_0} \to T_{r_0} \to 0$ gives an exact sequence

$$H^0(H,T_{r_0}) \to H^1(H,T_r) \to H^1(H,T_{r+r_0}).$$

Let $\tilde{C}'$ and $\tilde{c}_1,\tilde{c}_2,\tilde{c}_3 \in \tilde{C}'$ be the images of C' and $c_1,c_2,c_3 \in C'$ in $H^1(H,T_{r+r_0})$, let $\tilde{C}$ be the image of $\tilde{C}'$ in $H^1(H_{r+r_0},T_{r+r_0})$ and apply the relevant part of the previous proposition to get a $\tilde{g} \in X_{r+r_0}(\tilde{C}) = \mathrm{Gal}(H_{r+r_0}(\tilde{C})/H_{r+r_0})$. Since C' is $\mathrm{Gal}(H/F)$-stable, so is $\tilde{C}'$, $\tilde{C}$ is stable under $\mathrm{Gal}(H_{r+r_0}/F)$ and $H_{r+r_0}(\tilde{C})$ is a Galois extension of F. Pick a prime $\tilde{\mathcal{L}} \nmid p$ of $H_{r+r_0}(\tilde{C})$ unramified over F with Frobenius $\mathrm{Fr}_{\tilde{\mathcal{L}}} = \tilde{g}\gamma$ in $\mathrm{Gal}(H_{r+r_0}(\tilde{C})/F)$. By definition of r_0, $H'_r \subset H_{r+r_0}$. Let $\mathcal{L} \mid \lambda \mid \ell$ be the primes of H'_r, H and F below $\tilde{\mathcal{L}}$. The Frobenius of $\mathcal{L}$ in $\mathrm{Gal}(H'_r/F)$ equals γ, thus $\mathcal{L}$ is a Kolyvagin prime for H'_r. In the commutative diagram

$$\begin{array}{ccccccc} C' & \xrightarrow{\mathrm{loc}_\lambda} & H^1_f(H_\lambda,T_r) & \xrightarrow{\mathrm{ev}(\gamma^2)} & (T_r)_{\gamma^2-1} & \xrightarrow{\mathrm{pr}_\epsilon} & (T_r)^{\mathrm{fr}}_{\gamma-\epsilon} \\ \downarrow{\scriptstyle \pi^{r_0}} & & \downarrow{\scriptstyle \pi^{r_0}} & & \downarrow{\scriptstyle \pi^{r_0}} & & \downarrow{\scriptstyle \pi^{r_0}} \\ \tilde{C}' & \xrightarrow{\mathrm{loc}_\lambda} & H^1_f(H_\lambda,T_{r+r_0}) & \xrightarrow{\mathrm{ev}(\gamma^2)} & (T_{r+r_0})_{\gamma^2-1} & \xrightarrow{\mathrm{pr}_\epsilon} & (T_{r+r_0})^{\mathrm{fr}}_{\gamma-\epsilon} \end{array}$$

the first and second row are respectively $\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}$ and $\mathrm{loc}^{\mathrm{fr}}_{\tilde{g},\epsilon}$, thus

$$\exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda} c_i\right) \geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\bullet} c_i\right) \geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\tilde{g},\bullet} \tilde{c}_i\right)$$

$$\mathrm{length}\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda}\left(\langle c_1,c_2\rangle\right)\right) \geq \mathrm{length}\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\bullet}\left(\langle c_1,c_2\rangle\right)\right) \geq \mathrm{length}\left(\mathrm{loc}^{\mathrm{fr}}_{\tilde{g},\bullet}\left(\langle \tilde{c}_1,\tilde{c}_2\rangle\right)\right).$$

where the first inequalities come from Lemma 2.4. Since plainly

$$\exp(c_i) \geq \exp(\tilde{c}_i) \geq \exp(c_i) - r_0 \quad \text{and} \quad d(c_1, c_2) \geq d(\tilde{c}_1, \tilde{c}_2) \geq d(c_1, c_2) - r_0$$

it remains to establish that also

$$\max\left\{\text{length}\tilde{f}\left(\langle \tilde{c}_1, \tilde{c}_2\rangle\right)\right\} \geq \max\left\{\text{length} f\left(\langle c_1, c_2\rangle\right)\right\} - 2r_0$$

with the max respectively taken over

$$\begin{array}{rcl} \tilde{f} & \in & \mathrm{Hom}_{R_r}\left(\tilde{C}', (T_{r+r_0})^{\mathrm{fr}}_{\gamma-1} \oplus (T_{r+r_0})^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1} \\ \text{and} \quad f & \in & \mathrm{Hom}_{R_r}\left(C', (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}\right)^{\gamma-1} \end{array}$$

and we leave this to the reader. □

3.2.5. We need a more manageable formula for the constant

$$\begin{array}{rcl} \mathcal{C}(c_1, c_2) & = & \max\left\{\text{length} f\left(\langle c_1, c_2\rangle\right) : f \in \mathcal{Z}\right\} \\ & = & \max\left\{\text{length}(f_+, f_-)\left\langle(\overline{c}_{1,+}, \overline{c}_{1,-}), (\overline{c}_{2,+}, \overline{c}_{2,-})\right\rangle : f_\pm \in \mathcal{Z}(\pm)\right\} \end{array}$$

where $\mathcal{Z}(\pm) = \mathrm{Hom}_{R_r}\left(C'_{\gamma\mp1}, (T_r)^{\mathrm{fr}}_{\gamma\mp1}\right)$. Set $D = \langle \overline{c}_1, \overline{c}_2\rangle \subset C'_{\gamma-1} \oplus C'_{\gamma+1}$ and

$$D_\pm = D \cap C'_{\gamma\mp1} \subset D^\pm = \langle \overline{c}_{1,\pm}, \overline{c}_{2,\pm}\rangle \subset C'_{\gamma\mp1}.$$

Define $\overline{D} \subset \overline{D}(+) \oplus \overline{D}(-) \subset \overline{C}'(+) \oplus \overline{C}'(-)$ by

$$\overline{D} = \frac{D}{D_+ \oplus D_-}, \quad \overline{D}(\pm) = \frac{D^\pm}{D_\pm}, \quad \overline{C}'(\pm) = \frac{C'_{\gamma\mp1}}{D_\pm}.$$

Then $\overline{D}(+) \simeq \overline{D} \simeq \overline{D}(-)$ and $\overline{D} = \{(x, \theta(x)) : x \in \overline{D}(+)\}$ inside $\overline{D}(+) \oplus \overline{D}(-)$, where $\theta : \overline{D}(+) \to \overline{D}(-)$ is an isomorphism. Since $\overline{D}(+)$ is spanned by two elements, $\overline{D}(+) \simeq R_a \oplus R_b$ with $0 \leq a \leq b \leq r$. Fix morphisms $\overline{f}_+ : \overline{D}(+) \to (T_r)^{\mathrm{fr}}_{\gamma-1}$ with kernel $R_a \subset \overline{D}(+)$ and $\overline{f}_- : \overline{D}(-) \to (T_r)^{\mathrm{fr}}_{\gamma+1}$ with kernel $\theta(R_b) \subset \overline{D}(-)$. Since $(T_r)^{\mathrm{fr}}_{\gamma-1}$ and $(T_r)^{\mathrm{fr}}_{\gamma+1}$ are both free of rank one over R_r, such morphisms do exist and they extend to morphisms $\overline{f}_\pm : \overline{C}'(\pm) \to (T_r)^{\mathrm{fr}}_{\gamma\mp1}$, which we view as morphisms $f_\pm : C'_{\gamma\mp1} \to (T_r)^{\mathrm{fr}}_{\gamma\mp1}$. Then for $f = (f_+, f_-) \in \mathcal{Z} = \mathcal{Z}(+) \oplus \mathcal{Z}(-)$, we have

$$f\left(\langle c_1, c_2\rangle\right) = \overline{f}(\overline{D}) \quad \text{in} \quad (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1} \quad \text{with} \quad \overline{f}(\overline{D}) \simeq \overline{D}$$

where the morphism $\overline{f} = (\overline{f}_+, \overline{f}_-) : \overline{C}'(+) \oplus \overline{C}'(-) \to (T_r)^{\mathrm{fr}}_{\gamma-1} \oplus (T_r)^{\mathrm{fr}}_{\gamma+1}$ is induced by f, and is injective on $\overline{D}$ by construction. It follows that

$$\mathcal{C}(c_1, c_2) \geq \text{length}(\overline{D}) = \text{length}(D^+) + \text{length}(D^-) - \text{length}(D).$$

4. Kolyvagin's method à la Nekovář

We present here a variant of Kolyvagin's original argument, following the robust and flexible method devised by Nekovář in [49], see also [24, 27].

4.1. **The statement.**

4.1.1. *Assumptions.* Suppose that we are given

- a finite subextension H of $E[\infty]/E$,
- a self-dual $\mathrm{Gal}(H/F)$-stable Selmer structure $\mathcal{F}$ on V over H,
- a subset $\mathcal{P}_{\mathcal{F}}$ of $\mathcal{P} = \mathcal{P}_1(T, \gamma)$ as in 2.4.1,
- a weak Kolyvagin system $\kappa = (\kappa(\mathfrak{n}))_{\mathfrak{n} \in \mathcal{N}_{\mathcal{F}}}$ for $(T, H, \mathcal{F}, \mathcal{P}_F)$,

$$\kappa(\mathfrak{n}) \in H^1_{\mathcal{F}^{\mathfrak{n}}}\left(H, T_{r(\mathfrak{n})}\right) \otimes \mathbb{G}(\mathfrak{n}), \qquad \mathfrak{n} \in \mathcal{N}_{\mathcal{F}},$$

- a character $\chi : \mathrm{Gal}(H/E) \to R^\times$.

Suppose in addition to the assumption $\mathbf{BI}_0$ of section 3.1.1 that

- The assumptions $\mathbf{BI}_1$ and $\mathbf{BI}_2$ hold for the fixed field H' of $\ker(\chi)$.

4.1.2. For an $R[\mathrm{Gal}(H/E)]$-module M, we denote by $M(\chi)$ the R-submodule of M on which $\mathrm{Gal}(H/E)$ acts by χ and for $m \in M$, we set

$$m(\chi) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathrm{Gal}(H/E)} \chi(\sigma)^{-1} \sigma m \in M(\chi).$$

If M is an $R[\mathrm{Gal}(H/F)]$-module, then for every $m \in M$,

$$\gamma \cdot (m(\chi)) = (\gamma \cdot m)(\chi^{-1}) \quad \text{in} \quad \gamma \cdot M(\chi) = M(\chi^{-1}).$$

Let $\kappa_\circ \in H^1_{\mathcal{F}}(H, V)$ be the image of the basic class $\kappa(1) \in H^1_{\mathcal{F}}(H, T)$ of κ.

Theorem 4.1. *With the above assumptions and notations,*

$$\kappa_\circ(\chi) \neq 0 \quad \text{in} \quad H^1_{\mathcal{F}}(H, V)(\chi) \quad \Longrightarrow \quad H^1_{\mathcal{F}}(H, V)(\chi) = \Phi \cdot \kappa_\circ(\chi).$$

4.1.3. Replacing $(H, \chi, \mathcal{F}, \kappa)$ by $(H', \chi', \mathcal{F}', \kappa')$ where $\chi' : \mathrm{Gal}(H'/E) \hookrightarrow R^\times$ is the primitive character induced by χ, $\mathcal{F}' = \mathrm{res}^* \mathcal{F}$ and $\kappa' = \mathrm{cores}(\kappa)$ as in 2.4.5, we immediately reduce to the case where $H = H'$, i.e. $\chi : \mathrm{Gal}(H/E) \hookrightarrow R^\times$ is a primitive character. Note that the restriction map indeed induces an isomorphism

$$H^1_{\mathcal{F}'}(H', V')(\chi') \simeq H^1_{\mathcal{F}}(H, V)^{\mathrm{Gal}(H/H')}(\chi') = H^1_{\mathcal{F}}(H, V)(\chi)$$

which maps $\kappa'_\circ(\chi')$ to $\kappa_\circ(\chi)$. Suppose from now on that $H' = H$ and $\kappa_\circ(\chi) \neq 0$.

4.1.4. Since $H^1_{\mathcal{F}}(H, V) = H^1_{\mathcal{F}}(H, T) \otimes \Phi$, the χ-component $\kappa(1)(\chi)$ of the basic class spans a free rank one R-submodule of $H^1_{\mathcal{F}}(H, T)(\chi)$ and we have to show that the corresponding quotient is finite. Since $H^1_{\mathcal{F}}(H, T)$ is finitely generated over R, it is sufficient to establish that there is a constant $c^?(\chi) < \infty$ such that

$$\forall r \gg 0 : \qquad \exp\left(\frac{H^1_{\mathcal{F}}(H, T)(\chi)}{R \cdot \kappa(1)(\chi)}\right)_r \leq c^?(\chi). \tag{4.1}$$

This is equivalent to

$$\forall r \gg 0 : \qquad \exp\left(\frac{H^1_{\mathcal{F}}(H, T)(\chi)_r}{R_r \cdot \kappa(1)(\chi)_r}\right) \leq c^?(\chi)$$

where $\kappa(1)(\chi)_r$ is the image of $\kappa(1)(\chi)$ in $H^1_{\mathcal{F}}(H, T)(\chi)_r$, and we know that

$$\exp\left(\kappa(1)(\chi)_r\right) \geq r - c(\chi) \quad \text{with} \quad c(\chi) \stackrel{\text{def}}{=} \exp\left(\frac{H^1_{\mathcal{F}}(H, T)(\chi)}{R \cdot \kappa(1)(\chi)}\right)_{\text{tors}} < \infty.$$

4.1.5. Consider now the following commutative diagram:

$$\begin{array}{ccccc}
H^1_{\mathcal{F}}(H,T)(\chi)_r & \xrightarrow{(1_r)} & H^1_{\mathcal{F}}(H,T)_r(\chi) & \xrightarrow{(2_r)} & H^1_{\mathcal{F}}(H,T_r)(\chi) \\
& \searrow^{(1'_r)} & \big\downarrow & & \big\downarrow \\
& & H^1_{\mathcal{F}}(H,T)_r & \xrightarrow{(2'_r)} & H^1_{\mathcal{F}}(H,T_r) \\
& & \big\downarrow{\scriptstyle (3_r)} & & \big\downarrow \\
& & H^1(H,T)_r & \xrightarrow{(2''_r)} & H^1(H,T_r)
\end{array}$$

The morphism $(1'_r)$ sits in the exact sequence

$$0 \to M(\chi)^r \to M^r \to (M/M(\chi))^r \to M(\chi)_r \xrightarrow{(1'_r)} M_r \to (M/M(\chi))_r \to 0$$

where $M = H^1_{\mathcal{F}}(H,T)$ is a finitely generated R-module, thus

$$\exp\left(\ker(1_r)\right) \leq c'(\chi) \stackrel{\text{def}}{=} \exp\left(\frac{H^1_{\mathcal{F}}(H,T)}{H^1_{\mathcal{F}}(H,T)(\chi)}\right)_{\text{tors}} < \infty.$$

The morphism $(2''_r)$ is injective. The morphism (3_r) is also injective, by definition of the Selmer structure $\mathcal{F}$ on T – it is induced from the Selmer structure $\mathcal{F}$ on V. The commutativity of the bottom square (which comes from the definition of the Selmer structure $\mathcal{F}$ on T_r) then implies that $(2'_r)$ is injective, and so is therefore also its restriction (2_r) to the χ-components. It follows that

$$\begin{aligned}
\exp\left(\frac{H^1_{\mathcal{F}}(H,T)(\chi)_r}{R\cdot\kappa(1)(\chi)_r}\right) &\leq \exp\left(\frac{H^1_{\mathcal{F}}(H,T_r)(\chi)}{R_r\cdot\kappa_r(1)(\chi)}\right) + c'(\chi) \\
\exp\left(\kappa_r(1)(\chi)\right) &\geq \exp\left(\kappa(1)(\chi)_r\right) - c'(\chi) \\
&\geq r - c(\chi) - c'(\chi)
\end{aligned}$$

where $\kappa_r(1)$ is the image of $\kappa(1)$ inside $H^1_{\mathcal{F}}(H,T_r)$.

4.1.6. In the next three sections, we will establish the following proposition.

Proposition 4.2. *There is a constant $d^?(\chi) < \infty$ such that for every $r \geq 1$ and every weak Kolyvagin system $\kappa_r = (\kappa_r(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_{\mathcal{F},r}}$ of level r,*

(1) *If $\chi^2 = 1$, then*

$$\exp\left(\frac{H^1_{\mathcal{F}}(H,T_r)(\chi)}{R_r\cdot\kappa_r(1)(\chi)}\right) \leq 2\left(r - \exp\left(\kappa_r(1)(\chi)\right)\right) + d^?(\chi).$$

(2) *If $\chi^2 \neq 1$, then*

$$\exp\left(\frac{H^1_{\mathcal{F}}(H,T_r)(\chi)}{R_r\cdot\kappa_r(1)(\chi)}\right) \leq 5\left(r - \exp\left(\kappa_r(1)(\chi)\right)\right) + d^?(\chi).$$

Applying the proposition for $r \gg 0$ with the weak Kolyvagin system of level r induced by our fixed Kolyvagin system, we obtain the desired inequality (4.1) with

$$c^?(\chi) \stackrel{\text{def}}{=} d^?(\chi) + \begin{cases} 2c(\chi) + 3c'(\chi) & \text{if } \chi^2 = 1, \\ 5c(\chi) + 6c'(\chi) & \text{if } \chi^2 \neq 1. \end{cases}$$

4.1.7. We simplify our notations as follows: we fix a weak Kolyvagin system of level r as in Proposition 4.2, but denote it by $\kappa'_r = (\kappa'_r(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_{\mathcal{F},r}}$ to set $\kappa(\mathfrak{n}) = \kappa'_r(\mathfrak{n})(\chi)$, so that $\kappa(\mathfrak{n}) \in H^1_{\mathcal{F}^{\mathfrak{n}}}(H, T_r)(\chi) \otimes \mathbb{G}(\mathfrak{n})$ for all $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$. We define

$$e \stackrel{\text{def}}{=} \exp(\kappa(1))$$

and assume that the proposition is not trivially true, i.e.

$$\begin{cases} 2(r-e) + d^?(\chi) < r & \text{if } \chi^2 = 1 \\ 5(r-e) + d^?(\chi) < r & \text{if } \chi^2 \neq 1 \end{cases}$$

The constant $d^?(\chi) \geq 0$ will be defined in due time. We also choose for each $\ell \in \mathcal{P}_{\mathcal{F}}$ a generator ζ_ℓ of $\mathbb{G}(\ell)$ and set $\tilde{\kappa}(\mathfrak{n}) = \tilde{\kappa}'_r(\mathfrak{n})(\chi)$ where $\tilde{\kappa}'_r$ is the ζ-Kolyvagin system corresponding to κ'_r, so that $\kappa(\mathfrak{n}) = \tilde{\kappa}(\mathfrak{n}) \otimes (\otimes_{\ell|\mathfrak{n}}\zeta_\ell)$ in $H^1_{\mathcal{F}^n}(H, T_r)(\chi) \otimes \mathbb{G}(\mathfrak{n})$ for every $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$. We denote by $\nu(\mathfrak{n})$ the number of prime divisors $\ell \mid \mathfrak{n}$ of $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$.

4.1.8. For any $R_r[\mathrm{Gal}(H/F)]$-module M, any $m \in M$ and $\epsilon \in \{\pm 1\}$, we set

$$m^\epsilon = (1+\epsilon\gamma)m \quad \text{in} \quad M^\epsilon = M^{\gamma-\epsilon} \quad \text{and} \quad m_\epsilon = \text{image of } m \text{ in } M_\epsilon = M_{\gamma-\epsilon}.$$

Thus $2m = m^+ + m^-$, $\exp(m^\epsilon) \leq \exp(m_\epsilon) \leq \exp(m)$ and

$$\exp(m) - v_\Phi(2) \leq \exp(2m) \leq \max\left\{\exp(m^\pm)\right\} \leq \max\left\{\exp(m_\pm)\right\} \leq \exp(m)$$

4.2. **Annihilation relations.**

4.2.1. The proof of Proposition 4.2 relies on the following annihilation relation. For every $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$, combining Tate's global duality from section 1.4.7 with the morphisms from section 2.2.4, we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1_{\mathcal{F}_{\mathfrak{n}}}(H,T_r) & \longrightarrow & H^1_{\mathcal{F}}(H,T_r) & \xrightarrow{\oplus \mathrm{loc}_\lambda} & \oplus_{\lambda|\mathfrak{n}} H^1_f(H_\lambda, T_r) \\
 & & & & & \updownarrow \\
0 \longrightarrow & H^1_{\mathcal{F}^*(1)}(H,T_r^*(1)) & \longrightarrow & H^1_{\mathcal{F}^*(1)^{\mathfrak{n}}}(H,T_r^*(1)) & \xrightarrow{\oplus \mathrm{loc}^s_\lambda} & \oplus_{\lambda|\mathfrak{n}} H^1_s(H_\lambda, T_r^*(1)) \\
 & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & H^1_{\mathcal{F}}(H,T_r) & \longrightarrow & H^1_{\mathcal{F}^{\mathfrak{n}}}(H,T_r) & \xrightarrow{\oplus \mathrm{loc}^s_\lambda} & \oplus_{\lambda|\mathfrak{n}} H^1_s(H_\lambda, T_r)
\end{array}$$

Here, exactness of the first two lines has the same meaning as in 1.4.7, the bottom vertical maps have kernels and cokernels of exponent $\leq \exp(\Xi')$, and the whole diagram is $\mathrm{Gal}(H/F)$-equivariant. Twisting the bottom two lines by $\mathbb{G}(\mathfrak{n})$, we obtain yet another commutative diagram with similar properties,

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1_{\mathcal{F}_{\mathfrak{n}}}(H,T_r) & \longrightarrow & H^1_{\mathcal{F}}(H,T_r) & \xrightarrow{\oplus \mathrm{loc}_\lambda} & \oplus_{\lambda|\mathfrak{n}} H^1_f(H_\lambda, T_r) \\
 & & & & & \updownarrow \\
0 \longrightarrow & H^1_{\mathcal{F}^*(1)}(\cdots)\otimes\mathbb{G}(\mathfrak{n}) & \longrightarrow & H^1_{\mathcal{F}^*(1)^{\mathfrak{n}}}(\cdots)\otimes\mathbb{G}(\mathfrak{n}) & \xrightarrow{\oplus \mathrm{loc}^s_\lambda} & \oplus_{\lambda|\mathfrak{n}} H^1_s(\cdots)\otimes\mathbb{G}(\mathfrak{n}) \\
 & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & H^1_{\mathcal{F}}(H,T_r)\otimes\mathbb{G}(\mathfrak{n}) & \longrightarrow & H^1_{\mathcal{F}^{\mathfrak{n}}}(H,T_r)\otimes\mathbb{G}(\mathfrak{n}) & \xrightarrow{\oplus \mathrm{loc}^s_\lambda} & \oplus_{\lambda|\mathfrak{n}} H^1_s(H_\lambda, T_r)\otimes\mathbb{G}(\mathfrak{n})
\end{array}$$

in which the first vertical arrow now refers to the sum of the twisted perfect pairings

$$H^1_f(H_\lambda, T_r) \times H^1_s(H_\lambda, T_r^*(1)) \otimes \mathbb{G}(\mathfrak{n}) \to R_r(\mathfrak{n}), \quad R_r(\mathfrak{n}) \stackrel{\text{def}}{=} R_r \otimes \mathbb{G}(\mathfrak{n}).$$

Let us denote by

$$[-,-]^{\mathfrak{n}}_\lambda : H^1_f(H_\lambda, T_r) \times H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}) \to R_r(\mathfrak{n})$$

the induced (non-perfect) pairing. Then for every

$$a \in H^1_{\mathcal{F}}(H, T_r)(\chi) \quad \text{and} \quad b \in H^1_{\mathcal{F}^{\mathfrak{n}}}(H, T_r)(\chi^{-1}) \otimes \mathbb{G}(\mathfrak{n})$$

we have, for any choice $(\lambda_\ell)_{\ell|\mathfrak{n}}$ of primes $\lambda_\ell \mid \ell \mid \mathfrak{n}$ of H,

$$\begin{aligned}
0 &\overset{(1)}{=} \sum_{\lambda|\mathfrak{n}} [\mathrm{loc}_\lambda(a), \mathrm{loc}^s_\lambda(b)]^{\mathfrak{n}}_\lambda \\
&\overset{(2)}{=} \sum_{\ell|\mathfrak{n}} \sum_{\sigma \in \mathrm{Gal}(H/E)} \left[\mathrm{loc}_{\sigma\lambda_\ell}(a), \mathrm{loc}^s_{\sigma\lambda_\ell}(b)\right]^{\mathfrak{n}}_{\sigma\lambda_\ell} \\
&\overset{(3)}{=} \sum_{\ell|\mathfrak{n}} \sum_{\sigma \in \mathrm{Gal}(H/E)} \sigma\left[\mathrm{loc}_{\lambda_\ell}(\sigma^{-1}a), \mathrm{loc}^s_{\lambda_\ell}\left(\sigma^{-1}b\right)\right]^{\mathfrak{n}}_{\lambda_\ell} \\
&\overset{(4)}{=} \sum_{\ell|\mathfrak{n}} \sum_{\sigma \in \mathrm{Gal}(H/E)} \left[\mathrm{loc}_{\lambda_\ell}(\chi(\sigma)^{-1}a), \mathrm{loc}^s_{\lambda_\ell}(\chi(\sigma)b)\right]^{\mathfrak{n}}_{\lambda_\ell} \\
&\overset{(5)}{=} \sum_{\ell|\mathfrak{n}} \sum_{\sigma \in \mathrm{Gal}(H/E)} \left[\chi(\sigma)^{-1}\mathrm{loc}_{\lambda_\ell}(a), \chi(\sigma)\mathrm{loc}^s_{\lambda_\ell}(b)\right]^{\mathfrak{n}}_{\lambda_\ell} \\
&\overset{(6)}{=} \sum_{\ell|\mathfrak{n}} \sum_{\sigma \in \mathrm{Gal}(H/E)} \left[\mathrm{loc}_{\lambda_\ell}(a), \mathrm{loc}^s_{\lambda_\ell}(b)\right]^{\mathfrak{n}}_{\lambda_\ell} \\
&= [H : E] \cdot \sum_{\ell|\mathfrak{n}} \left[\mathrm{loc}_{\lambda_\ell}(a), \mathrm{loc}^s_{\lambda_\ell}(b)\right]^{\mathfrak{n}}_{\lambda_\ell}
\end{aligned}$$

in $R_r(\mathfrak{n})$, using global duality for (1), the fact that for each $\ell \mid \mathfrak{n}$, $\ell\mathcal{O}_E$ splits completely in H/E for (2), the $\mathrm{Gal}(H/F)$-equivariance of our diagrams for (3), the triviality of the action of $\mathrm{Gal}(H/E)$ on $\mathbb{G}(\mathfrak{n})$ and $R_r(\mathfrak{n})$ for (4), the R_r-linearity of the localizations maps for (5), and the R_r-bilinearity of the pairings for (6).

4.2.2. It follows that for $\mathfrak{n}\ell \in \mathcal{N}_{\mathcal{F},r}$ and any $\lambda \mid \ell$, for every

$$c \in H^1_{\mathcal{F}_{\mathfrak{n}}}(H, T_r)(\chi) \quad \text{and} \quad c' \in H^1_{\mathcal{F}^{\mathfrak{n}\ell}}(H, T_r)(\chi^{-1}) \otimes \mathbb{G}(\mathfrak{n}\ell)$$

we have the following annihilation relation:

$$[H : E] \cdot [\mathrm{loc}_\lambda(c), \mathrm{loc}^s_\lambda(c')]^{\mathfrak{n}\ell}_\lambda = 0 \quad \text{in} \quad R_r(\mathfrak{n}\ell) \simeq R_r.$$

4.2.3. Suppose that $\mathcal{L}$ is a Kolyvagin prime for H'_r above λ of H and ℓ of F. Combining the commutative diagrams from section 1.3.4 and 2.2.3 with the morphism ξ of section 2.2.4, we obtain a commutative diagram of twisted pairings

$$\begin{array}{ccccc}
H^1(H_\lambda, T_r) & \times & H^1(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell) & \longrightarrow & R_r(\mathfrak{n}\ell) \\
\uparrow & & \downarrow & & \| \\
H^1_f(H_\lambda, T_r) & \times & H^1_s(H_\lambda, T_r) \otimes \mathbb{G}(\mathfrak{n}\ell) & \longrightarrow & R_r(\mathfrak{n}\ell) \\
\downarrow\simeq & & \downarrow\simeq & & \uparrow\simeq \\
(T_r)_{\gamma^2-1} & \times & (T_r)^{\gamma^2-1} \otimes \mathbb{G}(\mathfrak{n}) & \longrightarrow & R_r(\mathfrak{n})(1) \\
\downarrow & & \uparrow & & \| \\
(T_r)^{\mathrm{fr}}_{\gamma^2-1} & \times & (T_r)^{\gamma^2-1}_{\mathrm{fr}} \otimes \mathbb{G}(\mathfrak{n}) & \longrightarrow & R_r(\mathfrak{n})(1) \\
\downarrow & & \uparrow & & \| \\
(T_r)^{\mathrm{fr}}_{\gamma-\epsilon} & \times & (T_r)^{\gamma+\epsilon}_{\mathrm{fr}} \otimes \mathbb{G}(\mathfrak{n}) & \longrightarrow & R_r(\mathfrak{n})(1)
\end{array}$$

The second line is our pairing $[-,-]_{\lambda}^{\mathfrak{n}\ell}$, and the last three pairings, denoted by

$$\langle -,-\rangle_{\gamma^2-1}^{\mathfrak{n}}, \quad \langle -,-\rangle_{\gamma^2-1}^{\mathrm{fr},\mathfrak{n}} \quad \text{and} \quad \langle -,-\rangle_{\gamma-\epsilon}^{\mathrm{fr},\mathfrak{n}}$$

are twisted from the pairings of section 2.2.3.

4.3. **Proof of Proposition 4.2 when $\chi^2 = 1$.** Then for any $R_r[\mathrm{Gal}(H/F)]$-module M, the χ-component $M(\chi)$ is an $R_r[\mathrm{Gal}(H/F)]$-submodule of M.

4.3.1. For $\mathfrak{n} \in \mathcal{N}_{\mathcal{F},r}$, we set $\kappa^{\pm}(\mathfrak{n}) = (1 \pm \gamma)\kappa(\mathfrak{n})$ and $\tilde{\kappa}^{\pm}(\mathfrak{n}) = (1 \pm \gamma)\tilde{\kappa}(\mathfrak{n})$. Thus

$$\kappa^{\pm}(\mathfrak{n}) = \tilde{\kappa}^{\pm\nu(\mathfrak{n})}(\mathfrak{n}) \otimes (\otimes_{\ell|\mathfrak{n}}\zeta_\ell) \quad \text{in} \quad H^1_{\mathcal{F}^{\mathfrak{n}}}(H, T_r)(\chi) \otimes \mathbb{G}(\mathfrak{n})$$

and $2\kappa(\mathfrak{n}) = \kappa^+(\mathfrak{n}) + \kappa^-(\mathfrak{n})$. We define or choose $\epsilon \in \{\pm 1\}$ such that

$$e - v_\Phi(2) \leq \exp(2\kappa(1)) \leq \max\left\{\exp\left(\kappa^+(1)\right), \exp\left(\kappa^-(1)\right)\right\} = \exp(\kappa^\epsilon(1)) \leq e.$$

We fix $a \in H^1_{\mathcal{F}}(H, T_r)(\chi)$ such that $a^{-\epsilon} = (1 - \epsilon\gamma)a$ has maximal exponent.

4.3.2. We first apply Proposition 3.5 (1) to

$$c_1 = \kappa^\epsilon(1) \quad \text{and} \quad c_2 = a^{-\epsilon} \quad \text{in} \quad C' = H^1_{\mathcal{F}}(H, T_r)(\chi).$$

We obtain a Kolyvagin primes $\mathcal{L}$ of H'_r above $\lambda \mid \ell$ of H and F with

$$\begin{aligned} \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\bullet}(\kappa^\epsilon(1))\right) &\geq \exp(\kappa^\epsilon(1)) - \mathcal{C}_1 \geq e - \mathcal{C}_1 - v_\Phi(2) \\ \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\bullet}(a^{-\epsilon})\right) &\geq \exp(a^{-\epsilon}) - \mathcal{C}_2. \end{aligned}$$

Since $\mathrm{loc}^{\mathrm{fr}}_{\lambda,-\epsilon}(\kappa^\epsilon(1)) = 0$ and $\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(a^{-\epsilon}) = 0$, it follows that

$$\begin{aligned} \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(\kappa(1))\right) \geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(\kappa^\epsilon(1))\right) &\geq e - \mathcal{C}_1 - v_\Phi(2), \\ \text{and} \qquad \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,-\epsilon}(a^{-\epsilon})\right) &\geq \exp(a^{-\epsilon}) - \mathcal{C}_2. \end{aligned} \tag{4.2}$$

Then by the Kolyvagin system relations, the last commutative diagram of section 2.3.4 and Lemma 2.3, the first of these inequalities implies

$$\begin{aligned} \exp\left(\mathrm{loc}^s_\lambda(\kappa^\epsilon(\ell))\right) &\geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(\kappa(1))\right) - e(Q,\epsilon) \\ &\geq e - \mathcal{C}_1 - e(Q,\epsilon) - v_\Phi(2). \end{aligned} \tag{4.3}$$

We now apply the annihilation relation of section 4.2.2 to

$$c = a^{-\epsilon} \in H^1_{\mathcal{F}}(H, T_r)(\chi) \quad \text{and} \quad c' = \kappa^\epsilon(\ell) \in H^1_{\mathcal{F}^\ell}(H, T_r)(\chi^{-1}) \otimes \mathbb{G}(\ell).$$

We obtain:

$$[H : E] \cdot \left[\mathrm{loc}_\lambda(a^{-\epsilon}), \mathrm{loc}^s_\lambda(\kappa^\epsilon(\ell))\right]^\ell_\lambda = 0 \quad \text{in} \quad R_r(\ell).$$

Taking into account remark 2.8 and the diagram of section 4.2.3, this yields

$$[H : E] \cdot \left\langle \mathrm{loc}^{\mathrm{fr}}_{\lambda,-\epsilon}(a^{-\epsilon}), \mathrm{loc}^s_\lambda(\kappa^\epsilon(\ell))\right\rangle^{\mathrm{fr}}_{\gamma+\epsilon} = 0 \quad \text{in} \quad R_r(1).$$

It then follows from 2.2.5 that

$$\exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,-\epsilon}(a^{-\epsilon})\right) + \exp\left(\mathrm{loc}^s_\lambda(\kappa^\epsilon(\ell))\right) \leq r + v_\Phi([H:E]) + e(\Xi,\epsilon) \tag{4.4}$$

Putting (4.2) (4.3) and (4.4) together and using the definition of a, we obtain

$$\exp\left(b^{-\epsilon}\right) \leq r - e + \mathcal{C}_1 + \mathcal{C}_2 + e(Q,\epsilon) + e(\Xi,\epsilon) + v_\Phi(2[H:E]) \tag{4.5}$$

for every $b \in H^1_{\mathcal{F}}(H, T_r)(\chi)$.

