MORPHING ETALE SPACES

CHRISTOPHE CORNUT

Abstract. We give a simple description of the category of sheaves on the small etale site of an irreducible scheme whose local rings are geometrically unibranch and henselian, which affords a characterization of representable sheaves.

1. Introduction

Let S be a scheme, $\iota : Z \hookrightarrow S$ a closed immersion with complementary open immersion $j : U \hookrightarrow S$. By Grothendieck's gluing theorem [2, IV, 9.5], the category $\mathsf{Shv}(S_{et})$ of sheaves on the small etale site of S is equivalent to the category of triples (B_Z, B_U, ℓ_B) where $B_Z \in \mathsf{Shv}(Z_{et})$, $B_U \in \mathsf{Shv}(U_{et})$, and $\ell_B : B_Z \to \iota^* j_* B_U$ is a morphism in $\mathsf{Shv}(Z_{et})$. The equivalence takes $B \in \mathsf{Shv}(S_{et})$ to $B_Z = \iota^* B$, $B_U = j^* B$, and $\ell_B = \iota^*(u_B)$ where $u_B : B \to j_* j^* B$ is the unit of the adjunction

$$j^* : \mathsf{Shv}(S_{et}) \to \mathsf{Shv}(U_{et}) : j_*.$$

This gets particularly simple when Z and U are punctual schemes, corresponding to the closed and generic points s and η of a 1-dimensional irreducible local scheme S: by the topological invariance of etale sheaves, the first two components of our triples may then be viewed as etale sheaves on the corresponding residue fields, i.e. as sets equipped with a smooth action of the corresponding absolute Galois groups. The description of the connecting morphism ℓ_B is however somewhat trickier.

When S is the spectrum of a henselian discrete valuation ring $\mathcal{O}$, we arrive at the following picture. Let K be the fraction field of $\mathcal{O}$, K^{sep} a separable closure of K, K^{nr} the maximal unramified extension of K in K^{sep}, $\mathcal{O}^{nr}$ the integral closure of $\mathcal{O}$ in K^{nr}, $G = \mathrm{Gal}(K^{sep}/K)$ the Galois group and $I = \mathrm{Gal}(K^{sep}/K^{nr})$ the inertia subgroup, so that $G/I = \mathrm{Gal}(K^{nr}/K) \simeq \mathrm{Gal}(k^{sep}/k)$ where k^{sep} is the residue field of $\mathcal{O}^{nr}$, a separable closure of the residue field k of $\mathcal{O}$. Then $\mathsf{Shv}(S_{et})$ is equivalent to the category of morphisms of smooth G-sets $\ell_B : B_{\overline{s}} \to B_{\overline{\eta}}$ where I acts trivially on $B_{\overline{s}}$. A sheaf B is mapped to the localization morphism between its stalks at the geometric points $\overline{s}$ and $\overline{\eta}$ of $S = \mathrm{Spec}(\mathcal{O})$ which are respectively determined by

$$\begin{tikzcd}[column sep=small] & \mathcal{O}^{nr} \arrow[dr, two heads] & \\ \mathcal{O} \arrow[ur, hook] \arrow[dr, two heads] & & k^{sep} \\ & k \arrow[ur, hook] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & \mathcal{O}^{nr} \arrow[dr, hook] & & \\ \mathcal{O} \arrow[ur, hook] \arrow[dr, hook] & & K^{nr} \arrow[r, hook] & K^{sep}. \\ & K \arrow[ur, hook] & & \end{tikzcd}$$

2000 *Mathematics Subject Classification.* 14A20, 14A25.

When $B = \mathrm{Hom}_S(-, X)$ for some $X \in S_{et}$, we obtain the morphism

$$\begin{tikzcd} X(k^{sep}) \arrow[r, "\ell_B"] & X(K^{sep}) \\ X(\mathcal{O}^{nr}) \arrow[u, "\simeq"] \arrow[r] & X(K^{nr}) \arrow[u, hook] \end{tikzcd}$$

When X belongs to the strictly full subcategory S_{set} of separated etale S-schemes, the morphism ℓ_B is injective by the valuative criterion of separatedness. In general, covering X by affines, we find that ℓ_B is injective on G-orbits, i.e.

$$\forall x \in X(k^{sep}), \quad \forall g \in G: \qquad \ell_B(gx) = \ell_B(x) \iff gx = x.$$

It turns out that the converse implications hold. Namely, a sheaf $B \in \mathsf{Shv}(S_{et})$ is representable by some $X \in S_{set}$ (resp. by some $X \in S_{et}$) if and only if $\ell_B : B_{\overline{s}} \to B_{\overline{\eta}}$ is injective (resp. injective on orbits). In particular, there are adjunctions

$$(-)_{set} : \mathsf{Shv}(S_{et}) \longleftrightarrow S_{set} : \mathrm{yon}$$

$$(-)_{et} : \mathsf{Shv}(S_{et}) \longleftrightarrow S_{et} : \mathrm{yon}$$

where yon is the Yoneda embedding, and the left adjoints $(-)_{set}$ and $(-)_{et}$ correspond to the functors which map a G-morphism $\ell_B : B_{\overline{s}} \to B_{\overline{\eta}}$ to respectively

$$(\ell_B)_{set} : \mathrm{Im}(B_{\overline{s}} \to B_{\overline{\eta}}) \xhookrightarrow{\mathrm{inc}} B_{\overline{\eta}}$$

$$(\ell_B)_{et} : \mathrm{Im}(B_{\overline{s}} \to B_{\overline{\eta}} \times G\backslash B_{\overline{s}}) \xrightarrow{p_1} B_{\overline{\eta}}$$

The modest goal of this paper is to explain and generalize this to the case where S is irreducible with geometrically unibranch henselian local rings, e.g. $S = \mathrm{Spec}(\mathcal{O})$ where $\mathcal{O}$ is an arbitrary henselian valuation ring. Sections 2 to 6 investigate the relations between various notions of etale objects over S: etale algebraic spaces, etale sheaves, fet-sheaves (which are sheaves on the subsite S_{fet} of S_{et} whose objects are finite etale over opens of S), Zariski sheaves of finite etale sheaves, Zariski sheaves of π-sets or G-sets. Under good assumptions on S, we arrive at a fairly concrete strictly full subcategory $\mathsf{Shv}^{\star}_G(S_{Zar})$ of the category of Zariski sheaves of G-sets, which is equivalent to all categories previously considered. We explore its features in section 7, and collect our findings in section 8. Section 9 spells them out when $S = \mathrm{Spec}(\mathcal{O})$ for a henselian valuation ring $\mathcal{O}$.

2. From etale algebraic spaces to etale sheaves

We fix a big fppf site $(\mathsf{Sch}/S)_{fppf}$ as defined in [6, Tag 021L] and let S_{et} be the corresponding small etale site, whose underlying category is the strictly full subcategory of X's in $(\mathsf{Sch}/S)_{fppf}$ which are etale over S, equipped with the induced topology. The embedding $S_{et} \hookrightarrow (\mathsf{Sch}/S)_{fppf}$ induces a morphism of sites

$$\theta : (\mathsf{Sch}/S)_{fppf} \to S_{et},$$

whence an adjunction between the corresponding pull-back and push-out functors on sheaves, both of which 2-commute with the Yoneda embeddings:

$$\begin{tikzcd} & S_{et} \arrow[dl, bend right, "\text{yon}"'] \arrow[dr, bend left, "\text{yon}"] & \\ \mathsf{Shv}(S_{et}) \arrow[rr, shift left, "\theta^*"] & & \mathsf{Shv}((\mathsf{Sch}/S)_{fppf}) \arrow[ll, shift left, "\theta_*"] \end{tikzcd}$$

Let $\mathsf{AlgSp}(S)$ be the strictly full subcategory of algebraic spaces in $\mathsf{Shv}((\mathsf{Sch}/S)_{fppf})$, and $\mathsf{AlgSp}_{et}(S)$ the strictly full subcategory of algebraic spaces etale over S.

Proposition 1. *The above adjunction restricts to mutually inverse equivalences*

$$\theta^* : \mathsf{Shv}(S_{et}) \longleftrightarrow \mathsf{AlgSp}_{et}(S) : \theta_*$$

and induces an adjunction

$$\text{inc} : \mathsf{AlgSp}_{et}(S) \longleftrightarrow \mathsf{Shv}((\mathsf{Sch}/S)_{fppf}) : \theta^*\theta_*.$$

Proof. It is sufficient to establish that for $A \in \mathsf{AlgSp}_{et}(S)$ and $B \in \mathsf{Shv}(S_{et})$,

(1) The counit $\theta^*\theta_* A \to A$ is an isomorphism.
(2) $\theta^* B$ belongs to $\mathsf{AlgSp}_{et}(S)$ and the unit $B \to \theta_*\theta^* B$ is an isomorphism.