4.3.3. Fix $b \in H^1_{\mathcal{F}}(H,T_r)(\chi)$ with $b^\epsilon \in H^1_{\mathcal{F}_\ell}(H,T_r)(\chi)$ and apply Proposition 3.5 to

$$c_1 = \tilde{\kappa}^{-\epsilon}(\ell) \quad \text{and} \quad c_2 = b^\epsilon \quad \text{in} \quad C' = H^1_{\mathcal{F}^\ell}(H,T_r)(\chi).$$

We obtain a Kolyvagin prime $\mathcal{L}'$ of H'_r above $\lambda' \mid \ell'$ of H and F, with $\ell' \neq \ell$ and

$$\begin{aligned} \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',\bullet}\left(\tilde{\kappa}^{-\epsilon}(\ell)\right)\right) &\geq \exp\left(\tilde{\kappa}^{-\epsilon}(\ell)\right) - \mathcal{C}_1, \\ \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',\bullet}\left(b^\epsilon\right)\right) &\geq \exp\left(b^\epsilon\right) - \mathcal{C}_2. \end{aligned}$$

As above and using (4.3) for a lower bound on $\exp\left(\tilde{\kappa}^{-\epsilon}(\ell)\right) = \exp\left(\kappa^\epsilon(\ell)\right)$, this yields

$$\begin{aligned} \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',-\epsilon}\left(\tilde{\kappa}(\ell)\right)\right) &\geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',-\epsilon}\left(\tilde{\kappa}^{-\epsilon}(\ell)\right)\right) \\ &\geq e - 2\mathcal{C}_1 - e(Q,\epsilon) - v_\Phi(2), \\ \text{and} \quad \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',\epsilon}\left(b^\epsilon\right)\right) &\geq \exp\left(b^\epsilon\right) - \mathcal{C}_2. \end{aligned} \tag{4.6}$$

By the Kolyvagin system relations,

$$\begin{aligned} \mathrm{loc}^s_{\lambda'}\left(\kappa^\epsilon(\ell\ell')\right) &= (1+\epsilon\gamma)\cdot\left(\Phi_{\lambda'} \circ \mathrm{loc}_{\lambda'}\left(\tilde{\kappa}(\ell)\otimes\zeta_\ell\right)\right) \\ &= (1+\epsilon\gamma)\cdot\left(\left(\Phi_{\lambda'} \circ \mathrm{loc}_{\lambda'}\left(\tilde{\kappa}(\ell)\right)\right)\otimes\zeta_\ell\right) \\ &= \left((1-\epsilon\gamma)\left(\Phi_{\lambda'} \circ \mathrm{loc}_{\lambda'}\left(\tilde{\kappa}(\ell)\right)\right)\right)\otimes\zeta_\ell \end{aligned}$$

so it follows from (4.6), section 2.3.4 and Lemma 2.3 that

$$\begin{aligned} \exp\left(\mathrm{loc}^s_{\lambda'}\left(\kappa^\epsilon(\ell\ell')\right)\right) &\geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',-\epsilon}\left(\tilde{\kappa}(\ell)\right)\right) - e(Q,-\epsilon) \\ &\geq e - 2\mathcal{C}_1 - e(Q,\epsilon) - e(Q,-\epsilon) - v_\Phi(2) \end{aligned} \tag{4.7}$$

We now apply the annihilation relation of section 4.2.2 to

$$c = b^\epsilon \in H^1_{\mathcal{F}_\ell}(H,T_r)(\chi) \quad \text{and} \quad c' = \kappa^\epsilon(\ell\ell') \in H^1_{\mathcal{F}^{\ell\ell'}}(H,T_r)(\chi^{-1}) \otimes \mathbb{G}(\ell\ell').$$

We obtain:

$$[H:E]\cdot\left[\mathrm{loc}_{\lambda'}(b^\epsilon), \mathrm{loc}^s_{\lambda'}\left(\kappa^\epsilon(\ell\ell')\right)\right]^{\ell\ell'}_{\lambda'} = 0 \quad \text{in} \quad R_r(\ell\ell').$$

Taking into account remark 2.8 and the diagram of section 4.2.3, this yields

$$[H:E]\cdot\left\langle \mathrm{loc}^{\mathrm{fr}}_{\lambda',\epsilon}(b^\epsilon), \mathrm{loc}^s_{\lambda'}\left(\kappa^\epsilon(\ell\ell')\right)\right\rangle^{\mathrm{fr},\ell}_{\gamma-\epsilon} = 0 \quad \text{in} \quad R_r(\ell)(1).$$

It then follows from 2.2.5 that

$$\exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda',\epsilon}(b^\epsilon)\right) + \exp\left(\mathrm{loc}^s_{\lambda'}\left(\kappa^\epsilon(\ell\ell')\right)\right) \leq r + v_\Phi([H:E]) + e(\Xi,-\epsilon). \tag{4.8}$$

Putting (4.6), (4.7) and (4.8) together, we obtain

$$\exp(b^\epsilon) \leq r - e + 2\mathcal{C}_1 + \mathcal{C}_2 + e(\Xi,-\epsilon) + e(Q,\epsilon) + e(Q,-\epsilon) + v_\Phi(2[H:E]). \tag{4.9}$$

4.3.4. Let now c be any element of $H^1_{\mathcal{F}}(H,T_r)(\chi)$. For $i = r - e + \mathcal{C}_1 + v_\Phi(2)$,

$$\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}\left(\pi^i c\right) \in R_r \cdot \mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}\left(\kappa(1)\right) \quad \text{in} \quad (T_r)^{\mathrm{fr}}_{\gamma-\epsilon}$$

by (4.2), so there is an $\alpha \in R_r$ such that

$$\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(c_1) = 0 \quad \text{in} \quad (T_r)^{\mathrm{fr}}_{\gamma-\epsilon} \quad \text{for} \quad c_1 = \pi^i c - \alpha\kappa(1) \in H^1_{\mathcal{F}}(H,T_r)(\chi).$$

It then follows from Lemma 2.3 that

$$\exp(\mathrm{loc}_\lambda(c_1)_\epsilon) \leq e(\epsilon) \quad \text{in} \quad H^1_f(H_\lambda,T_r)_{\gamma-\epsilon} \simeq (T_r)_{\gamma-\epsilon}.$$

Since $\exp(\mathrm{loc}_\lambda(c_1^\epsilon)) = \exp(\mathrm{loc}_\lambda(c_1)^\epsilon) \leq \exp(\mathrm{loc}_\lambda(c_1)_\epsilon)$, we obtain

$$\mathrm{loc}_\lambda(b^\epsilon) = 0 \quad \text{in} \quad H^1_f(H_\lambda,T_r) \quad \text{for} \quad b = \pi^{e(\epsilon)}c_1 \in H^1_{\mathcal{F}}(H,T_r)(\chi).$$

Since $b^\epsilon \in H^1_{\mathcal{F}}(H,T_r)(\chi)$, it follows that actually $b^\epsilon \in H^1_{\mathcal{F}_\ell}(H,T_r)(\chi)$. Since

$$\exp(b) \leq \max\left\{\exp(b^\epsilon), \exp(b^{-\epsilon})\right\} + v_\Phi(2)$$

we now find using (4.9) and (4.5) that

$$\exp(b) \leq r - e + 2\mathcal{C}_1 + \mathcal{C}_2 + e(\Xi,\pm) + e(Q,\pm) + v_\Phi(4[H:E])$$

where $e(\star,\pm) = e(\star,+) + e(\star,-)$. Since $b = \pi^{e(\epsilon)+i}c - \pi^{e(\epsilon)}\alpha\kappa(1)$, it follows that

$$\exp\left(\frac{H^1_{\mathcal{F}}(H,T_r)(\chi)}{R_r\kappa(1)}\right) \leq 2(r-e) + d^?(\chi)$$

with

$$\begin{aligned} d^?(\chi) &\overset{\text{def}}{=} 3\mathcal{C}_1 + \mathcal{C}_2 + e(\Xi,\pm) + e(Q,\pm) + e(\epsilon) + v_\Phi(8[H:E]) \\ &= 4c(T,H) + 4z(V,H) + 4r_0 + e(\Xi,\pm) + e(Q,\pm) + e(\epsilon) \\ &\quad + v_\Phi(2^{15}[H:E]) + \delta_{\{2\}}(p). \end{aligned}$$

4.4. **Proof of Proposition 4.2 when $\chi^2 \neq 1$.** Suppose that $\chi^2 \neq 1$ and define

$$e(\chi) \overset{\text{def}}{=} \exp\left(\frac{R}{\langle \chi^2(\sigma) - 1 : \sigma \in \mathrm{Gal}(H/E)\rangle}\right)$$

Thus $e(\chi) = 0$ unless the residue character $\chi^2 : \mathrm{Gal}(H/E) \to R_1^\times$ is trivial. If the residue character is trivial, the order of χ^2 is a non-trivial power of p, so its image contains μ_p, and $e(\chi) \leq v_\Phi(p)/(p-1)$. For any $R[\mathrm{Gal}(H/E)]$-module M,

$$\exp\left(M(\chi) \cap M(\chi^{-1})\right) \leq e(\chi).$$

If M is an $R[\mathrm{Gal}(H/F)]$-module, then for $m \in M(\chi)$, $\epsilon \in \{\pm 1\}$ and $i \geq 0$,

$$\pi^i m^\epsilon = 0 \iff \pi^i \epsilon m = \pi^i \gamma m \Longrightarrow \pi^i m \in M(\chi) \cap M(\chi^{-1}) \Longrightarrow \pi^{i+e(\chi)} m = 0$$

and therefore $\exp(m^\epsilon) \geq \exp(m) - e(\chi)$.

4.4.1. We first apply Proposition 3.5 (1) to

$$c_1 = \kappa^+(1) \quad \text{and} \quad c_2 = \kappa^-(1) \quad \text{in} \quad C' = H^1_{\mathcal{F}}(H,T_r)(\chi) + H^1_{\mathcal{F}}(H,T_r)(\chi^{-1}).$$

to obtain a Kolyvagin prime $\mathcal{L}$ for H'_r above $\lambda \mid \ell$ of H and F with

$$\exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(\kappa(1))\right) \geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(\kappa^\epsilon(1))\right) \geq \exp(\kappa^\epsilon(1)) - \mathcal{C}_2 \geq e - e(\chi) - \mathcal{C}_2 \tag{4.10}$$

for $\epsilon \in \{\pm 1\}$. From the Kolyvagin system relations, we obtain

$$\exp\left(\mathrm{loc}^s_\lambda(\kappa^\epsilon(\ell))\right) \geq \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda,\epsilon}(\kappa(1))\right) - e(Q,\epsilon) \geq e - e(\chi) - e(Q,\epsilon) - \mathcal{C}_2. \tag{4.11}$$

It follows that

$$\exp\left(\mathrm{loc}^s_\lambda(\kappa(\ell))\right), \exp\left(\mathrm{loc}^s_\lambda(\gamma\kappa(\ell))\right) \geq e - e(\chi) - e'(Q,\pm) - \mathcal{C}_2 - v_\Phi(2) \tag{4.12}$$

where $e'(Q,\pm) \overset{\text{def}}{=} \max\{e(Q,+), e(Q,-)\}$. Applying the annihilation relations to

$$c = a \in H^1_{\mathcal{F}}(H,T_r)(\chi) \quad \text{and} \quad c' = \gamma\kappa(\ell) \in H^1_{\mathcal{F}^\ell}(H,T_r)(\chi^{-1}) \otimes \mathbb{G}(\ell),$$

we thus obtain:

$$[H:E] \cdot [\mathrm{loc}_\lambda(a), \mathrm{loc}^s_\lambda(\gamma\kappa(\ell))]^\ell_\lambda = 0 \quad \text{in} \quad R_r(\ell).$$

Taking into account remark 2.8 and the diagram of section 4.2.3, this yields

$$[H:E] \cdot \left\langle \mathrm{loc}^{\mathrm{fr}}_\lambda(a), \mathrm{loc}^s_\lambda(\gamma\kappa(\ell))\right\rangle^{\mathrm{fr}}_{\gamma^2-1} = 0 \quad \text{in} \quad R_r(1).$$

It then follows from 2.2.5 that $\pi^i \mathrm{loc}^{\mathrm{fr}}_\lambda(a) \in R_r \cdot \mathrm{loc}^{\mathrm{fr}}_\lambda(\kappa(1))$ for

$$i \geq 2(r-e) + e(\Xi) + 2e(\chi) + 2\mathcal{C}_2 + e'(Q,\pm) + v_\Phi(2[H:E]).$$

By Lemma 2.2, there is an $\alpha \in R_r$ such that $\mathrm{loc}_\lambda(b) = 0$ for $b = \pi^i a - \alpha\kappa(1)$, where

$$i = 2(r-e) + e(\Xi) + e(\gamma) + 2e(\chi) + 2\mathcal{C}_2 + e'(Q,\pm) + v_\Phi(2[H:E]). \tag{4.13}$$

Since $b \in H^1_{\mathcal{F}}(H,T_r)(\chi)$, it actually belongs to $H^1_{\mathcal{F}_\ell}(H,T_r)(\chi)$.

4.4.2. We now apply the second part of Proposition 3.5 to

$$c_1 = \kappa(1), \quad c_2 = \tilde{\kappa}(\ell), \quad c_3 = b \quad \text{in} \quad C' = H^1_{\mathcal{F}^\ell}(H,T_r)(\chi) + H^1_{\mathcal{F}^\ell}(H,T_r)(\chi^{-1}).$$

We obtain a Kolyvagin prime $\mathcal{L}'$ for H'_r above $\lambda' \mid \ell'$ of H and F with $\ell' \neq \ell$ and

$$\begin{array}{rcl} \max\left\{d(\star), \mathrm{length}\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda'}\left(\langle\kappa(1),\tilde{\kappa}(\ell)\rangle\right)\right)\right\} & \geq & \mathcal{C}(\star) - 2\mathcal{C}_1, \\ \text{and} \quad \exp\left(\mathrm{loc}^{\mathrm{fr}}_{\lambda'}(b)\right) & \geq & \exp(b) - \mathcal{C}_3. \end{array} \tag{4.14}$$

Here $d(\star) = \max\left\{\exp\left(\overline{\kappa(1)}\right), \exp\left(\overline{\tilde{\kappa}(\ell)}\right)\right\}$ where

$$\overline{\kappa(1)} = (\kappa(1)_+, \kappa(1)_-) \quad \text{and} \quad \overline{\tilde{\kappa}(\ell)} = (\tilde{\kappa}(\ell)_+, \tilde{\kappa}(\ell)_-)$$

are the images of $\kappa(1)$ and $\tilde{\kappa}(\ell)$ in $C'_{\gamma-1} \oplus C'_{\gamma+1}$ and by section 3.2.5,

$$\begin{array}{rcl} \mathcal{C}(\star) & \geq & \mathrm{length}(D^+) + \mathrm{length}(D^-) - \mathrm{length}(D) \\ & \geq & \mathrm{length}\left(\mathrm{loc}_{\lambda,+}\left(D^+\right)\right) + \mathrm{length}\left(\mathrm{loc}_{\lambda,-}\left(D^-\right)\right) - (e+r) \end{array}$$

where $D = \left\langle \overline{\kappa(1)}, \overline{\tilde{\kappa}(\ell)} \right\rangle$ and $D^\epsilon = \langle \kappa(1)_\epsilon, \tilde{\kappa}(\ell)_\epsilon \rangle \subset C'_{\gamma-\epsilon}$ for $\epsilon \in \{\pm 1\}$, with

$$\mathrm{loc}_{\lambda,\epsilon} : C'_{\gamma-\epsilon} \to H^1(H_\lambda, T_r)_{\gamma-\epsilon}$$

induced by $\mathrm{loc}_\lambda : C' \to H^1(H_\lambda, T_r)$. Note that

$$\exp\left(\mathrm{loc}^s_{\lambda,\epsilon}\left(\tilde{\kappa}(\ell)_\epsilon\right)\right) = \exp\left(\mathrm{loc}^s_\lambda\left(\tilde{\kappa}(\ell)\right)_\epsilon\right) \geq \exp\left(\mathrm{loc}^s_\lambda\left(\tilde{\kappa}(\ell)\right)^\epsilon\right) = \exp\left(\mathrm{loc}^s_\lambda\left(\tilde{\kappa}^\epsilon(\ell)\right)\right).$$

Using respectively (4.10) and (4.11), we find

$$\begin{array}{rcl} \exp\left(\mathrm{loc}_{\lambda,\epsilon}\left(\kappa(1)_\epsilon\right)\right) & \geq & e - e(\chi) - \mathcal{C}_2 \quad \text{and} \\ \exp\left(\mathrm{loc}^s_{\lambda,\epsilon}\left(\tilde{\kappa}(\ell)_\epsilon\right)\right) & \geq & e - e(\chi) - \mathcal{C}_2 - e(Q,-\epsilon) \end{array}$$

It follows easily, using the split finite/singular exact sequence, that

$$\mathrm{length}\left(\mathrm{loc}_{\lambda,\epsilon}\left(D^\epsilon\right)\right) \geq 2(e - e(\chi) - \mathcal{C}_2) - e(Q,-\epsilon)$$

and therefore

$$\mathcal{C}(\star) \geq 3e - r - 4\left(e(\chi) + \mathcal{C}_2\right) - e(Q,\pm). \tag{4.15}$$

4.4.3. We will verify later on that we may assume that

$$3e \geq 2r + 4\left(e(\chi) + \mathcal{C}_2\right) + e(Q, \pm) + 2\mathcal{C}_1. \tag{4.16}$$

Then $\mathcal{C}(\star) - 2\mathcal{C}_1 \geq r$ by (4.15), thus

$$\text{length}\left(\text{loc}^{\text{fr}}_{\lambda'}\left(\langle \kappa(1), \tilde{\kappa}(\ell)\rangle\right)\right) \geq 3e - r - 4\left(e(\chi) + \mathcal{C}_2\right) - e(Q, \pm) - 2\mathcal{C}_1$$

by (4.14) and (4.15). By the Kolyvagin system relations and Lemma 2.2,

$$\text{length}\left(\text{loc}^{s}_{\lambda'}\left(\langle \kappa(\ell') \otimes \zeta_\ell, \kappa(\ell\ell')\rangle\right)\right) \geq \text{length}\left(\text{loc}^{\text{fr}}_{\lambda'}\left(\langle \kappa(1), \tilde{\kappa}(\ell)\rangle\right)\right) - 2e(Q)$$

We define an R_r-module $\mathcal{L}$ by the formula

$$\mathcal{L} = \gamma\left\langle \kappa(\ell') \otimes \zeta_\ell, \kappa(\ell\ell')\right\rangle \subset H^1_{\mathcal{F}^{\ell\ell'}}(H, T_r)(\chi^{-1}) \otimes \mathbb{G}(\ell\ell').$$

Since $\text{loc}^s_{\lambda'} \circ \gamma = \gamma \circ \text{loc}^s_{\lambda'}$, the previous two inequalities give

$$\text{length}\left(\text{loc}^{s}_{\lambda'}\left(\mathcal{L}\right)\right) \geq 3e - r - 4\left(e(\chi) + \mathcal{C}_2\right) - e(Q, \pm) - 2\left(\mathcal{C}_1 + e(Q)\right) \tag{4.17}$$

We now apply the annihilation relations of section 4.2.2 twice, with

$$c = b \in H^1_{\mathcal{F}_\ell}(H, T_r)(\chi)$$

and

$$c' = \{\gamma \cdot \kappa(\ell') \otimes \zeta_\ell \text{ or } \kappa(\ell\ell')\} \in H^1_{\mathcal{F}^{\ell\ell'}}(H, T_r)(\chi^{-1}) \otimes \mathbb{G}(\ell\ell').$$

We obtain:

$$[H : E] \cdot [\text{loc}_{\lambda'}(b), \text{loc}^s_{\lambda'}(\mathcal{L})]^{\ell\ell'}_{\lambda'} = 0 \quad \text{in} \quad R_r(\ell\ell').$$

Taking into account remark 2.8 and the diagram of section 4.2.3, this yields

$$[H : E] \cdot \left\langle \text{loc}^{\text{fr}}_{\lambda'}(b), \text{loc}^s_{\lambda'}(\mathcal{L})\right\rangle^{\text{fr},\ell}_{\gamma^2 - 1} = 0 \quad \text{in} \quad R_r(\ell)(1).$$

By section 2.2.5,

$$\text{length}\left(\text{loc}_{\lambda'}\left(\mathcal{L}\right)\right) \leq 2r + e(\Xi) + v_\Phi([H : E]) - \exp\left(\text{loc}^{\text{fr}}_{\lambda'}(b)\right). \tag{4.18}$$

Then (4.14), (4.17) and (4.18) together imply

$$\begin{aligned} \exp(b) \quad \leq \quad & 3(r - e) + 2\mathcal{C}_1 + 4\mathcal{C}_2 + \mathcal{C}_3 + v_\Phi([H : E]) \\ & + e(\Xi) + 4e(\chi) + e(Q, \pm) + 2e(Q) \end{aligned}$$

Since $b = \pi^i a - \alpha\kappa(1)$ with i as in (4.13), we find that

$$\exp\left(\frac{H^1_{\mathcal{F}}(H, T_r)(\chi)}{R_r \cdot \kappa(1)}\right) \leq 5(r - e) + d^?(\chi)$$

with

$$\begin{aligned} d^?(\chi) \quad & \stackrel{\text{def}}{=} \quad 2\mathcal{C}_1 + 6\mathcal{C}_2 + \mathcal{C}_3 + v_\Phi(2[H : E]^2) \\ & \qquad + 2e(\Xi) + e(\gamma) + 6e(\chi) + 2e(Q) \\ & \qquad + e(Q, \pm) + e'(Q, \pm) \\ & = \quad 9\left(c(T, H) + z(V, H) + r_0\right) + v_\Phi\left(2^{28}[H : E]^2\right) \\ & \qquad + 7\delta_{\{2\}}(p) + \delta_{\{3\}}(p) \\ & \qquad + 2e(\Xi) + e(\gamma) + 6e(\chi) + 2e(Q) + e(Q, \pm) + e'(Q, \pm). \end{aligned}$$

We may now revisit the assumption we made at the beginning of this subsection. If

$$5(r - e) + d^?(\chi) \geq r,$$

Proposition 4.2 says nothing. Otherwise 4.16 holds and we are done.

Part 2. A new Euler System

5. The Shimura variety and its special cycles

5.1. **The ambient group.** Let F be a totally real number field and pick $\mathfrak{f}_\circ$ in

$$\mathfrak{F} \overset{\text{def}}{=} \mathrm{Spec}(F)(\overline{\mathbb{Q}}) = \mathrm{Spec}(F)(\mathbb{C}) = \mathrm{Spec}(F)(\mathbb{R}).$$

Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Fix a positive integer $n > 0$ and let $(\mathcal{V}, \varphi)$ be a quadratic F-vector space of odd dimension $2n+1$. Put

$$(\mathcal{V}_\mathfrak{f}, \varphi_\mathfrak{f}) \overset{\text{def}}{=} (\mathcal{V}, \varphi) \otimes_{F,\mathfrak{f}} \mathbb{R}$$

for $\mathfrak{f} \in \mathfrak{F}$, and suppose that

$$\mathrm{sign}(\mathcal{V}_\mathfrak{f}, \varphi_\mathfrak{f}) = \begin{cases} (2n-1, 2) & \text{if } \mathfrak{f} = \mathfrak{f}_\circ, \\ (2n+1, 0) \text{ or } (0, 2n+1) & \text{if } \mathfrak{f} \neq \mathfrak{f}_\circ. \end{cases}$$

Set $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathbf{SO}(\mathcal{V}, \varphi)$. This is a $\mathbb{Q}$-simple algebraic group over $\mathbb{Q}$ whose points on a commutative $\mathbb{Q}$-algebra R are given by $\mathbf{G}(R) = SO(\mathcal{V}_R, \varphi_R)$. In particular,

$$\mathbf{G}_\mathbb{R} = \prod_{\mathfrak{f} \in \mathfrak{F}} \mathbf{G}_\mathfrak{f} \quad \text{with} \quad \mathbf{G}_\mathfrak{f} = \mathbf{SO}(\mathcal{V}_\mathfrak{f}, \varphi_\mathfrak{f}) \simeq \begin{cases} \mathbf{SO}(2n-1, 2) & \text{if } \mathfrak{f} = \mathfrak{f}_\circ, \\ \mathbf{SO}(2n+1) & \text{if } \mathfrak{f} \neq \mathfrak{f}_\circ. \end{cases}$$

We set $(\mathcal{V}_\circ, \varphi_\circ) = (\mathcal{V}_{\mathfrak{f}_\circ}, \varphi_{\mathfrak{f}_\circ})$, $\mathbf{G}_\circ = \mathbf{G}_{\mathfrak{f}_\circ}$ and $\mathbf{G}^\circ = \prod_{\mathfrak{f} \neq \mathfrak{f}_\circ} \mathbf{G}_\mathfrak{f}$, so that $\mathbf{G}_\mathbb{R} = \mathbf{G}_\circ \times \mathbf{G}^\circ$.

5.2. **The ambient domain.** Let $\mathcal{X}$ be the space of oriented negative $\mathbb{R}$-planes in $(\mathcal{V}_\circ, \varphi_\circ)$: the set of all pairs $x = (D_x, \theta_x)$ where D_x is a 2-dimensional anisotropic negative $\mathbb{R}$-subspace of $(\mathcal{V}_\circ, \varphi_\circ)$ and θ_x is an orientation on D_x. The action of $\mathbf{G}_\circ(\mathbb{R}) = SO(\mathcal{V}_\circ, \varphi_\circ)$ on $\mathcal{V}_\circ$ induces a transitive action of $\mathbf{G}(\mathbb{R}) = \mathbf{G}_\circ(\mathbb{R}) \times \mathbf{G}^\circ(\mathbb{R})$ on $\mathcal{X}$ and the stabilizer of x is a maximal connected and compact subgroup, namely

$$K_x = \left(SO(D_x) \times SO(D_x^\perp)\right) \times \mathbf{G}^\circ(\mathbb{R}) \subset \mathbf{G}_\circ(\mathbb{R}) \times \mathbf{G}^\circ(\mathbb{R}).$$

We denote by $x \mapsto \overline{x}$ the $\mathbf{G}(\mathbb{R})$-equivariant map given by $\overline{x} = (D_x, \theta_x^{-1})$.

5.3. **The ambient Shimura datum.** Put $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_{m,\mathbb{C}})$, let $\mathbf{S}^1 \subset \mathbf{S}$ be the kernel of the norm $N : \mathbf{S} \twoheadrightarrow \mathbf{G}_{m,\mathbb{R}}$, and let $\nu : \mathbf{S} \twoheadrightarrow \mathbf{S}^1$ be the surjective homomorphism which maps $z \in \mathbb{C}^\times = \mathbf{S}(\mathbb{R})$ to $\nu(z) = z/\overline{z} \in \mathbf{S}^1(\mathbb{R})$. Viewing the orientation θ_x on the anisotropic plane D_x as being given by an isomorphism

$$\theta_x : \mathbf{S}^1 \xrightarrow{\simeq} \mathbf{SO}(D_x)$$

we define a morphism $h_x : \mathbf{S} \to \mathbf{G}_\mathbb{R}$ by $h_x = u_x \circ \nu$ where

$$u_x : \mathbf{S}^1 \xrightarrow{\theta_x} \mathbf{SO}(D_x) \hookrightarrow \mathbf{SO}(D_x) \times \mathbf{SO}(D_x^\perp) \hookrightarrow \mathbf{G}_\circ \hookrightarrow \mathbf{G}_\mathbb{R}.$$

Then $h_{\overline{x}} = h_x^{-1}$ and $h_{gx} = \mathrm{Int}(g) \circ h_x$ for all $g \in \mathbf{G}(\mathbb{R})$ and $x \in \mathcal{X}$. The $\mathbf{G}(\mathbb{R})$-equivariant map $x \mapsto h_x$ identifies $\mathcal{X}$ with a $\mathbf{G}(\mathbb{R})$-conjugacy class of morphisms $h : \mathbf{S} \to \mathbf{G}_\mathbb{R}$ satisfying the axioms $SV1-3$ of [43]. Since $\mathbf{G}$ is adjoint, these morphisms also trivially satisfy the auxiliary axioms $SV4-6$.

5.4. **The subgroup.** Let E be a totally imaginary quadratic extension of F which splits $(\mathcal{V},\varphi)$, i.e. such that the quadratic space $(\mathcal{V},\varphi)\otimes_F E$ over E contains a totally isotropic E-subspace of dimension n. Then by Proposition 9.2, $(\mathcal{V},\varphi)$ contains E-hermitian F-hyperplanes, and all of them are conjugate under $\mathbf{G}(\mathbb{Q}) = SO(\mathcal{V},\varphi)$. We fix once and for all such an E-hermitian F-hyperplane $(\mathcal{W},\psi)\subset(\mathcal{V},\varphi)$ and set

$$\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{U}(\mathcal{W},\psi) \quad \text{inside} \quad \mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{SO}(\mathcal{V},\varphi).$$

This is a reductive $\mathbb{Q}$-subgroup scheme of $\mathbf{G}$. We also define $\mathbf{T} = \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_{m,E}$, $\mathbf{Z} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}_{m,F}$, let $\mathbf{T}^1\subset\mathbf{T}$ be the kernel of the norm $N:\mathbf{T}\twoheadrightarrow\mathbf{Z}$, and denote by $\mathbf{H}^1\subset\mathbf{H}$ the kernel of $\det:\mathbf{H}\twoheadrightarrow\mathbf{T}^1$, so that $\mathbf{H}^1 = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{SU}(\mathcal{W},\psi)$. We will also view $\mathbf{T}^1$ as the center of $\mathbf{H}$. For $\mathfrak{f}\in\mathfrak{F}$, we set $(\mathcal{W}_\mathfrak{f},\psi_\mathfrak{f}) = (\mathcal{W},\psi)\otimes_{F,\mathfrak{f}}\mathbb{R}$. This is an hermitian space over $E_\mathfrak{f} = E\otimes_{F,\mathfrak{f}}\mathbb{R}$. There are compatible decompositions

$$\mathbf{H}_\mathbb{R} = \prod_{\mathfrak{f}\in\mathfrak{F}}\mathbf{H}_\mathfrak{f} \quad\text{with}\quad \mathbf{H}_\mathfrak{f} = \mathbf{U}(\mathcal{W}_\mathfrak{f},\psi_\mathfrak{f}) \simeq \begin{cases}\mathbf{U}(n-1,1) & \text{if } \mathfrak{f}=\mathfrak{f}_\circ,\\ \mathbf{U}(n) & \text{if } \mathfrak{f}\neq\mathfrak{f}_\circ,\end{cases}$$

$$\mathbf{H}^1_\mathbb{R} = \prod_{\mathfrak{f}\in\mathfrak{F}}\mathbf{H}^1_\mathfrak{f} \quad\text{with}\quad \mathbf{H}^1_\mathfrak{f} = \mathbf{SU}(\mathcal{W}_\mathfrak{f},\psi_\mathfrak{f}) \simeq \begin{cases}\mathbf{SU}(n-1,1) & \text{if } \mathfrak{f}=\mathfrak{f}_\circ,\\ \mathbf{SU}(n) & \text{if } \mathfrak{f}\neq\mathfrak{f}_\circ,\end{cases}$$

$$\mathbf{T}_\mathbb{R} = \prod_{\mathfrak{f}\in\mathfrak{F}}\mathbf{T}_\mathfrak{f} \quad\text{with}\quad \mathbf{T}_\mathfrak{f} = \mathrm{Res}_{E_\mathfrak{f}/\mathbb{R}}\mathbf{G}_{m,E_\mathfrak{f}} \simeq \mathbf{S},$$

$$\text{and}\quad \mathbf{T}^1_\mathbb{R} = \prod_{\mathfrak{f}\in\mathfrak{F}}\mathbf{T}^1_\mathfrak{f} \quad\text{with}\quad \mathbf{T}^1_\mathfrak{f}\simeq\mathbf{S}^1.$$

We set $E_\circ = E_{\mathfrak{f}_\circ}$, $(\mathcal{W}_\circ,\psi_\circ) = (\mathcal{W}_{\mathfrak{f}_\circ},\psi_{\mathfrak{f}_\circ})$, $\mathbf{X}_\circ = \mathbf{X}_{\mathfrak{f}_\circ}$ and $\mathbf{X}^\circ = \prod_{\mathfrak{f}\neq\mathfrak{f}^\circ}\mathbf{X}_\mathfrak{f}$ for every $\mathbf{X}$ in $\{\mathbf{H},\mathbf{H}^1,\mathbf{T},\mathbf{T}^1\}$, so that $\mathbf{X} = \mathbf{X}_\circ\times\mathbf{X}^\circ$. We also set

$$\mathfrak{E} = \mathrm{Spec}(E)(\overline{\mathbb{Q}}) = \mathrm{Spec}(E)(\mathbb{C}),\quad \mathrm{Gal}(E/F) = \{\mathrm{Id},\star\},\quad \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{Id},c\}$$

and denote by $\{\mathfrak{f}^+,\mathfrak{f}^-\}$ the fiber of the restriction map $\mathfrak{E}\twoheadrightarrow\mathfrak{F}$ over $\mathfrak{f}\in\mathfrak{F}$. This means that we fix a CM type on E/F: a section $\mathfrak{f}\mapsto\mathfrak{f}^+$ of $\mathfrak{E}\twoheadrightarrow\mathfrak{F}$. The involution

$$\mathfrak{e}\mapsto\bar{\mathfrak{e}} = \mathfrak{e}\circ\star = c\circ\mathfrak{e}$$

of $\mathfrak{E}$ exchanges $\mathfrak{f}^+$ and $\mathfrak{f}^-$ for every $\mathfrak{f}\in\mathfrak{F}$.

5.5. **The subdomains.** Let $\mathcal{Y}$ be the space of all negative $E_\circ$-lines in $(\mathcal{W}_\circ,\psi_\circ)$. Each $y\in\mathcal{Y}$ yields a negative $\mathbb{R}$-plane D_y in $(\mathcal{V}_\circ,\varphi_\circ)$, equipped with an isomorphism

$$\mathbf{T}^1_\circ \simeq \mathbf{U}(y,\psi_\circ|y) = \mathbf{SO}(D_y,\varphi_\circ|D_y).$$

For each choice of sign $\pm$, the embedding $\mathfrak{f}^\pm_\circ : E\hookrightarrow\mathbb{C}$ yields an isomorphism $\mathfrak{f}^\pm_\circ : E_\circ\simeq\mathbb{C}$ which induces isomorphisms $\mathfrak{f}^\pm_\circ:\mathbf{T}_\circ\simeq\mathbf{S}$ and $\mathfrak{f}^\pm_\circ:\mathbf{T}^1_\circ\simeq\mathbf{S}^1$. Composing the inverse of the latter with the displayed isomorphism, we obtain an orientation

$$\theta^\pm_y : \mathbf{S}^1 \xrightarrow{\simeq} \mathbf{SO}(D_y).$$

The action of $\mathbf{H}_\circ(\mathbb{R})$ on $\mathcal{W}_\circ$ induces a transitive action of $\mathbf{H}(\mathbb{R}) = \mathbf{H}_\circ(\mathbb{R})\times\mathbf{H}^\circ(\mathbb{R})$ on $\mathcal{Y}$ and the stabilizer of y is a maximal connected and compact subgroup, namely

$$K'_y = \left(U(y)\times U(y^\perp)\right)\times\mathbf{H}^\circ(\mathbb{R}) \subset \mathbf{H}_\circ(\mathbb{R})\times\mathbf{H}^\circ(\mathbb{R}).$$

The map $y\mapsto y^\pm = (D_y,\theta^\pm_y)$ is an $\mathbf{H}(\mathbb{R})$-equivariant bijection from $\mathcal{Y}$ to a subdomain $\mathcal{Y}^\pm$ of $\mathcal{X}$ and one checks easily that $\mathcal{X}(\mathbf{H}) = \mathcal{Y}^+\coprod\mathcal{Y}^-$ where

$$\mathcal{X}(\mathbf{H}) = \{x\in\mathcal{X} : h_x:\mathbf{S}\to\mathbf{G}_\mathbb{R} \text{ factors through } \mathbf{H}_\mathbb{R}\hookrightarrow\mathbf{G}_\mathbb{R}\}.$$

The involution $x \mapsto \overline{x}$ of $\mathcal{X}$ exchanges the two components $\mathcal{Y}^+$ and $\mathcal{Y}^-$ of $\mathcal{X}(\mathbf{H})$.

5.6. **A subtorus.** Let $\mathcal{B} = (w_1, \cdots, w_n)$ be an orthogonal E-basis of $(\mathcal{W}, \psi)$. Then

$$\mathbf{T}(\mathcal{B}) \overset{\text{def}}{=} \mathrm{Res}_{F/\mathbb{Q}}\left(\mathbf{U}(Ew_1) \times \cdots \times \mathbf{U}(Ew_n)\right) \subset \mathbf{H} \subset \mathbf{G}$$

is a maximal $\mathbb{Q}$-subtorus of both $\mathbf{H}$ and $\mathbf{G}$, with

$$\mathbf{T}(\mathcal{B}) = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{U}(Ew_1) \times \cdots \times \mathrm{Res}_{F/\mathbb{Q}}\mathbf{U}(Ew_n) \simeq (\mathbf{T}^1)^n.$$

There is an $\mathrm{Aut}(\mathbb{C})$-equivariant isomorphism

$$\{f : \mathfrak{E} \to \mathbb{Z} | f(\mathfrak{e}) + f(\bar{\mathfrak{e}}) = 0 \text{ for all } \mathfrak{e} \in \mathfrak{E}\} \xrightarrow{\simeq} X_*(\mathbf{T}^1)$$

given by $f \mapsto \mu_f$ where for all $z \in \mathbb{C}^\times$ and $\mathfrak{e} \in \mathfrak{E}$,

$$\mu_f(z) : \mathfrak{e} \mapsto z^{f(\mathfrak{e})} \quad \text{in} \quad \mathbf{T}^1(\mathbb{C}) \subset \mathbf{T}(\mathbb{C}) = (E \otimes \mathbb{C})^\times = (\prod_{\mathfrak{e} \in \mathfrak{E}} \mathbb{C})^\times = \mathcal{F}(\mathfrak{E}, \mathbb{C}^\times).$$

We also denote by $f \mapsto \mu_f$ the corresponding $\mathrm{Aut}(\mathbb{C})$-equivariant isomorphism

$$\{f : \mathfrak{E} \to \mathbb{Z}^n | f(\mathfrak{e}) + f(\bar{\mathfrak{e}}) = 0 \text{ for all } \mathfrak{e} \in \mathfrak{E}\} \xrightarrow{\simeq} X_*(\mathbf{T}(\mathcal{B})). \tag{5.1}$$

Let $S_n^+ = \{\pm 1\}^n \rtimes S_n$ be the group of signed permutations, acting on $\mathbb{Z}^n$ by

$$(\epsilon_\bullet \sigma)(\lambda_1, \cdots, \lambda_n) = (\epsilon_1 \lambda_{\sigma^{-1}(1)}, \cdots, \epsilon_n \lambda_{\sigma^{-1}(n)}).$$

The Weyl groups of $\mathbf{H}$ and $\mathbf{G}$ with respect to $\mathbf{T}(\mathcal{B})$ are then respectively equal to

$$W(\mathbf{H}, \mathcal{B}) = \mathcal{F}(\mathfrak{E}, S_n) \subset W(\mathbf{G}, \mathcal{B}) = \mathcal{F}(\mathfrak{E}, S_n^+)$$

acting on $X_*(\mathbf{T}(\mathcal{B})) \subset \mathcal{F}(\mathfrak{E}, \mathbb{Z}^n)$ by $(w \cdot f)(\mathfrak{e}) = w(\mathfrak{e}) \cdot f(\mathfrak{e})$. Here $\mathcal{F}(A, B)$ is the set of all maps from A to B.

5.7. **The reflex fields.** Let $\mathcal{B}_\circ = (w_{1,\circ}, \cdots, w_{n,\circ})$ be the orthogonal $E_\circ$-basis of $(\mathcal{W}_\circ, \psi_\circ)$ obtained from $\mathcal{B}$ by base change along $\mathfrak{f}_\circ : F \hookrightarrow \mathbb{R}$. Since the signature of $(\mathcal{W}_\circ, \psi_\circ)$ equals $(n-1, 1)$, there is a unique i in $\{1, \cdots, n\}$ with $\psi_\circ(w_{i,\circ}, w_{i,\circ}) < 0$. Then $w_{i,\circ}$ spans a negative $E_\circ$-line $y_\mathcal{B}$ in $(\mathcal{W}_\circ, \psi_\circ)$, giving rise to *special* points $y_\mathcal{B}^\pm \in \mathcal{Y}^\pm$: the corresponding morphisms $h_\mathcal{B}^\pm : \mathbf{S} \to \mathbf{H}_\mathbb{R}$ (or $\mathbf{G}_\mathbb{R}$) factor through $\mathbf{T}(\mathcal{B})_\mathbb{R} \hookrightarrow \mathbf{H}_\mathbb{R}$. The induced cocharacters $\mu_\mathcal{B}^\pm : \mathbf{G}_{m,\mathbb{C}} \to \mathbf{T}(\mathcal{B})_\mathbb{C}$ defined by

$$\mu_\mathcal{B}^\pm = (h_\mathcal{B}^\pm)_\mathbb{C} \circ \iota : \mathbf{G}_{m,\mathbb{C}} \to \mathbf{G}_{m,\mathbb{C}} \times \mathbf{G}_{m,\mathbb{C}} \simeq \mathbf{S}_\mathbb{C} \to \mathbf{T}(\mathcal{B})_\mathbb{C}$$

where $\iota(z) = (z, 1)$ correspond under the identification (5.1) to the functions

$$f_\mathcal{B}^\pm : \mathfrak{E} \to \mathbb{Z}^n, \quad \mathfrak{e} \mapsto \begin{cases} (0, \cdots, 0, +1, 0, \cdots, 0) & \text{if } \mathfrak{e} = \mathfrak{f}_\circ^\pm, \\ (0, \cdots, 0, -1, 0, \cdots, 0) & \text{if } \mathfrak{e} = \mathfrak{f}_\circ^\mp, \\ (0, \cdots, 0) & \text{if } \mathfrak{e} \notin \{\mathfrak{f}_\circ^+, \mathfrak{f}_\circ^-\}, \end{cases}$$

with the non-trivial coefficients at the i-th place. The reflex field $E(\mathbf{G}, \mathcal{X})$ is the field of definition of the $\mathbf{G}(\mathbb{C})$-conjugacy class of $\mu_\mathcal{B}^+ : \mathbf{G}_{m,\mathbb{C}} \to \mathbf{G}_\mathbb{C}$. By a standard argument, it is also the field of definition of the $W(\mathbf{G}, \mathcal{B})$-orbit of $\mu_\mathcal{B}^+ \in X_*(\mathbf{T}(\mathcal{B}))$. Similarly, the reflex field $E(\mathbf{H}, \mathcal{Y}^\pm)$ is the field of definition of the $W(\mathbf{H}, \mathcal{B})$-orbit of $\mu_\mathcal{B}^\pm \in X_*(\mathbf{T}(\mathcal{B}))$. Given the above description of the $W(\mathbf{G}, \mathcal{B}) \rtimes \mathrm{Aut}(\mathbb{C})$-module $X_*(\mathbf{T}(\mathcal{B}))$, we find that our reflex fields are respectively given by

$$E(\mathbf{G}, \mathcal{X}) = \mathfrak{f}_\circ(F) \subset \mathbb{R} \quad \text{and} \quad E(\mathbf{H}, \mathcal{Y}^\pm) = \mathfrak{f}_\circ(E) \subset \mathbb{C}$$

where $\mathfrak{f}_\circ(E)$ is the quadratic extension $\mathfrak{f}_\circ^+(E) = \mathfrak{f}_\circ^-(E)$ of $\mathfrak{f}_\circ(F)$.

5.8. **The reflex norms.** Note that the cocharacters $\mu_{\mathcal{B}}^{\pm} : \mathbf{G}_{m,\mathbb{C}} \to \mathbf{T}(\mathcal{B})_{\mathbb{C}}$ are themselves already defined over $\mathfrak{f}_{\circ}(E)$. We set $\mathfrak{f}_{\circ}\mathbf{T} = \mathrm{Res}_{\mathfrak{f}_{\circ}(E)/\mathbb{Q}}(\mathbf{G}_{m,\mathfrak{f}_{\circ}(E)})$ and define the reflex norm $r_{\mathcal{B}}^{\pm} : \mathfrak{f}_{\circ}\mathbf{T} \to \mathbf{T}(\mathcal{B})$ as the composition

$$\mathfrak{f}_{\circ}\mathbf{T} = \mathrm{Res}_{\mathfrak{f}_{\circ}(E)/\mathbb{Q}}\left(\mathbf{G}_{m,\mathfrak{f}_{\circ}(E)}\right) \longrightarrow \mathrm{Res}_{\mathfrak{f}_{\circ}(E)/\mathbb{Q}}\left(\mathbf{T}(\mathcal{B})_{\mathfrak{f}_{\circ}(E)}\right) \longrightarrow \mathbf{T}(\mathcal{B})$$

where the first morphism is induced by $\mu_{\mathcal{B}}^{\pm} : \mathbf{G}_{m,\mathfrak{f}_{\circ}(E)} \to \mathbf{T}(\mathcal{B})_{\mathfrak{f}_{\circ}(E)}$ and the second one is the norm. The isomorphism of fields $\mathfrak{f}_{\circ}^{+} : E \to \mathfrak{f}_{\circ}(E)$ yields an isomorphism of tori $\mathfrak{f}_{\circ}^{+} : \mathbf{T} \to \mathfrak{f}_{\circ}\mathbf{T}$ and the composition $r_{\mathcal{B}}^{\pm} \circ \mathfrak{f}_{\circ}^{+} : \mathbf{T} \to \mathbf{T}(\mathcal{B})$ is given by

$$r_{\mathcal{B}}^{\pm} \circ f_{\circ}^{+} = (1, \cdots, 1, \nu^{\pm 1}, 1, \cdots, 1), \qquad \mathbf{T} \to \mathbf{T}(\mathcal{B}) \simeq (\mathbf{T}^{1})^{n}$$

where the non-trivial coefficient is at the i-th place and $\nu : \mathbf{T} \to \mathbf{T}^{1}$ maps z in $\mathbf{T}$ to $z/z^{\star}$ in $\mathbf{T}^{1}$. Similarly, the cocharacter $\det(\mu^{\pm})$ defined by

$$\mathbf{G}_{m,\mathbb{C}} \xrightarrow{\mu_{\mathcal{B}}^{\pm}} \mathbf{T}(\mathcal{B})_{\mathbb{C}} \hookrightarrow \mathbf{H}_{\mathbb{C}} \xrightarrow{\det} \mathbf{T}_{\mathbb{C}}^{1}$$

is independent of $\mathcal{B}$ and defined over $\mathfrak{f}_{\circ}(E)$, with corresponding reflex norm

$$r^{\pm} : \mathfrak{f}_{\circ}\mathbf{T} \to \mathbf{T}^{1}, \quad r^{\pm} \circ \mathfrak{f}_{\circ}^{+} = \nu^{\pm 1} : \mathbf{T} \to \mathbf{T}^{1}.$$

These formulas suggest to identify the abstract number fields F and E with the embedded reflex fields $\mathfrak{f}_{\circ}(F)$ and $\mathfrak{f}_{\circ}(E)$, using respectively $\mathfrak{f}_{\circ} : F \to \mathfrak{f}_{\circ}(F)$ and $\mathfrak{f}_{\circ}^{+} : E \to \mathfrak{f}_{\circ}(E)$. We shall do so in the sequel. In particular, $\mathfrak{f}_{\circ}\mathbf{T} = \mathbf{T}$.

5.9. **The Shimura varieties.** For a neat compact open subgroup K of $\mathbf{G}(\mathbb{A}_f)$, we denote by $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ the corresponding Shimura variety. It is a quasi-projective smooth algebraic variety over the reflex field $E(\mathbf{G}, \mathcal{X}) = F$ with complex points

$$\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash \left(\mathbf{G}(\mathbb{A}_f)/K \times \mathcal{X}\right).$$

These varieties are proper precisely when (V, φ) is anisotropic – this is for instance the case when $F \neq \mathbb{Q}$[2]. For $g \in \mathbf{G}(\mathbb{A}_f)$ and neat compact open subgroups K_1 and K_2 of $\mathbf{G}(\mathbb{A}_f)$ such that $g^{-1}K_1 g \subset K_2$, there is a finite étale cover

$$[\cdot g] : \mathrm{Sh}_{K_1}(\mathbf{G}, \mathcal{X}) \to \mathrm{Sh}_{K_2}(\mathbf{G}, \mathcal{X})$$

given by $\mathbf{G}(\mathbb{Q})(hK_1, x) \mapsto \mathbf{G}(\mathbb{Q})(hgK_2, x)$ on complex points. Taking $g = 1$, we obtain a projective system $\mathrm{Sh}(\mathbf{G}, \mathcal{X}) = (\mathrm{Sh}_K(\mathbf{G}, \mathcal{X}))_K$ of algebraic varieties over F, indexed by neat compact open subgroups K of $\mathbf{G}(\mathbb{A}_f)$, with finite étale transition maps, equipped with a continuous right action of $\mathbf{G}(\mathbb{A}_f)$.

There is a similar projective system, indexed by neat compact open subgroups K' of $\mathbf{H}(\mathbb{A}_f)$, of smooth algebraic varieties over the reflex field $E(\mathbf{H}, \mathcal{Y}^{\pm}) = E$, with

$$\mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^{\pm})(\mathbb{C}) = \mathbf{H}(\mathbb{Q}) \backslash \left(\mathbf{H}(\mathbb{A}_f)/K' \times \mathcal{Y}^{\pm}\right).$$

For $K' \subset K$, the map given by $\mathbf{H}(\mathbb{Q}) \cdot (gK', y) \mapsto \mathbf{G}(\mathbb{Q}) \cdot (gK, y)$ is a finite morphism

$$\mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^{\pm}) \longrightarrow \mathrm{Sh}_K(\mathbf{G}, \mathcal{X}) \times_{\mathrm{Spec}(F)} \mathrm{Spec}\,(E).$$

These Shimura varieties are equidimensional of respective dimensions

$$\dim \mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^{\pm}) = n - 1 \quad \text{and} \quad \dim \mathrm{Sh}_K(\mathbf{G}, \mathcal{X}) = 2n - 1.$$

[2]When $F = \mathbb{Q}$ and $n > 1$, (V, φ) is always isotropic and our varieties are not proper.

5.10. **Complex conjugation.** By the conjecture of Langlands established by Milne and Shih in [45] and [19, V] (see also [42]), the action of complex conjugation on $\mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C})$ is given by $\mathbf{G}(\mathbb{Q})(gK,x)\mapsto\mathbf{G}(\mathbb{Q})(gK,\overline{x})$ and the anti-holomorphic map $x\mapsto\overline{x}$ on $\mathcal{X}$ yields a cartesian diagram

$$\begin{array}{ccccc}\mathrm{Sh}_{K'}(\mathbf{H},\mathcal{Y}^{\pm}) & \longrightarrow & \mathrm{Sh}_K(\mathbf{G},\mathcal{X})\times_{\mathrm{Spec}(F)}\mathrm{Spec}\,(E) & \longrightarrow & \mathrm{Spec}(E)\\ \downarrow & & \downarrow{\scriptstyle(\mathrm{Id},\star)} & & \downarrow{\scriptstyle\star}\\ \mathrm{Sh}_{K'}(\mathbf{H},\mathcal{Y}^{\mp}) & \longrightarrow & \mathrm{Sh}_K(\mathbf{G},\mathcal{X})\times_{\mathrm{Spec}(F)}\mathrm{Spec}\,(E) & \longrightarrow & \mathrm{Spec}(E).\end{array}$$

5.11. **The reciprocity law.** Let F^{ab} and E^{ab} be the maximal abelian extensions of F and E in $\overline{\mathbb{Q}}$. Let $E[\infty]$ be the subfield of E^{ab} which is fixed by the image of the transfer map. Class field theory gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc}1 & \longrightarrow & \mathbf{Z}(\mathbb{A}_f) & \longrightarrow & \mathbf{T}(\mathbb{A}_f) & \xrightarrow{\nu} & \mathbf{T}^1(\mathbb{A}_f) & \longrightarrow & 1\\ & & {\scriptstyle\mathrm{Art}_F}\downarrow & & {\scriptstyle\mathrm{Art}_E}\downarrow & & {\scriptstyle\mathrm{Art}^1_E}\downarrow & & \\ 1 & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/E) & \longrightarrow & \mathrm{Gal}(E[\infty]/E) & \longrightarrow & 1\end{array}$$

where Art_F and Art_E are the Artin reciprocity maps (matching geometric Frobeniuses to uniformizers), and the last vertical map induces an isomorphism

$$\mathrm{Art}^1_E:\mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q})\xrightarrow{\simeq}\mathrm{Gal}(E[\infty]/E).$$

The reciprocity law for special points tells us that for every $g\in\mathbf{G}(\mathbb{A}_f)$ and $\mathcal{B}$ as above, for any $\sigma\in\mathrm{Aut}(\mathbb{C},E)$ and $\lambda\in\mathbf{T}(\mathbb{A}_f)$ such that $\sigma|E^{\mathrm{ab}}=\mathrm{Art}_E(\lambda)$, we have

$$\sigma\cdot[gK,y^{\pm}_{\mathcal{B}}]=[r^{\pm}_{\mathcal{B}}(\lambda)gK,y^{\pm}_{\mathcal{B}}]\quad\text{in}\quad\mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C})$$

In particular, $[gK,y^{\pm}_{\mathcal{B}}]$ is defined over $E[\infty]$ and for every $\lambda\in\mathbf{T}^1(\mathbb{A}_f)$,

$$\mathrm{Art}^1_E(\lambda)\cdot[gK,y^{\pm}_{\mathcal{B}}]=[\iota^{\pm}_{\mathcal{B}}(\lambda)gK,y^{\pm}_{\mathcal{B}}]$$

where the morphism $\iota^{\pm}_{\mathcal{B}}:\mathbf{T}^1\hookrightarrow\mathbf{T}(\mathcal{B})\subset\mathbf{H}\subset\mathbf{G}$ is given by

$$\mathbf{T}^1\ni\lambda\mapsto(1,\cdots,1,\lambda^{\pm1},1,\cdots,1)\in\mathbf{T}(\mathcal{B})\simeq(\mathbf{T}^1)^n.$$

The same statement also holds when $g\in\mathbf{H}(\mathbb{A}_f)$ for the corresponding special points

$$[gK',y^{+}_{\mathcal{B}}]\in\mathrm{Sh}_{K'}\left(\mathbf{H},\mathcal{Y}^{+}\right)(\mathbb{C})\quad\text{and}\quad[gK',y^{-}_{\mathcal{B}}]\in\mathrm{Sh}_{K'}\left(\mathbf{H},\mathcal{Y}^{-}\right)(\mathbb{C}).$$

5.12. **The connected components.** The reciprocity law for connected components, described in [18] and corrected in [43] also follows from the above reciprocity laws. Since $\mathbf{H}^1$ is simply connected (being isomorphic to $\mathbf{SL}_n^{\mathfrak{F}}$ over any algebraically closed extension of $\mathbb{Q}$) with $\mathbf{H}^1(\mathbb{R})$ not compact, we know that $\mathbf{H}^1(\mathbb{R})$ and thus also $\mathcal{Y}$ are connected, and that $\mathbf{H}^1(\mathbb{Q})$ is dense in $\mathbf{H}^1(\mathbb{A}_f)$. It follows that

$$\pi_0\left(\mathrm{Sh}_{K'}(\mathbf{H},\mathcal{Y}^{\pm})\right)=\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)/K'=\mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{H}(\mathbb{A}_f)/K'$$

and using the determinant map $\det:\mathbf{H}\to\mathbf{T}^1$, also

$$\pi_0\left(\mathrm{Sh}_{K'}(\mathbf{H},\mathcal{Y}^{\pm})\right)\simeq\mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q})\det(K').$$

Then for all $\sigma\in\mathrm{Aut}(\mathbb{C}/E)$ and $\lambda\in\mathbf{T}^1(\mathbb{A}_f)$ such that $\mathrm{Art}^1_E(\lambda)=\sigma|E[\infty]$,

$$\sigma\cdot C=\lambda^{\pm1}C\quad\text{for all }C\in\pi_0\left(\mathrm{Sh}_{K'}(\mathbf{H},\mathcal{Y}^{\pm})\right).$$

5.13. **The normalizer of H in G.** If $g \in \mathbf{G}$ commutes with $\mathbf{T}^1 = Z(\mathbf{H})$, it fixes $\mathcal{W}^\perp = \{v \in V \text{ s.t. } \forall t \in \mathbf{T}^1, tv = v\}$, therefore also $\mathcal{W}$, and acts E-linearly on it, which implies that $g|\mathcal{W}$ also preserves the E-hermitian pairing ψ, and therefore has determinant 1, so that $g|\mathcal{W}^\perp$ must be the identity, and finally g belongs to $\mathbf{H}$. Thus $\mathbf{H} = Z_G(\mathbf{T}^1)$, from which easily follows that $\mathbf{T}^1$ and $\mathbf{H}$ have the same normalizer $\mathbf{N}$ in $\mathbf{G}$. The kernel of the inner action of $\mathbf{N}$ on $\mathbf{T}^1$ equals $\mathbf{H}$, and its image is a finite étale commutative group of automorphisms of $\mathbf{T}^1$. The resulting action on

$$X_*(\mathbf{T}^1) \simeq \{f : \mathfrak{E} \to \mathbb{Z} \text{ s.t. } \forall \mathfrak{e} \in \mathfrak{E},\ f(\mathfrak{e}) + f(\bar{\mathfrak{e}}) = 0\}$$

yields an isomorphism of finite groups with $\mathrm{Aut}(\mathbb{C})$-actions

$$(\mathbf{N}/\mathbf{H})(\mathbb{C}) \simeq \mathcal{F}(\mathfrak{F}, \{\pm 1\})$$

where $g : \mathfrak{F} \to \{\pm 1\}$ acts on $f : \mathfrak{E} \to \mathbb{Z}$ by $(gf)(\mathfrak{e}) = g(\mathfrak{e}|F)f(\mathfrak{e})$. In particular,

$$(\mathbf{N}/\mathbf{H})(\mathbb{Q}) \simeq \{\pm 1\} \simeq \mathbf{N}(\mathbb{Q})/\mathbf{H}(\mathbb{Q}).$$

To any orthogonal E-basis $\mathcal{B} = (w_1, \cdots, w_n)$ of $(\mathcal{W}, \psi)$ as before, we may attach the element $n_{\mathcal{B}} \in \mathbf{G}(\mathbb{Q}) = SO(\mathcal{V}, \varphi)$ of order 2 which is defined by

$$n_{\mathcal{B}}(\lambda_1 w_1 + \cdots + \lambda_n w_n + x) = \lambda_1^\star w_1 + \cdots + \lambda_n^\star w_n + (-1)^n x$$

for all $\lambda_j \in E$ and $x \in \mathcal{W}^\perp$. Then $n_{\mathcal{B}}$ actually belongs to $\mathbf{N}(\mathbb{Q}) - \mathbf{H}(\mathbb{Q})$. Moreover, $n_{\mathcal{B}} \cdot y_{\mathcal{B}}^{\pm} = y_{\mathcal{B}}^{\mp}$ in $\mathcal{X}$. Thus $n_{\mathcal{B}}$ exchanges the two components $\mathcal{Y}^+$ and $\mathcal{Y}^-$ of $\mathcal{X}(\mathbf{H})$, and conjugation by $n_{\mathcal{B}}$ yields a commutative square of Shimura data

$$\begin{array}{ccc} (\mathbf{H}, \mathcal{Y}^\pm) & \hookrightarrow & (\mathbf{G}, \mathcal{X}) \\ \downarrow & & \downarrow \\ (\mathbf{H}, \mathcal{Y}^\mp) & \hookrightarrow & (\mathbf{G}, \mathcal{X}) \end{array}$$

For $K' \subset K$, there is a corresponding commutative square of E-morphisms,

$$\begin{array}{ccc} \mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^\pm) & \to & \mathrm{Sh}_K(\mathbf{G}, \mathcal{X}) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{n_{\mathcal{B}} K' n_{\mathcal{B}}}(\mathbf{H}, \mathcal{Y}^\mp) & \to & \mathrm{Sh}_{n_{\mathcal{B}} K n_{\mathcal{B}}}(\mathbf{G}, \mathcal{X}) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E) \end{array}$$

in which the second vertical isomorphism is base changed from the F-morphism

$$[\cdot n_{\mathcal{B}}] : \mathrm{Sh}_K(\mathbf{G}, \mathcal{X}) \to \mathrm{Sh}_{n_{\mathcal{B}} K n_{\mathcal{B}}}(\mathbf{G}, \mathcal{X}).$$

5.14. **The cycles.** For $g \in \mathbf{G}(\mathbb{A}_f)$, we denote by $\mathcal{Z}_K(g)$ the image of $gK \times \mathcal{Y}^+$ in

$$\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash (\mathbf{G}(\mathbb{A}_f)/K \times \mathcal{X}).$$

If $K' = \mathbf{H}(\mathbb{A}_f) \cap gKg^{-1}$, then $\mathcal{Z}_K(g)$ is also the image of the connected component

$$\mathbf{H}(\mathbb{Q}) \cdot (K' \times \mathcal{Y}^+) \quad \text{of} \quad \mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^+)(\mathbb{C}) = \mathbf{H}(\mathbb{Q}) \backslash \left(\mathbf{H}(\mathbb{A}_f)/K' \times \mathcal{Y}^+\right)$$

under the following finite E-morphism:

$$\mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^+) \to \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, \mathcal{X}) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E) \xrightarrow{[\cdot g]} \mathrm{Sh}_K(\mathbf{G}, \mathcal{X}) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E).$$

Since the geometrical connected components of $\mathrm{Sh}_{K'}(\mathbf{H}, \mathcal{Y}^+)$ are irreducible and defined over $E[\infty]$, we may also view $\mathcal{Z}_K(g)$ as an irreducible closed subset – or an integral subvariety – of codimension n in $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})_{E[\infty]}$. We define

$$\mathcal{Z}_K(\mathbf{H}) = \{\mathcal{Z}_K(g) : g \in \mathbf{G}(\mathbb{A}_f)\}.$$

Proposition 5.1. *The map $g \mapsto \mathcal{Z}_K(g)$ induces a bijection*

$$\mathcal{Z}_K(\bullet) : \mathbf{H}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K \xrightarrow{\simeq} \mathcal{Z}_K(\mathbf{H}).$$

Proof. The only non-trivial assertion is the injectivity of the map. So let g_1 and g_2 be in $\mathbf{G}(\mathbb{A}_f)$ with $\mathcal{Z}_K(g_1) = \mathcal{Z}_K(g_2)$, i.e. $\mathbf{G}(\mathbb{Q})(g_1K, \mathcal{Y}^+) = \mathbf{G}(\mathbb{Q})(g_2K, \mathcal{Y}^+)$. Then

$$\forall y \in \mathcal{Y}^+ : \qquad \exists \gamma \in \mathbf{G}(\mathbb{Q}) \cap g_1 K g_2^{-1} \text{ such that } y \in \gamma\mathcal{Y}^+.$$

Therefore $\mathcal{Y}^+ = \cup_{\gamma \in \Gamma(g_1,g_2)}(\mathcal{Y}^+ \cap \gamma\mathcal{Y}^+)$ with $\Gamma(g_1, g_2) = \mathbf{G}(\mathbb{Q}) \cap g_1 K g_2^{-1}$, which is a countable set. By the Baire category theorem, there is a $\gamma \in \Gamma(g_1, g_2)$ such that $\mathcal{Y}^+ \cap \gamma\mathcal{Y}^+$ contains a non-empty open subspace of $\mathcal{Y}^+$. Since $\mathcal{Y}^+$ and $\gamma\mathcal{Y}^+$ are totally geodesic connected submanifolds of $\mathcal{X}$, it must be that $\mathcal{Y}^+ = \gamma\mathcal{Y}^+$. Thus for any negative $E_\circ$-line D in $(\mathcal{W}_\circ, \psi_\circ)$, the image $\gamma_\circ$ of γ in $\mathbf{G}_\circ(\mathbb{R}) = SO(\mathcal{V}_\circ, \varphi_\circ)$ maps D to a negative $E_\circ$-line γD of $(\mathcal{W}_\circ, \psi_\circ)$, and the map $\gamma : D \to \gamma D$ is $E_\circ$-linear. Since these negative lines span $\mathcal{W}_\circ$, $\gamma_\circ$ preserves $\mathcal{W}_\circ$ and acts $E_\circ$-linearly on it, i.e. $\gamma_\circ \in \mathbf{H}_\circ(\mathbb{R}) = U(\mathcal{W}_\circ, \psi_\circ)$. Thus γ belongs to the intersection $\mathbf{H}(\mathbb{Q})$ of $\mathbf{G}(\mathbb{Q})$ and

$$\mathbf{H}_\circ(\mathbb{R}) \times \mathbf{G}^\circ(\mathbb{R}) \subset \mathbf{G}_\circ(\mathbb{R}) \times \mathbf{G}^\circ(\mathbb{R}) = \mathbf{G}(\mathbb{R})$$

in $\mathbf{G}(\mathbb{R})$. Therefore $g_1 \in \mathbf{H}(\mathbb{Q})g_2K$, as was to be shown. □

5.15. **The Galois action.** Since K is open in $\mathbf{G}(\mathbb{A}_f)$, we also have

$$\mathcal{Z}_K(\bullet) : \overline{\mathbf{H}(\mathbb{Q})}\backslash\mathbf{G}(\mathbb{A}_f)/K \xrightarrow{\simeq} \mathcal{Z}_K(\mathbf{H})$$

where $\overline{\mathbf{H}(\mathbb{Q})}$ is the closure of $\mathbf{H}(\mathbb{Q})$ in $\mathbf{G}(\mathbb{A}_f)$. Since the derived group $\mathbf{H}^1$ of $\mathbf{H}$ is simply connected, $\mathbf{H}^1(\mathbb{Q})$ is dense in $\mathbf{H}^1(\mathbb{A}_f)$ by strong approximation. Since E is a CM extension of F, $\mathbf{T}^1(\mathbb{Q})$ is discrete and thus closed in $\mathbf{T}^1(\mathbb{A}_f)$. It follows that

$$\mathbf{H}(\mathbb{Q}) \cdot \mathbf{H}^1(\mathbb{A}_f) \subset \overline{\mathbf{H}(\mathbb{Q})} \subset (\det)^{-1}\left(\mathbf{T}^1(\mathbb{Q})\right) = \mathbf{H}(\mathbb{Q}) \cdot \mathbf{H}^1(\mathbb{A}_f)$$

i.e. $\overline{\mathbf{H}(\mathbb{Q})} = \mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f)$ in $\mathbf{H}(\mathbb{A}_f)$, and finally

$$\mathcal{Z}_K(\bullet) : \mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/K \xrightarrow{\simeq} \mathcal{Z}_K(\mathbf{H}).$$

Plainly, $\overline{\mathbf{H}(\mathbb{Q})} = \mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f)$ is a normal subgroup of $\mathbf{N}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)$ and the quotient group therefore acts on $\mathcal{Z}_K(\mathbf{H})$ by left multiplication in $\mathbf{G}(\mathbb{A}_f)$. This quotient group is a generalized dihedral extension

$$1 \to \left(\frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})} \simeq \frac{\mathbf{H}(\mathbb{A}_f)}{\mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f)}\right) \to \frac{\mathbf{N}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)}{\mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f)} \to \left(\frac{\mathbf{N}(\mathbb{Q})}{\mathbf{H}(\mathbb{Q})} \simeq \{\pm 1\}\right) \to 1$$

with a natural splitting arising from $\mathbf{N}(\mathbb{Q}) \hookrightarrow \mathbf{N}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)$, giving rise to

$$\det{}^\sharp : \mathbf{N}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(\mathbb{Q})\mathbf{H}^1(\mathbb{A}_f) \xrightarrow{\simeq} \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \rtimes \{\pm 1\}$$

which is isomorphic to the dihedral Galois extension

$$1 \to \mathrm{Gal}(E[\infty]/E) \to \mathrm{Gal}(E[\infty]/F) \to \mathrm{Gal}(E/F) \to 1$$

endowed with the canonical splitting given by complex conjugation,

$$\mathrm{Gal}(E[\infty]/F) \simeq \mathrm{Gal}(E[\infty]/E) \rtimes \{1, c\}.$$

We denote the resulting extension of $\mathrm{Art}^1_E : \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \xrightarrow{\simeq} \mathrm{Gal}(E[\infty]/E)$ by

$$\mathrm{Art}^1_{E/F} : \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \rtimes \{\pm 1\} \xrightarrow{\simeq} \mathrm{Gal}(E[\infty]/F).$$

Proposition 5.2. *For all $g \in \mathbf{G}(\mathbb{A}_f)$, $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$ and $\lambda \in \mathbf{N}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)$ such that $\mathrm{Art}^1_{E/F} \circ \det^\sharp(\lambda) = \sigma|E[\infty]$, we have $\sigma \cdot \mathcal{Z}_K(g) = \mathcal{Z}_K(\lambda g)$.*

Proof. For $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$, this is 5.12. On the other hand $c \cdot \mathcal{Z}_K(g)$ is the image of $gK \times \mathcal{Y}^-$ in $\mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C})$. But for any λ in $\mathbf{N}(\mathbb{Q}) \smallsetminus \mathbf{H}(\mathbb{Q})$, $\mathcal{Y}^- = \lambda\mathcal{Y}^+$, thus

$$\mathbf{G}(\mathbb{Q}) \cdot (gK \times \mathcal{Y}^-) = \mathbf{G}(\mathbb{Q})\lambda \cdot (gK \times \mathcal{Y}^+) = \mathbf{G}(\mathbb{Q}) \cdot (\lambda gK \times \mathcal{Y}^+)$$

and therefore $c \cdot \mathcal{Z}_K(g) = \mathcal{Z}_K(\lambda g)$. This proves the proposition. □

5.16. **Schwartz spaces.** We now define a sequence of smooth left $\mathbf{G}(\mathbb{A}_f)$-modules

$$\hat{\mathcal{S}}(\mathbf{H}) \twoheadrightarrow \mathcal{S}(\mathbf{H}) \hookrightarrow \mathcal{Z}^n(\mathbf{G})$$

and the corresponding sequences of $\mathcal{H}_K$-modules

$$\hat{\mathcal{S}}_K(\mathbf{H}) \twoheadrightarrow \mathcal{S}_K(\mathbf{H}) \hookrightarrow \mathcal{Z}^n_K(\mathbf{G}).$$

We pass from one to the others by applying the covariant functors

$$H^0(K,-) \simeq \mathrm{Hom}_{\mathbb{Z}[\mathbf{G}(\mathbb{A}_f)]}\left(\mathbb{Z}[\mathbf{G}(\mathbb{A}_f)/K],-\right)$$

or taking the colimits over all neat compact open subgroups K of $\mathbf{G}(\mathbb{A}_f)$. Here

$$\mathcal{H}_K \stackrel{\text{def}}{=} \mathrm{End}_{\mathbb{Z}[\mathbf{G}(\mathbb{A}_f)]}\left(\mathbb{Z}[\mathbf{G}(\mathbb{A}_f)/K]\right)$$

is the usual Hecke algebra. We start with the following spaces:

$$\hat{\mathcal{S}}(\mathbf{H}) \stackrel{\text{def}}{=} \mathcal{S}\left(\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)\right) \quad \text{and} \quad \mathcal{S}(\mathbf{H}) \stackrel{\text{def}}{=} \mathcal{S}\left(\overline{\mathbf{H}(\mathbb{Q})}\backslash\mathbf{G}(\mathbb{A}_f)\right)$$

where $\mathcal{S}(X)$ is the Schwartz space of all locally constant and compactly supported $\mathbb{Z}$-valued functions $s : X \to \mathbb{Z}$, and the smooth left action of $\mathbf{G}(\mathbb{A}_f)$ is given by $(g \cdot s)(x) = s(xg)$. Since $\overline{\mathbf{H}(\mathbb{Q})} = \mathbf{H}(\mathbb{Q}) \cdot \mathbf{H}^1(\mathbb{A}_f)$ and $\mathbf{H}(\mathbb{Q})/\mathbf{H}^1(\mathbb{Q}) \simeq \mathbf{T}^1(\mathbb{Q})$ is discrete in $\mathbf{T}^1(\mathbb{A}_f) \simeq \mathbf{H}(\mathbb{A}_f)/\mathbf{H}^1(\mathbb{A}_f)$, the right $\mathbf{G}(\mathbb{A}_f)$-equivariant projection $\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f) \twoheadrightarrow \overline{\mathbf{H}(\mathbb{Q})}\backslash\mathbf{G}(\mathbb{A}_f)$ is also left $\mathbf{T}^1(\mathbb{A}_f)$-equivariant, its fibers are the $\mathbf{T}^1(\mathbb{Q})$-orbits in $\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)$, and the latter are discrete. This projection thus induces a surjective $\mathbf{T}^1(\mathbb{A}_f) \times \mathbf{G}(\mathbb{A}_f)$-equivariant morphism $\hat{\mathcal{S}}(\mathbf{H}) \twoheadrightarrow \mathcal{S}(\mathbf{H})$. The $\mathbf{T}^1(\mathbb{A}_f)$-action on $\mathcal{S}(\mathbf{H})$ is trivial on $\mathbf{T}^1(\mathbb{Q})$, and therefore descends to a smooth left action of Gal_E on $\mathcal{S}(\mathbf{H})$, which extends to a smooth left action of Gal_F as follows:

$$(\sigma \cdot s)(x) = s(h^{-1}x)$$

for $\sigma \in \mathrm{Gal}_F$ and $h \in \mathbf{N}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)$ with $\mathrm{Art}^1_{E/F} \circ \det^\sharp(h) = \sigma|_{E[\infty]}$. In particular,

$$\mathcal{S}_K(\mathbf{H}) = \mathcal{S}\left(\mathbf{H}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K\right)$$

is also equipped with a smooth left action of Gal_F. By propositions 5.1 and 5.2,

$$s \mapsto \mathcal{Z}_K(s) \stackrel{\text{def}}{=} \sum s(g)\mathcal{Z}_K(g)$$

defines an injective Gal_F-equivariant morphism from $\mathcal{S}_K(\mathbf{H})$ to the group

$$\mathcal{Z}^n_K(\mathbf{G}) = \mathcal{Z}^n\left(\mathrm{Sh}_K(\mathbf{G},\mathcal{X}) \times_F \overline{\mathbb{Q}}\right)$$

of cycles of codimension n on $\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}$, i.e. the free abelian group spanned by the irreducible closed subsets of codimension n in $\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}$. For $g \in \mathbf{G}(\mathbb{A}_f)$ and neat compact open subgroups K_1 and K_2 of $\mathbf{G}(\mathbb{A}_f)$ with $g^{-1}K_1 g \subset K_2$, the diagram

$$\begin{array}{ccc} \mathcal{S}_{K_2}(\mathbf{H}) & \longrightarrow & \mathcal{Z}^n_{K_2}(\mathbf{G}) \\ {\scriptstyle g\cdot}\downarrow & & \downarrow{\scriptstyle [\cdot g]^*} \\ \mathcal{S}_{K_1}(\mathbf{H}) & \longrightarrow & \mathcal{Z}^n_{K_1}(\mathbf{G}) \end{array}$$

is commutative, where $[\cdot g]^*$ is the flat pull-back morphism on cycles induced by

$$[\cdot g] : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}.$$

It follows that the collection of Gal_F-equivariant morphisms $\mathcal{S}_K(\mathbf{H}) \hookrightarrow \mathcal{Z}^n_K$ glue to a $\mathbf{G}(\mathbb{A}_f) \times \mathrm{Gal}_F$-equivariant morphism from $\mathcal{S}(\mathbf{H})$ to

$$\mathcal{Z}^n(\mathbf{G}) \overset{\text{def}}{=} \lim_{K} \mathcal{Z}^n_K(\mathbf{G}).$$

Finally, the morphism $\mathcal{Z}^n_K(\mathbf{G}) \to \mathcal{Z}^n(\mathbf{G})$ is injective and identifies $\mathcal{Z}^n_K(\mathbf{G})$ with the K-invariants in $\mathcal{Z}^n(\mathbf{G})$ by étale descent for cycles.

6. The Main theorem

This section needs to be cleaned. I do not really know how to address the non-proper case: when $F = \mathbb{Q}$, I probably need to use Intersection cohomology. Also, it would be nice to pin down a specific Galois representation using automorphic constructions. Finally, some of my assumptions (6.3.9, 6.3.10) are actually conjectures, which I could try to establish here or elsewhere.