For $X \in S_{et}$, we denote by $H_X \in \mathsf{Shv}((\mathsf{Sch}/S)_{fppf})$ and $h_X \in \mathsf{Shv}(S_{et})$ the images of X under the Yoneda embeddings. Thus $\theta_* H_X = h_X$ and $\theta^* h_X \simeq H_X$.

Since A is an etale algebraic space over S, there is a $U \in S_{et}$ and a section $a \in A(U)$ such that the corresponding morphism $a : H_U \to A$ is etale surjective, i.e. relatively representable by etale surjective morphisms of schemes. In particular, $a : H_U \to A$ is an epimorphism in $\mathsf{Shv}((\mathsf{Sch}/S)_{et})$. So it is also an epimorphism in $\mathsf{Shv}((\mathsf{Sch}/S)_{fppf})$, and $\theta_* a$, which is the morphism $a : h_U \to \theta_* A$ corresponding to a in $\theta_* A(U) = A(U)$, is an epimorphism in $\mathsf{Shv}(S_{et})$. By general properties of topoi, these epimorphisms induce isomorphisms $H_U/\Sigma \simeq A$ and $h_U/\sigma \simeq \theta_* A$, where $\Sigma = H_U \times_A H_U$ and $\sigma = h_U \times_{\theta_* A} h_U$ are the induced equivalence relations on H_U and h_U. Since θ_* is a right adjoint, it commutes with all limits, so $\sigma = \theta_*\Sigma$. Since $a : H_U \to A$ is etale surjective, so are both projections $\Sigma \to H_U$. In particular, Σ is representable by a scheme R which is etale over U, hence etale over S, i.e. $\Sigma = H_R$ and $\sigma = h_R$ with $R \in S_{et}$. Since θ^* is exact, $\theta^*\theta_* A$ is the quotient of $\theta^* h_U \simeq H_U$ by $\theta^* h_R \simeq H_R$, i.e. $\theta^*\theta_* A \simeq H_U/H_R \simeq A$. One checks that the isomorphism $\theta^*\theta_* A \to A$ thus constructed is the counit of our adjunction, and this proves (1).

Fix a set $\mathcal{S}$ of generators of S_{et}, for instance the set of all standard etale affine schemes over affine open subschemes of S. Let $\mathcal{B}$ be the set of pairs (X, x) with $X \in \mathcal{S}$ and $x \in B(X)$. Set $U = \coprod_{(X,x)\in\mathcal{B}} X$, so that $U \in S_{et}$. Let $b : h_U \to B$ be the morphism of etale sheaves on S corresponding to the section $b \in B(U)$ whose restriction to the (X, x)-component X of U equals $x \in B(X)$. Then $b : h_U \to B$ is an epimorphism in $\mathsf{Shv}(S_{et})$. By general properties of topoi, it induces an isomorphism $h_U/\sigma \simeq B$ where $\sigma = h_U \times_B h_U$ is the equivalence relation on h_U induced by b. By lemma 2 below applied to the diagonal of B, the etale sheaf σ is representable by an open subscheme R of $U \times_S U$, so $R \in S_{et}$ is an etale equivalence relation on $U \in S_{et}$ and $\sigma = h_R$. Since $B \simeq h_U/h_R$ and θ^* is exact, $\theta^* B \simeq H_U/H_R$, which belongs to $\mathsf{AlgSp}_{et}(S)$. As above, $\theta_*\theta^* B \simeq h_U/h_R \simeq B$, and the isomorphism $B \to \theta_*\theta^* B$ thus constructed is the unit of our adjunction. This proves (2). ∎

Lemma 2. *Any monomorphism of* $\mathsf{Shv}(S_{et})$ *is representable by open immersions.*

Proof. Let B' be a subsheaf of $B \in \mathsf{Shv}(S_{et})$, fix $T \in S_{et}$ and $b \in B(T)$. Since B' is a Zariski subsheaf of B, there is a largest open U of T such that $b|_U \in B'(U)$. Let $f : T' \to T$ be any morphism in S_{et} such that $c = f^*b \in B(T')$ belongs to $B'(T')$. Let $V = f(T')$ be the image of f. Since f is etale, V is open in T, $f : T' \twoheadrightarrow V$ is an etale covering, and $T'' = T' \times_V T'$ equals $T' \times_T T'$. Let $p_i : T'' \to T'$ be the projections. Since $p_1^* f^* b = p_2^* f^* b$ in $B(T'')$, $p_1^* c = p_2^* c$ in $B'(T'')$; since B' is a sheaf, $c = f^* b'$ for some $b' \in B'(V)$; since B is a sheaf and $f^* b' = c = f^*(b|_V)$ in $B(T')$, $b' = b|_V$ in $B(V)$. So $b|_V = b'$ belongs to $B'(V)$ and $V \subset U$. It follows that $U \hookrightarrow T$ represents $B' \times_B T \hookrightarrow T$, which proves the lemma. ☐

For later use, we also record here the following consequence.

Proposition 3. *If $B = \cup B_i$ in $\mathsf{Shv}(S_{et})$ with B_i representable by (X_i, b_i), $X_i \in S_{et}$, $b_i \in B_i(X_i)$, then B is representable by (X, b) for some $X \in S_{et}$, $b \in B(X)$, $B_i \hookrightarrow B$ is representable by an open immersion $X_i \hookrightarrow X$, with $X = \cup X_i$ and $b|_{X_i} = b_i$.*

Proof. This follows from the previous lemma and a variant of [6, Tag 01JJ]. ☐

We denote by α the restriction of θ_* to $\mathsf{AlgSp}_{et}(S)$, an equivalence of categories

$$\alpha : \mathsf{AlgSp}_{et}(S) \to \mathsf{Shv}(S_{et}).$$

Remark 4. Let $A \in \mathsf{AlgSp}_{et}(S)$ and $B = \alpha(A) \in \mathsf{Shv}(S_{et})$. If A is a scheme, i.e. A is representable by an S-scheme X, then $X \to S$ is etale, i.e. $X \in S_{et}$, and X represents B. Conversely if B is representable by $X \in S_{et}$, then X, viewed as an algebraic space over S, is etale over S and $\alpha(X) \simeq B$, so $A \simeq X$, i.e. A is a scheme.

Proposition 5. *For a morphism $f : S' \to S$, there is a 2-commutative diagram*

$$\begin{tikzcd}
\mathsf{Shv}(S_{et}) \arrow[r, shift left, "\theta^*"] \arrow[d, "f^*"'] & \mathsf{AlgSp}_{et}(S) \arrow[l, shift left, "\alpha"] \arrow[r, hook, "\mathrm{inc}"] \arrow[d, "f^*"] & \mathsf{Shv}((\mathsf{Sch}/S)_{fppf}) \arrow[d, "f^*"] \\
\mathsf{Shv}(S'_{et}) \arrow[r, shift left, "\theta^*"] & \mathsf{AlgSp}_{et}(S') \arrow[l, shift left, "\alpha"] \arrow[r, hook, "\mathrm{inc}"] & \mathsf{Shv}((\mathsf{Sch}/S')_{fppf})
\end{tikzcd}$$

with right adjoint 2-commutative diagram

$$\begin{tikzcd}
\mathsf{Shv}(S_{et}) \arrow[r, shift left, "\theta^*"] & \mathsf{AlgSp}_{et}(S) \arrow[l, shift left, "\alpha"] & \mathsf{Shv}((\mathsf{Sch}/S)_{fppf}) \arrow[l, "\theta^*\theta_*"'] \\
\mathsf{Shv}(S'_{et}) \arrow[r, shift left, "\theta^*"] \arrow[u, "f_*"] & \mathsf{AlgSp}_{et}(S') \arrow[l, shift left, "\alpha"] \arrow[u, "f_*^{et}"'] & \mathsf{Shv}((\mathsf{Sch}/S')_{fppf}) \arrow[l, "\theta^*\theta_*"'] \arrow[u, "f_*"']
\end{tikzcd}$$