6.1. p-adic Abel-Jacobi Extensions. Reference: [29, 48].

6.1.1. Fix a prime number p. Let X be an algebraic variety over some field k with $pk \neq 0$. Let $\overline{X} = X \times_k k^{\mathrm{sep}}$ where k^{sep} is a separable closure of k.

6.1.2. For $i \in \mathbb{N}$, let $(\overline{X}_i, \subset)$ be the ordered set of all closed subschemes $\iota_Z : Z \hookrightarrow \overline{X}$ which are equidimensional of dimension $\dim Z = i$. For each such Z, there is an exact sequence in p-adic étale homology

$$0 \to H_{2i+1}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to H_{2i+1}\left(\overline{X} - Z, \mathbb{Z}_p(i)\right) \to H_{2i}\left(Z, \mathbb{Z}_p(i)\right) \xrightarrow{\iota_{Z,*}} H_{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to \cdots,$$

and a fundamental class $\eta_Z \in H_{2i}\left(Z, \mathbb{Z}_p(i)\right)$ whose image in $H_{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right)$ is the cycle class $\mathrm{cl}(Z) = \iota_{Z,*}(\eta_Z)$. Taking inductive limits over the cofiltered set $(\overline{X}_i, \subset)$, we obtain an exact sequence of $\mathbb{Z}_p[\mathrm{Gal}_k]$-modules

$$0 \to H_{2i+1}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to \varinjlim H_{2i+1}\left(\overline{X} - Z, \mathbb{Z}_p(i)\right) \to \varinjlim H_{2i}\left(Z, \mathbb{Z}_p(i)\right) \xrightarrow{\iota_*} H_{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to \cdots,$$

together with Gal_k-equivariant morphisms $\eta : \mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p \to \varinjlim H_{2i}\left(Z, \mathbb{Z}_p(i)\right)$ and

$$\mathrm{cl} = \iota_* \circ \eta : \qquad \mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p \to H_{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right),$$

where $\mathcal{Z}_i(\star)$ is the group of cycles of dimension i on $\star$. The pull-back of

$$0 \to H_{2i+1}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to \varinjlim H_{2i+1}\left(\overline{X} - Z, \mathbb{Z}_p(i)\right) \to \ker(\iota_*) \to 0$$

along the restriction of η to $\left(\mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p\right)_0 \overset{\text{def}}{=} \ker(\mathrm{cl})$ is an extension

$$0 \to H_{2i+1}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to \star \to \left(\mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p\right)_0 \to 0$$

of continuous Gal_k-modules. For each $k \subset k' \subset k^{\mathrm{sep}}$, we have a cartesian diagram

$$\begin{array}{ccccc}
\left(\mathcal{Z}_i(X \times_k k') \otimes \mathbb{Z}_p\right)_0 & \xrightarrow{\simeq} & H^0\left(k', \left(\mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p\right)_0\right) & \hookrightarrow & \left(\mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p\right)_0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}_i(X \times_k k') \otimes \mathbb{Z}_p & \xrightarrow{\simeq} & H^0\left(k', \mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p\right) & \hookrightarrow & \mathcal{Z}_i(\overline{X}) \otimes \mathbb{Z}_p
\end{array}$$

The p-adic Abel-Jacobi map is the connecting homomorphism

$$\mathrm{AJ}_p : \left(\mathcal{Z}_i(X \times_k k') \otimes \mathbb{Z}_p\right)_0 \to H^1\left(k', H_{2i+1}\left(\overline{X}, \mathbb{Z}_p(i)\right)\right).$$

All these constructions behave bi-variantly with respect to finite flat morphisms.

6.1.3. Suppose from now on that X is smooth and equidimensional of dimension d. Using the duality isomorphisms $H_i(Z, \mathbb{Z}_p(j)) \simeq H_Z^{2d-i}(X, \mathbb{Z}_p(d-j))$, we may replace the étale homology groups by the (continuous) étale cohomology groups throughout the above discussion. We obtain the p-adic Abel-Jacobi extension

$$0 \to H^{2i-1}\left(\overline{X}, \mathbb{Z}_p(i)\right) \to \star \to \left(\mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p\right)_0 \to 0$$

where $\mathcal{Z}^i(\overline{X}) = \mathcal{Z}_{d-i}(\overline{X})$ is the group of cycles of codimension i in X and

$$\left(\mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p\right)_0 \stackrel{\text{def}}{=} \ker\left(\text{cl} : \mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p \to H^{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right)\right)$$

is the kernel of the cycle class map in p-adic étale cohomology. For $k \subset k' \subset k^{\text{sep}}$, we now have a cartesian diagram

$$\begin{array}{ccccc}
\left(\mathcal{Z}^i(X \times_k k') \otimes \mathbb{Z}_p\right)_0 & \xrightarrow{\simeq} & H^0\left(k', \left(\mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p\right)_0\right) & \hookrightarrow & \left(\mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p\right)_0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}^i(X \times_k k') \otimes \mathbb{Z}_p & \xrightarrow{\simeq} & H^0\left(k', \mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p\right) & \hookrightarrow & \mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p
\end{array}$$

and the corresponding p-adic Abel-Jacobi map

$$\text{AJ}_p : \left(\mathcal{Z}^i(X \times_k k') \otimes \mathbb{Z}_p\right)_0 \to H^1\left(k', H^{2i-1}\left(\overline{X}, \mathbb{Z}_p(i)\right)\right)$$

Again, all these constructions behave bi-variantly with respect to finite morphisms of smooth schemes (which are automatically flat by the miracle flatness theorem).

6.1.4. In this smooth and equidimensional case, the p-adic Abel-Jacobi map may also be retrieved from the Hochschild-Serre (or Leray) spectral sequence

$$E_2^{i,j} = H^i\left(k', H^j\left(\overline{X}, \mathbb{Z}_p(n)\right)\right) \Longrightarrow H^{i+j}\left(X \times_k k', \mathbb{Z}_p(n)\right)$$

which degenerates at E_2 and gives a morphism

$$\ker\left(H^{2i}\left(X \times_k k', \mathbb{Z}_p(i)\right) \to H^{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right)\right) \to H^1\left(k', H^{2i-1}\left(\overline{X}, \mathbb{Z}_p(n)\right)\right).$$

Then, one just needs to use the commutativity of the cycles maps

$$\begin{array}{ccc}
\mathcal{Z}^i(X \times_k k') \otimes \mathbb{Z}_p & \hookrightarrow & \mathcal{Z}^i(\overline{X}) \otimes \mathbb{Z}_p \\
\text{cl}\downarrow & & \downarrow\text{cl} \\
H^{2i}\left(X \times_k k', \mathbb{Z}_p(i)\right) & \longrightarrow & H^{2i}\left(\overline{X}, \mathbb{Z}_p(i)\right)
\end{array}$$

to obtain the p-adic Abel-Jacobi map

$$\text{AJ}_p : \left(\mathcal{Z}^i(X \times_k k') \otimes \mathbb{Z}_p\right)_0 \to H^1\left(k', H^{2i-1}\left(\overline{X}, \mathbb{Z}_p(i)\right)\right).$$

6.1.5. Suppose that k is a finite extension of $\mathbb{Q}_\ell$ for some prime number ℓ. Set

$$H^{2i-1}(i) \stackrel{\text{def}}{=} H^{2i-1}\left(\overline{X}, \mathbb{Q}_p(i)\right) = H^{2i-1}\left(\overline{X}, \mathbb{Z}_p(i)\right) \otimes \mathbb{Q}_p.$$

It is known that the $\mathbb{Q}_p$-linear p-adic Abel-Jacobi map

$$\text{AJ}_p \otimes \mathbb{Q}_p : \left(\mathcal{Z}^i(X \times_k k') \otimes \mathbb{Q}_p\right)_0 \to H^1\left(k', H^{2i-1}(i)\right)$$

lands into the Bloch-Kato subspace (1.2.9), defined by

$$H^1_f\left(k', H^{2i-1}(i)\right) \stackrel{\text{def}}{=} \begin{cases} \ker\left(H^1\left(k', H^{2i-1}(i)\right) \to H^1\left(k'^{\text{ur}}, H^{2i-1}(i)\right)\right) & \ell \neq p, \\ \ker\left(H^1\left(k', H^{2i-1}(i)\right) \to H^1\left(k', H^{2i-1}(i) \otimes B_{\text{crys}}\right)\right) & \ell = p, \end{cases}$$

in the following cases: X is proper and smooth over k, and either X has potentially good reduction or the suitable version of the purity conjecture for the monodromy filtration on $H^{2i-1}(i)$ holds, see [48]. Then also

$$\mathrm{AJ}_p\left(\left(\mathscr{Z}^i(X\times_k k')\otimes\mathbb{Z}_p\right)_0\right)\subset H^1_f\left(k',H^{2i-1}\left(\overline{X},\mathbb{Z}_p(i)\right)\right)$$

where the right-hand side is defined by the cartesian diagram

$$\begin{array}{ccc} H^1_f\left(k',H^{2i-1}\left(\overline{X},\mathbb{Z}_p(i)\right)\right) & \hookrightarrow & H^1\left(k',H^{2i-1}\left(\overline{X},\mathbb{Z}_p(i)\right)\right) \\ \downarrow & & \downarrow \\ H^1_f\left(k',H^{2i-1}\left(\overline{X},\mathbb{Q}_p(i)\right)\right) & \hookrightarrow & H^1\left(k',H^{2i-1}\left(\overline{X},\mathbb{Q}_p(i)\right)\right). \end{array}$$

Remark 6.1. When $\ell\neq p$, the above assumptions on X actually imply that

$$H^1_f\left(k',H^{2i-1}\left(\overline{X},\mathbb{Q}_p(i)\right)\right)=H^1\left(k',H^{2i-1}\left(\overline{X},\mathbb{Q}_p(i)\right)\right)=0.$$

See [48, Proposition 2.5]. It then follows that

$$H^1_f\left(k',H^{2i-1}\left(\overline{X},\mathbb{Z}_p(i)\right)\right)=H^1\left(k',H^{2i-1}\left(\overline{X},\mathbb{Z}_p(i)\right)\right),$$

and this is a torsion $\mathbb{Z}_p$-module.

6.1.6. Suppose that $\sigma: k^{\mathrm{sep}}\hookrightarrow\mathbb{C}$ and X is smooth. Then the diagram

$$\begin{array}{ccccc} \mathscr{Z}^i(X\times_k\mathbb{C}) & = & \mathscr{Z}^i(X\times_k\mathbb{C}) & \hookleftarrow & \mathscr{Z}^i(X\times_k k') \\ {\scriptstyle \mathrm{cl}\otimes 1}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^{2i}_B\left(X(\mathbb{C}),\mathbb{Z}(i)\right)\otimes\mathbb{Z}_p & \xrightarrow{\simeq} & H^{2i}\left(X\times_k\mathbb{C},\mathbb{Z}_p(i)\right) & \xleftarrow{\simeq} & H^{2i}\left(\overline{X},\mathbb{Z}_p(i)\right) \end{array}$$

shows that $\left(\mathscr{Z}^i(X\times_k k')\otimes\mathbb{Z}_p\right)_0=\mathscr{Z}^i_0(X\times_k k')\otimes\mathbb{Z}_p$ where

$$\mathscr{Z}^i_0(X\times_k k')\overset{\mathrm{def}}{=}\ker\left(\mathscr{Z}^i(X\times_k k')\xrightarrow{\mathrm{cl}}H^{2i}_B\left(X(\mathbb{C}),\mathbb{Z}(i)\right)\right)$$

is the kernel of the class map in Betti cohomology.

6.1.7. Suppose that S is the spectrum of an henselian discrete valuation ring, with generic point t and special point s of residue characteristic $\neq p$. Fix an algebraic closure of $k(t)$, giving rise to an algebraic closure of $k(s)$, corresponding to geometric points $\bar{t}$ above t and $\bar{s}$ above s. For a subextension $k(t')$ of $k(\bar{t})/k(t)$, we denote by $k(s')$ the residue field of the normal closure of $k(t)$ in $k(t')$. Let $\mathscr{X}\to S$ be a proper and smooth morphism. For $\star\in\{s,t,\bar{s},\bar{t},s',t'\}$, we denote by $\mathscr{X}_\star\to\star$ the corresponding fiber, $\mathscr{X}_\star=\mathscr{X}\times_S\mathrm{Spec}(k(\star))$. There is a commutative diagram

$$\begin{array}{ccccc} \mathscr{Z}^i(\mathscr{X}_{t'})\otimes\mathbb{Z}_p & \xrightarrow{\mathrm{cl}} & H^{2i}\left(\mathscr{X}_{t'},\mathbb{Z}_p(i)\right) & \longrightarrow & H^{2i}\left(\mathscr{X}_{\bar{t}},\mathbb{Z}_p(i)\right) \\ {\scriptstyle \mathrm{red}}\downarrow & & \uparrow & & \uparrow\simeq \\ \mathscr{Z}^i(\mathscr{X}_{s'})\otimes\mathbb{Z}_p & \xrightarrow{\mathrm{cl}} & H^{2i}\left(\mathscr{X}_{s'},\mathbb{Z}_p(i)\right) & \longrightarrow & H^{2i}\left(\mathscr{X}_{\bar{s}},\mathbb{Z}_p(i)\right) \end{array}$$

described in [1, X App 7.13-15], where the first vertical arrow is the specialization map of *loc. cit*, see also [21, 20.3]. It restricts to a morphism

$$\mathrm{red}:\left(\mathscr{Z}^i(\mathscr{X}_{t'})\otimes\mathbb{Z}_p\right)_0\to\left(\mathscr{Z}^i(\mathscr{X}_{s'})\otimes\mathbb{Z}_p\right)_0$$

which in turns fits into a commutative diagram of Abel-Jacobi maps

$$\begin{array}{ccc} \left(\mathscr{Z}^i(\mathscr{X}_{t'})\otimes\mathbb{Z}_p\right)_0 & \xrightarrow{\mathrm{AJ}_p} & H^1\left(k(t'),H^{2i-1}\left(\mathscr{X}_{t'},\mathbb{Z}_p(i)\right)\right) \\ {\scriptstyle \mathrm{red}}\downarrow & & \uparrow \\ \left(\mathscr{Z}^i(\mathscr{X}_{s'})\otimes\mathbb{Z}_p\right)_0 & \xrightarrow{\mathrm{AJ}_p} & H^1\left(k(s'),H^{2i-1}\left(\mathscr{X}_{s'},\mathbb{Z}_p(i)\right)\right) \end{array}$$

The commutativity of this diagram follows from the compatibility of the specialization map with the Leray spectral sequence. Note that this commutativity proves the case where $\ell \neq p$ and X has good reduction in 6.1.5.

6.2. Derived distributions.

6.2.1. Let K be a neat compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Applying 6.1 to our smooth variety $X = \mathrm{Sh}_K = \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ over $k = F$ with $i = n$, we obtain our p-adic Abel-Jacobi extension, an exact sequence of $\mathcal{H}_K \otimes \mathbb{Z}_p[\mathrm{Gal}_F]$-modules

$$0 \to H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right) \to \star \to \mathcal{Z}^n_{K,0} \otimes \mathbb{Z}_p \to 0. \tag{6.1}$$

Here $\mathcal{Z}^n_{K,0}$ is defined within the following $\mathcal{H}_K$-equivariant cartesian diagram

$$\begin{array}{ccccccc}
\mathcal{S}_{K,0} & \hookrightarrow & \mathcal{Z}^n_{K,0} = \mathcal{Z}^n_0(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}) & \hookrightarrow & \mathcal{Z}^n_0\left(\mathrm{Sh}_K \times_F \mathbb{C}\right) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{S}_K & \hookrightarrow & \mathcal{Z}^n_K = \mathcal{Z}^n(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}) & \hookrightarrow & \mathcal{Z}^n\left(\mathrm{Sh}_K \times_F \mathbb{C}\right) & \xrightarrow{\mathrm{cl}} & H^{2n}_B\left(\mathrm{Sh}_K(\mathbb{C}), \mathbb{Z}(n)\right).
\end{array}$$

Note that we have simplified our notations: $\mathcal{Z}^n_K = \mathcal{Z}^n_K(\mathbf{G})$, $\mathcal{S}_K = \mathcal{S}_K(\mathbf{H})$.

6.2.2. We denote by $\mathfrak{e}$ an element of the annihilator of the $\mathcal{H}_K$-submodule

$$\mathcal{S}_K/\mathcal{S}_0 \hookrightarrow \mathcal{Z}^n_K/\mathcal{Z}^n_{K,0} \hookrightarrow H^{2n}_B\left(\mathrm{Sh}_K(\mathbb{C}), \mathbb{Z}(n)\right)$$

in the center of $\mathcal{H}_K$. Multiplication by $\mathfrak{e}$ thus defines an $\mathcal{H}_K[\mathrm{Gal}_F]$-linear morphism

$$\mathfrak{e}\mathcal{Z} : \mathcal{S}_K \to \mathcal{Z}^n_{K,0}.$$

Pulling back our p-adic Abel-Jacobi extension (6.1) through $\mathfrak{e}\mathcal{Z} \otimes \mathbb{Z}_p$, we obtain a derived p-adic distribution, namely an exact sequence of $\mathcal{H}_K \otimes \mathbb{Z}_p[\mathrm{Gal}_F]$-modules

$$0 \to H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right) \to \star \to \mathcal{S}_K \otimes \mathbb{Z}_p \to 0. \tag{6.2}$$

Remark 6.2. Our favorite choice of $\mathfrak{e}$ would be an integral multiple of the odd projector defined by Morel and Suh in [47], provided that the latter does exist.

6.2.3. We denote by $\mathscr{H}$ a subring of $\mathcal{H}_K$. We also fix a finite extension Φ of $\mathbb{Q}_p$ with ring of integers R, and a quotient V of the $\mathscr{H} \otimes \Phi[\mathrm{Gal}_F]$-module

$$H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \Phi(n)\right) = H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right) \otimes \Phi.$$

We denote by T the $\mathscr{H}[\mathrm{Gal}_F]$-stable R-lattice in V which is the image of

$$H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, R(n)\right) = H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right) \otimes R.$$

Pushing-out (6.2)$\otimes R$ along the $\mathscr{H} \otimes R[\mathrm{Gal}_F]$-equivariant projection

$$H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right) \otimes R \twoheadrightarrow T$$

we obtain an exact sequence of $\mathscr{H} \otimes R[\mathrm{Gal}_F]$-modules

$$0 \to T \to \star \to \mathcal{S}_K \otimes R \to 0. \tag{6.3}$$

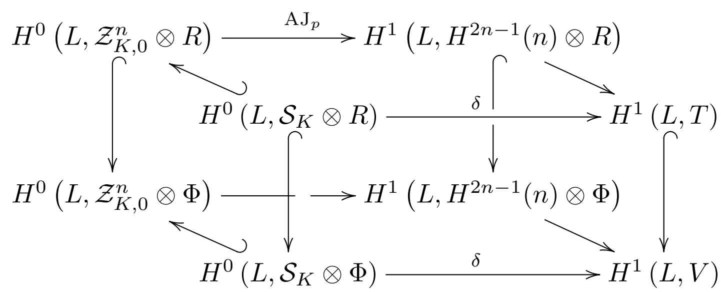
Pushing-out (6.2)$\otimes\Phi$ along the $\mathscr{H} \otimes \Phi[\mathrm{Gal}_F]$-equivariant projection

$$H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right) \otimes \Phi \twoheadrightarrow V$$

we obtain (6.3)$\otimes\Phi$, which is an exact sequence of $\mathscr{H} \otimes \Phi[\mathrm{Gal}_F]$-modules

$$0 \to V \to \star \to \mathcal{S}_K \otimes \Phi \to 0. \tag{6.4}$$

6.2.4. For any subextension L of $E[\infty]/E$, we obtain a commutative diagram of $\mathscr{H} \otimes R[\mathrm{Gal}(L/F)]$ and $\mathscr{H} \otimes \Phi[\mathrm{Gal}(L/F)]$-equivariant connecting homomorphisms



where $H^{2n-1}(n) = H^{2n-1}\left(\mathrm{Sh}_K \times_F \overline{\mathbb{Q}}, \mathbb{Z}_p(n)\right)$.

6.3. **The main theorem and its assumptions.**

6.3.1. We first require that for finite subextensions L of $E[\infty]/E$, the image of the above Abel-Jacobi map is contained in the Bloch-Kato Selmer group:

$$\delta\left(H^0\left(L, \mathcal{S}_K \otimes \Phi\right)\right) \subset H^1_f(L, V) \subset H^1(L, V).$$

The commutativity of the front face of the previous diagram shows that then also

$$\delta\left(H^0\left(L, \mathcal{S}_K \otimes R\right)\right) \subset H^1_f(L, T) \subset H^1(L, T).$$

Remark 6.3. At least when Sh_K is proper (e.g. when $F \neq \mathbb{Q}$), this should follow from the conjectures mentioned in 6.1.5, by the commutativity of the lower face of the previous diagram and the functoriality of the Bloch-Kato local conditions. In particular, the required local conditions are satisfied at all places v of L where Sh_K has good reduction (with $v \nmid p$ or $v \mid p$). At places $v \nmid p$ of L, our assumption would also follow from suitable properties of the $\Phi[\mathrm{Gal}_L]$-representation V alone. For instance if V is pure of weight -1, then $H^1_f(L_v, V) = H^1(L_v, V) = 0$ as in [48, 2.5]. In particular, our assumption holds for all L when $F \neq \mathbb{Q}$, V is pure of weight -1, and our Shimura variety Sh_K has good reduction at all places $v \mid p$ of F.

6.3.2. We can now state our main theorem.

Theorem 6.4. *Let L be a finite subextension of $E[\infty]/E$, let $\chi : \mathrm{Gal}(L/E) \to R^\times$ be a character. Under the assumptions listed in 6.3.1, 6.3.3- 6.3.4 and 6.3.9-6.3.10, suppose that the χ-component of $\delta(z) \in H^1_f(L, V)$ is non-zero for some element $z \in H^0(L, \mathcal{S}_K \otimes \Phi)$. Then the χ-component of $H^1_f(L, V)$ has dimension one:*

$$\delta(z)(\chi) \neq 0 \quad \Longrightarrow \quad H^1_f(L, V)(\chi) = \Phi \cdot \delta(z)(\chi) \simeq \Phi.$$

We list and comment our assumptions in the next subsections. We will deduce Theorem 6.4 from Theorem 4.1, which means that we have to import here the assumptions required in the Kolyvagin system argument. They are listed in 6.3.3-6.3.4. The remaining assumptions 6.3.9-6.3.10 are unrelated to the Kolyvagin system argument, but crucial for the construction of an Euler system extending $\delta(z)$.

6.3.3. We require that our continuous Galois representation

$$\rho : \mathrm{Gal}_F \to \mathrm{Aut}_\Phi(V)$$

satisfies the assumptions of 2.1.4. Since we already know (by construction) that V is geometric, i.e. unramified at all but finitely many places v of F and de Rham at all $v \mid p$, this boils down to the existence of a perfect ε-symmetric pairing $\langle -,- \rangle : V \times V \to \Phi(1)$ (for some $\varepsilon \in \{\pm 1\}$) such that $\langle T, T \rangle \subset R(1)$.

Remark 6.5. At least when $X = \mathrm{Sh}_K$ is proper (for instance when $F \neq \mathbb{Q}$), this assumption is innocuous and $\varepsilon = -1$: since X is proper, smooth and equidimensional of dimension $2n-1$, Poincaré duality gives a symplectic perfect pairing

$$\langle -,- \rangle : H^{2n-1}\left(\overline{X}, \mathbb{Z}_p(n)\right) \times H^{2n-1}\left(\overline{X}, \mathbb{Z}_p(n)\right) \to H^{2(2n-1)}\left(\overline{X}, \mathbb{Z}_p(2n)\right) \xrightarrow{\mathrm{Tr}} \mathbb{Z}_p(1).$$

Our subring $\mathscr{H}$ of $\mathcal{H}_K$ – to be specified later – should act by self-adjoint operators for this pairing. We may then take V to be any non-degenerate $\mathscr{H} \otimes \Phi[\mathrm{Gal}_F]$-stable subspace of $H^{2n-1}\left(\overline{X}, \mathbb{Z}_p(n)\right) \otimes \Phi$, equipped with the symplectic perfect pairing induced by $\langle -,- \rangle \otimes \Phi$, realized as the quotient of $H^{2n-1}\left(\overline{X}, \mathbb{Z}_p(n)\right) \otimes \Phi = V \oplus V^\perp$ by the $\mathscr{H} \otimes \Phi[\mathrm{Gal}_F]$-stable orthogonal complement $V^\perp$ of V.

6.3.4. We next require the "big image assumptions" of section 2.2 and 3.1.1, i.e.:

- $\mathbf{BI}_0$ There is a $\gamma \in \mathrm{Gal}_F$ with $\gamma = \tau$ on $E(\mu_{p^\infty})$ and $\dim_\Phi V^{\gamma-1} = 1$.
- $\mathbf{BI}_1$ The image of $\rho : \mathrm{Gal}_{H'} \to \mathrm{Aut}_\Phi(V)$ contains a non-trivial homothety.
- $\mathbf{BI}_2$ The representation V of $\mathrm{Gal}_{H'}$ is absolutely irreducible.

Here H' is the fixed field of $\ker(\chi) \subset \mathrm{Gal}(L/E)$, where $\chi : \mathrm{Gal}(L/E) \to R^\times$ is the target character in Theorem 6.4. Note that $\mathbf{BI}_2$ *a fortiori* holds for the representation V of Gal_F. It implies $\mathrm{End}_{\Phi[\mathrm{Gal}_F]}(V) = \Phi$ and $\mathrm{End}_{R[\mathrm{Gal}_F]}(T) = R$. It follows that the action of $\mathscr{H}$ on T or V is given by a ring homomorphism $\mathscr{H} \to R$.

Remark 6.6. There are many results pertaining to analogous "big image" properties for Galois representations, from the pioneering work of Serre in [58] to the more recent findings in [60, 3.2.4] or [2, 3], which are particularly relevant here.

6.3.5. For each prime number ℓ such that $K = K^\ell K_\ell$ for some hyperspecial compact open subgroup K_ℓ of $\mathbf{G}(\mathbb{Q}_\ell)$ and for each place $v \mid \ell$ of the reflex field $F = E(\mathbf{G}, \mathcal{X})$, Blasius and Rogawski [6] defined a Hecke polynomial $P_v(t)$ with coefficients in the local spherical Hecke algebra $\mathcal{H}_\ell = \mathcal{H}(\mathbf{G}(\mathbb{Q}_\ell)//K_\ell)$, and they conjectured that: (1) the Shimura variety $X = \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ has good reduction at v (in some sense); and (2) $P_v(\mathrm{Fr}_v) \equiv 0$ on the (intersection) cohomology of X, where Fr_v is the (geometric, as usual) Frobenius.

6.3.6. In our case, $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}} \underline{\mathbf{G}}$ for $\underline{\mathbf{G}} = SO(V, \varphi)$ over F, $\mathbf{G}(\mathbb{Q}_\ell) = \prod_{v|\ell} \underline{\mathbf{G}}(F_v)$ and $K_\ell \subset \mathbf{G}(\mathbb{Q}_\ell)$ is hyperspecial if and only if $K_\ell = \prod_{v|\ell} K_v$ with K_v hyperspecial in $\underline{\mathbf{G}}(F_v)$. We then have $\mathcal{H}_\ell = \otimes_{v|\ell} \mathcal{H}_v$ where $\mathcal{H}_v = \mathcal{H}(\underline{\mathbf{G}}(F_v)//K_v)$, and the Hecke polynomial $P_v(t)$ belongs to $\mathcal{H}_v[t]$ – see section 10.2, where these polynomials are defined and partially computed. This suggests that a slightly more general conjecture could be true: for any finite place v of the reflex field $F = E(\mathbf{G}, \mathcal{X})$ such that $K = K^v K_v$ in $\mathbf{G}(\mathbb{A}_f) = \underline{\mathbf{G}}(\mathbb{A}_{F,f}) = \underline{\mathbf{G}}(\mathbb{A}^v_{F,f}) \times \underline{\mathbf{G}}(F_v)$ for some hyperspecial compact open subgroup K_v of $\underline{\mathbf{G}}(F_v)$, (1′) the Shimura variety X has good reduction at v and (2′) $P_v(\mathrm{Fr}_v) \equiv 0$ on the (intersection) cohomology of X.

6.3.7. The first part of the Blasius-Rogawski conjecture has been established for all Shimura varieties of abelian type by Kisin [31, 32], and this includes our Shimura variety $X = \mathrm{Sh}(\mathbf{G}, \mathcal{X})$. The second part of the Blasius-Rogawski conjecture has been established for many PEL Shimura varieties [20, 14, 34], and then also for a few non-PEL ones [13, 50], but to the best of our knowledge, it is not yet known in our case when $n > 1$. The refined conjecture should not be much harder.

On the other hand in most of the known cases, the relation $P_v(\mathrm{Fr}_v) = 0$ is first established in a suitable *ad-hoc* ring of correspondences *modulo rien du tout*[3] on the special fibers — and it thus *a fortiori* holds in the actual quotient of that ring that (presumably) acts on the cohomology, as predicted by the Blasius-Rogawski conjecture (this last step being usually left to the reader).

6.3.8. For all but finitely many primes ℓ of F, we have the following properties: (1) $K = K^\ell K_\ell$ with K_ℓ hyperspecial in $\underline{\mathbf{G}}(F_\ell)$, (2) X has good reduction at ℓ, and (3) V is unramified at ℓ. Let U be a sufficiently small open subset of $\mathrm{Spec}(\mathcal{O}_F)$ where these properties hold and let $\mathscr{X}$ be the smooth model of X over U given by (2), i.e. constructed by Kisin. For the subring $\mathscr{H}$ of $\mathcal{H}_K$, we only require that it contains the local spherical algebra $\mathcal{H}_\ell$, for every prime $\ell \in U$ which is inert in E/F. For instance, we could take $\mathscr{H} = \mathcal{H}_K$, or $\mathscr{H} = \otimes'_\ell \mathcal{H}_\ell$, the commutative restricted tensor product of the $\mathcal{H}_\ell$'s for every prime $\ell \in U$ which is inert in E/F.

6.3.9. For each prime $\ell \in U$ which is inert in E/F, we require that the characteristic polynomial of Fr_ℓ acting on the Φ-vector space V is equal to the image of $P_\ell(t) \in \mathcal{H}_\ell[t]$ in $R[t] \subset \Phi[t]$, under the morphism $\mathcal{H}_\ell \to R$ giving the action of $\mathcal{H}_\ell$ on T and V. This implies that $P_\ell(\mathrm{Fr}_\ell) = 0$ on V, as predicted by the Blasius-Rogawski conjecture. But this also fixes the Φ-dimension of V: $\dim_\Phi V = \deg P_\ell = 2n$. Moreover, we will see that $P_\ell(\pm 1) \neq 0$, thus $\mathrm{Fr}_\ell^2 - 1$ is an isomorphism on V.

Remark 6.7. According to standard conjectures (Langlands, Kottwitz, Arthur...) and at least when $X = \mathrm{Sh}_K$ is proper (for instance when $F \neq \mathbb{Q}$), every single Jordan-Holder factor of $H^{2n-1}(\overline{X}, \Phi(n))$ should be symplectic, pure of weight -1, and of Φ-dimension $2n$ for any sufficiently large coefficient field Φ.

6.3.10. For ℓ as above, pick a place of $\overline{\mathbb{Q}}$ above ℓ, giving rise to geometric generic and special points $\bar{t}$ and $\bar{s}$ of U at ℓ, and to a geometric reduction map

$$\mathrm{red} : \mathcal{Z}^n\left(\mathscr{X}_{\bar{t}}\right) \to \mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}\right)$$

between the groups of cycles of codimension n on the geometric generic and special fibers of $\mathscr{X}$ at ℓ, as explained in [1, X App 7.13-16] or [21, §20.3] (cf. also 6.1.7). We finally require that this reduction map is trivial on the submodule

$$\mathfrak{e}\mathcal{Z}\left(P_\ell(\mathrm{Fr}_\ell) \cdot \mathcal{S}_K^I\right) \subset \mathfrak{e}\mathcal{Z}\left(\mathcal{S}_K\right) \subset \mathcal{Z}_{K,0}^n \subset \mathcal{Z}_K^n \subset \mathcal{Z}^n\left(\mathscr{X}_{\bar{t}}\right)$$

where I is the inertia group. This assumption does not depend upon the chosen extension of ℓ, and true when n equals 1. It is crucial for our purposes to require that the above vanishing occurs in the torsion-free group $\mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}\right)$.

Remark 6.8. We conjecture that this vanishing assumption is always true. More generally, note that the above reduction map factors through the coinvariants of I in $\mathcal{Z}^n(\mathscr{X}_{\bar{t}})$, and we may conjecture that the induced morphism

$$\mathrm{red} : \mathcal{Z}^n\left(\mathscr{X}_{\bar{t}}\right)_I \to \mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}\right)$$

[3] [1, X App 7.13.11].

vanishes on the image of $P_\ell(\mathrm{Fr}_\ell)$ acting on $\mathcal{Z}^n\left(\mathscr{X}_{\bar{t}}\right)_I$. Now let $\mathscr{X}_{\bar{s}}^\circ$ be the μ-ordinary (open) locus in $\mathscr{X}_{\bar{s}}$ as defined in [59], and consider the μ-ordinary reduction map

$$\mathrm{red}^\circ : \mathcal{Z}^n\left(\mathscr{X}_{\bar{t}}\right)_I \to \mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}\right) \twoheadrightarrow \mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}^\circ\right)$$

obtained by composing red with the restriction map from $\mathscr{X}_{\bar{s}}$ to $\mathscr{X}_{\bar{s}}^\circ$. It seems likely that the strategies developed thus far to attack the Blasius-Rogawski conjecture [20, 14, 34] could yield the following variant: the μ-ordinary reduction map red° vanishes on the image of $P_\ell(\mathrm{Fr}_\ell)$. When n is even, we should also have

$$\mathrm{red}\left(\mathcal{Z}\left(\mathcal{S}_K(\mathbf{H})\right)\right) \cap \ker\left(\mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}\right) \twoheadrightarrow \mathcal{Z}^n\left(\mathscr{X}_{\bar{s}}^\circ\right)\right) = 0$$

in which case the reduction map red would already vanish on $P_\ell(\mathrm{Fr}_\ell)\mathcal{Z}\left(\mathcal{S}_K(\mathbf{H})\right)_I$. When n is odd, the last displayed equation is false (already for $n = 1$) and a proof of our conjecture would require a more detailed analysis along the lines of [34].

7. The Kolyvagin system

In this section, we prove Theorem 6.4 as follows. Plainly, we may assume that (1) $z \in \mathcal{S}_K(\mathbf{G}) \otimes R$ (as opposed to $\mathcal{S}_K(\mathbf{G}) \otimes \Phi$) and (2) L is the field of definition $E(z)$ of z (as opposed to $E(z) \subset L$). We will then construct a finite extension H of L in $E[\infty]$, such that a non-zero multiple of $\delta(z) \in H^1_f(H, T)$ extends to a strong Kolyvagin system $\kappa = (\kappa(\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}}$. Theorem 6.4 then follows from Theorem 4.1.

7.1. **The Euler system.**

7.1.1. *Adeles.* For a finite set S of finite places of F, we write

$$\mathbb{A}_{F,f} \overset{\mathrm{def}}{=} \mathbb{A}_f \otimes F = F_S \times \mathbb{A}_{F,f}^S \quad \text{with} \quad F_S \overset{\mathrm{def}}{=} \prod_{\mathfrak{q}\in S} F_{\mathfrak{q}} \quad \text{and} \quad \mathbb{A}_{F,f}^S \overset{\mathrm{def}}{=} \prod_{\mathfrak{p}\notin S}{}' F_{\mathfrak{p}}.$$

7.1.2. *Groups over F.* Set $\underline{\mathbf{G}} = \mathbf{SO}(\mathcal{V}, \varphi)$, $\underline{\mathbf{H}} = \mathbf{U}(\mathcal{W}, \psi)$, $\underline{\mathbf{T}} = \mathrm{Res}_{E/F}\mathbb{G}_{m,E}$, let $\underline{\mathbf{T}}^1 \subset \underline{\mathbf{T}}$ be the kernel of the norm map $N : \underline{\mathbf{T}} \to \mathbb{G}_{m,F}$, let $\underline{\mathbf{H}}^1 \subset \underline{\mathbf{H}}$ be the kernel of the determinant map $\det : \underline{\mathbf{H}} \to \underline{\mathbf{T}}^1$, and define $\nu : \underline{\mathbf{T}} \twoheadrightarrow \underline{\mathbf{T}}^1$ by $z \mapsto z/\bar{z}$. For any $\mathbf{X} \in \{\mathbf{G}, \mathbf{H}, \mathbf{H}^1, \mathbf{T}, \mathbf{T}^1\}$, we thus have $\mathbf{X} = \mathrm{Res}_{F/\mathbb{Q}}(\underline{\mathbf{X}})$ and

$$\mathbf{X}(\mathbb{A}_f) = \underline{\mathbf{X}}(\mathbb{A}_{F,f}) = \underline{\mathbf{X}}(F_S) \times \underline{\mathbf{X}}(\mathbb{A}_{F,f}^S) \quad \text{with} \quad \underline{\mathbf{X}}(\mathbb{A}_{F,f}^S) = \prod_{\mathfrak{p}\notin S}{}' \underline{\mathbf{X}}(F_{\mathfrak{p}}).$$

7.1.3. *Bad places.* Fix a neat compact open subgroup K of $\mathbf{G}(\mathbb{A}_f)$ and some element z of $\mathcal{S}_K(\mathbf{H}) \otimes R$. We may then also find (and fix) a finite set $S = S(K, z)$ of finite places of F with the following properties:

1. $K = K_S \times K^S$ where K_S is a compact open subgroup of $\underline{\mathbf{G}}(F_S)$, K^S is a compact open subgroup of $\underline{\mathbf{G}}(\mathbb{A}_{F,f}^S)$ and $K^S = \prod_{\mathfrak{p}\notin S} K_{\mathfrak{p}}$ where $K_{\mathfrak{p}}$ is hyperspecial in $\underline{\mathbf{G}}(F_{\mathfrak{p}})$ and $K_{\mathfrak{p}} \cap \underline{\mathbf{H}}(F_{\mathfrak{p}})$ is hyperspecial in $\underline{\mathbf{H}}(F_{\mathfrak{p}})$.
2. $z = \sum a_i \mathcal{Z}_K(g_i)$ with $a_i \in R$, $g_i = (g_{i,S}, 1)$ in $\underline{\mathbf{G}}(\mathbb{A}_{F,f}) = \underline{\mathbf{G}}(F_S) \times \underline{\mathbf{G}}(\mathbb{A}_{F,f}^S)$.
3. S contains any other bad places coming from 6.3.9 or dividing $2p$.