Proof. Consider the commutative diagram of morphisms of sites

$$\begin{tikzcd}
(\mathsf{Sch}/S)_{fppf} \arrow[r, "\theta"] & S_{et} \\
(\mathsf{Sch}/S')_{fppf} \arrow[r, "\theta"'] \arrow[u, "f"] & S'_{et} \arrow[u, "f"']
\end{tikzcd}$$

whose underlying continuous functors, going in opposite directions, are given by

$$f(X) = X \times_S S' \qquad \text{and} \qquad \theta(X) = X.$$

The corresponding pull-back and push-out functors respectively give the forward and backward outer rectangles of our diagrams, and the 2-commutativity of all remaining possible squares follow from proposition 1. $\square$

Remark 6. While $f^* : \mathsf{AlgSp}_{et}(S) \to \mathsf{AlgSp}_{et}(S')$ is the restriction of the eponymous functor on fppf sheaves, its right adjoint $f^{et}_* : \mathsf{AlgSp}_{et}(S') \to \mathsf{AlgSp}_{et}(S)$ is given by

$$\begin{tikzcd}
\mathsf{Shv}(S_{et}) \arrow[r, "\theta^*"] & \mathsf{AlgSp}_{et}(S) & \mathsf{Shv}((\mathsf{Sch}/S)_{fppf}) \arrow[l, "\theta^*\theta_*"'] \\
\mathsf{Shv}(S'_{et}) \arrow[u, "f_*"] & \mathsf{AlgSp}_{et}(S') \arrow[l, "\alpha"'] \arrow[u, "f^{et}_*"'] \arrow[r, hook, "\mathrm{inc}"] & \mathsf{Shv}((\mathsf{Sch}/S')_{fppf}) \arrow[u, "f_*"']
\end{tikzcd}$$

Let $\overline{s} \to S$ be a geometric point of S over $s \in S$, $k(s, \overline{s})$ the separable closure of $k(s)$ in $k(\overline{s})$, $\mathcal{O}^{sh}_{S,\overline{s}}$ the strict henselization of $\mathcal{O}_{S,s}$ with respect to $k(s) \hookrightarrow k(\overline{s})$. Then $k(s, \overline{s})$ is the residue field of $\mathcal{O}^{sh}_{S,\overline{s}}$ and the action of $\Gamma(s) = \mathrm{Gal}(k(s, \overline{s})/k(s))$ on $k(s, \overline{s})$ lifts uniquely to a continuous action on the local ring $\mathcal{O}^{sh}_{S,\overline{s}}$. Evaluation at $\overline{s}$ gives points of the topoi $\mathsf{Shv}((\mathsf{Sch}/S)_{fppf})$ and $\mathsf{Shv}(S_{et})$, with stalks

$$\mathsf{Shv}(S_{et}) \xrightarrow{\theta^*} \mathsf{Shv}((\mathsf{Sch}/S)_{fppf}) \xrightarrow{(-)(\overline{s})} \mathsf{Set}_{\mathrm{Aut}(\overline{s}/s)} \xrightarrow{\text{forget}} \mathsf{Set}$$

where for any group H, Set_H is the category of sets with a left action of H.

Proposition 7. *For A in $\mathsf{AlgSp}_{et}(S)$ with image $B = \alpha(A)$ in $\mathsf{Shv}(S_{et})$, there are functorial $\mathrm{Aut}(\overline{s}/s)$-equivariant isomorphisms of $\Gamma(s)$-sets*

$$B_{\overline{s}} = \varinjlim_{(X,x)\in S_{et}(\overline{s})} B(X) \simeq A(\mathcal{O}^{sh}_{S,\overline{s}}) \simeq A\left(k(s, \overline{s})\right) \simeq A(\overline{s})$$

where $S_{et}(\overline{s})$ is the category of pairs (X, x) with $X \in S_{et}$, $x \in X(\overline{s})$.

Proof. For $(X, x) \in S_{et}(\overline{s})$, there is a canonical factorization of $x : \overline{s} \to X$ as

$$\overline{s} \twoheadrightarrow \mathrm{Spec}\left(k(s, \overline{s})\right) \hookrightarrow \mathrm{Spec}\left(\mathcal{O}^{sh}_{S,\overline{s}}\right) \to X.$$

Evaluating on A and taking colimits gives an $\mathrm{Aut}(\overline{s}/s)$-equivariant sequence

$$\varinjlim_{(X,x)\in S_{et}(\overline{s})} B(X) \to A(\mathcal{O}^{sh}_{S,\overline{s}}) \to A\left(k(s, \overline{s})\right) \to A(\overline{s}).$$

We have to show that all maps are bijections. In the colimit, we may restrict the indexing category to the full initial category $U^{af}_{et}(\overline{s})$ of pairs (X, x) where X is affine over some fixed affine neighborhood U of s in S. Then by [6, Tag 04GW],

$$\mathrm{Spec}\left(\mathcal{O}^{sh}_{S,\overline{s}}\right) = \varprojlim_{(X,x)\in U^{af}_{et}(\overline{s})} X \quad \text{in} \quad \mathsf{Sch}/S.$$

On the other hand, A is locally of finite presentation over S by [6, Tag 0468], so

$$\varinjlim_{(X,x)\in S_{et}(\overline{s})} B(X) \xrightarrow{\simeq} A(\mathcal{O}^{sh}_{S,\overline{s}})$$

by [6, Tag 01ZC]. For $A = \mathrm{Hom}_S(-, X)$ with $X \in S_{et}$, we have isomorphisms

$$A\left(\mathcal{O}^{sh}_{S,\overline{s}}\right) \xrightarrow{\simeq} A\left(k(s, \overline{s})\right) \xrightarrow{\simeq} A(\overline{s})$$

by [4, 18.5.4.4] for the first map, and using that the fiber $X_s \to s$ is a disjoint union of spectra of finite separable extensions of $k(s)$ for the second map. For a general A, choose a presentation $A \simeq (U/R)_{fppf}$ with $U, R \in S_{et}$. It is now sufficient to establish that for any $\mathcal{O}$ in $\{\mathcal{O}^{sh}_{S,\overline{s}}, k(s, \overline{s}), k(\overline{s})\}$, the map $U(\mathcal{O}) \to A(\mathcal{O})$ identifies $A(\mathcal{O})$ with the quotient of $U(\mathcal{O})$ by the equivalence relation $R(\mathcal{O})$. Since

$U \times_A U = R$ as presheaves on Sch/S, $U(\mathcal{O})/R(\mathcal{O}) \to A(\mathcal{O})$ is injective. Since $U \to A$ is etale surjective and $\mathcal{O}$ is strictly henselian, it is also surjective. $\square$

3. From etale sheaves to finite etale sheaves

Definition 8. A morphism of schemes $f : X \to Y$ is *fet* if it factors as $X \to U \hookrightarrow Y$ where $f' : X \to U$ is finite etale and $U \hookrightarrow Y$ is an open immersion.

Note that $f'(X)$ is then clopen in U, and open in Y. In particular, we may always take $U = f(X)$. Thus $f : X \to Y$ is fet if and only if $f(X)$ is open in Y and $f : X \to f(X)$ is finite etale. A fet morphism is separated and etale, and an etale morphism f is fet if and only if $X \to f(X)$ is finite. Fet morphisms are plainly stable under arbitrary base change. A morphism $f : X \to Y$ between a fet S-scheme X and a separated etale S-scheme Y is fet, and its image $f(X)$ is fet over S. Indeed if U is the image of X in S, then f factors as $X \to Y_U \hookrightarrow Y$. Since X is finite etale over U and Y_U is separated etale over U, $X \to Y_U$ is finite etale and $f(X)$ is finite etale over U, so $X \to Y$ is fet and $f(X) \to S$ is fet.

Definition 9. We denote by S_{fet} the site whose underlying category is the strictly full subcategory of fet S-schemes in S_{et}, equipped with the induced topology.

By [1, III, Corollaire 3.3] and the next lemma, coverings in S_{fet} are just coverings in S_{et}, i.e. jointly surjective families of morphisms in S_{fet} with fixed target.

Lemma 10. *The category S_{fet} is stable under fiber products.*

Proof. Let $X_1 \to X_3$ and $X_2 \to X_3$ be morphisms in S_{fet}, U_i the image of X_i in S, and $V = U_1 \cap U_2$. The cartesian diagram

$$\begin{tikzcd}
X_1 \times_{X_3} X_2 \arrow[r] \arrow[d] & X_{2,V} \arrow[r] \arrow[d] & X_2 \arrow[d] \\
X_{1,V} \arrow[r] \arrow[d] & X_{3,V} \arrow[r, hook] \arrow[d, hook] & X_{3,U_2} \arrow[d, hook] \\
X_1 \arrow[r] & X_{3,U_1} \arrow[r, hook] & X_3
\end{tikzcd}$$

shows that $X_1 \times_{X_3} X_2$ is finite etale over $X_{3,V} = X_3 \times_{U_3} V$ which is finite etale over V, so $X_1 \times_{X_3} X_2$ is finite etale over V and indeed fet over S. $\square$

We denote by $\mathsf{Shv}(S_{fet})$ the category of fet-sheaves, i.e. sheaves on S_{fet}, and let

$$\beta : \mathsf{Shv}(S_{et}) \to \mathsf{Shv}(S_{fet})$$

be the restriction functor. This is usually badly behaved.