7.1.4. Let $\mathcal{O}_F^S$ be the ring of S-units in F, so that $\mathrm{Spec}(\mathcal{O}_F^S) = \mathrm{Spec}(\mathcal{O}_F) \smallsetminus S$. Then all of the above reductive groups over F extend to reductive groups over $\mathrm{Spec}(\mathcal{O}_F^S)$, with $\underline{\mathbf{G}}(\mathcal{O}_{F,\mathfrak{p}}) = K_\mathfrak{p}$ and $\underline{\mathbf{H}}(\mathcal{O}_{F,\mathfrak{p}}) = K_\mathfrak{p} \cap \underline{\mathbf{H}}(F_\mathfrak{p})$ for $\mathfrak{p} \notin S$. The stabilizer of z in $\mathbf{H}(\mathbb{A}_f)$ contains $\mathbf{H}(\mathbb{Q}) \cdot K_{H,z}$ where $K_{\mathbf{H},z} = K_{\mathbf{H},z,S} \times K_{\mathbf{H}}^S$ with

$$K_{\mathbf{H},z,S} = \left(\bigcap_i g_{i,S} K_S g_{i,S}^{-1}\right) \cap \underline{\mathbf{H}}(F_S) \quad \text{and} \quad K_{\mathbf{H}}^S = K^S \cap \underline{\mathbf{H}}(\mathbb{A}_{F,f}^S) = \prod_{\mathfrak{p} \notin S} \underline{\mathbf{H}}(\mathcal{O}_{F,\mathfrak{p}}).$$

The stabilizer of z in $\mathbf{T}^1(\mathbb{A}_f)$ contains $\mathbf{T}^1(\mathbb{Q}) \cdot U_z$ where $U_z = U_{z,S} \times U^S(1)$ with

$$U_{z,S} = \det(K_{\mathbf{H},z,S}) \quad \text{and} \quad U^S(1) = \det(K_{\mathbf{H}}^S) = \prod_{\mathfrak{p} \notin S} \underline{\mathbf{T}}^1(\mathcal{O}_{F,\mathfrak{p}}).$$

We lift $z \in \mathcal{S}_K(\mathbf{H}) \otimes R$ to $\hat{z} \in \hat{\mathcal{S}}_K(\mathbf{H}) \otimes R$ by the formula

$$\hat{z} = \sum a_i \mathbf{1}_{\mathbf{H}^1(\mathbb{A}_f) g_i K}.$$

The stabilizer of $\hat{z}$ in $\mathbf{H}(\mathbb{A}_f)$ (resp. $\mathbf{T}^1(\mathbb{A}_f)$) contains $K_{\mathbf{H},z}$ (resp. U_z).

7.1.5. Since $\mathbf{T}^1(\mathbb{Q})$ is discrete in $\mathbf{T}^1(\mathbb{A}_f)$, $\mathbf{T}^1(\mathbb{Q}) \cap U_z$ is finite. We choose a compact open subgroup U_S of $U_{z,S}$ such that the image of $\mathbf{T}^1(\mathbb{Q}) \cap U_z$ in $U_{z,S}$ has trivial intersection with U_S. Set $U(1) \stackrel{\text{def}}{=} U_S \times U^S(1)$. Thus $\mathbf{T}^1(\mathbb{Q}) \cap U(1) = \{1\}$ and since $U(1) \subset U_z$, the field of definition $E(z)$ of z is contained in the subextension H of $E[\infty]/E$ fixed by $\mathrm{Art}_E^1(U(1)) \simeq U(1)$. More generally, for any nonzero ideal $\mathfrak{n}$ of $\mathcal{O}_F$ which is prime to S, we denote by $H[\mathfrak{n}]$ the subextension of $E[\infty]/E$ fixed by $\mathrm{Art}_E^1(U(\mathfrak{n})) \simeq U(\mathfrak{n})$, where $U(\mathfrak{n}) \stackrel{\text{def}}{=} U_S \times U^S(\mathfrak{n})$ with $U^S(\mathfrak{n}) = \prod_{\mathfrak{p} \notin S} U_\mathfrak{p}^1(v_\mathfrak{p}(\mathfrak{n}))$ and

$$U_\mathfrak{p}^1(c) \stackrel{\text{def}}{=} \left\{\lambda/\overline{\lambda} : \lambda \in (\mathcal{O}_{F,\mathfrak{p}} + \mathfrak{p}^c \mathcal{O}_{E,\mathfrak{p}})^\times\right\} \subset U_\mathfrak{p}^1(0) = \underline{\mathbf{T}}^1(\mathcal{O}_{F,\mathfrak{p}})$$

for $\mathfrak{p} \notin S$ and $c \in \mathbb{N}$. Thus $H[1] = H$ and for $\mathfrak{m} \mid \mathfrak{n}$, we have $H[\mathfrak{m}] \subset H[\mathfrak{n}]$ with

$$\mathrm{Gal}\left(H[\mathfrak{n}]/H[\mathfrak{m}]\right) \simeq \frac{U(\mathfrak{m})}{U(\mathfrak{n})} \simeq \frac{U^S(\mathfrak{m})}{U^S(\mathfrak{n})} \simeq \prod_{\mathfrak{p} \mid \frac{\mathfrak{n}}{\mathfrak{m}}} \frac{U_\mathfrak{p}^1(v_\mathfrak{p}(\mathfrak{m}))}{U_\mathfrak{p}^1(v_\mathfrak{p}(\mathfrak{n}))}$$

We define the following Galois groups:

$$G(\mathfrak{n}) \stackrel{\text{def}}{=} \mathrm{Gal}\left(H[\mathfrak{n}]/H\right) \subset \mathcal{G}(\mathfrak{n}) \stackrel{\text{def}}{=} \mathrm{Gal}\left(H[\mathfrak{n}]/E\right) \subset \mathcal{G}^+(\mathfrak{n}) \stackrel{\text{def}}{=} \mathrm{Gal}\left(H[\mathfrak{n}]/F\right).$$

7.1.6. We denote by $\mathcal{P}$ the set of primes ℓ of F which are inert in E/F and do not belong to S, and we denote by $\mathcal{N}$ the set of square-free products of elements of $\mathcal{P}$. For ℓ in $\mathcal{P}$, we denote by $\mathbb{F}(\ell) \stackrel{\text{def}}{=} \mathcal{O}_F/\ell$ and $\mathbb{E}(\ell) \stackrel{\text{def}}{=} \mathcal{O}_E/\ell\mathcal{O}_E$ the residue fields of $\mathcal{O}_F$ and $\mathcal{O}_E$ at ℓ. The morphism $\nu : \underline{\mathbf{T}}(\mathcal{O}_{F,\ell}) \to \underline{\mathbf{T}}^1(\mathcal{O}_{F,\ell})$ identifies $U_\ell^1(0)/U_\ell^1(1)$ with $\mathcal{O}_{E,\ell}^\times/(\mathcal{O}_{F,\ell} + \ell\mathcal{O}_{E,\ell})^\times$ and the reduction map $\mathcal{O}_{E,\ell} \to \mathbb{E}(\ell)$ identifies the latter with the finite cyclic group $\mathbb{G}(\ell) \stackrel{\text{def}}{=} \mathbb{E}(\ell)^\times/\mathbb{F}(\ell)^\times$. For any $\mathfrak{n}$ in $\mathcal{N}$, we thus obtain

$$G(\mathfrak{n}) = \prod_{\ell \mid \mathfrak{n}} \mathrm{Gal}(H[\mathfrak{n}]/H[\mathfrak{n}/\ell]) \simeq \prod_{\ell \mid \mathfrak{n}} G(\ell) \simeq \prod_{\ell \mid \mathfrak{n}} \frac{U_\ell^1(0)}{U_\ell^1(1)} \simeq \prod_{\ell \mid \mathfrak{n}} \frac{\mathbb{E}(\ell)^\times}{\mathbb{F}(\ell)^\times} = \prod_{\ell \mid \mathfrak{n}} \mathbb{G}(\ell).$$

7.1.7. Recall that $\hat{\mathcal{S}}(\mathbf{H}) = \mathcal{S}\left(\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)\right)$. The product decomposition

$$\mathbf{H}^1(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f) = \mathbf{H}^1(F_S)\backslash\mathbf{G}(F_S) \times \prod_{\mathfrak{p}\notin S}{}' \underline{\mathbf{H}}^1(F_\mathfrak{p})\backslash\underline{\mathbf{G}}(F_\mathfrak{p})$$

where $\prod'$ is the restricted product yields a tensor product decomposition

$$\hat{\mathcal{S}}(\mathbf{H}) = \mathcal{S}\left(\underline{\mathbf{H}}^1(F_S)\backslash\underline{\mathbf{G}}(F_S)\right) \otimes \otimes'_{\mathfrak{p}\notin S}\mathcal{S}\left(\underline{\mathbf{H}}^1(F_\mathfrak{p})\backslash\underline{\mathbf{G}}(F_\mathfrak{p})\right)$$

where $\otimes'$ is the restricted tensor product, with respect to the spherical functions

$$\circ_\mathfrak{p} = \mathbf{1}_{\underline{\mathbf{H}}^1(F_\mathfrak{p})\cdot\underline{\mathbf{G}}(\mathcal{O}_{F,\mathfrak{p}})} \in \mathcal{S}\left(\underline{\mathbf{H}}^1(F_\mathfrak{p})\backslash\underline{\mathbf{G}}(F_\mathfrak{p})\right).$$

Taking K-invariants, we obtain an analogous decomposition

$$\hat{\mathcal{S}}_K(\mathbf{H}) = \mathcal{S}\left(\underline{\mathbf{H}}^1(F_S)\backslash\underline{\mathbf{G}}(F_S)/K_S\right) \otimes \otimes'_{\mathfrak{p}\notin S}\mathcal{S}\left(\underline{\mathbf{H}}^1(F_\mathfrak{p})\backslash\underline{\mathbf{G}}(F_\mathfrak{p})/\underline{\mathbf{G}}(\mathcal{O}_{F,\mathfrak{p}})\right).$$

By construction, $\hat{z} = \hat{z}_S \otimes \circ^S(1)$ in $\hat{\mathcal{S}}_K(\mathbf{H}) \otimes R$ with

$$\begin{array}{rcll} \hat{z}_S & = & \sum a_i \mathbf{1}_{\underline{\mathbf{H}}^1(F_S)g_{i,S}K_S} & \text{in} \quad \mathcal{S}\left(\underline{\mathbf{H}}^1(F_S)\backslash\underline{\mathbf{G}}(F_S)/K_S\right) \otimes R, \\ \circ^S(1) & = & \otimes'_{\mathfrak{p}\notin S}\circ_\mathfrak{p} & \text{in} \quad \otimes'_{\mathfrak{p}\notin S}\mathcal{S}\left(\underline{\mathbf{H}}^1(F_\mathfrak{p})\backslash\underline{\mathbf{G}}(F_\mathfrak{p})/\underline{\mathbf{G}}(\mathcal{O}_{F,\mathfrak{p}})\right). \end{array}$$

7.1.8. The above decomposition of $\hat{\mathcal{S}}_K(\mathbf{H})$ is compatible with the decomposition

$$\mathbf{T}^1(\mathbb{A}_f) = \underline{\mathbf{T}}^1(F_S) \times \prod_{\mathfrak{p}\notin S}{}' \underline{\mathbf{T}}^1(F_\mathfrak{p})$$

and with the similar decomposition of the Hecke algebra,

$$\mathcal{H}_K = \mathcal{H}_K^S \otimes \otimes'_{\mathfrak{p}\notin S}\mathcal{H}_\mathfrak{p} \quad \text{where} \quad \left\{ \begin{array}{lcl} \mathcal{H}_K^S & \overset{\text{def}}{=} & \mathrm{End}_{\mathbb{Z}[\underline{\mathbf{G}}(F_S)]}\left(\mathbb{Z}\left[\underline{\mathbf{G}}(F_S)/K_S\right]\right), \\ \mathcal{H}_\mathfrak{p} & \overset{\text{def}}{=} & \mathrm{End}_{\mathbb{Z}[\underline{\mathbf{G}}(F_\mathfrak{p})]}\left(\mathbb{Z}\left[\underline{\mathbf{G}}(F_\mathfrak{p})/\underline{\mathbf{G}}(\mathcal{O}_{F,\mathfrak{p}})\right]\right). \end{array} \right.$$

7.1.9. Suppose that $\mathfrak{p} \notin S$. The well-known structure of the local spherical Hecke algebra $\mathcal{H}_\mathfrak{p}$ is reviewed in section 10.1 below: we have

$$\mathcal{H}_\mathfrak{p} = \mathbb{Z}\left[c_{\mathfrak{p},1}, \cdots, c_{\mathfrak{p},n}\right]$$

for a well-defined collection $\{c_{\mathfrak{p},i}\} \in \mathcal{H}_\mathfrak{p}$ of algebraically independent elements. The work of Sakellaridis [54, 55] on spherical varieties yields a great deal of information about the main features of $\mathcal{S}\left(\underline{\mathbf{H}}^1(F_\mathfrak{p})\backslash\underline{\mathbf{G}}(F_\mathfrak{p})/\underline{\mathbf{G}}(\mathcal{O}_{F,\mathfrak{p}})\right)$ as a module over the group algebra $\mathcal{H}_\mathfrak{p}[\mathbf{T}^1(F_\mathfrak{p})]$. But it does not seem to imply the following fine, slightly weird but key result, whose proof will be given in section 8 below. We set

$$T_\mathfrak{p} \overset{\text{def}}{=} c_{\mathfrak{p},n} \in \mathcal{H}_\mathfrak{p}.$$

Theorem 7.1. *For every $\ell \in \mathcal{P}$, there is an element*

$$\circ_\ell^\flat \in \mathcal{S}\left(\underline{\mathbf{H}}^1(F_\ell)\backslash\underline{\mathbf{G}}(F_\ell)/\underline{\mathbf{G}}(\mathcal{O}_{F,\ell})\right)^{U_\ell^1(1)}$$

such that

$$\mathrm{Tr}_\ell(\circ_\ell^\flat) = T_\ell \cdot \circ_\ell \quad \text{in} \quad \mathcal{S}\left(\underline{\mathbf{H}}^1(F_\ell)\backslash\underline{\mathbf{G}}(F_\ell)/\underline{\mathbf{G}}(\mathcal{O}_{F,\ell})\right)^{U_\ell^1(0)}.$$

Here Tr_ℓ is the usual trace operator which sums over $U_\ell^1(0)/U_\ell^1(1) \simeq \mathbb{G}(\ell)$.

7.1.10. For $\mathfrak{n} \in \mathcal{N}$, we denote by $z(\mathfrak{n}) \in \mathcal{S}_K(\mathbf{H}) \otimes R$ the image of $\hat{z}(\mathfrak{n}) \in \hat{\mathcal{S}}_K(\mathbf{H}) \otimes R$,

$$\hat{z}(\mathfrak{n}) \stackrel{\text{def}}{=} \hat{z}_S \otimes \circ^S(\mathfrak{n}) \quad \text{with} \quad \circ^S(\mathfrak{n}) \stackrel{\text{def}}{=} \left(\otimes_{\ell|\mathfrak{n}} \circ_\ell^\flat\right) \otimes \left(\otimes'_{\mathfrak{p}\nmid\mathfrak{n}} \circ_\mathfrak{p}\right).$$

Proposition 7.2. *We have $z(1) = z$. For every $\mathfrak{n} \in \mathcal{N}$, $z(\mathfrak{n})$ is defined over $H[\mathfrak{n}]$ and for every $\ell \in \mathcal{P}$ which does not divide $\mathfrak{n}$,*

$$\mathrm{Tr}_\ell(z(\mathfrak{n}\ell)) = T_\ell \cdot z(\mathfrak{n}) \quad \textit{in} \quad H^0\left(H[\mathfrak{n}], \mathcal{S}_K(\mathbf{H}) \otimes R\right)$$

Here Tr_ℓ is the usual trace operator which sums over $\mathrm{Gal}(H[\mathfrak{n}\ell]/H[\mathfrak{n}]) \simeq \mathbb{G}(\ell)$.

Proof. This immediately follows from the definitions. □

7.2. The Kolyvagin system.

7.2.1. We keep the choices and notations of the previous section. Recall that the subring $\mathscr{H}$ of $\mathcal{H}_K$ contains $\mathcal{H}_\ell$ for all $\ell \in \mathcal{P}$, and acts on V and T through a morphism $\mathscr{H} \to R$. Let $\mathcal{H} = \mathscr{H} \otimes R$, which is an augmented R-algebra.

7.2.2. We now consider the following "universal" $\mathcal{H}[\mathrm{Gal}(E[\infty]/F)]$-module:

$$\mathbf{S} \stackrel{\text{def}}{=} \varinjlim \mathbf{S}(\mathfrak{n}) \quad \text{with} \quad \mathbf{S}(\mathfrak{n}) \stackrel{\text{def}}{=} \frac{\oplus_{\mathfrak{m}|\mathfrak{n}} \mathcal{H}[\mathcal{G}^+(\mathfrak{m})] \cdot \mathbf{z}(\mathfrak{m})}{\langle \mathrm{Tr}_\ell\left(\mathbf{z}(\mathfrak{m}\ell)\right) - T_\ell \cdot \mathbf{z}(\mathfrak{m}) : \mathfrak{m}\ell \mid \mathfrak{n}\rangle}.$$

Here $\mathfrak{n}$ runs through the filtered set $\mathcal{N}$ ordered by divisibility, $\mathcal{H}[\mathcal{G}^+(\mathfrak{m})] \cdot \mathbf{z}(\mathfrak{m})$ is the free $\mathcal{H}[\mathcal{G}^+(\mathfrak{m})]$-module with basis $\mathbf{z}(\mathfrak{m})$, viewed as an $\mathcal{H}[\mathrm{Gal}(E[\infty]/F)]$-module, the denominator is the $\mathcal{H}[\mathrm{Gal}(E[\infty]/F)]$-submodule of $\oplus_{\mathfrak{m}|\mathfrak{n}} \mathcal{H}[\mathcal{G}^+(\mathfrak{m})] \cdot \mathbf{z}(\mathfrak{m})$ spanned by the elements $\mathrm{Tr}_{\mathfrak{m}\ell}\left(\mathbf{z}(\mathfrak{m}\ell)\right) - T_\ell \cdot \mathbf{z}(\mathfrak{m})$ for all pairs $(\mathfrak{m}, \ell) \in \mathcal{N} \times \mathcal{P}$ such that $\mathfrak{m}\ell \mid \mathfrak{n}$, and the transition morphisms $\mathbf{S}(\mathfrak{n}) \to \mathbf{S}(\mathfrak{n}')$ for $\mathfrak{n} \mid \mathfrak{n}'$ are induced by the obvious morphism which takes $\mathbf{z}(\mathfrak{m})$ (for $\mathfrak{m} \mid \mathfrak{n}$) to $\mathbf{z}(\mathfrak{m})$ (with $\mathfrak{m} \mid \mathfrak{n}'$). We also denote by $\mathbf{z}(\mathfrak{n}) \in \mathbf{S}$ the image of $\mathbf{z}(\mathfrak{n}) \in \mathbf{S}(\mathfrak{n})$ under $\mathbf{S}(\mathfrak{n}) \to \mathbf{S}$.

Proposition 7.3. *There is a unique $\mathcal{H}[\mathrm{Gal}(E[\infty]/F)]$-equivariant morphism*

$$\mathbf{d} : \mathbf{S} \to \mathcal{S}_K(\mathbf{H}) \otimes R$$

which takes $\mathbf{z}(\mathfrak{n})$ to $z(\mathfrak{n})$ for every $\mathfrak{n} \in \mathcal{N}$.

Proof. This immediately follows from the definitions. □

Proposition 7.4. *For every $\mathfrak{n} \in \mathcal{N}$, $\mathbf{S}(\mathfrak{n})$ is a finite free $\mathcal{H}$-module.*

Proof. Let $\nu(\mathfrak{m})$ be the number of prime factors of $\mathfrak{m} \mid \mathfrak{n}$. For $i \leq \nu(\mathfrak{n})$, let $\mathrm{Fil}^i \mathbf{S}(\mathfrak{n})$ be the $\mathcal{H}[\mathrm{Gal}_F]$-submodule of $\mathbf{S}(\mathfrak{n})$ spanned by $\{\mathbf{z}(\mathfrak{m}) : \mathfrak{m} \mid \mathfrak{n}, \nu(\mathfrak{m}) \leq i\}$. Thus

$$0 = \mathrm{Fil}^{-1}\mathbf{S}(\mathfrak{n}) \subset \mathrm{Fil}^0\mathbf{S}(\mathfrak{n}) \subset \cdots \subset \mathrm{Fil}^{\nu(\mathfrak{n})}\mathbf{S}(\mathfrak{n}) = \mathbf{S}(\mathfrak{n}).$$

It is now sufficient to establish that $\mathbf{S}^i(\mathfrak{n}) \stackrel{\text{def}}{=} \mathrm{Fil}^i\mathbf{S}(\mathfrak{n})/\mathrm{Fil}^{i-1}\mathbf{S}(\mathfrak{n})$ is a finite free $\mathcal{H}$-module for $0 \leq i \leq \nu(\mathfrak{n})$. One checks that (with obvious notations)

$$\begin{aligned}
\mathbf{S}^i(\mathfrak{n}) &\simeq \oplus_{\mathfrak{m}|\mathfrak{n}, \nu(m)=i} \frac{\mathcal{H}[\mathcal{G}^+(\mathfrak{m})] \cdot \mathbf{z}(\mathfrak{m})}{\langle \mathrm{Tr}_\ell\left(\mathbf{z}(\mathfrak{m})\right) : \ell \mid \mathfrak{m}\rangle} \\
\frac{\mathcal{H}[\mathcal{G}^+(\mathfrak{m})] \cdot \mathbf{z}(\mathfrak{m})}{\langle \mathrm{Tr}_\ell\left(\mathbf{z}(\mathfrak{m})\right) : \ell \mid \mathfrak{m}\rangle} &\simeq \mathcal{H} \otimes_\mathbb{Z} \frac{\mathbb{Z}[\mathcal{G}^+(\mathfrak{m})]}{\langle \mathrm{Tr}_\ell : \ell \mid \mathfrak{m}\rangle} \\
\frac{\mathbb{Z}[\mathcal{G}^+(\mathfrak{m})]}{\langle \mathrm{Tr}_\ell : \ell \mid \mathfrak{m}\rangle} &\simeq \mathbb{Z}[\mathcal{G}^+(\mathfrak{m})] \otimes_{\mathbb{Z}[G(\mathfrak{m})]} \frac{\mathbb{Z}[G(\mathfrak{m})]}{\langle \mathrm{Tr}_\ell : \ell \mid \mathfrak{m}\rangle} \\
\frac{\mathbb{Z}[G(\mathfrak{m})]}{\langle \mathrm{Tr}_\ell : \ell \mid \mathfrak{m}\rangle} &\simeq \otimes_{\ell|\mathfrak{m}} \frac{\mathbb{Z}[\mathbb{G}(\ell)]}{\langle \mathrm{Tr}_\ell\rangle}
\end{aligned}$$

Since $\mathbb{Z}[\mathbb{G}(\ell)]/\langle \mathrm{Tr}_\ell\rangle \simeq \mathbb{Z}^{|\mathbb{F}(\ell)|}$ and $\mathbb{Z}[\mathcal{G}^+(\mathfrak{m})]$ is free of rank $[H:F]$ over $\mathbb{Z}[G(\mathfrak{m})]$, it follows that $\mathbf{S}^i(\mathfrak{n})$ is free of rank $[H:F]\sum_{\mathfrak{m}|\mathfrak{n},\nu(\mathfrak{m})=i}|\mathcal{O}_F/\mathfrak{m}|$ over $\mathcal{H}$. □

7.2.3. Pulling back the extension 6.3 through $\mathbf{d}:\mathbf{S}\to\mathcal{S}_K(\mathbf{H})$ and $\mathbf{S}(\mathfrak{n})\to\mathbf{S}$, we obtain a commutative diagram of $\mathcal{H}[\mathrm{Gal}_F]$-modules with exact rows

$$\begin{array}{ccccccccc}
0 & \to & T & \to & \mathbf{E}(\mathfrak{n}) & \to & \mathbf{S}(\mathfrak{n}) & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & T & \to & \mathbf{E} & \to & \mathbf{S} & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & T & \to & \star & \to & \mathcal{S}_K(\mathbf{H})\otimes R & \to & 0
\end{array}$$

Let $\mathcal{I}(\mathfrak{n})\subset\mathcal{H}$ be the ideal spanned by $|\mathbb{F}(\ell)|+1$ and T_ℓ for $\ell\mid\mathfrak{n}$. Since $\mathbf{S}(\mathfrak{n})$ is finite free over $\mathcal{H}$, tensoring with $\overline{\mathcal{H}}(\mathfrak{n})\overset{\mathrm{def}}{=}\mathcal{H}/\mathcal{I}(\mathfrak{n})$ yields another exact sequence

$$\begin{array}{ccccccccc}
0 & \to & T & \to & \mathbf{E}(\mathfrak{n}) & \to & \mathbf{S}(\mathfrak{n}) & \to & 0\\
 & & \downarrow{\scriptstyle \mathrm{proj}} & & \downarrow & & \downarrow & & \\
0 & \to & \overline{T}(\mathfrak{n}) & \to & \overline{\mathbf{E}}(\mathfrak{n}) & \to & \overline{\mathbf{S}}(\mathfrak{n}) & \to & 0
\end{array}$$

where $\overline{T}(\mathfrak{n})\overset{\mathrm{def}}{=}T/\mathcal{I}(\mathfrak{n})T$, $\overline{\mathbf{E}}(\mathfrak{n})\overset{\mathrm{def}}{=}\mathbf{E}(\mathfrak{n})/\mathcal{I}(\mathfrak{n})\mathbf{E}(\mathfrak{n})$ and $\overline{\mathbf{S}}(\mathfrak{n})\overset{\mathrm{def}}{=}\mathbf{S}(\mathfrak{n})/\mathcal{I}(\mathfrak{n})\mathbf{S}(\mathfrak{n})$.

7.2.4. The coboundary maps in the cohomology over H or $H[\mathfrak{n}]$ of the previous four exact sequences yield the following commutative diagram of $\mathcal{H}[\mathcal{G}^+(\mathfrak{n})]$-modules:

$$\begin{array}{ccccccc}
\overline{\mathbf{S}}(\mathfrak{n})^{G(\mathfrak{n})} & \hookrightarrow & \overline{\mathbf{S}}(\mathfrak{n}) & \leftarrow & \mathbf{S}(\mathfrak{n}) & \xrightarrow{\mathbf{d}} & H^0\left(H[\mathfrak{n}],\mathcal{S}_K(\mathbf{H})\otimes R\right)\\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta}\\
H^1_{\mathrm{res}_\mathfrak{n}^* f}\left(H,\overline{T}(\mathfrak{n})\right) & \xrightarrow{\mathrm{res}} & H^1_f\left(H[\mathfrak{n}],\overline{T}(\mathfrak{n})\right) & \xleftarrow{\mathrm{proj}} & H^1_f\left(H[\mathfrak{n}],T\right) & = & H^1_f\left(H[\mathfrak{n}],T\right)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H^1\left(H,\overline{T}(\mathfrak{n})\right) & \xrightarrow{\mathrm{res}} & H^1\left(H[\mathfrak{n}],\overline{T}(\mathfrak{n})\right) & \xleftarrow{\mathrm{proj}} & H^1\left(H[\mathfrak{n}],T\right) & = & H^1\left(H[\mathfrak{n}],T\right)
\end{array}$$

Here f is the Galois invariant Bloch-Kato Selmer structure and $\mathrm{res}_\mathfrak{n}^* f$ is its inverse image under the restriction map (from H to $H[\mathfrak{n}]$), so that the bottom-left square is cartesian. We have used the assumption in 6.3.1 to factor the coboundary vertical maps through the Bloch-Kato Selmer groups in the second line of our diagram.

7.2.5. For each $\ell\in\mathcal{P}$, fix a generator ζ_ℓ of the cyclic group $\mathbb{G}(\ell)$. For each $\mathfrak{n}\in\mathcal{N}$, define Kolyvagin's derivative operator $\partial_\mathfrak{n}\in\mathbb{Z}[G(\mathfrak{n})]$ by the following formula:

$$\partial_\mathfrak{n}\overset{\mathrm{def}}{=}\prod_{\ell|\mathfrak{n}}\partial_\ell,\quad \partial_\ell\overset{\mathrm{def}}{=}\sum_{i=1}^{N(\ell)} i\tau_\ell^i\in\mathbb{Z}[\mathbb{G}(\ell)]\simeq\mathbb{Z}[G(\ell)]\subset\mathbb{Z}[G(\mathfrak{n})].$$

Here we have used the isomorphism $\mathbb{G}(\ell)\simeq G(\ell)$ and the canonical embedding

$$G(\ell)=\mathrm{Gal}\left(H[\ell]/H\right)\xrightarrow{\simeq}\mathrm{Gal}\left(H[\mathfrak{n}]/H[\mathfrak{n}/\ell]\right)\hookrightarrow\mathrm{Gal}\left(H[\mathfrak{n}]/H\right)=G(\mathfrak{n})$$

to view ζ_ℓ as an element τ_ℓ of $G(\mathfrak{n})$. We could also write

$$\partial_\mathfrak{n}=\otimes_{\ell|\mathfrak{n}}\partial_\ell\quad\text{in}\quad\mathbb{Z}[G(\mathfrak{n})]\simeq\otimes_{\ell|\mathfrak{n}}\mathbb{Z}[G(\ell)]$$

where the isomorphism is induced by $G(\mathfrak{n})\simeq\prod_{\ell|\mathfrak{n}}G(\ell)$. We have

$$(\tau_\ell-1)\partial_\ell=(|\mathbb{F}(\ell)|+1)-\mathrm{Tr}_\ell\quad\text{in}\quad\mathbb{Z}[G(\ell)]$$

where $\mathrm{Tr}_\ell = \sum_{i=0}^{|\mathbb{F}(\ell)|} \tau_\ell^i$ is the usual trace. It follows that

$$(\tau_\ell - 1)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}) = (|\mathbb{F}(\ell)| + 1)\partial_{\mathfrak{n}/\ell}\mathbf{z}(\mathfrak{n}) - T_\ell \partial_{\mathfrak{n}/\ell}\mathbf{z}(\mathfrak{n}/\ell)$$

in $\mathbf{S}(\mathfrak{n})$ or $\mathbf{S}$. In particular, if $\overline{z}(\mathfrak{n}) \in \overline{\mathbf{S}}(\mathfrak{n})$ is the image of $z(\mathfrak{n}) \in \mathbf{S}(\mathfrak{n})$, then

$$\partial_\mathfrak{n}\overline{\mathbf{z}}(\mathfrak{n}) \in H^0\left(G(\mathfrak{n}), \overline{\mathbf{S}}(\mathfrak{n})\right) = H^0\left(H, \overline{\mathbf{S}}(\mathfrak{n})\right).$$

Applying the coboundary map $\delta : H^0\left(H, \overline{\mathbf{S}}(\mathfrak{n})\right) \to H^1_{\mathrm{res}_\mathfrak{n}^* f}\left(H, \overline{T}(\mathfrak{n})\right)$, we obtain

$$\tilde{\kappa}(\mathfrak{n}) \in H^1_{\mathrm{res}_\mathfrak{n}^* f}\left(H, \overline{T}(\mathfrak{n})\right), \qquad \tilde{\kappa}(\mathfrak{n}) \overset{\mathrm{def}}{=} \delta\left(\partial_\mathfrak{n}\overline{\mathbf{z}}(\mathfrak{n})\right).$$

For $\mathfrak{n} = 1$, $\mathcal{I}(1) = 0$, $\overline{T}(1) = T$ and we have

$$\tilde{\kappa}(1) = \delta(\mathbf{z}(1)) = \delta(z) \in H^1_f(H, T).$$

For any $\mathfrak{n} \in \mathcal{N}$, we have the following relation:

$$\begin{array}{ccccc} H^1_{\mathrm{res}_\mathfrak{n}^* f}\left(H, \overline{T}(\mathfrak{n})\right) & \xrightarrow{\mathrm{res}} & H^1_f\left(H[\mathfrak{n}], \overline{T}(\mathfrak{n})\right) & \xleftarrow{\mathrm{proj}} & H^1_f\left(H[\mathfrak{n}], T\right) \\ \tilde{\kappa}(\mathfrak{n}) & \longmapsto & \partial_\mathfrak{n}\delta(\overline{\mathbf{z}}(\mathfrak{n})) & \longleftarrow & \partial_\mathfrak{n}\delta(\mathbf{z}(\mathfrak{n})) \end{array}$$

7.2.6. For every finite place v of H, let $c(v) \in \mathbb{N}$ be the constant from Proposition 1.1 (if $v \nmid p$) or 1.2 (if $v \mid p$) for $K = H_v$. Since $c(v) = 0$ when V is unramified at $v \nmid p$,

$$c \overset{\mathrm{def}}{=} \max\{c(v)\} \in \mathbb{N}.$$

If $v \nmid \mathfrak{n} \in \mathcal{N}$, then $H[\mathfrak{n}]/H$ is unramified at v, thus

$$\mathfrak{m}^{c(v)} H^1_{\mathrm{res}_\mathfrak{n}^* f}\left(H_v, \overline{T}(\mathfrak{n})\right) \subset H^1_f\left(H_v, \overline{T}(\mathfrak{n})\right) \subset H^1_{\mathrm{res}_\mathfrak{n}^* f}\left(H_v, \overline{T}(\mathfrak{n})\right).$$

It follows that for every finite place $v \nmid \mathfrak{n}$ of H,

$$\pi^c \operatorname{loc}_v\left(\tilde{\kappa}(\mathfrak{n}))\right) \in H^1_f\left(H_v, \overline{T}(\mathfrak{n})\right). \tag{7.1}$$

7.2.7. Let $v \mid \mathfrak{n}$ be a finite place of H, say $v \mid \ell \in \mathcal{P}$ with $\ell \mid \mathfrak{n}$, and let w be a place of $H[\mathfrak{n}]$ above v. The decomposition and inertia subgroups at $\ell\mathcal{O}_E$ in $\mathcal{G}(\mathfrak{n}) = \mathrm{Gal}(H[\mathfrak{n}]/E)$ are both equal to $\mathrm{Gal}(H[\mathfrak{n}]/H[\mathfrak{n}/\ell]) \simeq G(\ell)$, i.e. $H_v = E_\ell$ and $H[\mathfrak{n}]_w$ is a totally ramified extension of E_ℓ with Galois group $G(\ell) \simeq \mathbb{G}(\ell)$, which is Galois an dihedral over F_ℓ. With the notations of 1.3.1, $H[\mathfrak{n}]_w = E_\ell(1)$. We have

$$\mathrm{res}_w \circ \operatorname{loc}_v\left(\tilde{\kappa}(\mathfrak{n})\right) = \partial_\ell \cdot \mathrm{proj} \circ \operatorname{loc}_w\left(\delta(\partial_{\mathfrak{n}/\ell}\mathbf{z}(\mathfrak{n}))\right) \in \partial_\ell \cdot H^1_f\left(H[\mathfrak{n}]_w, \overline{T}(\mathfrak{n})\right).$$

But T is unramified at $\ell \notin S$, so

$$H^1_f\left(H[\mathfrak{n}]_w, \overline{T}(\mathfrak{n})\right) = H^1_{ur}\left(H[\mathfrak{n}]_w, \overline{T}(\mathfrak{n})\right) = H^1_{ur}\left(H_v, \overline{T}(\mathfrak{n})\right) \simeq \overline{T}(\mathfrak{n})_{\mathrm{Fr}_\ell^2 - 1}$$

on which $\partial_\ell = \sum_{i=1}^{|\mathbb{F}(\ell)|} i\tau_\ell^i$ acts by $\sum_{i=1}^{|\mathbb{F}(\ell)|} i = \frac{|\mathbb{F}(\ell)| \cdot (|\mathbb{F}(\ell)| + 1)}{2}$, thus

$$\mathrm{res}_w\left(2 \operatorname{loc}_v\left(\tilde{\kappa}(\mathfrak{n})\right)\right) = 0 \quad \text{in} \quad H^1\left(H[\mathfrak{n}]_w, \overline{T}(\mathfrak{n})\right)$$

since $|\mathbb{F}(\ell)| + 1 \in \mathcal{I}(\mathfrak{n})$, and therefore

$$2 \operatorname{loc}_v\left(\tilde{\kappa}(\mathfrak{n})\right) \in H^1_t\left(H_v, \overline{T}(\mathfrak{n})\right). \tag{7.2}$$

7.2.8. Fix $\mathfrak{n}\ell \in \mathcal{N}$ and choose an embedding $\overline{F} \hookrightarrow \overline{F}_\ell$. Let $\lambda \mid w \mid v \mid \ell\mathcal{O}_E \mid \ell$ be the induced places of $H[\mathfrak{n}\ell]$, $H[\mathfrak{n}]$, H, E and F. The corresponding completions are

$$F_\ell \subset E_\ell = H_v = H[\mathfrak{n}]_w \subset E_\ell(1) = H[\mathfrak{n}\ell]_\lambda \subset E_\ell^{\mathrm{ur}}(1) \subset \overline{F}_\ell$$

where $E_\ell(1)$ is also the completion of $H[\ell]$, and $E_\ell^{\mathrm{ur}}(1)$ is the maximal unramified extension of $E_\ell(1)$, namely the compositum of $E_\ell(1)$ with the maximal unramified extension $F_\ell^{\mathrm{ur}} = E_\ell^{\mathrm{ur}}$ of F_ℓ and E_ℓ. Since T is unramified at $\ell \notin S$ and $\ell \nmid p$,

$$\mathrm{loc}_\lambda \circ \delta\left(\mathbf{S}(\mathfrak{n}\ell)\right) \subset H^1_f\left(E_\ell(1), T\right) = H^1_{ur}\left(E_\ell(1), T\right)$$

by 7.2.4. It then follows from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & H^0\left(E_\ell^{\mathrm{ur}}(1), T\right) & \to & H^0\left(E_\ell^{\mathrm{ur}}(1), \mathbf{E}(\mathfrak{n}\ell)\right) & \to & H^0\left(E_\ell^{\mathrm{ur}}(1), \mathbf{S}(\mathfrak{n}\ell)\right) & \xrightarrow{\delta=0} & H^1\left(E_\ell^{\mathrm{ur}}(1), T\right) & \\
 & \| & & \hookdownarrow & & \| & & & \\
0 \longrightarrow & T & \longrightarrow & \mathbf{E}(\mathfrak{n}\ell) & \longrightarrow & \mathbf{S}(\mathfrak{n}\ell) & \longrightarrow & 0 &
\end{array}$$

that $H^0\left(E_\ell^{\mathrm{ur}}(1), \mathbf{E}(\mathfrak{n}\ell)\right) = \mathbf{E}(\mathfrak{n}\ell)$, i.e. the second line of our diagram is an exact sequence of $\mathcal{H}[\mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/F_\ell)]$-modules. Since $\mathbf{S}(\mathfrak{n}\ell)$ if finite free over $\mathcal{H}$, we may choose an $\mathcal{H}$-equivariant splitting $\mathbf{E}(\mathfrak{n}\ell) = T \oplus \mathbf{S}(\mathfrak{n}\ell)$. The Galois action on $\mathbf{E}(\mathfrak{n}\ell)$ is then given in matrix form on $T \oplus \mathbf{S}(\mathfrak{n}\ell)$ by a formula

$$\sigma \mapsto \begin{pmatrix} \sigma|T & f(\sigma, -) \\ & \sigma|\mathbf{S}(\mathfrak{n}\ell) \end{pmatrix} \quad \text{with} \quad f : \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/F_\ell) \times \mathbf{S}(\mathfrak{n}\ell) \to T.$$

The function f is continuous in the first variable, $\mathcal{H}$-linear in the second variable, and satisfies the following cocycle condition: for every $\sigma_1, \sigma_2 \in \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/F_\ell)$,

$$f(\sigma_1\sigma_2, -) = \sigma_1 f(\sigma_2, -) + f(\sigma_1, \sigma_2 -). \tag{7.3}$$

On $\mathrm{Gal}(E_\ell^{\mathrm{ur}}(1), E_\ell) = \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell(1)) \times \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell^{\mathrm{ur}})$, this becomes

$$f(\sigma_1\sigma_2, -) = \begin{cases} \sigma_1 f(\sigma_2, -) + f(\sigma_1, -) & \text{for } \sigma_1, \sigma_2 \in \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell(1)), \\ f(\sigma_2, -) + f(\sigma_1, \sigma_2 -) & \text{for } \sigma_1, \sigma_2 \in \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell^{\mathrm{ur}}). \end{cases}$$

The second formula implies that

$$f \equiv 0 \quad \text{on} \quad \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell^{\mathrm{ur}}) \times \mathrm{im}\left(\mathbf{S}(\mathfrak{n}) \to \mathbf{S}(\mathfrak{n}\ell)\right).$$

7.2.9. Let $\tilde{\kappa}'(\mathfrak{n}) \in H^1\left(H, \overline{T}(\mathfrak{n}\ell)\right)$ be the image of $\tilde{\kappa}(\mathfrak{n}) \in H^1\left(H, \overline{T}(\mathfrak{n})\right)$ under the morphism induced by the projection $\overline{T}(\mathfrak{n}) \twoheadrightarrow \overline{T}(\mathfrak{n}\ell)$ coming from $\mathcal{I}(\mathfrak{n}) \subset \mathcal{I}(\mathfrak{n}\ell)$. It follows from the definitions and the above discussion that the localizations

$$\mathrm{loc}_v\left(\tilde{\kappa}'(\mathfrak{n})\right), \mathrm{loc}_v\left(\tilde{\kappa}(\mathfrak{n}\ell)\right) \in H^1\left(H_v, \overline{T}(\mathfrak{n}\ell)\right)$$

are inflated from the cohomology classes (recall that $E_\ell = H_v$)

$$\tilde{\kappa}'_v(\mathfrak{n}),\ \tilde{\kappa}_v(\mathfrak{n}\ell) \in H^1\left(\mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell), \overline{T}(\mathfrak{n}\ell)\right)$$

which are respectively represented by the cocycles

$$\sigma \mapsto \overline{f}\left(\sigma, \partial_{\mathfrak{n}}\mathbf{z}(\mathfrak{n})\right), \sigma \mapsto \overline{f}\left(\sigma, \partial_{\mathfrak{n}\ell}\mathbf{z}(\mathfrak{n}\ell)\right) \qquad \sigma \in \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/E_\ell)$$

where $\overline{f}$ is the reduction of f modulo $\mathcal{I}(\mathfrak{n}\ell)$,

$$\overline{f} : \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/F_\ell) \times \mathbf{S}(\mathfrak{n}\ell) \to \mathrm{Gal}(E_\ell^{\mathrm{ur}}(1)/F_\ell) \times \overline{\mathbf{S}}(\mathfrak{n}\ell) \to \overline{T}(\mathfrak{n}\ell).$$

7.2.10. Recall from our assumptions in 6.3.8 and 6.3.9 that the characteristic polynomial of the geometric Frobenius Fr_ℓ acting on the free R-module T is the image of the Hecke polynomial $P_\ell(t) \in \mathcal{H}_\ell[t]$ under the morphism

$$\mathcal{H}_\ell \hookrightarrow \mathscr{H} \hookrightarrow \mathcal{H} = \mathscr{H} \otimes R \twoheadrightarrow R.$$

We now view $P_\ell(t)$ as a polynomial in $\mathcal{H}[t]$. Since $T_\ell = c_{\ell,n}$ and

$$P_\ell(t) \equiv \sum_{k=0}^{n} (-1)^{\frac{k(k+1)}{2}} t^k (t^2-1)^{n-k} c_{\ell,k} \bmod (|\mathbb{F}(\ell)|+1)\,\mathcal{H}[t]$$

by 10.2.6, it follows that

$$P_\ell(t) \equiv (t^2-1) Q_\ell(t) \quad \text{in} \quad \overline{\mathcal{H}}(\mathfrak{n}\ell)[t]$$

with

$$Q_\ell(t) = \sum_{k=0}^{n-1} (-1)^{\frac{k(k+1)}{2}} t^k (t^2-1)^{n-1-k} c_{\ell,k} \quad \text{in} \quad \mathcal{H}[t].$$

Since $P_\ell(\mathrm{Fr}_\ell) = 0$ on T and $\overline{T}(\mathfrak{n}\ell)$, we obtain the finite/singular homomorphism

$$\begin{array}{ccc} H^1_f\left(H_v, \overline{T}(\mathfrak{n}\ell)\right) & \xrightarrow{\varphi_\ell} & H^1_s\left(H_v, \overline{T}(\mathfrak{n}\ell)\right) \\ {\scriptstyle \mathrm{ev}(\mathrm{Fr}_\ell^2)}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \mathrm{ev}(\tau_\ell)} \\ \overline{T}(\mathfrak{n}\ell)_{\mathrm{Fr}_\ell^2-1} & \xrightarrow{Q_\ell(\mathrm{Fr}_\ell)} & \overline{T}(\mathfrak{n}\ell)^{\mathrm{Fr}_\ell^2-1} \end{array}$$

We denote by ι_ℓ the involution induced by $-(-1)^{\frac{n(n+1)}{2}} \mathrm{Fr}_\ell^{-n}$ on this diagram.

Proposition 7.5. *We have* $\mathrm{loc}_v(\tilde{\kappa}'(\mathfrak{n})) \in H^1_f\left(H_v, \overline{T}(\mathfrak{n}\ell)\right)$ *and*

$$\iota_\ell \varphi_\ell \left(\mathrm{loc}_v(\tilde{\kappa}'(\mathfrak{n}))\right) = \mathrm{loc}_v^s(\tilde{\kappa}(\mathfrak{n}\ell)) \quad \textit{in} \quad H^1_s\left(H_v, \overline{T}(\mathfrak{n}\ell)\right)$$

where loc_v^s *is the composition of the localization map* $H^1(H,-) \to H^1(H_v,-)$ *with the projection to the singular quotient* $H^1(H_v,-) \twoheadrightarrow H^1_s(H_v,-)$.

We will prove this in the next two subsections.

7.2.11. Set $\sigma_\ell = \mathrm{Fr}_\ell$ and $Q'_\ell(t) = -(-1)^{\frac{n(n+1)}{2}} t^{-n} Q_\ell(t)$. We have to show that

$$Q'_\ell(\sigma_\ell)\overline{f}\left(\sigma_\ell^2, \partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) = \overline{f}\left(\tau_\ell, \partial_{\mathfrak{n}\ell}\mathbf{z}(\mathfrak{n}\ell)\right) \quad \text{in} \quad \overline{T}(\mathfrak{n}\ell).$$

This is equivalent to

$$Q'_\ell(\sigma_\ell) f\left(\sigma_\ell^2, \partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) - f\left(\tau_\ell, \partial_{\mathfrak{n}\ell}\mathbf{z}(\mathfrak{n}\ell)\right) \in \mathcal{I}(\mathfrak{n}\ell)T.$$

Since $\sigma_\ell^2 - 1$ is injective on T and $\mathcal{I}(\ell) \subset \mathcal{I}(\mathfrak{n}\ell)$, it is sufficient to establish that

$$(\sigma_\ell^2-1)Q'_\ell(\sigma_\ell) f\left(\sigma_\ell^2, \partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) - (\sigma_\ell^2-1) f\left(\tau_\ell, \partial_{\mathfrak{n}\ell}\mathbf{z}(\mathfrak{n}\ell)\right) \in (\sigma_\ell^2-1)\mathcal{I}(\ell)T.$$

Since $\sigma_\ell^2\tau_\ell = \tau_\ell\sigma_\ell^2$, the cocycle relation 7.3 for f implies

$$\sigma_\ell^2 f(\tau_\ell,*) + f(\sigma_\ell^2, \tau_\ell *) = f(\sigma_\ell^2\tau_\ell,*) = f(\tau_\ell\sigma_\ell^2,*) = \tau_\ell f(\sigma_\ell^2,*) + f(\tau_\ell, \sigma_\ell^2 *)$$

and since $\sigma_\ell^2 \equiv 1$ on $\mathbf{S}(\mathfrak{n}\ell)$ and $\tau_\ell \equiv 1$ on T, we find that

$$(\sigma_\ell^2-1) f(\tau_\ell,*) = f(\sigma_\ell^2, (1-\tau_\ell)*).$$

In particular,

$$\begin{aligned} (\sigma_\ell^2-1) f\left(\tau_\ell, \partial_{\mathfrak{n}\ell}\mathbf{z}(\mathfrak{n}\ell)\right) &= f\left(\sigma_\ell^2, (1-\tau_\ell)\partial_\ell\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}\ell)\right) \\ &= f\left(\sigma_\ell^2, \left(\mathrm{Tr}_\ell - (|\mathbb{F}(\ell)|+1)\right)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}\ell)\right) \\ &= f\left(\sigma_\ell^2, T_\ell\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}) - (|\mathbb{F}(\ell)|+1)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}\ell)\right) \end{aligned}$$

Let us now write $P'_\ell(t) = -(-1)^{\frac{n(n+1)}{2}} t^{-n} P_\ell(t)$, so that

$$\begin{array}{rcl} P_\ell(t) & = & (t^2-1)Q_\ell(t) + (-1)^{\frac{n(n+1)}{2}} \left(T_\ell t^n - (|\mathbb{F}(\ell)|+1)R_\ell(t)\right) \\ \text{and} \quad P'_\ell(t) & = & (t^2-1)Q'_\ell(t) - \left(T_\ell - (|\mathbb{F}(\ell)|+1)t^{-n}R_\ell(t)\right) \end{array}$$

for some $R_\ell(t) \in \mathcal{H}[t]$. Since $P'_\ell(\sigma_\ell) \equiv 0$ on T,

$$(\sigma_\ell^2 - 1)Q'_\ell(\sigma_\ell) \equiv T_\ell - (|\mathbb{F}(\ell)|+1)\sigma_\ell^{-n} R_\ell(\sigma_\ell) \text{ on } T.$$

The cocycle relation 7.3 for f implies that

$$f(\sigma_\ell^2, *) : \mathbf{S}(\mathfrak{n}\ell) \to T_{\sigma_\ell^2 - 1}$$

is σ_ℓ-equivariant. It follows that

$$\begin{array}{rcl} (\sigma_\ell^2-1)Q'_\ell(\sigma_\ell) f\left(\sigma_\ell^2, \partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) & = & \left(T_\ell - (|\mathbb{F}(\ell)|+1)\sigma_\ell^{-n}R_\ell(\sigma_\ell)\right) f\left(\sigma_\ell^2, \partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) \\ & \equiv & f\left(\sigma_\ell^2, \left(T_\ell - (|\mathbb{F}(\ell)|+1)\sigma_\ell^{-n}R_\ell(\sigma_\ell)\right)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) \\ & \equiv & f\left(\sigma_\ell^2, T_\ell\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}) - (|\mathbb{F}(\ell)|+1)\sigma_\ell^{-n}R_\ell(\sigma_\ell)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) \end{array}$$

modulo $(\sigma_\ell^2-1)\mathcal{I}(\ell)T$. We thus have to show that $f(\sigma_\ell^2, *) \in (\sigma_\ell^2-1)\mathcal{I}(\ell)T$ for

$$\begin{array}{rcl} * & = & \left(T_\ell\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}) - (|\mathbb{F}(\ell)|+1)\sigma_\ell^{-n}R_\ell(\sigma_\ell)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n})\right) - \left(T_\ell\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}) - (|\mathbb{F}(\ell)|+1)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}\ell)\right) \\ & = & (|\mathbb{F}(\ell)|+1)\left(\sigma_\ell^{-n}R_\ell(\sigma_\ell)\partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}) - \partial_\mathfrak{n}\mathbf{z}(\mathfrak{n}\ell)\right) \end{array}$$

and since $|\mathbb{F}(\ell)|+1$ belongs to $\mathcal{I}(\ell)$, it is sufficient to establish that

$$f\left(\sigma_\ell^2, R_\ell(\sigma_\ell)g\mathbf{z}(\mathfrak{n}) - \sigma_\ell^n g\mathbf{z}(\mathfrak{n}\ell)\right) = 0 \quad \text{in} \quad T_{\sigma_\ell^2-1}$$

for every $g \in G(\mathfrak{n})$, and we may assume that $g = 1$.

7.2.12. Since $\ell \notin S$, $X = \mathrm{Sh}_K$ has good reduction at ℓ. Let $\mathcal{X}$ be the smooth model over $\mathcal{O}_{F,\ell}$, as constructed by Kisin, with generic fiber X and special fiber $\mathcal{X}_{\mathbb{F}(\ell)}$. Writing $\mathcal{Z}^n(\star)$ for the group of n-codimensional cycles on $\star$ and $\mathcal{Z}^n_0(\star)$ for the subgroup of cohomologically trivial cycles, we have a commutative diagram

$$\begin{array}{ccccccc} \mathbf{S}(\mathfrak{n}) & \longrightarrow & \mathcal{Z}^n_0(\mathcal{X}_{E_\ell}) \otimes R & \xrightarrow{\mathrm{AJ}_p} & H^1_f\left(E_\ell, H^{2n-1}\left(\mathcal{X}_{\overline{F}_\ell}, R(n)\right)\right) & \longrightarrow & H^1_f(E_\ell, T) \\ \downarrow & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \mathbf{S}(\mathfrak{n}\ell) & \longrightarrow & \mathcal{Z}^n_0(\mathcal{X}_{E_\ell(1)}) \otimes R & \xrightarrow{\mathrm{AJ}_p} & H^1_f\left(E_\ell(1), H^{2n-1}\left(\mathcal{X}_{\overline{F}_\ell}, R(n)\right)\right) & \longrightarrow & H^1_f(E_\ell(1), T) \\ \downarrow & & \downarrow \mathrm{red} & & \uparrow \simeq & & \uparrow \simeq \\ \mathbf{S}(\mathfrak{n}\ell)_{G(\ell)} & \longrightarrow & \mathcal{Z}^n_0(\mathcal{X}_{\mathbb{E}(\ell)}) \otimes R & \xrightarrow{\mathrm{AJ}_p} & H^1\left(\mathbb{E}(\ell), H^{2n-1}\left(\mathcal{X}_{\overline{\mathbb{F}}(\ell)}, R(n)\right)\right) & \longrightarrow & H^1(\mathbb{E}(\ell), T) \end{array}$$

We want to show that the middle row is trivial on

$$R_\ell(\sigma_\ell)\mathbf{z}(\mathfrak{n}) - \sigma_\ell^n\mathbf{z}(\mathfrak{n}\ell).$$

It is sufficient to show that

$$\begin{array}{rrcl} & \mathrm{red}\left(R_\ell(\sigma_\ell)z(\mathfrak{n})\right) & = & \mathrm{red}\left(\sigma_\ell^n z(\mathfrak{n}\ell)\right) \\ \text{inside} & \mathcal{Z}^n_0(\mathcal{X}_{\mathbb{E}(\ell)}) \otimes R & \subset & \mathcal{Z}^n(\mathcal{X}_{\mathbb{E}(\ell)}) \otimes R. \end{array}$$

Changing v to γv, we may assume that $\gamma = 1$. We have:

$$\mathrm{red}\left(T_\ell z(\mathfrak{n})\right) = \mathrm{red}\left(\mathrm{Tr}_\ell z(\mathfrak{n}\ell)\right) = (|\mathbb{F}(\ell)|+1)\mathrm{red}\left(z(\mathfrak{n}\ell)\right).$$

Since $\mathcal{Z}^n(\mathcal{X}_{\mathbb{E}(\ell)}) \otimes R$ is torsion free, it is sufficient to show that

$$\mathrm{red}\left(\left((|\mathbb{F}(\ell)|+1)R_\ell(\sigma_\ell) - T_\ell\sigma_\ell^n\right)z(\mathfrak{n})\right) = 0.$$

Since $\sigma_\ell^2 = 1$ on $\mathcal{Z}^n(\mathcal{X}_{\mathbb{E}(\ell)})$, it is sufficient to show that

$$\mathrm{red}_v\left(P_\ell(\sigma_\ell) z(\mathfrak{n})\right) = 0.$$

This follows from our final assumption 6.3.10.

7.2.13. For every $\mathfrak{n} \in \mathcal{N}$, we now define

$$\begin{aligned} \kappa(\mathfrak{n}) &\stackrel{\text{def}}{=} \pi^c \prod_{\ell \mid \mathfrak{n}} \left(-(-1)^{\frac{n(n+1)}{2}} \mathrm{Fr}_\ell^{-n} \right) \cdot \tilde{\kappa}(\mathfrak{n}) \otimes \left(\bigotimes_{\ell \mid \mathfrak{n}} \zeta_\ell \right) \\ &\in H^1\left(H, \overline{T}(\mathfrak{n})\right) \otimes \mathbb{G}(\mathfrak{n}). \end{aligned}$$

Then by 7.2.6 (7.1) and 7.2.7 (7.2)

$$\kappa(\mathfrak{n}) \in H^1_{\mathcal{F}^{\mathfrak{n}}}\left(H, \overline{T}(\mathfrak{n})\right) \otimes \mathbb{G}(\mathfrak{n}) \quad \text{and} \quad 2\kappa(\mathfrak{n}) \in H^1_{\mathcal{F}(\mathfrak{n})}\left(H, \overline{T}(\mathfrak{n})\right) \otimes \mathbb{G}(\mathfrak{n}).$$

Moreover by Proposition 7.5 for $\mathfrak{n}\ell \in \mathcal{N}$ and every place v of H above ℓ,

$$\Phi_v \circ \mathrm{loc}_v\left(\kappa(\mathfrak{n})\right) = \mathrm{loc}_v^s\left(\kappa(\mathfrak{n}\ell)\right).$$

Thus $\kappa = (\kappa(\mathfrak{n}))_{\mathfrak{n} \in \mathcal{N}}$ is a weak Kolyvagin system for (T, γ, H) with

$$\kappa(1) = \pi^c \tilde{\kappa}(1) = \pi^c \delta(z) \in H^1_f(H, T).$$

Theorem 6.4 now follows from Theorem 4.1.

8. The tame distribution relations

8.1. **Notations.**

8.1.1. Suppose now that F is a local field of mixed characteristic $(0, p)$ with $p \neq 2$, E is an unramified quadratic extension of F, (V, φ) is a quadratic space over F of dimension $2n+1$ and Witt index n, and (W, ψ) is a fixed E-hermitian F-hyperplane in (V, φ). We let D be the orthogonal complement of W in $V = W \perp D$. Multiplying φ by a suitable constant, we may and do assume that $(D, \varphi|_D) \simeq \langle 1 \rangle$[4]. We let $\mathbb{E}$ and $\mathbb{F}$ be the residue fields of E and F and set $q = |\mathbb{F}|$. We consider the groups

$$G \stackrel{\text{def}}{=} SO(V, \varphi) \supset H \stackrel{\text{def}}{=} U(W, \psi) \supset H^1 \stackrel{\text{def}}{=} SU(W, \psi).$$

8.1.2. Let $\mathcal{I}$ be the Bruhat-Tits building of G, realized as the set of all self-dual F-norms on V [11, 17]. Let $\mathcal{I}_0 \subset \mathcal{I}$ be the Bruhat-Tits building of $SO(W, \varphi|_W)$, realized as the subset of $\mathcal{I}$ made of those self-dual F-norms whose restriction to D (or W) is also self-dual. For $\alpha \in \mathcal{I}$ and $\lambda \in \mathbb{R}$, we denote by $S(\alpha, \lambda)$ the "sphere of radius q^λ", as defined in [17, 1.5.3]. It is a finite dimensional $\mathbb{F}$-vector space which is equipped with a non-degenerate quadratic form when $\lambda \in \frac{1}{2}\mathbb{Z}$ [17, 3.12]. Let

$$\begin{aligned} \mathcal{I}(\tfrac{1}{2}) &\stackrel{\text{def}}{=} \left\{ \alpha \in \mathcal{I} : S(\alpha, \lambda) = 0 \text{ for } \lambda \in \mathbb{R} \setminus \tfrac{1}{2}\mathbb{Z} \right\} \\ \text{and} \quad \mathcal{I}(\tfrac{1}{4}) &\stackrel{\text{def}}{=} \left\{ \alpha \in \mathcal{I} : S(\alpha, \lambda) = 0 \text{ for } \lambda \in \mathbb{R} \setminus \tfrac{1}{4}\mathbb{Z} \right\} \end{aligned}$$

be the set of vertices and edges in $\mathcal{I}$, as defined in [17, 3.10 & 3.11]. The G-orbits in $\mathcal{I}(\frac{1}{2})$ and $\mathcal{I}(\frac{1}{4})$ are classified by a reduced type map [17, 3.8] defined by

$$\mathrm{rtype}(\alpha) = (a, b, c) \iff \dim_{\mathbb{F}} \begin{pmatrix} S(\alpha, 0) \\ S(\alpha, \frac{1}{4}) \\ S(\alpha, \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 2a+1 \\ b \\ 2c \end{pmatrix}$$

[4] This will simplify our description of various sets related to Bruhat-Tits buildings.

which yields bijections

$$\begin{array}{ccc} G\backslash\mathcal{I}(\frac{1}{2}) & \xrightarrow{\text{rtype}} & \left\{(a,c)\in\mathbb{N}^2 : a+c=n\right\} \\ \big\downarrow & & \big\downarrow \\ G\backslash\mathcal{I}(\frac{1}{4}) & \xrightarrow{\text{rtype}} & \left\{(a,b,c)\in\mathbb{N}^2 : a+b+c=n\right\} \end{array}$$

The target and source of an edge of reduced type (a,b,c) are vertices of reduced types $(a+b,c)$ and $(a,b+c)$, respectively. For $k\in\{0,\cdots,n\}$, we denote by $\mathbb{I}_k\subset\mathcal{I}(\frac{1}{2})$ the G-orbit of vertices of reduced type $(n-k,k)$, and by $\mathbb{I}_{0,k}\in\mathcal{I}(\frac{1}{4})$ the G-orbit of edges of reduced type $(n-k,k,0)$. Our assumptions imply that $\mathbb{I}_0$ is the unique G-orbit of hyperspecial points in $\mathcal{I}$. For any $k\in\{0,\cdots,n\}$, the G-equivariant target and source maps $\pi_k^0:\mathbb{I}_{0,k}\twoheadrightarrow\mathbb{I}_0$ and $\pi_0^k:\mathbb{I}_{0,k}\twoheadrightarrow\mathbb{I}_k$ have finite fibers and they identify $\mathbb{I}_{0,k}$ with a subset of $\mathbb{I}_0\times\mathbb{I}_k$. More precisely for $x\in\mathbb{I}_0$ (resp. $x\in\mathbb{I}_k$) with stabilizer G_x in G, there is a G_x-equivariant bijection between the fiber $(\pi_k^0)^{-1}(x)$ (resp. $(\pi_0^k)^{-1}(x)$) and the set of all k-dimensional totally isotropic $\mathbb{F}$-subspaces of $S(x,0)$ (resp. $S(x,\frac{1}{2})$), see [17, 3.13].

8.1.3. As explained in [16, §5-6], there is a G-invariant vector valued distance

$$\mathbf{d}:\mathcal{I}\times\mathcal{I}\to\mathbf{C}^{\mathbb{R}}(G)$$

which induces a genuine CAT(0)-distance $d=|\mathbf{d}|$ on $\mathcal{I}$,

$$d:\mathcal{I}\times\mathcal{I}\to\mathbb{R}_+$$

The universal closed Weyl cone $\mathbf{C}^{\mathbb{R}}(G)$ of our group $G=SO(V,\varphi)$ is here given by

$$\mathbf{C}^{\mathbb{R}}(G)\simeq\{(\lambda_1,\cdots,\lambda_n)\in\mathbb{R}^n:\lambda_1\geq\cdots\geq\lambda_n\geq 0\}$$

with the length function $|-|:\mathbf{C}^{\mathbb{R}}(G)\to\mathbb{R}_+$ and partial order $\preceq$ given by

$$|(\lambda_1,\cdots,\lambda_n)|=\sqrt{\lambda_1^2+\cdots+\lambda_n^2}$$

$$(\lambda_1,\cdots,\lambda_n)\preceq(\lambda_1',\cdots,\lambda_n')\iff\forall i\in\{1,\cdots,n\}:\ \lambda_1+\cdots+\lambda_i\leq\lambda_1'+\cdots+\lambda_i'.$$

It contains the submonoid $\mathbf{C}^{\mathbb{Z}}(G)$ of conjugacy classes of cocharacters of G,

$$\mathbf{C}^{\mathbb{Z}}(G)\simeq\{(\lambda_1,\cdots,\lambda_n)\in\mathbb{Z}^n:\lambda_1\geq\cdots\geq\lambda_n\geq 0\}.$$

The vector valued distance $\mathbf{d}$ maps $\mathbb{I}_0\times\mathbb{I}_0$ to $\mathbf{C}^{\mathbb{Z}}(G)$ and induces a bijection

$$\mathbf{d}:G\backslash\left(\mathbb{I}_0\times\mathbb{I}_0\right)\xrightarrow{\simeq}\mathbf{C}^{\mathbb{Z}}(G).$$

8.2. **The spherical Hecke algebra.**

8.2.1. We define the spherical Hecke algebra $\mathcal{H}$ of G as follows:

$$\mathcal{H}\overset{\text{def}}{=}\mathrm{End}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_0]\right).$$

Thus $\mathcal{H}$ is free as a $\mathbb{Z}$-module with basis $\left\{c_\mu:\mu\in\mathbf{C}^{\mathbb{Z}}(G)\right\}$ where

$$c_\mu(x)\overset{\text{def}}{=}\sum_{\mathbf{d}(x,z)=\mu}z$$

for any $x\in\mathbb{I}_0\subset\mathbb{Z}[\mathbb{I}_0]$. For $0\leq k\leq n$, we set $c_k=c_\mu$ with $\mu=(1^k,0^{n-k})$. Thus $c_0=1$ is the unit of $\mathcal{H}$ and it follows from the Satake isomorphism (see below) that

$$\mathcal{H}=\mathbb{Z}[c_1,\cdots,c_n]$$

is the commutative polynomial algebra on the independent variables $\{c_1,\cdots,c_n\}$.

8.2.2. We will also consider the following elements of $\mathcal{H}$. For $0 \leq k \leq n$, we set

$$c'_k \stackrel{\text{def}}{=} c_k^0 \circ c_0^k \in \mathcal{H} = \mathrm{End}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_0]\right)$$

where the morphisms

$$c_0^k \in \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_0], \mathbb{Z}[\mathbb{I}_k]\right) \quad \text{and} \quad c_k^0 \in \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_k], \mathbb{Z}[\mathbb{I}_0]\right)$$

are the correspondences respectively defined by

$$c_0^k \stackrel{\text{def}}{=} (\pi_0^k)_* \circ (\pi_k^0)^* \quad \text{and} \quad c_k^0 \stackrel{\text{def}}{=} (\pi_k^0)_* \circ (\pi_0^k)^*$$

where the morphisms

$$\begin{array}{lcllcl} (\pi_k^0)^* & \in & \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_0], \mathbb{Z}[\mathbb{I}_{0,k}]\right) & (\pi_0^k)^* & \in & \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_k], \mathbb{Z}[\mathbb{I}_{0,k}]\right) \\ (\pi_0^k)_* & \in & \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_{0,k}], \mathbb{Z}[\mathbb{I}_k]\right) & (\pi_k^0)_* & \in & \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}[\mathbb{I}_{0,k}], \mathbb{Z}[\mathbb{I}_0]\right) \end{array}$$

are the pull-back and push-forward maps induced by the G-equivariant projections with finite fibers $\pi_k^0 : \mathbb{I}_{0,k} \twoheadrightarrow \mathbb{I}_0$ and $\pi_0^k : \mathbb{I}_{0,k} \twoheadrightarrow \mathbb{I}_k$. Equivalently, we have

$$c_0^k(x) = \sum_{y \in \mathbb{I}_k(x)} y \quad \text{and} \quad c_k^0(y) = \sum_{z \in \mathbb{I}_0(y)} z$$

for any $x \in \mathbb{I}_0 \subset \mathbb{Z}[\mathbb{I}_0]$ and $y \in \mathbb{I}_k \subset \mathbb{Z}[\mathbb{I}_k]$, where

$$\mathbb{I}_k(x) \stackrel{\text{def}}{=} \pi_0^k\left(\left(\pi_k^0\right)^{-1}(x)\right) \quad \text{and} \quad \mathbb{I}_0(y) \stackrel{\text{def}}{=} \pi_0^k\left(\left(\pi_k^0\right)^{-1}(y)\right)$$

are the supports of $c_0^k(x)$ and $c_k^0(y)$. A simple calculation shows that

$$\mathbb{I}_k(x) = \left\{ y' \in \mathbb{I}_k : \mathbf{d}(x, y') = \left(\left(\tfrac{1}{2}\right)^k, 0^{n-k}\right)\right\},$$

$$\mathbb{I}_0(y) = \left\{ x' \in \mathbb{I}_0 : \mathbf{d}(y, x') = \left(\left(\tfrac{1}{2}\right)^k, 0^{n-k}\right)\right\}.$$

In particular for any $x \in \mathbb{I}_0$, $y \in \mathbb{I}_k(x)$ and $z \in \mathbb{I}_0(y)$,

$$\mathbf{d}(x, z) \preceq \mathbf{d}(x, y) + \mathbf{d}(y, z) = (1^k, 0^{n-k})$$

thus $\mathbf{d}(x, z) = (1^i, 0^{n-i})$ for some $i \in \{0, \cdots, k\}$. It follows that

$$c'_k = n_{k,k} \cdot c_k + \cdots + n_{k,0} \cdot c_0$$

for some integers $n_{k,i} \in \mathbb{N}$ that we shall now compute.

8.2.3. Recall that $q = |\mathbb{F}|$. For $d \in \mathbb{N}$, we define

$$\begin{array}{rcl} \pi(d) & \stackrel{\text{def}}{=} & (q^d - 1) \cdots (q - 1) \\ \sigma(d) & \stackrel{\text{def}}{=} & (q^d + 1) \cdots (q + 1) \\ \tau(d) & \stackrel{\text{def}}{=} & (q^d - (-1)^d) \cdots (q + 1) \\ \pi_2(d) & \stackrel{\text{def}}{=} & \pi(d)\sigma(d) \end{array}$$

with the conventions $\pi(1) = \sigma(1) = \tau(1) = \pi_2(1) = 1$. For $d = d_1 + \cdots + d_s$,

$$\frac{\pi(d)}{\pi(d_1) \cdots \pi(d_s)}, \quad \frac{\tau(d)}{\tau(d_1) \cdots \tau(d_s)}, \quad \frac{\pi_2(d)}{\pi_2(d_1) \cdots \pi_2(d_s)} \quad \text{and} \quad \frac{\tau(2d)}{\pi_2(d)}$$

are positive integers.

8.2.4. For a given $i \in \{0, \cdots, k\}$, the coefficient $n_{k,i}$ may be computed as follows. Fix $x, z \in \mathbb{I}_0$ with $\mathbf{d}(x,z) = (1^i, 0^{n-i})$. Then $n_{k,i}$ is the number of y's in $\mathbb{I}_k(x)$ such that $z \in \mathbb{I}_0(y)$. For $i = k$, we should have $\mathbf{d}(x,z) = \mathbf{d}(x,y) + \mathbf{d}(y,z)$, which implies that $d(x,z) = d(x,y) + d(y,z)$ since $\mathbf{d}(x,y) = \mathbf{d}(y,z)$, therefore y ought to be the middle point of the segment $[x,z]$ of $\mathcal{I}$. Conversely if y is the middle point of $[x,z]$, then $y \in \mathbb{I}_k(x)$ and $z \in \mathbb{I}_0(y)$, thus $n_{k,k} = 1$. Note that more generally for any $i \in \{0, \cdots, k\}$ and $\{x,y,z\}$ as above, there is an apartment of $\mathcal{I}$ containing the 1-dimensional facets $]x,y[$ and $]y,z[$, which thus also contains $\{x,y,z\}$.