Example 11. Let $\mathcal{O}$ be the local ring of $\mathbb{Z}[X]/(X^2+1)$ at the prime $P = (X-2)$ above $p = 5$. Take $S = \operatorname{Spec}(\mathbb{Z}_{(p)})$, $X = \operatorname{Spec}(\mathcal{O})$, X' the S-scheme obtained by gluing two copies of X along its generic fiber Y, $\iota : Y \hookrightarrow X$ and $a, b : X \hookrightarrow X'$ the corresponding open embeddings. Let $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ and $a, b : \mathcal{X} \to \mathcal{X}'$ be the induced morphisms between the corresponding representable etale sheaves on S. Then $\beta(\iota)$ is an isomorphism while ι is not, so β does not reflect isomorphisms; $\beta(a) = \beta(b)$ while $a \neq b$, so β is not faithful; and the nontrivial automorphism of $\beta(\mathcal{Y}) = \beta(\mathcal{X})$ does not lift to any morphism of $\mathcal{X}$, so β is not full. Note that $\mathbb{Z}_{(p)}$ is normal, local... but not henselian: this is necessary by corollary 15 below.

Definition 12. We say that S is locally henselian (resp. locally strictly henselian) if $\mathcal{O}_{S,s}$ is henselian (resp. strictly henselian) for every $s \in S$.

Example 13. The spectrum of an absolutely integrally closed ring is locally strictly henselian [6, Tag 0DCS]. An integral normal scheme with separably closed function field is locally strictly henselian [6, Tag 09Z9]. A punctual scheme is locally henselian [6, Tag 06RS]. A henselian valuation ring is locally henselian: this follows from Gabber's criterion for henselian pairs [6, Tag 09XI].

Proposition 14. *Suppose that S is locally henselian. Then any object of S_{et} has a Zariski covering by objects of S_{fet}.*

Proof. Let $f : X \to S$ be an etale morphism, $x \in X$, $s = f(x)$. We have to find an open neighborhood U of x in X such that $f : U \to f(U)$ is finite etale. Shrinking S and X, we may assume that both are affine, in which case f is affine, with finite fibers. Let $X(s) \to S(s)$ be the base change of f to $S(s) = \mathrm{Spec}(\mathcal{O}_{S,s})$. By [5, 2.3.2], there is a clopen decomposition $X(s) = X(s)^f \coprod X(s)'$ with $X(s)^f \to S(s)$ finite (and etale) and $X(s)'_s = \emptyset$ – so $x \in X(S)^f$. By [3, §8], shrinking S further around s, we may assume that it comes from a clopen decomposition $X = X^f \coprod X'$ with X^f finite over S. Then $U = X^f$ is the desired neighborhood of x in X. □

Corollary 15. *If S is locally henselian, then β is an equivalence of categories.*

Proof. This now follows from the comparison lemma of [1, III, Théorème 4.1]. □

Corollary 16. *Suppose that S is locally henselian. Then $B \in \mathsf{Shv}(S_{et})$ is representable if and only if $\beta(B) \in \mathsf{Shv}(S_{fet})$ is a union of representable subpresheaves.*

Proof. Suppose $B = \mathrm{Hom}_{S_{et}}(-, X)$ for some $X \in S_{et}$. Let $X = \cup X_i$ be a Zariski covering of X by objects $X_i \in S_{fet}$ and set $B_i = \mathrm{Hom}_{S_{et}}(-, X_i)$, a subsheaf of B. Then $B = \cup B_i$ in $\mathsf{Shv}(S_{et})$, so $\beta(B) = \cup \beta(B_i)$ in $\mathsf{Shv}(S_{fet})$, with $\beta(B_i) \in \mathsf{Shv}(S_{fet})$ representable by $X_i \in S_{fet}$. Suppose conversely that $B' = \beta(B)$ is a union of representable subpresheaves: there is a collection of objects $X_i \in S_{fet}$ and sections $b_i \in B'(X_i)$ inducing monomorphisms $(-)^* b_i : \mathrm{Hom}_{S_{fet}}(-, X_i) \hookrightarrow B'$ with image $B'_i \subset B'$ such that $B' = \cup B'_i$ in $\mathsf{Shv}(S_{fet})$. Then the sections $b_i \in B(X_i)$ induce monomorphisms $(-)^* b_i : \mathrm{Hom}_{S_{et}}(-, X_i) \hookrightarrow B$ with image $B_i \subset B$ such that $B = \cup B_i$ in $\mathsf{Shv}(S_{et})$. Note that $b_i \in B_i(X_i)$ and (X_i, b_i) represents B_i. Then by proposition 3, B is representable by a pair (X, b), $X \in S_{et}$, $b \in B(X)$, with $B_i \subset B$ representable by an open embedding $X_i \hookrightarrow X$, with $X = \cup X_i$ and $b|_{X_i} = b_i$. □

Proposition 17. *Let $f : X \to S$ be an integral morphism with X irreducible and S locally henselian. Then $f(X)$ is closed and $f : X \to f(X)$ is an homeomorphism.*

Proof. Since f is universally closed [6, Tag 01WM], we just have to show that it is injective. Fix $s \in f(S)$. Our assumptions on f are stable under base change to $\mathrm{Spec}(\mathcal{O}_{S,s})$, so we may assume that S is local henselian with closed point s. By [6, Tag 09XI], any idempotent of $\Gamma(X_s, \mathcal{O}_{X_s})$ lifts to an idempotent of $\Gamma(X, \mathcal{O}_X)$. Since X is connected, it follows that X_s is connected. But X_s is also totally disconnected by [6, Tag 00GS and Tag 04MG], so it must be a single point. □

4. From fet-sheaves to Zariski Fet-sheaves

Let S_{Zar} be the usual Zariski site of S. For $U \in S_{Zar}$, let Fet_U be the category of finite etale U-schemes, which we view as a strictly full subcategory of S_{fet} and equip with the induced topology. Since Fet_U is stable under fiber products, coverings in Fet_U still correspond to jointly surjective families of morphisms, and for $V \subset U$, the base change functor $\mathsf{Fet}_U \to \mathsf{Fet}_V$ is continuous. Plainly $S_{fet} = \cup_U \mathsf{Fet}_U$. For $V = \emptyset$, $\mathsf{Fet}_\emptyset$ is the initial object $\emptyset$ of S_{fet}, $\mathsf{Shv}(\mathsf{Fet}_\emptyset)$ is the punctual category with a single sheaf $B_\emptyset$ whose direct image under $\emptyset \to U$ is the final object of $\mathsf{Shv}(\mathsf{Fet}_U)$.

Definition 18. A Zariski Fet-sheaf is given by the following data:

(1) For $U \in S_{Zar}$, a sheaf $B_U \in \mathsf{Shv}(\mathsf{Fet}_U)$,
(2) For $j_V^U : V \hookrightarrow U$ in S_{Zar}, a morphism $r_U^V : B_U \to (j_V^U)_*(B_V)$ in $\mathsf{Shv}(\mathsf{Fet}_U)$.

Here $(j_V^U)_*(B_V)(X) = B_V(X_V)$ for $X \in \mathsf{Fet}_U$. These are required to satisfy

(1) the cocycle relation $r_U^W = (j_V^U)_*(r_V^W) \circ r_U^V$ for $W \subset V \subset U$,
(2) the sheaf-like condition that for any covering $U = \cup U_i$ in S_{Zar},

$$B_U = \ker \left(\prod_i (j_{U_i}^U)_* B_{U_i} \rightrightarrows \prod_{i,j} (j_{U_{i,j}}^U)_* B_{U_{i,j}} \right)$$

in $\mathsf{Shv}(\mathsf{Fet}_U)$, where $U_{i,j} = U_i \cap U_j$ as usual.

A morphism from $((B_U), (r_U^V))$ to $((B'_U), (r'^V_U))$ is given by a collection $b = (b_U)_U$ of morphisms $b_U : B_U \to B'_U$ in $\mathsf{Shv}(\mathsf{Fet}_U)$, such that for any $V \subset U$, the diagram

$$\begin{tikzcd} B_U \arrow[r, "b_U"] \arrow[d, "r_U^V"'] & B'_U \arrow[d, "r'^V_U"] \\ (j_V^U)_*(B_V) \arrow[r, "(j_V^U)_*(b_V)"] & (j_V^U)_*(B'_V) \end{tikzcd}$$

is commutative in $\mathsf{Shv}(\mathsf{Fet}_U)$. We denote by $\mathsf{Shv}_{\mathsf{Fet}}(S_{Zar})$ the category thus defined.

Remark 19. We may also describe objects of $\mathsf{Shv}_{\mathsf{Fet}}(S_{Zar})$ as pairs $((B_U), (\tilde{r}_U^V))$ where for $j_V^U : V \hookrightarrow U$ in S_{Zar}, $\tilde{r}_U^V : (j_V^U)^* B_U \to B_V$ is a morphism in $\mathsf{Shv}(\mathsf{Fet}_V)$. The cocycle relation becomes $\tilde{r}_U^W = \tilde{r}_V^W \circ (j_W^V)^*(\tilde{r}_U^V)$, but the formulation of the sheaf-like condition requires passing back to the adjoint morphisms $B_U \to (j_V^U)_* B_V$.

Proposition 20. *There is an equivalence of categories*

$$\gamma_1 : \mathsf{Shv}(S_{fet}) \to \mathsf{Shv}_{\mathsf{Fet}}(S_{Zar}), \qquad B \mapsto ((B_U), (r_U^V))$$

where B_U is the restriction of B to Fet_U and for $V \subset U$ and $X \in \mathsf{Fet}_U$,

$$(r_U^V)_X : B_U(X) = B(X) \xrightarrow{\text{res}} B(X_V) = B_V(X_V) = (j_V^U)_*(B_V)(X).$$

Proof. Any X in S_{fet} belongs to Fet_U where U is the image of X in S, so γ_1 **is faithful**: for a morphism $b : B \to B'$ in $\mathsf{Shv}(S_{fet})$, $b_X : B(X) \to B'(X)$ equals $b_{U,X} : B_U(X) \to B'_U(X)$. A morphism $f : Y \to X$ in S_{fet} factors as $Y \to X_V \hookrightarrow X$ where $V \subset U$ is the image of Y in S and $f' : Y \to X_V$ is a morphism in Fet_V, so γ_1 **is full**: for any morphism $(b_U) : \gamma_1(B) \to \gamma_1(B')$, the formula $b_X = b_{U,X}$ defines a morphism of presheaves $b : B \to B'$ since in the commutative diagram

$$\begin{tikzcd} B(X) \arrow[r, equal] \arrow[d, "b_X"'] & B_U(X) \arrow[r, "(r_U^V)_X"] \arrow[d, "b_{U,X}"'] & B_V(X_V) \arrow[r, "\text{res}_{f'}"] \arrow[d, "b_{V,X_V}"] & B_V(Y) \arrow[r, equal] \arrow[d, "b_{V,Y}"] & B(Y) \arrow[d, "b_Y"] \\ B'(X) \arrow[r, equal] & B'_U(X) \arrow[r, "(r'^V_U)_X"] & B'_V(X_V) \arrow[r, "\text{res}_{f'}"] & B'_V(Y) \arrow[r, equal] & B'(Y) \end{tikzcd}$$

the compositions of the horizontal maps are the restrictions along $f : Y \to X$ on B and B'; plainly, $\gamma_1(b) = (b_U)$. Finally, we will see in lemma 21 below that any covering $\{X_i \to X\}$ in S_{fet} has a refinement $\{X'_{j,k} \to X_{U_j} \to X\}$ where $U = \cup U_j$ is a Zariski covering and for each index j, $\{X'_{j,k} \to X_{U_j}\}$ is a covering in the full subcategory Fet_{U_j}. It follows that γ_1 **is essentially surjective**.

Indeed for $((B_U), (r_U^V))$ in $\mathsf{Shv}_{\mathsf{Fet}}(S_{Zar})$, we may set $B(X) = B_U(X)$ and for $f : Y \to X$, define $\mathrm{res}_f : B(X) \to B(Y)$ by the first line of the above diagram. If $g : Z \to Y$ is another morphism in S_{fet}, $W \subset V$ the image of Z and $g' : Z \to Y_W$ the induced morphism, the definition of $\mathsf{Shv}_{\mathsf{Fet}}(S_{Zar})$ yields a commutative diagram

$$\begin{tikzcd}
B(X) \arrow[rrrr, "\mathrm{res}_f"] \arrow[dr, equal] & & & & B(Y) \arrow[dddd, "\mathrm{res}_g"] \\
& B_U(X) \arrow[r, "(r_U^V)_X"] \arrow[dr, "(r_U^W)_X"'] & B_V(X_V) \arrow[r, "\mathrm{res}_{f'}"] \arrow[d, "(r_V^W)_{X_V}"] & B_V(Y) \arrow[d, "(r_V^W)_Y"] \arrow[ur, equal] & \\
& & B_W(X_W) \arrow[r, "\mathrm{res}_{f'}"] \arrow[dr, "\mathrm{res}_{(f\circ g)'}"'] & B_W(Y_W) \arrow[d, "\mathrm{res}_{g'}"] & \\
& & & B_W(Z) \arrow[dr, equal] & \\
& & & & B(Z)
\end{tikzcd}$$

whose outer triangle gives $\mathrm{res}_{f\circ g} = \mathrm{res}_g \circ \mathrm{res}_f$. We have thus defined a presheaf B on S_{fet}, and it remains to establish that it satisfies the sheaf property with respect to (1) horizontal coverings $X = \cup X_{U_i}$, $U = \cup U_i$, and (2) vertical coverings $\{f_j : X_j \to X\}$ entirely occurring in Fet_U. For (1), this follows from the sheaf-like condition that we have imposed on $((B_U), (r_V^U))$. For (2), we have to show that

$$B(X) \stackrel{?}{=} \ker\left(\prod_i B(X_i) \rightrightarrows \prod_{i,j} B(X_i \times_X X_j)\right) \quad \text{in} \quad \mathsf{Set}.$$

Unwinding the definitions, we have to show that

$$B_U(X) \stackrel{?}{=} \ker\left(\prod_i B_{U_i}(X_i) \rightrightarrows \prod_{i,j} B_{U'_{i,j}}(X_i \times_X X_j)\right) \quad \text{in} \quad \mathsf{Set}$$

where $U_i \subset U$ is the image of X_i and $U'_{i,j} \subset U_{i,j}$ is the image of $X_i \times_X X_j$. Since all schemes in sight are finite etale over U, any $V \in \{U_i, U'_{i,j}\}$ is actually clopen in U. Our sheaf-like condition for the resulting Zariski covering $U = V \coprod V'$ shows that $(r_U^V, r_U^{V'})$ induces an isomorphism $B_U \simeq (j_V^U)_*(B_V) \times (j_{V'}^U)_*(B_{V'})$. Thus

$$\begin{array}{rrcl}
(r_U^{U_i})_{X_i} : & B_U(X_i) & \to & B_{U_i}(X_i) \\
\text{and} \quad (r_U^{U'_{i,j}})_{X_i \times_X X_j} : & B_U(X_i \times_X X_j) & \to & B_{U'_{i,j}}(X_i \times_X X_j)
\end{array}$$

are bijections. The desired equality now becomes the sheaf condition

$$B_U(X) = \ker\left(\prod_i B_U(X_i) \rightrightarrows \prod_{i,j} B_U(X_i \times_X X_j)\right) \quad \text{in} \quad \mathsf{Set}$$

for the sheaf $B_U \in \mathsf{Shv}(\mathsf{Fet}_U)$ with respect to the covering $\{X_i \to X\}$. So we have constructed a $B \in \mathsf{Shv}(S_{fet})$. For any open U of S and $Y \in \mathsf{Fet}_U$ with image $V \subset U$, we have $B|_{\mathsf{Fet}_U}(Y) = B(Y) = B_V(Y)$ by definition of B, and we define

$$\left(B_U(Y) \xrightarrow{b_{U,Y}} B|_{\mathsf{Fet}_U}(Y) \right) = \left(B_U(Y) \xrightarrow{(r_U^V)_Y} B_V(Y) \right).$$

We have just seen that, as V is clopen in U, $b_{U,Y}$ is a bijection. One checks that $(b_{U,Y})_Y$ defines an isomorphism $b_U : B_U \to B|_{\mathsf{Fet}_U}$ in $\mathsf{Shv}(\mathsf{Fet}_U)$, and that $(b_U)_U$ defines an isomorphism $((B_U), (r_U^V)) \to \gamma_1(B)$. So γ_1 is essentially surjective. $\square$

Lemma 21. *Any covering $\{X_i \to X\}$ in S_{fet} has a refinement of the form*

$$\{X_{j,k} \to X_{U_j} \hookrightarrow X\}$$

where $X_{j,k}$ is open in some X_i, $U = \cup_j U_j$ is a Zariski covering of the image U of X in S and for each index j, $\{X_{j,k} \to X_{U_j}\}$ is a finite covering in Fet_{U_j}.