Fix $x \in \mathbb{I}_0$ and $i \in \{0, \cdots, k\}$. There is a bijection $y \leftrightarrow x_y$ between $\mathbb{I}_k(x)$ and the set of all k-dimensional isotropic $\mathbb{F}$-subspaces of $S(x,0)$. Thus by [9, 9.4.1],

$$\deg(c_0^k) = |\mathbb{I}_k(x)| = \frac{\pi(n)}{\pi(k)\pi(n-k)} \cdot \frac{\sigma(n)}{\sigma(n-k)}.$$

Fix $y \in \mathbb{I}_k(x)$. Likewise, there is a bijection $z \leftrightarrow y_z$ between $\mathbb{I}_0(y)$ and the set of all k-dimensional (i.e. maximal) isotropic $\mathbb{F}$-subspaces of $S(y, \frac{1}{2})$, thus by [9, 9.4.1],

$$\deg(c_k^0) = |\mathbb{I}_0(y)| = 2 \cdot \sigma(k-1).$$

Under this bijection, a simple calculation in any apartment containing $\{x,y,z\}$ shows that $\mathbf{d}(x,z) = (1^i, 0^{n-i})$ if and only if the $\mathbb{F}$-dimension of $y_x \cap y_z$ equals $k-i$. By [9, 9.4.2], the number of all such z's (for a fixed y) equals

$$q^{\frac{i(i-1)}{2}} \cdot \frac{\pi(k)}{\pi(i)\pi(k-i)}.$$

It follows that the degree of $n_{k,i} \cdot c_i$ equals

$$\deg(n_{k,i} \cdot c_i) = q^{\frac{i(i-1)}{2}} \cdot \frac{\pi(n)}{\pi(i)\pi(k-i)\pi(n-k)} \cdot \frac{\sigma(n)}{\sigma(n-k)}.$$

In particular since $n_{k,k} = 1$,

$$\deg(c_k) = q^{\frac{k(k-1)}{2}} \cdot \frac{\pi(n)}{\pi(k)\pi(n-k)} \cdot \frac{\sigma(n)}{\sigma(n-k)}.$$

Since also $\deg(n_{k,i} \cdot c_i) = n_{k,i} \cdot \deg(c_i)$, we find that

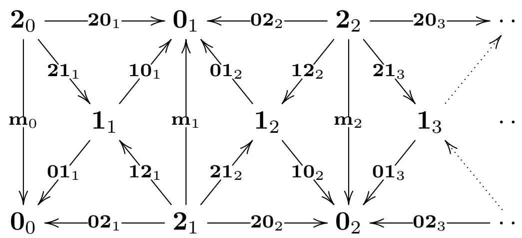
$$n_{k,i} = \frac{\pi(n-i)}{\pi(n-k)\pi(k-i)} \cdot \frac{\sigma(n-i)}{\sigma(n-k)} = \frac{\pi_2(n-i)}{\pi_2(n-k)\pi_2(k-i)} \cdot \sigma(k-i).$$

In particular $\sigma(k-i) \mid n_{k,i}$ for all $i \in \{0, \cdots, k\}$ and thus

$$c'_k \equiv c_k \bmod (q+1)\mathcal{H}.$$

8.3. **The tame distribution relations, I.**

8.3.1. The H-orbits in $\mathcal{I}_0$ or $\mathcal{I}$ are classified in [17, 4.1.4 & 5.1.11] by invariants with values in various sets of positive divisors on $\overline{\mathcal{L}} = [0,1] \times \mathbb{R}_+$, where $[0,1] \simeq \mathbb{R}/\sim$ should be viewed as a quotient (or fundamental domain) of $\mathbb{R}$ for the action of the affine group spanned by the reflections $\{i + x \mapsto i - x : i \in \mathbb{Z}\}$. Various special elements of $\overline{\mathcal{L}}$ were given a name in [17, 4.2.1], which will be used in the sequel. For instance for any $i \in \mathbb{N}$, the element with coordinate (i,i) in $\overline{\mathcal{L}} = \mathbb{R}/\sim \times \mathbb{R}_+$ is denoted by $\mathbf{0}_i$. These labels are shown on the following depiction of $\overline{\mathcal{L}}$, which the reader may use to visualize some of the definitions and results in [17] and this paper:



The H-orbits of edges correspond to divisors supported on any of these points. The H-orbits of vertices correspond to divisors supported on $\{\mathbf{0}_i, \mathbf{1}_{i+1}, \mathbf{2}_i : i \in \mathbb{N}\}$. The invariants of the source and target of an edge are obtained from the invariant of the edge by moving all points which are not already supported on this subset to the source and target of the corresponding arrows. For any x in $\mathcal{I}$, the corresponding invariant is a degree n positive divisor $w = \omega(x)$ on $\overline{\mathcal{L}}$, subject to the following condition. Write $w = 2w_{\mathrm{reg}} + w_{sp}$ where w_{sp} is multiplicity free. Then, there should exist a sequence $0 \leq c_1 < c_2 < \cdots < c_s$ in $\mathbb{R}_+$ such that

$$w_{sp} = [\gamma_1, c_1] + \cdots + [\gamma_s, c_s] \quad \text{with} \quad \gamma_1 = c_1, \quad \gamma_{i+1} = \gamma_i + (-1)^i (c_{i+1} - c_i).$$

We denote by $\mathbb{N}[\overline{\mathcal{L}}]^n_D$ the set of all such divisors. With these notations, the invariant of the convex projection x_0 of x to $\mathcal{I}_0$ is given by $\omega(x_0) = \underline{w} = 2w_{reg} + \underline{w}_{sp}$ where

$$\underline{w}_{sp} = \begin{cases} \sum_{i \equiv 1 \bmod 2} 2[\gamma_i, c_i] & \text{if } n \equiv 0 \bmod 2, \\ \sum_{i \equiv 0 \bmod 2} 2[\gamma_i, c_i] + \mathbf{0}_0 & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

If x is a vertex, all the c_i's belong to $\frac{1}{2}\mathbb{N}$ and the reduced type is computed as follows: we have $x \in \mathbb{I}_k$ where $k = \kappa(\overline{w})$ for the linear form κ on $\mathbb{Z}[\{\mathbf{0}_i, \mathbf{1}_{i+1}, \mathbf{2}_i : i \in \mathbb{N}\}]$ with $\kappa(\mathbf{0}_i) = 0$, $\kappa(\mathbf{1}_{i+1}) = 1$ and $\kappa(\mathbf{2}_i) = 2$ for all $i \in \mathbb{N}$, where

$$\overline{w} = w_{reg} + \sum_{i \equiv n \bmod 2} [\gamma_i, c_i].$$

For instance: for $n = 3$, there is an H-orbit of edges $e \in \mathcal{I}(\frac{1}{4})$ with invariant $\omega(e) = \mathbf{10}_1 + \mathbf{m}_1 + \mathbf{02}_2$. The source x and target y of any such e have invariants $\omega(x) = \mathbf{1}_1 + \mathbf{2}_1 + \mathbf{2}_2$ and $\omega(y) = 3 \cdot \mathbf{0}_1$, respectively belonging to $\mathbb{I}_3$ and $\mathbb{I}_0$ – thus e belongs to $\mathbb{I}_{0,3}$. The projections of e, x and y have invariants $\omega(e_0) = \mathbf{0}_0 + 2 \cdot \mathbf{m}_1$, $\omega(x_0) = \mathbf{0}_0 + 2 \cdot \mathbf{2}_1$ and $\omega(y_0) = \mathbf{0}_0 + 2 \cdot \mathbf{0}_1$, with $x_0 \in \mathbb{I}_2$, $y_0 \in \mathbb{I}_0$ and $e_0 \in \mathbb{I}_{0,2}$.

8.3.2. We now consider the action of $\mathcal{H}$ on the H-coinvariants of $\mathbb{Z}[\mathbb{I}_0]$,

$$\mathbb{Z}[H \backslash \mathbb{I}_0] \simeq \frac{\mathbb{Z}[\mathbb{I}_0]}{\langle (1-h)x : h \in H, \, x \in \mathbb{Z}[\mathbb{I}_0] \rangle}.$$

Under the invariant $\omega : H \backslash \mathcal{I}(\frac{1}{2}) \xrightarrow{\simeq} \mathcal{V}^n_D$ of [17, 5.1.11], we have

$$H \backslash \mathbb{I}_0 \simeq \left\{ \sum_{i \in \mathbb{N}} d_i \cdot \mathbf{0}_i : \, d_i \in \mathbb{N}, \, \sum_{i \in \mathbb{N}} d_i = n \right\}.$$

We denote by $\circ \in H \backslash \mathbb{I}_0 \subset \mathbb{Z}[H \backslash \mathbb{I}_0]$ the basic class with $\omega(\circ) = n \cdot \mathbf{0}_0$.

8.3.3. We claim that for every $k \in \{1, \cdots, n\}$, we have

$$c_k \cdot \circ \equiv 0 \bmod (q+1)\mathbb{Z}[H \backslash \mathbb{I}_0].$$

Since $c'_k \equiv c_k \bmod (q+1)\mathcal{H}$, this is equivalent to showing that

$$c'_k \cdot \circ \equiv 0 \bmod (q+1)\mathbb{Z}[H \backslash \mathbb{I}_0].$$

Recall that $c'_k = c_k^0 \circ c_0^k$ in $\mathcal{H}$ and still denote by

$$c_0^k : \mathbb{Z}[H \backslash \mathbb{I}_0] \to \mathbb{Z}[H \backslash \mathbb{I}_k] \quad \text{and} \quad c_k^0 : \mathbb{Z}[H \backslash \mathbb{I}_k] \to \mathbb{Z}[H \backslash \mathbb{I}_0]$$

the morphisms induced by the G-equivariant morphisms c_0^k and c_k^0 on the H-coinvariants. We thus have a factorization

$$\left(\mathbb{Z}[H \backslash \mathbb{I}_0] \xrightarrow{c'_k} \mathbb{Z}[H \backslash \mathbb{I}_0] \right) = \left(\mathbb{Z}[H \backslash \mathbb{I}_0] \xrightarrow{c_0^k} \mathbb{Z}[H \backslash \mathbb{I}_k] \xrightarrow{c_k^0} \mathbb{Z}[H \backslash \mathbb{I}_0] \right)$$

and it is sufficient to establish that

$$c_0^k(\circ) \equiv 0 \bmod (q+1)\mathbb{Z}[H \backslash \mathbb{I}_k].$$

Recall that $c_0^k = (\pi_0^k)_* \circ (\pi_k^0)^*$, which yields yet another factorization

$$\left(\mathbb{Z}[H \backslash \mathbb{I}_0] \xrightarrow{c_0^k} \mathbb{Z}[H \backslash \mathbb{I}_k] \right) = \left(\mathbb{Z}[H \backslash \mathbb{I}_0] \xrightarrow{(\pi_k^0)^*} \mathbb{Z}[H \backslash \mathbb{I}_{0,k}] \xrightarrow{(\pi_0^k)_*} \mathbb{Z}[H \backslash \mathbb{I}_k] \right)$$

and again, it is sufficient to establish that

$$(\pi_k^0)^*(\circ) \equiv 0 \bmod (q+1)\mathbb{Z}[H \backslash \mathbb{I}_{0,k}].$$

We thus have to show that for any $x \in \circ \subset \mathbb{I}_0$,

$$\sum_{y \in \mathbb{I}_k(x)} H \cdot (x, y) \equiv 0 \bmod (q+1)\mathbb{Z}[H \backslash \mathbb{I}_{0,k}].$$

For any $y_1, y_2 \in \mathbb{I}_k(x)$, we have $H \cdot (x, y_1) = H \cdot (x, y_2)$ if and only if $H_x \cdot y_1 = H_x \cdot y_2$, where G_x and $H_x = H \cap G_x$ are the stabilizers of x in G and H. The last displayed equation is therefore equivalent to the following claim:

Every orbit of H_x on $\mathbb{I}_k(x)$ has order divisible by $q+1$.

Since the bijection $y \leftrightarrow x_y$ from $\mathbb{I}_k(x)$ to the space of all k-dimensional isotropic $\mathbb{F}$-subspaces of $S(x, 0)$ is G_x-equivariant, the previous claim is itself equivalent to the following assertion:

Every orbit of $U(n, \mathbb{F})$ on the space of all k-dimensional isotropic $\mathbb{F}$-subspaces of the standard representation of $SO(2n+1, \mathbb{F})$ has order divisible by $q+1$.

We give a short proof in 8.3.4 and a more elaborate one in 8.3.5-8.3.9.

8.3.4. Set $\mathbb{V} = S(x, 0)$ and $\mathbb{W} = S(x_0, 0)$. Let $\mathcal{H}_k(\mathbb{V})$ and $\mathcal{H}_k(\mathbb{W})$ be the spaces of k-dimensional isotropic $\mathbb{F}$-subspaces in $\mathbb{V}$ and $\mathbb{W}$. Thus $\mathcal{H}_k(\mathbb{W}) \subset \mathcal{H}_k(\mathbb{V})$ and $H \mapsto H \cap \mathbb{W}$ defines a $U(\mathbb{W})$-equivariant projection $\mathcal{H}_k(\mathbb{V}) \setminus \mathcal{H}_k(\mathbb{W}) \to \mathcal{H}_{k-1}(\mathbb{W})$. Note also that $\mathcal{H}_1(\mathbb{V}) \setminus \mathcal{H}_1(\mathbb{W})$ is a single $U(\mathbb{W})$-orbit, of order $q^{n-1}(q^n - (-1)^n)$, which is divisible by $q+1$. It is thus sufficient to show that

For $k \geq 1$, every $U(\mathbb{W})$-orbit in $\mathcal{H}_k(\mathbb{W})$ has order divisible by $q+1$.

This follows from corollary 9.4 and [17, 4.3.9].

8.3.5. Fix $x \in \mathbb{I}_0$ and let $v = (x, y) \in \mathcal{I}(\frac{1}{4})$ be any edge, so that $y \in \mathbb{I}_k(x)$ for some $k \in \{0, \cdots, n\}$. Let $H_v = H_x \cap H_y$ be the stabilizer of v in H. We have to show that the index $[H_x : H_v]$ is divisible by $q+1$ unless $k = 0$ (i.e. $y = x$).

8.3.6. Suppose first that y (and thus also v) belongs to $\mathcal{I}_0$. Then $[H_x : H_v]$ is computed in [17, 4.3.9] as follows. Let $m = \lfloor \frac{n}{2} \rfloor$ be the Witt index of the hermitian space defining H, let $\omega : H\backslash\mathcal{I}_0 \xrightarrow{\simeq} \mathbb{N}[\overline{\mathcal{L}}]^m$ be the invariant of [17, 4.1.4], inducing

$$\omega : H\backslash\mathcal{I}_0(\tfrac{1}{2}) \xrightarrow{\simeq} \mathcal{V}^m \quad \text{and} \quad \omega : H\backslash\mathcal{I}_0(\tfrac{1}{4}) \xrightarrow{\simeq} \mathcal{E}^m$$

as in [17, 4.2.3], where $\mathcal{I}_0(\frac{1}{2}) = \mathcal{I}_0 \cap \mathcal{I}(\frac{1}{2})$ and $\mathcal{I}_0(\frac{1}{4}) = \mathcal{I}_0 \cap \mathcal{I}(\frac{1}{4})$. Thus $\omega(x) = m \cdot \mathbf{0}_0$,

$$\omega(v) = a \cdot \mathbf{0}_0 + b \cdot \mathbf{m}_0 + c \cdot \mathbf{01}_1 + d \cdot \mathbf{02}_1$$

for some $a, b, c, d \in \mathbb{N}$ with $a + b + c + d = m$,

$$\omega(y) = a \cdot \mathbf{0}_0 + b \cdot \mathbf{2}_0 + c \cdot \mathbf{1}_1 + d \cdot \mathbf{2}_1$$

and $2b + c + 2d = k$. Then $[H_x : H_v] = q^? \star$ for some power $? \in \mathbb{N}$ with

$$\begin{aligned} \star &= \frac{\tau(\Delta_0 + 2m)}{\pi(b)\pi(c)\pi(d)\sigma(b)\sigma(d)\tau(\Delta_0 + 2a)} \\ &= \frac{\tau(\Delta_0 + 2m)}{\tau(\Delta_0 + 2a)\tau(2b)\tau(2c)\tau(2d)} \times \frac{\tau(2b)}{\pi(b)\sigma(b)} \times \frac{\tau(2c)}{\pi(c)\sigma(c)} \times \frac{\tau(2d)}{\pi(d)\sigma(d)} \times \sigma(c) \end{aligned}$$

where $\Delta_0 = 0$ if n is even and $\Delta_0 = 1$ if n is odd. In this product, the five terms are positive integers. Since moreover for every $i \geq 1$,

$$\frac{\tau(2i)}{\pi(i)\sigma(i)} = \prod_{j=0}^{i-1} (q^{2j+1} + 1) \equiv 0 \bmod (q + 1),$$

we find that indeed $q + 1 \mid [H_x : H_v]$ unless $b = c = d = 0$, i.e. $k = 0$.

8.3.7. In the general case, let $y_0 \in \mathcal{I}_0$ be the convex projection of y. Then $y_0 \in \mathcal{I}_0(\frac{1}{2})$ is a vertex and in fact, $v_0 = (x, y_0)$ is yet another edge, i.e. $y \in \mathbb{I}_{k'}(x)$ for some $k' \in \{0, \cdots, n\}$; actually, $k' \leq k$ since the convex projection $\mathcal{I} \twoheadrightarrow \mathcal{I}_0$ is non-expanding. Since the convex projection is H-equivariant, $H_y \subset H_{y_0}$ and thus

$$H_v = H_x \cap H_y \subset H_{v_0} = H_x \cap H_{y_0} \subset H_x.$$

The index $[H_x : H_{v_0}]$ therefore divides $[H_x : H_v]$. By the previous case, we may thus assume that $y_0 = x$. We then have $[H_x : H_v] = [H_{y_0} : H_y]$.

8.3.8. Section 5.4 of [17] yields an algorithm to compute these kind of indices, but in the simple situation at hand, we can skip most of it. Indeed, under the invariant $\omega : H\backslash\mathcal{I}(\frac{1}{4}) \xrightarrow{\simeq} \mathcal{V}_D^n$ of [17, 5.1.11], we have

$$\omega(v) = a \cdot \mathbf{0}_0 + b \cdot \mathbf{m}_0 + c \cdot \mathbf{01}_1 + d \cdot \mathbf{02}_1$$

for some $a, b, c, d \in \mathbb{N}$ with $a + b + c + d = n$, $b \equiv 0 \bmod 2$ and $(1 - c)d \equiv 0 \bmod 2$. Then $\omega(y) = a \cdot \mathbf{0}_0 + b \cdot \mathbf{2}_0 + c \cdot \mathbf{1}_1 + d \cdot \mathbf{2}_1$ and

$$\omega(y_0) = a' \cdot \mathbf{0}_0 + b' \cdot \mathbf{2}_0 + c' \cdot \mathbf{1}_1 + d' \cdot \mathbf{2}_1$$

with $b = b'$ and

$$(a', c', d') = \begin{cases} (a, c, d) & \text{if } (c, d) \equiv (0, 0) \bmod 2, \\ (a + 1, c - 1, d) & \text{if } (c, d) \equiv (1, 0) \bmod 2, \\ (a, c + 1, d - 1) & \text{if } (c, d) \equiv (1, 1) \bmod 2. \end{cases}$$

Under our assumption, $y_0 = x$, thus $\omega(y_0) = \omega(x) = n \cdot \mathbf{0}_0$. It follows that $b = b' = 0$, $c' = 0$ thus $c' \neq c + 1$, hence $d = d' = 0$ and either $c = 0$ or $c = 1$. In other words,

$$\omega(v) = n \cdot \mathbf{0}_0 \quad \text{or} \quad \omega(v) = (n - 1) \cdot \mathbf{0}_0 + \mathbf{01}_1.$$

In the first case, $y = x$ and $k = 0$. We may thus furthermore restrict our attention to the second case, where $\omega(y) = (n-1)\cdot \mathbf{0}_0 + \mathbf{1}_1$ and $k = 1$.

8.3.9. This finally corresponds to a simple case of [17, 5.4.6], itself a very special case of [17, 5.4.3]. Switching back to orbits of isotropic spaces in odd dimensional quadratic spaces over $\mathbb{F}$, the relevant orbit is the unique $U(n,\mathbb{F})$-orbit of isotropic $\mathbb{F}$-lines in a quadratic space of dimension $2n+1$ which are not contained in the hermitian $\mathbb{F}$-hyperplane defining $U(n,\mathbb{F}) \subset SO(2n+1,\mathbb{F})$. The stabilizer of any point in this orbit is isomorphic to $U(n-1,\mathbb{F})$, and the size of the orbit thus equals

$$\frac{|U(n,\mathbb{F})|}{|U(n-1,\mathbb{F})|} = q^{n-1}\frac{\tau(n)}{\tau(n-1)} = q^{n-1}\cdot (q^n - (-1)^n).$$

This is indeed again divisible by $q+1$.

8.4. The tame distribution relations, II.

8.4.1. Let $U^1 = U(1)(F)$. The determinant yields an isomorphism

$$\det : H/H^1 \xrightarrow{\simeq} U^1.$$

We now consider the $\mathcal{H}[U^1]$-module of H^1-coinvariants in $\mathbb{Z}[\mathbb{I}_0]$,

$$\mathbb{Z}[H^1\backslash \mathbb{I}_0] \simeq \frac{\mathbb{Z}[\mathbb{I}_0]}{\langle (1-h)x : h\in H^1,\, x\in \mathbb{Z}[\mathbb{I}_\circ]\rangle}.$$

8.4.2. By [17, 5.1.14], for any x in $\mathcal{I}$, there is a unique integer $c\in\mathbb{N}$ such that

$$\det(H_x) = U^1(c) \stackrel{\text{def}}{=} \left\{\lambda/\overline{\lambda} : \lambda\in\mathcal{O}_{E,c}^\times\right\}$$

where $\mathcal{O}_{E,c} = \mathcal{O}_F + \mathfrak{m}_F^c\mathcal{O}_E$ is the order of conductor c in $\mathcal{O}_E$. We call $c = c(x)$ the conductor of x. It is related to the invariant $\omega : H\backslash\mathcal{I} \xrightarrow{\simeq} \mathbb{N}[\overline{\mathcal{L}}]^n_D$ of [17, 5.1.11] by

$$c(x) = \min\{\lceil c_i\rceil : i = 1,\cdots,n\} \quad \text{where} \quad \omega(x) = \sum_{i=1}^n [\theta_i, c_i].$$

For any x in the U^1-stable $\mathbb{Z}$-basis $H^1\backslash\mathbb{I}_0$ of $\mathbb{Z}[H^1\backslash\mathbb{I}_0]$, the stabilizer U^1_x of x in U^1 thus equals $U^1(c)$ where $c = c(x)$, the conductor of x, is related to the invariant $\omega(x) = \sum n_i\cdot\mathbf{0}_i$ by the formula $c(x) = \min\{i : n_i\neq 0\}$. In particular, x is fixed by the whole of U^1 (i.e. x is already an H-orbit) if and only if $c(x) = 0$. The basic point $\circ\in H^1\backslash\mathbb{I}_0$ is the basic H-orbit with $\omega(\circ) = n\cdot\mathbf{0}_0$, thus $c(\circ) = 0$.

8.4.3. We consider the Hecke submodules of $U^1(0) = U^1$ and $U^1(1)$-invariants:

$$\mathbb{Z}[H^1\backslash\mathbb{I}_0]^{U^1(0)} \subset \mathbb{Z}[H^1\backslash\mathbb{I}_0]^{U^1(1)} \subset \mathbb{Z}[H^1\backslash\mathbb{I}_0]$$

The first $\mathcal{H}$-module has a natural $\mathbb{Z}$-basis $\{e_x : x\in H\backslash\mathbb{I}_0\}$, where $e_x = \sum_{y\in H^1\backslash x} y$ is the sum of the H^1-orbits in x, viewed as an H-orbit $x\subset\mathbb{I}_0$. We denote by

$$\mathrm{Tr} : \mathbb{Z}[H^1\backslash\mathbb{I}_0]^{U^1(1)} \longrightarrow \mathbb{Z}[H^1\backslash\mathbb{I}_0]^{U^1(0)}$$

the trace map along the finite cyclic group $U^1(0)/U^1(1)\simeq\mathbb{E}^\times/\mathbb{F}^\times$ of order $q+1$.

8.4.4. We claim that:

For every $k \in \{1, \cdots, n\}$*, there is a positive element*

$$\circ_k \in \mathbb{N}[H^1\backslash\mathbb{I}_0]^{U^1(1)} \subset \mathbb{Z}[H^1\backslash\mathbb{I}_0]^{U^1(1)}$$

such that

$$\mathrm{Tr}(\circ_k) = c_k \cdot \circ \quad \textit{in} \quad \mathbb{Z}[H^1\backslash\mathbb{I}_0]^{U^1(0)}.$$

Indeed, since $\circ$ is U^1-stable, so is $c_k \cdot \circ$ and we may thus write $c_k \cdot \circ = \sum i_{k,x} \cdot e_x$ with $x \in H\backslash\mathbb{I}_0$ and $i_{k,x} \in \mathbb{N}$. Passing to the U^1-coinvariants, this becomes

$$c_k \cdot \circ = \sum i_{k,x} s_x \cdot x \quad \text{with} \quad s_x = \left|U^1/U^1(c(x))\right|$$

where x is now viewed as an element of the $\mathbb{Z}$-basis $H\backslash\mathbb{I}_0$ of $\mathbb{Z}[H\backslash\mathbb{I}_0]$. We have seen that $q+1 \mid i_{k,x}s_x$ for all x. This is actually obvious for any x with $c(x) \geq 1$, since then $q+1 \mid s_x$; but for every x with $c(x) = 0$, $s_x = 1$ and thus $q+1 \mid i_{k,x}$. For any x with $c(x) \neq 0$, viewed as a U^1-orbit in $H^1\backslash\mathbb{I}_0$, pick any $U^1(1)$-orbit $\tilde{x}$ in x and view it as an element of $\mathbb{N}[H^1\backslash\mathbb{I}_0]^{U^1(1)}$. Then $\mathrm{Tr}(\circ_k) = c_k \cdot \circ$ with

$$\circ_k = \sum_{c(x)=0} \tfrac{i_{k,x}}{q+1} \cdot e_x + \sum_{c(x)>0} i_{k,x} \cdot \tilde{x}.$$

8.4.5. The element $\circ_k \in \mathbb{N}[H^1\backslash\mathbb{I}_0]^{U^1(1)}$ such that $\mathrm{Tr}(\circ_k) = c_k \cdot \circ$ is unique up to the choice of a $U^1(1)$-orbit $\tilde{x}$ in x, for each x as above with $c(x) \geq 1$ and $i_{k,x} \neq 0$. The support of $c_k \cdot \circ$ is contained in $\{a \cdot \mathbf{0}_0 + b \cdot \mathbf{0}_1 + c \cdot \mathbf{0}_2 : a + b + c = n\}$, and typically contains more than one element of the form $b \cdot \mathbf{0}_1 + c \cdot \mathbf{0}_2$ (with $b + c = n$), in which case even the U^1 orbit of $\circ_k$ is not uniquely determined by our relation.

Part 3. **Appendices**

9. Hermitian and quadratic spaces

9.1. **Conventions.** Let E be a quadratic extension of a field F of characteristic $\neq 2$. We adopt the following conventions regarding quadratic and hermitian spaces. A quadratic space over F is a finite dimensional F-vector space V equipped with a non-degenerate quadratic form $Q : V \to F$, or equivalently, with a non-degenerate symmetric bilinear form $\varphi : V \times V \to F$. Here φ and Q are related as follows:

$$\varphi(x,y) = Q(x+y) - Q(x) - Q(y) \quad \text{and} \quad 2Q(x) = \varphi(x,x).$$

An hermitian space over E is a finite dimensional E-vector space W equipped with a non-degenerate hermitian form $\psi : W \times W \to E$. The latter should be left semi-linear and $*$-symmetric, where $*$ is the non-trivial involution of E/F:

$$\psi(\lambda x, y) = \lambda^* \psi(x,y) = \psi(x, \lambda^* y) \quad \text{and} \quad \psi(y,x) = \psi(x,y)^*.$$

The underlying quadratic space is the finite dimensional F -vector space underlying W, equipped with the symmetric bilinear form and quadratic form given by

$$\varphi(x,y) = \mathrm{tr}_{E/F}\psi(x,y) = \psi(x,y) + \psi(y,x) \quad \text{and} \quad Q(x) = \psi(x,x).$$

We will use the following invariants for quadratic and hermitian spaces: dimension ($\dim_F$, in $\mathbb{N}$), discriminant (disc_F, in $F^\times/(F^\times)^2$), Witt index (witt_F, in $\mathbb{N}$), signatures (sign_v for $v : F \hookrightarrow \mathbb{R}$, in $\mathbb{N}^2$) and Hasse invariant (hasse_F, in $\mathrm{Br}(F)[2]$) for quadratic spaces; dimension ($\dim_E$, in $\mathbb{N}$), discriminant (disc_E, in $F^\times/\mathrm{N}_{E/F}(E^\times)$) and signatures ($\mathrm{sign}_v$ for $v : F \hookrightarrow \mathbb{R}$, in $\mathbb{N}^2$) for hermitian spaces, as defined in [57].

We denote by $\mathbf{O}(V,\varphi)$, $\mathbf{SO}(V,\varphi)$, $\mathbf{U}(W,\psi)$ and $\mathbf{SU}(W,\psi)$ the orthogonal, special orthogonal, unitary and special unitary reductive groups over F attached to (V,φ) and (W,ψ). We denote by $O(V,\varphi)$, $SO(V,\varphi)$, $U(W,\psi)$ and $SU(W,\psi)$ the corresponding groups of F-valued points. As usual for $\lambda_1,\cdots,\lambda_r\in F^\times$, we denote by $\langle\lambda_1,\cdots,\lambda_r\rangle$ the quadratic space $V=F^r$ with the quadratic form $Q(x_1,\cdots,x_r)=\sum\lambda_i x_i^2$. We denote by $\delta_{E/F}\in F^\times/(F^\times)^2$ the discriminant of E/F, i.e. the discriminant of the quadratic space $(E,\mathrm{N}_{E/F})$ over F. We fix an element $\eta\in E^\times$ with $\eta+\eta^*=0$. Thus $\eta^2=-\eta\eta^*$ lifts $\delta_{E/F}$ to $F^\times$, $E=F\oplus F\eta$ and $(E,\mathrm{N}_{E/F})\simeq\langle 1,-\eta^2\rangle$. We denote by (V_E,φ_E) the base change of (V,φ) to a quadratic space over E.

9.2. **Hermitian structures.** Let (V,φ) be a quadratic space over F. An E-hermitian structure on (V,φ) is an E-hermitian space (W,ψ) whose underlying quadratic space equals (V,φ). In other words, it is a pair (ι,ψ) where $\iota:E\to\mathrm{End}_F(V)$ is a morphism of F-algebras and $\psi:V\times V\to E$ is a non-degenerate E-hermitian form for the E-vector space structure on V defined by ι, such that $\varphi=\mathrm{tr}_{E/F}\circ\psi$ on V. Let $*$ be the involution of $\mathrm{End}_F(V)$ which is induced by φ. There is a bijection $(\iota,\psi)\mapsto\iota$ between E-hermitian structures on (V,φ) and morphisms $\iota:(E,*)\to(\mathrm{End}_F(V),*)$ of F-algebras with involutions. One retrieves ψ from ι as follows: for any $x,y\in V$, $\psi(x,y)$ is the unique element of E such that $\mathrm{tr}_{E/F}(\lambda\psi(x,y))$ equals $\varphi(x,\iota(\lambda)y)$ for every $\lambda\in E$.

Proposition 9.1. *There is an E-hermitian structure on (V,φ) if and only if*

$$\dim_F V=2n,\quad \mathrm{disc}_F(V,\varphi)=\delta_{E/F}^n\quad\textit{and}\quad \mathrm{witt}_E(V_E,\varphi_E)=n$$

for some $n\in\mathbb{N}$, in which case all such structures are conjugate under $O(V,\varphi)$.

Proof. The first claim is due to Milnor and Husemoller, see [46, Appendix 2] or [38]. The second claim is a reformulation of an even older result of Jacobson [28]. □

9.3. **Hermitian subspaces.** An E-hermitian subspace of (V,φ) is a non-degenerate F-subspace W of V together with an E-hermitian structure on $(W,\varphi|_W)$. An E-hermitian F-hyperplane in (V,φ) is an E-hermitian subspace whose F-codimension equals one.

Proposition 9.2. *There is an E-hermitian F-hyperplane in (V,φ) if and only if*

$$\dim_F V=2n+1\quad\textit{and}\quad \mathrm{witt}_E(V_E,\varphi_E)=n$$

for some $n\in\mathbb{N}$, in which case all such structures are conjugate under $SO(V,\varphi)$.

Proof. Suppose that $V=W\perp D$ for some E-hermitian F-hyperplane W of V. Set $n=\dim_E W$. Then $\dim_F V=2n+1$ and $\mathrm{witt}_E(V_E,\varphi_E)=n$ by Proposition 9.1. Conversely, suppose that $\dim_F V=2n+1$ and $\mathrm{witt}_E(V_E,\varphi_E)=n$ for some $n\in\mathbb{N}$. Set $\delta_V=\mathrm{disc}_F(V,\phi)$, $D'=\langle-\delta_V\delta^{n+1}\rangle$ and $W'=V\perp D'$. Then

$$\dim_F W'=2(n+1),\quad \mathrm{disc}_F W'=\delta^{n+1}\quad\text{and}\quad \mathrm{witt}_E W'_E=n+1$$

since $D'_E\simeq\langle-\delta_V\rangle$ and $V_E\simeq\langle 1,-1\rangle^n\perp\langle\delta_V\rangle$ by assumption. Proposition 9.1 then implies that W' is the quadratic F-space that underlies some E-hermitian structure on W'. For any such structure, the orthogonal complement of the anisotropic E-line spanned by D' is an E-hermitian F-hyperplane $W=(E\cdot D')^\perp$ in $V=D'^\perp$.

Finally, suppose that (W_1,ψ_1) and (W_2,ψ_2) are two E-hermitian F-hyperplanes in (V,φ). The F-quadratic spaces underlying W_1 and W_2 have the same dimension and discriminant, so their orthogonal F-complements also have the same discriminant; being one dimensional, they are actually isomorphic. It follows that W_1 and W_2 are also isomorphic as F-quadratic spaces (by Witt's theorem) and as E-hermitian spaces (by Jacobson's theorem). Any isomorphism $(W_1, \psi_1) \simeq (W_2, \psi_2)$, viewed as an isomorphism of the underlying quadratic spaces, extends (by Witt's theorem) to an automorphism of (V, φ), which we may easily arrange to have determinant 1. This yields an element of $SO(V, \varphi)$ conjugating (W_1, ψ_1) to (W_2, ψ_2). $\square$

9.4. **Totally isotropic F-subspaces of E-hermitians spaces.** Let (W, ψ) be an E-hermitian space with underlying quadratic space (V, φ) over F and Witt index $m = \mathrm{witt}_E(W, \psi)$. We say that a family $(e_{\pm 1}, \cdots, e_{\pm m})$ of elements in W is an almost Witt E-basis of (W, ψ) if $\psi(e_i, e_j) = 0$ for $j \neq -i$ in $\{\pm 1, \cdots, \pm m\}$ and $\eta\psi(e_i, e_{-i}) = 1$ for $i \in \{1, \cdots, m\}$. Such families exist, they are all conjugate under $U(W, \psi)$ and each of them spans over E a non-degenerate E-hermitian subspace of (W, ψ) whose orthogonal complement is anisotropic.

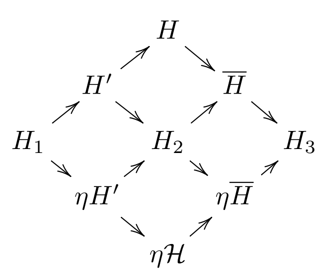
Proposition 9.3. *For any totally isotropic F-subspace H of (V, φ), there is an almost Witt E-basis $(e_{\pm 1}, \cdots, e_{\pm m})$ of (W, ψ) such that*

$$\{e_1, \cdots, e_{a+b+c}\} \cup \{\eta e_1, \cdots, \eta e_a\} \cup \left\{e_{-(a+b+1)}, \cdots, e_{-(a+b+c)}\right\}$$

is an F-basis of H, where $a, b, c \in \mathbb{N}$ are uniquely determined by

$$\dim_E \begin{pmatrix} H \cap \eta H \\ H \cap \eta H^{\perp} + H^{\perp} \cap \eta H \\ H + \eta H \end{pmatrix} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Proof. We consider the following diagram



$$H_1 = H \cap \eta H$$
$$H_2 = H \cap \eta H^{\perp} + H^{\perp} \cap \eta H$$
$$H_3 = H + \eta H$$
$$H' = H \cap H_2 = H \cap \eta H^{\perp}$$
$$\overline{H} = H + H_2$$

Note that H_1 and $H_2 = H_3 \cap H_3^{\perp}$ are totally isotropic E-subspaces of (W, ψ), H_3/H_2 is a non-degenerate E-subspace of $H_2^{\perp}/H_2$, $\overline{H}/H_2$ is a maximal totally isotropic F-subspace of H_3/H_2 and the formula $\theta = \mathrm{tr}_{E/F}(\eta\psi)$ defines a non-degenerate symplectic pairing on H_3/H_2 whose restriction to $\overline{H}/H_2$ is also non-degenerate. Any F-basis of H'/H_1 is an E-basis of H_2/H_1 and any symplectic F-basis of $H/H' \simeq \overline{H}/H_2$ is a Witt E-basis of H_3/H_2. Thus there is an F-basis

$$\{e_1, \cdots, e_a, \eta e_1, \cdots, \eta e_a\} \cup \{e_{a+1}, \cdots, e_{a+b}\} \cup \left\{e_{\pm(a+b+1)}, \cdots, e_{\pm(a+b+c)}\right\}$$

of H such that $\{e_1, \cdots, e_a\}$ is an E-basis of H_1, $\{e_1, \cdots, e_{a+b}\}$ is an E-basis of H_2, and $\left\{e_{\pm(a+b+1)}, \cdots, e_{\pm(a+b+c)}\right\}$ maps to a a Witt E-basis of H_3/H_2. Plainly,

$$\{e_1, \cdots, e_{a+b+c}\} \cup \left\{e_{-(a+b+1)}, \cdots, e_{-(a+b+c)}\right\}$$

extends to an almost Witt E-basis $(e_{\pm 1}, \cdots, e_{\pm m})$ of (W, ψ) as desired. $\square$

Corollary 9.4. *For every $k \in \{0, \cdots, 2m\}$, the previous proposition yields a bijection between the set of $U(W,\psi)$-orbits of totally isotropic F-subspaces of (V,φ) of F-dimension k and the set of triples $(a,b,c) \in \mathbb{N}$ with $2a+b+2c = k$ and $a+b+c \leq m$.*

10. The Hecke polynomial

Let F be a local field with ring of integers $\mathcal{O} \subset F$, uniformizer $\pi \in \mathcal{O}$ and residue field $k = \mathcal{O}/\pi\mathcal{O}$ with q_F elements. We normalize the valuation on F by $q_F\,|\pi| = 1$.