Proof. Let $\{f_i : X_i \to X\}$ be a covering in S_{fet}. Let $\pi : X \to S$ be the structural morphism and let U be its image. Since $\pi : X \to U$ is finite etale, there is an open partition $U = \coprod_{n \geq 1} U_n$ such that $X_n = \pi^{-1}(U_n)$ is finite flat of rank n over U_n. Set $X_{n,i} = \pi_i^{-1}(U_n)$, where $\pi_i : X_i \to S$ is the structural morphism. So $\{X_{n,i} \to X_n\}$ is a covering in $U_{n,fet}$. If $\{X_{n,j,k} \to X_{n,U_{n,j}} \hookrightarrow X_n\}$ is a refinement for $\{X_{n,i} \to X_n\}$ as desired, then so is $\{X_{n,j,k} \to X_{U_{n,j}} \hookrightarrow X\}$ for $\{X_i \to X\}$. We may thus assume that π is surjective of constant degree $n \geq 1$. We instead assume that π is surjective of degree *bounded* by $n \geq 1$ and argue by induction on n. If $n = 1$, then π is an isomorphism and the covering $\{X_i \twoheadrightarrow f_i(X_i) \hookrightarrow X\}$ is already of the desired form. In general, let S_i be the image of the structural morphism $\pi_i : X_i \to S$, so that f_i factors as $X_i \to X_{S_i} \hookrightarrow X$, where $f_i' : X_i \to X_{S_i}$ is a morphism between finite etale S_i-schemes, hence itself finite etale. The image Y_i of f_i' is clopen in X_{S_i}, so $X_{S_i} = Y_i \coprod Z_i$ with Y_i and Z_i finite etale over S_i. The image S_i' of Z_i in S_i is clopen in S_i, so $S_i = S_i' \coprod S_{i,*}$. Since $Y_i \to S_i$ is surjective by construction, the degree of the finite etale surjective morphism $Z_i \twoheadrightarrow S_i'$ is bounded by $n-1$. By our induction hypothesis, each one of the induced coverings $\{f_{i'}^{-1}(Z_i) \to Z_i\}$ has a refinement of the form $\{X_{i,j,k} \to Z_{i,S_{i,j}} \to Z_i\}$ where $S_i' = \cup_j S_{i,j}$ is a Zariski covering and $\{X_{i,j,k} \to Z_{i,S_{i,j}}\}$ is a finite covering in $\mathsf{Fet}_{S_{i,j}}$, with $X_{i,j,k}$ open in some $f_{i'}^{-1}(Z_i) \subset X_{i'}$. For any fixed i, $\{S_{i,j}\} \cup \{S_{i,*}\}$ is a Zariski covering of S_i, and since the S_i's cover S, we obtain a Zariski cover of S. Over $S_{i,j} \subset S_i' \subset S_i$, $X_{S_{i,j}} = Y_{i,S_{i,j}} \coprod Z_{i,S_{i,j}}$ has a finite covering in $\mathsf{Fet}_{S_{i,j}}$ obtained by adjoining to $\{X_{i,j,k} \to Z_{i,S_{i,j}} \hookrightarrow X_{S_{i,j}}\}$ the single morphism $\{X_{i,S_{i,j}} \twoheadrightarrow Y_{i,S_{i,j}} \hookrightarrow X_{S_{i,j}}\}$. Over $S_{i,*} \subset S_i$, $X_{S_{i,*}} = Y_{i,S_{i,*}}$ is covered by the single morphism $\{X_{i,S_{i,*}} \twoheadrightarrow X_{S_{i,*}}\}$. This yields a refinement of $\{X_i \to X\}$ of the desired form. $\square$

5. From Zariski Fet-sheaves to Zariski π-sheaves

We now assume that our base scheme S is *irreducible* with generic point η, and pick a geometric point $\overline{\eta}$ of S over η.

For a nonempty open U of S, U is irreducible hence connected, and the category of sheaves on Fet_U is equivalent to the category of smooth $\pi(U)$-sets, where the fundamental group $\pi(U)$ is the automorphism group $\pi(U, \overline{\eta})$ of the fiber functor $\mathsf{Fet}_U \to \mathsf{Set}$ which takes X to $X(\overline{\eta})$. Here $\pi(U)$ is equipped with the coarsest topology for which the actions of $\pi(U)$ on the finite discrete sets $X(\overline{\eta})$ are continuous, so $\pi(U)$ is a profinite group; and given a left $\pi(U)$-set Y, we say that $y \in Y$ is smooth if its stabilizer $\pi(U)_y$ is open in $\pi(U)$, we let Y^{sm} be the set of smooth points in Y, and we say that Y is smooth if $Y^{sm} = Y$, so that $Y \mapsto Y^{sm}$ is right adjoint to the inclusion of the strictly full subcategory $\mathsf{Set}^{sm}_{\pi(U)}$ of smooth $\pi(U)$-sets in the category $\mathsf{Set}_{\pi(U)}$ of all $\pi(U)$-sets. The fiber functor induces equivalences

$$\begin{tikzcd}
\mathsf{Fet}_U \arrow[r, "(-)(\overline{\eta})"] \arrow[d, hook, "\text{yon}"'] & \mathsf{Set}^{fsm}_{\pi(U)} \arrow[d, hook, "\text{inc}"] \\
\mathsf{Shv}(\mathsf{Fet}_U) \arrow[r, "(-)(\overline{\eta})"] & \mathsf{Set}^{sm}_{\pi(U)}
\end{tikzcd}$$

where $\mathsf{Set}^{fsm}_{\pi(U)}$ is the full subcategory of finite smooth $\pi(U)$-sets in $\mathsf{Set}^{sm}_{\pi(U)}$, and the bottom functor takes a sheaf $B_U \in \mathsf{Shv}(\mathsf{Fet}_U)$ to the smooth $\pi(U)$-set

$$\begin{aligned} B_U(\overline{\eta}) &= \varinjlim_{(X,x) \in \mathsf{Fet}_U(\overline{\eta})} B_U(X) \\ &= \varinjlim_{(X,x) \in \mathsf{Fet}^c_U(\overline{\eta})} B_U(X) \end{aligned}$$

Here $\mathsf{Fet}_U(\overline{\eta})$ is the category of pairs (X, x) with $X \in \mathsf{Fet}_U$ and $x \in X(\overline{\eta})$, on which $\pi(U)$ acts by $g \cdot (X, x) = (X, g \cdot x)$, and $\mathsf{Fet}^c_U(\overline{\eta})$ is the $\pi(U)$-stable strictly full initial subcategory where X is connected.

For nonempty opens $V \subset U$, the base change functor $\mathsf{Fet}_U \to \mathsf{Fet}_V$ is compatible with the fiber functors. It induces a continuous morphism $\pi(V) \to \pi(U)$, and the pull-back functor $(j^U_V)^* : \mathsf{Shv}(\mathsf{Fet}_U) \to \mathsf{Shv}(\mathsf{Fet}_V)$ corresponds to the restriction functor $\mathrm{res} : \mathsf{Set}^{sm}_{\pi(U)} \to \mathsf{Set}^{sm}_{\pi(V)}$. Accordingly, the data $((B_U), (\tilde{r}^V_U))$ which specifies a Zariski Fet-sheaf may now be viewed as a pair (C, ρ), where C is a presheaf of sets on nonempty opens of S, with each $C(U)$ equipped with an action ρ_U of $\pi(U)$, such that the restriction maps $C(U) \to C(V)$ are equivariant with respect to $\pi(V) \to \pi(U)$. We extend C to all opens by $C(\emptyset) = \{\star\}$. The sheaf condition on $((B_U), (r^V_U))$ unwinds to the following sheaf condition on C: for any Zariski covering $U = \cup U_i$ of a nonempty open U of S and any smooth $\pi(U)$-set Y,

$$\mathrm{Hom}_{\pi(U)}(Y, C(U)) = \ker \left(\prod_i \mathrm{Hom}_{\pi(U_i)}(Y, C(U_i)) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\pi(U_{i,j})}(Y, C(U_{i,j})) \right)$$

We denote by $\mathsf{Shv}_\pi(S_{Zar})$ the category of these *Zariski π-sheaves*, and let

$$\gamma_2 : \mathsf{Shv}_{\mathsf{Fet}}(S_{Zar}) \to \mathsf{Shv}_\pi(S_{Zar})$$

be the equivalence of categories just defined.

Remark 22. Irreducibility of S also implies that any X in S_{et} is locally connected. Indeed $X \to S$ is open, hence generizing, so the minimal points of X belong to X_η. Since $X \to S$ is etale, X_η is discrete, so its points are the minimal points of X, and they are locally finite in X. Therefore any quasi-compact open U of X has finitely many irreducible components, thus also finitely many connected components; being closed and disjoint, they must be open in U, hence also in X. So any point of X has a connected open neighborhood, and the connected components of X are open.

6. From Zariski π-sheaves to Zariski sheaves of G-sets

With assumptions as above, suppose moreover that S is *geometrically unibranch*. For $s \in S$, set $S(s) = \mathrm{Spec}(\mathcal{O}_{S,s})$, so that $S(s) = \varprojlim V$ where V runs through the affine open neighborhoods of s in S. For U open in S with $s \in U$, pull-back along

$$\eta \hookrightarrow S(s) \to U$$

and evaluation at $\overline{\eta}$ induce compatible fiber functors

$$\mathsf{Fet}_U \to \mathsf{Fet}_{S(s)} \to \mathsf{Fet}_\eta \to \mathsf{Set}$$

and the corresponding continuous morphisms between the fundamental groups

$$\pi(\eta) \to \pi(S(s)) \to \pi(U).$$

Our new assumption implies that the latter are surjective [6, Tag 0BQI]. The group $G = \pi(\eta)$ is the Galois group of the separable closure of $k(\eta)$ in $k(\overline{\eta})$. We set

$$I(s) = \ker\left(G \twoheadrightarrow \pi(S(s))\right) \quad \text{and} \quad I(U) = \ker\left(G \twoheadrightarrow \pi(U)\right).$$

Note that $I(\eta) = \{1\}$, i.e. $\pi(\eta) \simeq \pi(S(\eta))$ by [6, Tag 0BQN].

Proposition 23. (1) *For any $s \in S$, $I(s) = \cap_{s \in U} I(U)$.* (2) *For any nonempty open U of S, $I(U)$ is the closure of the subgroup of G generated by $\{I(s) : s \in U\}$.*

Proof. (1) Plainly $I(s) \subset \cap_{s\in U} I(U)$. Suppose conversely that $g \in \cap_{s \in U} I(U)$. Let X be a finite etale $S(s)$-scheme. By [3, 8.8.2.ii & 8.10.5.x] and [4, 17.7.8.ii], X extends to a finite etale U-scheme X' for some affine open neighborhood U of s in S. Since $g \in I(U)$, g acts trivially on $X'(\overline{\eta}) = X(\overline{\eta})$. Thus $g \in I(s)$.

(2) Let $I'(U)$ be the closure of the subgroup of G generated by $\{I(s) : s \in U\}$. Plainly $I'(U) \subset I(U)$. Conversely, let Ω be an open subgroup of G containing $I'(U)$. Then for all $s \in U$, $\Omega/I(s)$ is an open subgroup of $\pi(S(s)) = G/I(s)$, so there is a finite connected etale $S(s)$-scheme $X(s)$ and a point $x(s) \in X(s)(\overline{\eta})$ with stabilizer Ω in G. As above, we may and do extend $X(s)$ to a scheme X_s which is finite etale over some small neighborhood U_s of s in U. For $s, s' \in U$, there is a unique isomorphism between the restrictions of X_s and $X_{s'}$ over the intersection $U_s \cap U_{s'}$ which maps $x(s)$ to $x(s')$. It follows that the U_s-schemes X_s glue to a scheme X which is finite etale over U, and equipped with a G-equivariant isomorphism $G/\Omega \simeq X(\overline{\eta})$. Since $I(U)$ acts trivially on $X(\overline{\eta})$, we obtain $I(U) \subset \Omega$. Since $I'(U)$ is closed, it is the intersection of all such Ω's, thus also $I(U) \subset I'(U)$. □

Given the proposition, the following convention seems reasonable: we set

$$\pi(\emptyset) = G \quad \text{and} \quad I(\emptyset) = \{1\}.$$

For any open U of S, we identify $\mathsf{Set}_{\pi(U)}$ with the strictly full subcategory of Set_G where $I(U)$ acts trivially, and $\mathsf{Set}^{sm}_{\pi(U)}$ with $\mathsf{Set}_{\pi(U)} \cap \mathsf{Set}^{sm}_G$. Accordingly, we may now view a Zariski π-sheaf as a presheaf of G-sets, and the category $\mathsf{Shv}_\pi(S_{Zar})$ as the strictly full subcategory of $\mathsf{PreShv}_G(S_{Zar})$ whose objects are characterized by the following sheaf-like property: a presheaf of G-sets C on S_{Zar} belongs to $\mathsf{Shv}_\pi(S_{Zar})$ if and only if for every Zariski covering $U = \cup_i U_i$ in S_{Zar},

$$C(U) = \ker\left(\prod_i C(U_i) \rightrightarrows \prod_{i,j} C(U_{i,j})\right)^{sm,I(U)}.$$

Note that taking the empty covering of the empty open retrieves our condition that $C(\emptyset)$ is a singleton, now viewed as a terminal object in $\mathsf{Set}^{sm}_{\pi(\emptyset)} = \mathsf{Set}^{sm}_G$.

Applying the sheafification functor $a : \mathsf{PreShv}_G(S_{Zar}) \to \mathsf{Shv}_G(S_{Zar})$, we obtain

$$\gamma_3 : \mathsf{Shv}_\pi(S_{Zar}) \to \mathsf{Shv}_G(S_{Zar}).$$

Proposition 24. *There is an adjunction*

$$\gamma_3 : \mathsf{Shv}_\pi(S_{Zar}) \longleftrightarrow \mathsf{Shv}_G(S_{Zar}) : (-)^{sm,I}$$

$$\mathrm{Hom}_{\mathsf{Shv}_G(S_{Zar})}(\gamma_3(C), D) = \mathrm{Hom}_{\mathsf{Shv}_\pi(S_{Zar})}\left(C, D^{sm,I}\right)$$

for $C \in \mathsf{Shv}_\pi(S_{Zar})$ *and* $D \in \mathsf{Shv}_G(S_{Zar})$*, where for any open* U *of* S*,*

$$D^{sm,I}(U) = D(U)^{sm,I(U)}$$

The functor γ_3 *is fully faithful, and a Zariski sheaf of* G*-sets* $D \in \mathsf{Shv}_G(S_{Zar})$ *belongs to the essential image of* γ_3 *if and only if the following conditions hold:*

1. *For every quasi-compact open* U *of* S*,* $D(U)$ *is a smooth* G*-set.*
2. *For every* $s \in S$*,* $I(s)$ *acts trivially on the stalk* D_s *of* D *at* s*.*

Proof. The formula $D^{sm,I}(U) = D(U)^{sm,I(U)}$ defines a subpresheaf $D^{sm,I}$ of D since for opens $V \subset U$ of S, we have $I(V) \subset I(U)$, so $D(U) \to D(V)$ maps $D^{sm,I}(U)$ to $D(V)^{sm,I(U)} \subset D(V)^{sm,I(V)} = D^{sm,I}(V)$. If $U = \cup U_i$, then

$$\begin{aligned} D^{sm,I}(U) &= \ker\left(\prod_i D(U_i) \rightrightarrows \prod_{i,j} D(U_{i,j})\right)^{sm,I(U)} \\ &= \ker\left(\prod_i D(U_i)^{sm,I(U_i)} \rightrightarrows \prod_{i,j} D(U_{i,j})^{sm,I(U_{i,j})}\right)^{sm,I(U)} \\ &= \ker\left(\prod_i D^{sm,I}(U_i) \rightrightarrows \prod_{i,j} D^{sm,I}(U_{i,j})\right)^{sm,I(U)} \end{aligned}$$

using the sheaf property of D for the first equality, so $D^{sm,I}$ belongs to $\mathsf{Shv}_\pi(S_{Zar})$ and the functor $(-)^{sm,I} : \mathsf{Shv}_G(S_{Zar}) \to \mathsf{Shv}_\pi(S_{Zar})$ is well-defined. We have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Shv}_G(S_{Zar})}(\gamma_3(C), D) &= \mathrm{Hom}_{\mathsf{PreShv}_G(S_{Zar})}(C, D) \\ &= \mathrm{Hom}_{\mathsf{PreShv}_G(S_{Zar})}\left(C, D^{sm,I}\right) \\ &= \mathrm{Hom}_{\mathsf{Shv}_\pi(S_{Zar})}\left(C, D^{sm,I}\right) \end{aligned}$$

by the defining adjunction for γ_3, the fact that all $C(U)$'s are smooth G-sets fixed by $I(U)$, and the fact that $\mathsf{Shv}_\pi(S_{Zar})$ is a full subcategory of $\mathsf{PreShv}_G(S_{Zar})$.