10.1. **The Satake isomorphism.**

10.1.1. Let $\mathbf{G}$ be a reductive group over $\mathcal{O}$. By [23, XXVI 7.15] and Lang's theorem [36], $\mathbf{G}$ is quasi-split over $\mathcal{O}$: there is maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi decomposition $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. Let $\mathbf{S}$ be the maximal split subtorus of $\mathbf{T}$, $X_*(\mathbf{S})$ the group of cocharacters of $\mathbf{S}$ and $X_*^+(\mathbf{S}) \subset X_*(\mathbf{S})$ the cone of $\mathbf{B}$-dominant cocharacters, i.e. those which act on Lie($\mathbf{B}$) with non-negative weights. There are Cartan and Iwasawa decompositions

$$\mathbf{G}(F) = \coprod_{\mu \in X_*^+(\mathbf{S})} \mathbf{G}(\mathcal{O})\pi^\mu \mathbf{G}(\mathcal{O}) \quad \text{and} \quad \mathbf{G}(F) = \coprod_{\mu \in X_*(\mathbf{S})} \mathbf{U}(F)\pi^\mu \mathbf{G}(\mathcal{O})$$

where $\pi^\mu = \mu_F(\pi) \in \mathbf{S}(F)$ for $\mu : \mathbf{G}_{m,\mathcal{O}} \to \mathbf{S}$ in $X_*(\mathbf{S})$ [10, §4]. We set

$$\mathcal{H}(\mathbf{G}) \stackrel{\text{def}}{=} \mathrm{End}_{\mathbb{Z}[\mathbf{G}(F)]}\left(\mathbb{Z}[\mathbf{G}(F)/\mathbf{G}(\mathcal{O})]\right).$$

This is the Hecke algebra of $\mathbf{G}$. By Shapiro's lemma,

$$\mathcal{H}(\mathbf{G}) \simeq \mathrm{Hom}_{\mathbb{Z}[\mathbf{G}(\mathcal{O})]}\left(\mathbb{Z}, \mathbb{Z}[\mathbf{G}(F)/\mathbf{G}(\mathcal{O})]\right) \simeq \mathbb{Z}[\mathbf{G}(\mathcal{O})\backslash \mathbf{G}(F)/\mathbf{G}(\mathcal{O})].$$

Write $c_\mu = c_\mu^{\mathbf{G}} \in \mathcal{H}(\mathbf{G})$ for the Hecke operator corresponding to $\mathbf{G}(\mathcal{O})\pi^\mu\mathbf{G}(\mathcal{O})$. It acts on $\mathbb{Z}[\mathbf{G}(F)/\mathbf{G}(\mathcal{O})]$ by sending $[g\mathbf{G}(\mathcal{O})]$ to $\sum_{i \in I(\mu)} [gk_i\pi^\mu\mathbf{G}(\mathcal{O})]$ where

$$\mathbf{G}(\mathcal{O})\pi^\mu\mathbf{G}(\mathcal{O}) = \coprod_{i \in I(\mu)} k_i \pi^\mu \mathbf{G}(\mathcal{O}).$$

Restricting the $\mathbf{G}(F)$-actions to $\mathbf{B}(F)$ and passing to the $\mathbf{U}(F)$-coinvariants on the $\mathbb{Z}[\mathbf{B}(F)]$-module $\mathbb{Z}[\mathbf{G}(F)/\mathbf{G}(\mathcal{O})]$, we obtain a ring homomorphism

$$\mathrm{End}_{\mathbb{Z}[\mathbf{G}(F)]}\left(\mathbb{Z}[\mathbf{G}(F)/\mathbf{G}(\mathcal{O})]\right) \to \mathrm{End}_{\mathbb{Z}[\mathbf{T}(F)]}\left(\mathbb{Z}[\mathbf{U}(F)\backslash\mathbf{G}(F)/\mathbf{G}(\mathcal{O})]\right).$$

Since $\mathbf{T}(F)/\mathbf{T}(\mathcal{O}) \simeq \mathbf{U}(F)\backslash\mathbf{G}(F)/\mathbf{G}(\mathcal{O})$, this yields the twisted Satake transform

$$\mathcal{S}^\bullet : \mathcal{H}(\mathbf{G}) \to \mathcal{H}(\mathbf{T}).$$

It is given by $\mathcal{S}^\bullet(c_\mu^{\mathbf{G}}) = \sum n_{\mu,\nu} c_\nu^{\mathbf{T}}$ where for every $\nu \in X_*(\mathbf{S})$, $n_{\mu,\nu} = |I(\mu,\nu)|$ with

$$I(\mu,\nu) \stackrel{\text{def}}{=} \left\{ i \in I(\mu) : \mathbf{U}(F)\pi^\nu\mathbf{G}(\mathcal{O}) = \mathbf{U}(F)k_i\pi^\mu\mathbf{G}(\mathcal{O}) \right\}.$$

10.1.2. Let $\rho \in X^*(\mathbf{T})\otimes\mathbb{Q}$ be the half-sum of all positive roots of $(\mathbf{T},\mathbf{B})$. Let du be the left invariant Haar measure on $\mathbf{U}(F)$ giving $\mathbf{U}(\mathcal{O})$ volume 1. Let $\delta : \mathbf{T}(F) \to \mathbb{R}^\times_>$ be the continuous character such that $d(tut^{-1}) = \delta(t)d(u)$. It factors through $\mathbf{T}(F)/\mathbf{T}(\mathcal{O}) \simeq X_*(\mathbf{S})$ and the corresponding morphism $X_*(\mathbf{S}) \to \mathbb{R}^\times_>$ maps ν to

$$\delta(\pi^\nu) = |\det(\mathrm{ad}(\pi^\nu) : \mathrm{Lie}\,\mathbf{U}(F))| = q_F^{-\langle 2\rho,\nu\rangle}.$$

It follows that viewing our Hecke operators as bi-invariant functions on $\mathbf{G}(F)$,

$$\begin{aligned}\int_{\mathbf{U}(F)} c^{\mathbf{G}}_\mu(u\pi^\nu)du &= \sum_{i\in I(\mu)} \int_{\mathbf{U}(F)} \mathbf{1}_{k_i\pi^\mu\mathbf{G}(\mathcal{O})}(u\pi^\nu)du \\ &= \sum_{i\in I(\mu,\nu)} \int_{\mathbf{U}(F)} \mathbf{1}_{u_i\pi^\nu\mathbf{G}(\mathcal{O})}(u\pi^\nu)du \\ &= \sum_{i\in I(\mu,\nu)} \int_{\mathbf{U}(F)} \mathbf{1}_{\pi^\nu\mathbf{G}(\mathcal{O})\pi^{-\nu}}(u_i^{-1}u)du \\ &= n_{\mu,\nu}\cdot\delta(\pi^\nu).\end{aligned}$$

In the above equations, $u_i \in \mathbf{U}(F)$ with $u_i\pi^\nu\mathbf{G}(\mathcal{O}) = k_i\pi^\mu\mathbf{G}(\mathcal{O})$ for $i \in I(\mu,\nu)$. Thus more generally for every $f \in \mathcal{H}(\mathbf{G})$, $\mathcal{S}^\bullet(f) = \sum n_{f,\nu}c^{\mathbf{T}}_\nu$ in $\mathcal{H}(\mathbf{T})$ with

$$n_{f,\nu} = \delta^{-1}(\pi^\nu)\int_{\mathbf{U}(F)} f(u\pi^\nu)du = \int_{\mathbf{U}(F)} f(\pi^\nu u)du \quad \text{in} \quad \mathbb{Z}.$$

10.1.3. Set $h = 1$ if $\rho|_{\mathbf{S}} \in X^*(\mathbf{S})$ and $h = \frac{1}{2}$ otherwise, so that $\langle\rho,\nu\rangle \in h\mathbb{Z}$ for all $\nu \in X_*(\mathbf{S})$. The standard Satake transform is the ring homomorphism

$$\mathcal{S} : \mathcal{H}(\mathbf{G}) \otimes \mathbb{Z}[q_F^{\pm h}] \to \mathcal{H}(\mathbf{T}) \otimes \mathbb{Z}[q_F^{\pm h}]$$

which is obtained by composing $\mathcal{S}^\bullet$ with the ring automorphism of

$$\mathcal{H}(\mathbf{T}) \otimes \mathbb{Z}[q_F^{\pm h}] \simeq \mathbb{Z}[q_F^{\pm h}][X_*(\mathbf{S})]$$

mapping ν (or $c^{\mathbf{T}}_\nu$) to $q_F^{-\langle\rho,\nu\rangle}\nu$ (or $q_F^{-\langle\rho,\nu\rangle}c^{\mathbf{T}}_\nu$). Thus $\mathcal{S}(f) = \sum s_{f,\nu}c^{\mathbf{T}}_\nu$ with

$$s_{f,\nu} = \delta^{-1/2}(\pi^\nu)\int_{\mathbf{U}(F)} f(u\pi^\nu)du = \delta^{1/2}(\pi^\nu)\int_{\mathbf{U}(F)} f(\pi^\nu u)du \quad \text{in} \quad \mathbb{Z}[q_F^{\pm h}].$$

By [56], this standard Satake morphism does not depend upon the chosen Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$, and it induces a ring isomorphism

$$\mathcal{S}_{\mathbb{Q}} : \mathcal{H}(\mathbf{G}) \otimes \mathbb{Q}[q_F^{\pm h}] \xrightarrow{\simeq} \left(\mathcal{H}(\mathbf{T}) \otimes \mathbb{Q}[q_F^{\pm h}]\right)^{\mathbf{\Omega}(F)}$$

where $\mathbf{\Omega} = \mathbf{N_G}(\mathbf{T})/\mathbf{T}$ is the Weyl group of $\mathbf{T}$ in $\mathbf{G}$.

10.1.4. Fix an algebraic closure $\overline{F}$ of F, let F^{ur} be the maximal unramified extension of F in $\overline{F}$, so that $\Gamma \overset{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{ur}}/F) = \mathrm{Gal}(\overline{k}/k)$ where $\overline{k}$, an algebraic closure of k, is the residue field of the ring of integers $\mathcal{O}^{\mathrm{ur}}$ of F^{ur}. Let $\sigma \in \Gamma$ be the Frobenius of F. By [23, XXVI 7.15], $\mathbf{G}$ splits over F^{ur}. Let $1 \hookrightarrow \mathbf{G}^\vee \to {}^L\mathbf{G} \to \Gamma \to 1$ be the Langlands dual of $\mathbf{G}$ and fix a Γ-invariant pinning $(\mathbf{T}^\vee, \mathbf{B}^\vee, \cdots)$ of $\mathbf{G}^\vee$, so that ${}^L\mathbf{G} = \mathbf{G}^\vee \rtimes \Gamma$. The resulting Γ-equivariant isomorphism $X_*(\mathbf{T}) \simeq X^*(\mathbf{T}^\vee)$ identifies the Γ-group $\mathbf{\Omega}(\overline{F})$ with the Weyl group of $\mathbf{T}^\vee$ in $\mathbf{G}^\vee$, and the subgroup $X_*(\mathbf{S}) = X_*(\mathbf{T})^\Gamma$ of $X_*(\mathbf{T})$ with $X^*(\mathbf{S}^\vee)$, where $\mathbf{S}^\vee = \mathbf{T}^\vee/(\sigma-1)\mathbf{T}^\vee$. Thus

$$\mathbf{T}^\vee = \mathrm{Spec}(\mathbb{C}[X^*(\mathbf{T}^\vee)]) = \mathrm{Spec}(\mathbb{C}[X_*(\mathbf{T})]),$$

$$\mathbf{S}^\vee = \mathrm{Spec}(\mathbb{C}[X_*(\mathbf{S})]) \quad \text{and} \quad \mathbf{S}^\vee/\mathbf{\Omega}(F) = \mathrm{Spec}\left(\mathbb{C}[X_*(\mathbf{S})]^{\mathbf{\Omega}(F)}\right).$$

The morphisms $\mathbf{G}^\vee \hookleftarrow \mathbf{T}^\vee \twoheadrightarrow \mathbf{S}^\vee \twoheadrightarrow \mathbf{S}^\vee/\mathbf{\Omega}(F)$ also identify $\mathbb{C}[X_*(\mathbf{S})]^{\mathbf{\Omega}(F)}$ with the algebra of regular functions on $\mathbf{G}^\vee$ which are invariant under σ-conjugation. Note also that $\mathbb{C}[X_*(\mathbf{S})] \simeq \mathcal{H}(\mathbf{T}) \otimes \mathbb{C}$. Thus using the Satake isomorphism

$$\mathcal{S}_\mathbb{C} : \mathcal{H}(\mathbf{G}) \otimes \mathbb{C} \xrightarrow{\simeq} (\mathcal{H}(\mathbf{T}) \otimes \mathbb{C})^{\mathbf{\Omega}(F)}$$

any such function may be viewed as an element of $\mathcal{H}(\mathbf{G}) \otimes \mathbb{C}$.

10.2. The Hecke Polynomial.

10.2.1. Let μ be a conjugacy class of cocharacters of $\mathbf{G}_{\overline{F}}$. We also denote by $\mu \in X_*(\mathbf{T})$ the unique $\mathbf{B}_{\overline{F}}$-dominant cocharacter of $\mathbf{T}_{\overline{F}}$ in this conjugacy class. Both variants of μ have the same field of definition, a finite unramified extension $F(\mu) \subset F^{\mathrm{ur}}$ of F. Let $n(\mu) = [F(\mu) : F]$ be the degree of this extension. The isomorphism $\mathbf{X}_*(\mathbf{T}) \simeq \mathbf{X}^*(\mathbf{T}^\vee)$ maps μ to a $\mathbf{B}^\vee$-dominant character of $\mathbf{T}^\vee$ fixed by $\Gamma^{n(\mu)}$, which we also denote by μ. Let $r_\mu : \mathbf{G}^\vee \rtimes \Gamma^{n(\mu)} \to \mathbf{GL}(\mathcal{V}_\mu)$ be the unique irreducible representation of ${}^L(\mathbf{G}_{F(\mu)})$ whose restriction to $\mathbf{G}^\vee$ is the irreducible representation of highest weight μ, and such that $\Gamma^{n(\mu)}$ acts trivially on the (one-dimensional) highest weight space of $r_\mu|\mathbf{T}^\vee$. Consider for $g^\vee \in \mathbf{G}^\vee$ the characteristic polynomial

$$\det\left(X - q_F^{n(\mu)d(\mu)} r_\mu\left((g^\vee \rtimes \sigma)^{n(\mu)}\right)\right)$$

where $d(\mu) = \langle \rho, \mu \rangle$. Its coefficients, viewed as functions on $\mathbf{G}^\vee$, are regular functions on $\mathbf{G}^\vee$ which are invariant under σ-conjugation. We may thus define a unitary polynomial $P_\mu \in \mathcal{H}(\mathbf{G})_\mathbb{C}[X]$ by the following "formula" (compare with [6, §6])

$$\mathcal{S}_\mathbb{C}(P_\mu) = \det\left(X - q_F^{n(\mu)d(\mu)} r_\mu\left((g^\vee \rtimes \sigma)^{n(\mu)}\right)\right).$$

10.2.2. If $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_s$, then $\mathbf{T} = \mathbf{T}_1 \times \cdots \times \mathbf{T}_s$, $\mathbf{\Omega} = \mathbf{\Omega}_1 \times \cdots \times \mathbf{\Omega}_s$ and

$$\mathcal{H}(\mathbf{G}) = \mathcal{H}(\mathbf{G}_1) \otimes \cdots \otimes \mathcal{H}(\mathbf{G}_s), \quad \mathcal{H}(\mathbf{T}) = \mathcal{H}(\mathbf{T}_1) \otimes \cdots \otimes \mathcal{H}(\mathbf{T}_s)$$

with $\mathcal{S}^\bullet = \mathcal{S}_1^\bullet \otimes \cdots \otimes \mathcal{S}_s^\bullet$ and $\mathcal{S} = \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_s$. Moreover ${}^L\mathbf{G} = (\mathbf{G}_1^\vee \times \cdots \times \mathbf{G}_s^\vee) \rtimes \Gamma$ with the diagonal action and for $\mu = \mu_1 + \cdots + \mu_s$ in $X_*(\mathbf{T}) = X_*(\mathbf{T}_1) \oplus \cdots \oplus X_*(\mathbf{T}_s)$, $d(\mu) = d(\mu_1) + \cdots + d(\mu_s)$, $n(\mu) = \mathrm{lcm}\{n(\mu_1), \cdots, n(\mu_s)\}$ and $r_\mu = r_{\mu_1} \boxtimes \cdots \boxtimes r_{\mu_s}$ on $\mathbf{G}^\vee = \mathbf{G}_1^\vee \times \cdots \times \mathbf{G}_s^\vee$. In particular if $\mu_2 = \cdots = \mu_s = 0$, $d(\mu) = d(\mu_1)$, $n(\mu) = n(\mu_1)$ and $P_\mu = P_{\mu_1}$ with coefficients in the sub-algebra $\mathcal{H}(\mathbf{G}_1)_\mathbb{C}$ of $\mathcal{H}(\mathbf{G})_\mathbb{C}$.

10.2.3. If $\mathbf{G} = \mathrm{Res}_{\mathcal{O}'/\mathcal{O}}\mathbf{G}'$ for some unramified extension $\mathcal{O}'$ of $\mathcal{O}$ with fraction field $F' \subset F^{\mathrm{ur}}$ of degree $f = [F' : F]$, then $\mathbf{T} = \mathrm{Res}_{\mathcal{O}'/\mathcal{O}}\mathbf{T}'$, $\mathbf{\Omega} = \mathrm{Res}_{\mathcal{O}'/\mathcal{O}}\mathbf{\Omega}'$ and

$$\mathcal{H}(\mathbf{G}) = \mathcal{H}(\mathbf{G}'), \quad \mathcal{H}(\mathbf{T}) = \mathcal{H}(\mathbf{T}') \quad \text{with} \quad \mathcal{S}^\bullet_\mathbf{G} = \mathcal{S}^\bullet_{\mathbf{G}'} \quad \text{and} \quad \mathcal{S}_\mathbf{G} = \mathcal{S}_{\mathbf{G}'}.$$

Moreover, $\mathbf{G}^\vee = \mathrm{Ind}_{\Gamma'}^\Gamma(\mathbf{G}'^\vee)$ and $X_*(\mathbf{T}) = \mathrm{Ind}_{\Gamma'}^\Gamma X_*(\mathbf{T}')$ where $\Gamma' = \mathrm{Gal}(F^{\mathrm{ur}}/F')$. For μ in $X_*(\mathbf{T})$ with $\mu(\Gamma \setminus \Gamma') = 0$ and $\mu' = \mu(1) \in X_*(\mathbf{T}')$, $n(\mu) = f \cdot n(\mu')$, $d(\mu) = d(\mu')$, $\sigma' = \sigma^f$, $q_{F'} = q_F^f$, $\Gamma^{n(\mu)} = \Gamma'^{n(\mu')}$ and r_μ is the composition of evaluation at 1, $\mathbf{G}^\vee \rtimes \Gamma^{n(\mu)} \to \mathbf{G}'^\vee \rtimes \Gamma'^{n(\mu')}$, followed by $r_{\mu'}$. Thus for $g^\vee \in \mathbf{G}^\vee$,

$$\begin{aligned}
\mathcal{S}_\mathbb{C}(P_\mu) &= \det\left(X - q_F^{n(\mu)d(\mu)} r_\mu\left((g^\vee \rtimes \sigma)^{n(\mu)}\right)\right) \\
&= \det\left(X - q_{F'}^{n(\mu')d(\mu')} r_\mu\left(\left((g^\vee \sigma g^\vee \cdots \sigma^{f-1} g^\vee) \rtimes \sigma'\right)^{n(\mu')}\right)\right) \\
&= \det\left(X - q_{F'}^{n(\mu')d(\mu')} r_{\mu'}\left(\left((g^\vee(1) g^\vee(\sigma) \cdots g^\vee(\sigma^{f-1})) \rtimes \sigma'\right)^{n(\mu')}\right)\right)
\end{aligned}$$

On the other hand $X_*(\mathbf{S}) \hookrightarrow X_*(\mathbf{T})$ equals $X_*(\mathbf{S}') \hookrightarrow \mathrm{Ind}_{\Gamma'}^{\Gamma} X_*(\mathbf{S}') \hookrightarrow \mathrm{Ind}_{\Gamma'}^{\Gamma} X_*(\mathbf{T}')$ where the first map sends $\nu \in X_*(\mathbf{S}')$ to $\Gamma \ni \gamma \mapsto \nu \in X_*(\mathbf{S}')$, corresponding to the morphism $\mathbf{T}^\vee \to \mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathbf{S}'^\vee) \to \mathbf{S}'^\vee$ where the last map is the norm map which takes a Γ'-invariant map $s^\vee : \Gamma \to \mathbf{S}'^\vee$ to $s^\vee(1) \cdots s^\vee(\sigma^{f-1})$. It follows that $P_\mu = P_{\mu'}$.

10.2.4. Suppose from now on that $\mathbf{G}$ is split. Then $\mathbf{S} = \mathbf{T}$, $\mathbf{\Omega}(F) = \mathbf{\Omega}(\overline{F}) = \Omega$ is the full Weyl group of $\mathbf{S}^\vee = \mathbf{T}^\vee$ in $\mathbf{G}^\vee$ and Γ acts trivially on $\mathbf{G}^\vee$, i.e. ${}^L\mathbf{G} = \mathbf{G}^\vee \times \Gamma$. The isomorphism $X_*(\mathbf{S}) \simeq X^*(\mathbf{S}^\vee)$ identifies $X_*^+(\mathbf{S})$ with $X_+^*(\mathbf{S}^\vee)$, the cone of $\mathbf{B}^\vee$-dominant characters of $\mathbf{S}^\vee$. For $\nu \in X_*^+(\mathbf{S})$, let $r_\nu : \mathbf{G}^\vee \to \mathbf{GL}(\mathcal{V}_\nu)$ be the irreducible representation of highest weight $\nu \in X_+^*(\mathbf{S}^\vee)$ and for any algebraic representation $\mathcal{V}$ of $\mathbf{G}^\vee$, let $\mathrm{Tr}(\mathcal{V}) \in \mathbb{Z}[X^*(\mathbf{S}^\vee)]^\Omega$ be the character of $\mathcal{V}$. By [30, 40, 26] as explained in [25] or [39, §2.2], we have the following formula: for any $\lambda \in X_*^+(\mathbf{S})$

$$\mathrm{Tr}(\mathcal{V}_\lambda) = \sum_{\nu \le \lambda} q_F^{-\langle \rho, \nu \rangle} K_{\lambda,\nu}(q_F^{-1}) \mathcal{S}(c_\nu) \quad \text{in} \quad \mathbb{Z}[q_F^{\pm h}][X^*(\mathbf{S}^\vee)]^\Omega.$$

Here $\le$ is the usual dominance order on $X_*^+(\mathbf{S})$ and $K_{\lambda,\nu}(q) \in \mathbb{Z}[q]$ is defined by

$$K_{\lambda,\nu}(q) = \sum_{\omega \in \Omega} \epsilon(\omega) \mathcal{P}\left(\omega(\lambda + \rho^\vee) - (\nu + \rho^\vee); q\right)$$

where $\epsilon(\omega) = \det(\omega | X_*(\mathbf{S})) \in \{\pm 1\}$ for $\omega \in \Omega$, $\rho^\vee \in X_*(\mathbf{S}) \otimes \mathbb{Q}$ is the half-sum of the $\mathbf{B}$-positive coroots $R_+^\vee \subset X_*^+(\mathbf{S})$ and for any $\gamma \in X_*(\mathbf{S})$, $\mathcal{P}(\gamma; q) \in \mathbb{Z}[q]$ is Lusztig's q-analogue of Kostant's partition function, defined by the formal identity

$$\prod_{\alpha^\vee \in R_+^\vee} (1 - q e^{\alpha^\vee})^{-1} = \sum_{\gamma \in X_*(\mathbf{S})} \mathcal{P}(\gamma; q) e^\gamma.$$

For any $\mu \in X_*^+(\mathbf{S})$, $n(\mu) = 1$ and

$$\mathcal{S}_\mathbb{C}(P_\mu) = \det\left(X - q_F^{\langle \rho, \mu \rangle} r(g^\vee)\right) = \sum_{i=0}^{r(\mu)} (-1)^i q_F^{i \langle \rho, \mu \rangle} \mathrm{Tr}(\Lambda^i \mathcal{V}_\mu) \cdot X^{r(\mu) - i}$$

where $r(\mu) = \dim_\mathbb{C} \mathcal{V}_\mu$, therefore

$$P_\mu = \sum_{i=0}^{r(\mu)} \sum_{\lambda \in X_*^+(\mathbf{S})} \left(\sum_{\nu \le \lambda} (-1)^i q_F^{\langle \rho, i\mu - \nu \rangle} \mathrm{mult}(\mathcal{V}_\lambda, \Lambda^i \mathcal{V}_\mu) K_{\lambda,\nu}(q^{-1}) \right) \cdot c_\lambda X^{r(\mu) - i}$$

where $\mathrm{mult}(\mathcal{V}_\lambda, \Lambda^i \mathcal{V}_\mu)$ is the multiplicity of $\mathcal{V}_\lambda$ in $\Lambda^i \mathcal{V}_\mu$. For $(\nu, \lambda) \in X_*^+(\mathbf{S})$ with $\nu \le \lambda$ and $\mathrm{mult}(\mathcal{V}_\lambda, \Lambda^i \mathcal{V}_\mu) \ne 0$, $i\mu - \nu$ is a linear combination of coroots with coefficients in $\mathbb{N}$, thus $\langle \rho, i\mu - \nu \rangle \in \mathbb{N}$ by [23, XXI 3.5.1]. It follows that our Hecke polynomial $P_\mu \in (\mathcal{H} \otimes \mathbb{C})[X]$ actually has coefficients in $\mathcal{H} \otimes \mathbb{Z}[q_F^{\pm 1}]$.

10.2.5. When $\mathbf{G}$ is semi-simple, Brylinski [12] gives yet another formula for $K_{\lambda,\nu}$:

$$K_{\lambda,\nu}(q) = \sum_{j \ge 0} \dim_\mathbb{C} \left(J_e^j(\mathcal{V}_\lambda^\nu) / J_e^{j-1}(\mathcal{V}_\lambda^\nu) \right) q^j \quad \text{in} \quad \mathbb{Z}[q].$$

Here for any algebraic representation $\mathcal{V}$ of $\mathbf{G}^\vee$, $\mathcal{V}^\nu$ is the ν-eigenspace of $\mathcal{V}|_{\mathbf{S}^\vee}$ and $J_e^j(\mathcal{V}^\nu) = \ker(e^{j+1} : \mathcal{V}^\nu \to \mathcal{V})$ where e is a principal nilpotent of $\mathrm{Lie}(\mathbf{B}^\vee)$ compatible with Lie($\mathbf{S}^\vee$), as defined in *loc. cit.* It follows that for any $\mathcal{V}$,

$$\mathrm{Tr}(\mathcal{V}) = \sum_{\nu \in X_*^+(\mathbf{S})} q_F^{-\langle \rho, \nu \rangle} \left(\sum_{j \geq 0} \dim_{\mathbb{C}} \left(J_e^j(\mathcal{V}^\nu) / J_e^{j-1}(\mathcal{V}^\nu) \right) q_F^{-j} \right) \cdot \mathcal{S}(c_\nu).$$

We thus obtain the following formula for P_μ:

$$P_\mu = \sum_{i=0}^{r(\mu)} \sum_{\nu \in X_*^+(\mathbf{S})} \left(\sum_{j \geq 0} (-1)^i q_F^{\langle \rho, i\mu - \nu \rangle - j} \dim_{\mathbb{C}} \left(\frac{J_e^j((\Lambda^i \mathcal{V}_\mu)^\nu)}{J_e^{j-1}((\Lambda^i \mathcal{V}_\mu)^\nu)} \right) \right) \cdot c_\nu X^{r(\mu)-i}.$$

By [12, Lemma 2.3], for any $\mathcal{V}$ as above, $\nu \in X_*^+(\mathbf{S})$ and $j \in \mathbb{N}$, $e^j \mathcal{V}^\nu \subset \oplus_{\nu'} \mathcal{V}^{\nu'}$ where ν' runs through the weights of $\mathcal{V}$ such that $\langle \rho, \nu' - \nu \rangle = j$. It follows that $e^j \equiv 0$ on $(\Lambda^i \mathcal{V}_\mu)^\nu$ for $j > \langle \rho, i\mu - \nu \rangle$, i.e. $J_e^j((\Lambda^i \mathcal{V}_\mu)^\nu) = (\Lambda^i \mathcal{V}_\mu)^\nu$ for $j \geq \langle \rho, i\mu - \nu \rangle$. Our Hecke polynomial $P_\mu \in (\mathcal{H} \otimes \mathbb{Z}[q_F^{\pm 1}])[X]$ thus actually has coefficients in $\mathcal{H}$.

10.2.6. Let us compute this for the split adjoint group $\mathbf{G} = \mathbf{SO}(2n+1)$. Inside

$$X^*(\mathbf{S}) = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n \quad \text{and} \quad X_*(\mathbf{S}) = \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee,$$

with $\langle \alpha_i, \alpha_j^\vee \rangle = \delta_{i,j}$, the simple roots and coroots are given by

$$\alpha_1 - \alpha_2, \cdots, \alpha_2 - \alpha_3, \cdots, \alpha_{n-1} - \alpha_n, \alpha_n,$$
$$\alpha_1^\vee - \alpha_2^\vee, \cdots, \alpha_2^\vee - \alpha_3^\vee, \cdots, \alpha_{n-1}^\vee - \alpha_n^\vee, 2\alpha_n^\vee.$$

For $0 \leq k \leq n$, we set $c_k = c_{\mu_k}$ with $\mu_k = \alpha_1^\vee + \cdots + \alpha_k^\vee$ in

$$X_*^+(\mathbf{S}) = X_+^*(\mathbf{S}^\vee) = \left\{ \lambda_1 \alpha_1^\vee + \cdots + \lambda_n \alpha_n^\vee : \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \right\}.$$

The half-sum of positive roots of $\mathbf{G}$ is given by

$$\rho = \tfrac{2n-1}{2}\alpha_1 + \cdots + \tfrac{3}{2}\alpha_{n-1} + \tfrac{1}{2}\alpha_n.$$

We shall compute $P = P_\mu$ for $\mu = \mu_1$. The dual group is $\mathbf{G}^\vee = \mathbf{Sp}(\mathcal{V}, \psi)$, with

$$\mathcal{V} = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n \oplus \mathbb{C}v_{-n} \oplus \cdots \oplus \mathbb{C}v_{-1} \quad \text{and} \quad \psi(v_i, v_{-j}) = \mathrm{sign}(i)\delta_{i,j}.$$

In the indicated $\mathbb{C}$-basis of $\mathcal{V}$, $\mathbf{S}^\vee$ and $\mathbf{B}^\vee$ are respectively the subgroups of diagonal and upper triangular symplectic matrices, and for $1 \leq i \leq n$, $\mathbb{C}v_{\pm i}$ is the $\pm\alpha_i^\vee$-eigenspace of $\mathcal{V}$. Our principal nilpotent $e \in \mathrm{Lie}(\mathbf{B}^\vee)$ acts on $\mathcal{V}$ as follows:

$$v_i \mapsto \begin{cases} 0 & i = 1, \\ v_{i-1} & 1 < i \leq n, \\ v_n & i = -n, \\ -v_{i-1} & -n < i \leq -1. \end{cases}$$

The irreducible representation of $\mathbf{G}^\vee$ with highest weight $\mu = \alpha_1^\vee$ is the standard representation $r : \mathbf{G}^\vee \hookrightarrow \mathbf{GL}(\mathcal{V})$. Since $\langle \rho, \mu \rangle = \frac{2n-1}{2}$, we thus find that

$$P = \sum_{i=0}^{2n} (-1)^i q_F^{\frac{i(i-1)}{2}} h_i \cdot X^{2n-i} \quad \text{with} \quad \mathcal{S}_{\mathbb{C}}(h_i) = q_F^{\frac{i(2n-i)}{2}} \mathrm{Tr}(\Lambda^i \mathcal{V}).$$

Since $\Lambda^i \mathcal{V} \simeq \Lambda^{2n-i} \mathcal{V}$, $h_i = h_{2n-i}$ and we only need to compute h_i for $0 \leq i \leq n$. A basis of $\Lambda^i \mathcal{V}$ is made of $v_J = v_{J(1)} \wedge \cdots \wedge v_{J(i)}$, where $J \subset \{\pm 1, \cdots, \pm n\}$ is a subset of order i, viewed as an increasing map from $\{1, \cdots, i\}$ to the ordered set

$$\{1 \prec \cdots \prec n-1 \prec n \prec -n \prec -(n-1) \prec \cdots \prec -1\}.$$

The weight of v_J is $\alpha^\vee_{J'} = \sum_{j\in J'} \alpha^\vee_j$ where $J' = J \setminus -J$, thus $J = J' \coprod J''$ with $J' \cap -J' = \emptyset$ and $J'' = -J''$. This weight is dominant only for $J' = \{1, \cdots, k\}$ for some $k \in \{0, \cdots, n\}$ with $k \equiv i \bmod 2$, in which case it equals μ_k. Thus for $i \leq n$,

$$h_i = \sum_{0\leq k\leq_2 i} P^n_{i,k}(q_F) \cdot c_k = h_{2n-i}$$

where $k \leq_2 i$ stands for ($k \leq i$ and $k \equiv i \bmod 2$), with

$$P^n_{i,k}(q) = q^{\frac{(i-k)(2n-i-k)}{2}} \sum_{j\geq 0} \dim_{\mathbb{C}} \left(\frac{J^j_e((\Lambda^i \mathcal{V})^{\mu_k})}{J^{j-1}_e((\Lambda^i \mathcal{V})^{\mu_k})} \right) q^{-j} = P^n_{2n-i,k}(q).$$

A basis of $(\Lambda^i\mathcal{V})^{\mu_k}$ is made of the v_J's with $J = J_k(R) = \{1, \cdots, k\} \coprod J''$ where $J'' = R \coprod -R$ for some $R \subset \{k + 1, \cdots, n\}$ with $\frac{i-k}{2}$ elements. An elementary computation then shows that for any (i, k) with $0 \leq k \leq_2 i \leq_2 2n - k$,

$$P^n_{i,k}(q) = P^{n-1}_{i-1,k-1}(q) = \cdots = P^{n-k}_{i-k,0}(q).$$

We thus obtain the following formula:

$$\begin{aligned} P &= \sum_{k=0}^{n} \left(\sum_{k\leq_2 i\leq_2 2n-k} (-1)^i q_F^{\frac{i(i-1)}{2}} P^n_{i,k}(q_F) X^{2n-i} \right) c_k \\ &= \sum_{k=0}^{n} (-1)^k q_F^{\frac{k(k-1)}{2}} X^k \left(\sum_{\ell=0}^{n-k} q_F^{\ell(2k+2\ell-1)} Q^{n-k}_\ell(q_F)(X^2)^{n-k-\ell} \right) c_k \end{aligned}$$

where $Q^m_\ell = P^m_{2\ell,0} \in \mathbb{Z}[q]$ for $0 \leq \ell \leq m$,

$$Q^n_\ell(q) = q^{2\ell(n-\ell)} \sum_{j\geq 0} \dim_{\mathbb{C}} \left(\frac{J^j_e((\Lambda^{2\ell}\mathcal{V})^0)}{J^{j-1}_e((\Lambda^{2\ell}\mathcal{V})^0)} \right) q^{-j}.$$

Again, $Q^n_\ell = Q^n_{n-\ell}$. One computes easily that $Q^n_0 = 1 = Q^n_n$, $Q^2_1 = 1 + q^2$,

$$Q^3_1 = Q^3_2 = 1 + q^2 + q^4$$

$$Q^4_1 = Q^4_3 = 1 + q^2 + q^4 + q^6 \quad \text{and} \quad Q^4_2 = 1 + q^2 + 2q^4 + q^6 + q^8.$$

In fact by [22, 17.5], $\Lambda^{2\ell}\mathcal{V} = \oplus_{k=0}^{\ell} \mathcal{V}_{\mu_{2k}}$as a representation of $\mathbf{G}^\vee$, thus

$$Q^n_\ell(q) = q^{2\ell(n-\ell)} \sum_{k=0}^{\ell} K_{\mu_{2k},0}(q^{-1})$$

and since $K_{\mu_{2k},0}(q) \in \mathbb{Z}[q^2]$ by [37, 7.2], also $Q^m_\ell(q) \in \mathbb{Z}[q^2]$. In particular,

$$Q^n_\ell(-1) = Q^n_\ell(1) = \dim_{\mathbb{C}}(\Lambda^{2\ell}\mathcal{V})^0 = C^\ell_n$$

and therefore

$$P(X) \equiv \sum_{k=0}^{n} (-1)^{\frac{k(k+1)}{2}} X^k (X^2 - 1)^{n-k} c_k \bmod (q_F + 1).$$

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