The full faithfulness of γ_3 is equivalent to the unit $C \to \gamma_3(C)^{sm,I}$ being an isomorphism, which we now establish. It is a monomorphism since the sheaf-like condition on C implies that C is a separated presheaf, so already $C \to \gamma_3(C)$ is a monomorphism. For any section d of $D = \gamma_3(C)$ over U, there is a Zariski covering

$U = \cup U_i$ such that $d|_{U_i} = c_i$ in $D(U_i)$ for some $c_i \in C(U_i)$; then (c_i) belongs to the kernel K of $\prod_i C(U_i) \rightrightarrows \prod_{i,j} C(U_{i,j})$ by injectivity of $C(U_{i,j}) \to D(U_{i,j})$. If d moreover belongs to $D^{sm,I}(U)$, there is a compact open subgroup Ω of G containing $I(U)$ and fixing d; then Ω also fixes all $d|_{U_i}$'s, all c_i's by injectivity of $C(U_i) \to D(U_i)$, and (c_i) thus belongs to $K^{sm,I(U)}$, which is the image of $C(U)$ by the sheaf-like property of C; so there's a $c \in C(U)$ such that $c|_{U_i} = c_i = d|_{U_i}$ for all i, whence $d = c$ by the sheaf property of D. So $C(U) = D^{sm,I}(U)$ as claimed.

A sheaf of G-sets D belongs to the essential image of γ_3 if and only if the counit $c_D : \gamma_3(D^{sm,I}) \to D$ is an isomorphism. Note that c_D is always a monomorphism since $D^{sm,I}$ is a subpresheaf of D, so c_D is an isomorphism if and only if it is an epimorphism, i.e. for any U and $d \in D(U)$, there is a covering $U = \cup U_i$ such that the stabilizer $G_{d|_{U_i}}$ of $d|_{U_i}$ is open in G and contains $I(U_i)$. The necessity of our conditions is thus clear. Conversely, assume (1) and (2). Fix $s \in S$, $d_s \in D_s$, and lift d_s to $d \in D(U)$ for some affine neighborhood U of s. Let G_d and G_{d_s} be the stabilizers of d and d_s, so that $G_d \subset G_{d_s}$. By (1), G_d is open in G, so $[G_{d_s} : G_d]$ is finite. Pick a set of representatives $\sigma_1, \cdots, \sigma_n \in G_{d_s}$ for G_{d_s}/G_d. Since

$$(\sigma_1 d)_s = \cdots = (\sigma_n d)_s,$$

there is a smaller affine neighborhood $s \in U' \subset U$ where also

$$\sigma_1 d|_{U'} = \cdots = \sigma_n d|_{U'}.$$

Replacing U by U', we may thus assume that $G_{d_s} = G_d$. The G-orbit of d is finite and for $s \in V_1 \subset V_2 \subset U$, we have $I(V_1) \cdot d \subset I(V_2) \cdot d$. So shrinking U again, we may assume that $I(V) \cdot d = I(U) \cdot d$ for all $V \subset U$ with $s \in V$. This means that

$$I(V)/I(V)_d \simeq I(U)/I(U)_d$$

or equivalently:

$$I(U)_d/I(V)_d \simeq I(U)/I(V)$$

for all $V \subset U$ with $s \in V$. Taking limits over all such V's, we find that

$$I(U)_d/I(s)_d \simeq I(U)/I(s)$$

by proposition 23, or equivalently,

$$I(s)/I(s)_d \simeq I(U)/I(U)_d.$$

But by (2), $I(s) \subset G_{d_s}$ which equals G_d, so $I(s)_d = I(s)$, and $I(U)_d = I(U)$. In other words, $d \in D^{sm,I}(U)$. A fortiori, d_s belongs to the image of

$$c_{D,s} : \gamma_3(D^{sm,I})_s \to D_s.$$

Since d_s and s were arbitrary, c_D is surjective on stalks, hence an epimorphism, thus an isomorphism: $D \simeq \gamma_3(D^{sm,I})$ belongs to the essential image of γ_3. □

We denote by $\mathsf{Shv}^\star_G(S_{Zar})$ the strictly full subcategory of $\mathsf{Shv}_G(S_{Zar})$ defined in the previous proposition, so that γ_3 induces an equivalence of categories

$$\gamma_3 : \mathsf{Shv}_\pi(S_{Zar}) \to \mathsf{Shv}^\star_G(S_{Zar}).$$

Remark 25. If S is irreducible and geometrically unibranch, any X in S_{et} is locally irreducible. Indeed we have seen that any quasi-compact open U of X has finitely many irreducible components. Since U is etale over S, it is also geometrically unibranch, so its irreducible components are disjoint, hence open. It follows that the irreducible components of X are open, and equal to its connected components.

7. The category $\mathsf{Shv}_G^\star(S_{Zar})$

With assumptions and notations as in the previous section (S irreducible and geometrically unibranch), we review here various constructions related to Zariski sheaves of G-sets. We write Shv, $\mathsf{Shv}_G^\star$ etc… for $\mathsf{Shv}(S_{Zar})$, $\mathsf{Shv}_G^\star(S_{Zar})$ etc…

7.1. **Sheafification.** We start with the classical adjunction

$$a : \mathsf{PreShv}_G \longleftrightarrow \mathsf{Shv}_G : \mathrm{inc} \tag{7.1}$$

where a is the sheafification functor. For $D \in \mathsf{PreShv}_G$, the sheaf of sets underlying the sheaf of G-sets $a(D)$ is the sheafification of the presheaf of sets underlying D. For every $s \in S$, the unit $D \to a(D)$ induces an isomorphism of G-sets $D_s \to a(D)_s$.

7.2. **Smoothness for sheaves of G-sets.** Let D be a sheaf of G-sets. For a subgroup H of G, the formula $D^H(U) = D(U)^H$ defines a subsheaf D^H of D, with $(D^H)_s \subset (D_s)^H$ for all $s \in S$. We denote by D^{sm} the subsheaf of D defined by

$$D^{sm} = \cup D^\Omega \quad \text{in} \quad \mathsf{Shv}$$

where Ω runs through the compact open subgroups of G; this is a G-stable subsheaf of D with $D(U)^{sm} \subset D^{sm}(U)$ for every open U of S and $(D^{sm})_s \subset (D_s)^{sm}$ for every $s \in S$. We say that D is smooth if $D^{sm} = D$, and we denote by Shv_G^{sm} the strictly full subcategory of smooth sheaves of G-sets in Shv_G. We have adjunctions:

$$\mathrm{inc} : \mathsf{Shv}_G^{sm} \longleftrightarrow \mathsf{Shv}_G : (-)^{sm}. \tag{7.2}$$

The counit is the embedding $D^{sm} \hookrightarrow D$.

Lemma 26. *For a Zariski sheaf of G-sets D, the following are equivalent:*

1. *For every quasi-compact open U of S, $D(U)$ is a smooth G-set,*
2. *For every affine open U of S, $D(U)$ is a smooth G-set,*
3. *D is smooth, i.e. $D \in \mathsf{Shv}_G^{sm}$.*

They imply

4. *For every $s \in S$, D_s is a smooth G-set.*
5. *For every open U of S and $d \in D(U)$ the stabilizer G_d of d is closed in G.*

Proof. Plainly (1) ⇒ (2) ⇒ (4) and (4) ⇒ (5) by injectivity of $D(U) \to \prod_{s\in U} D_s$. If (2) holds, then $D^{sm}(U) = D(U)$ for all affine open U, hence for all U by the sheaf property for D^{sm} and D, so (2) ⇒ (3). Finally (3) ⇒ (1) by [6, Tag 0738]. □

7.3. **The $\star$-condition.** Plainly, $\mathsf{Shv}_G^\star \subset \mathsf{Shv}_G^{sm}$ and again, there is an adjunction

$$\mathrm{inc} : \mathsf{Shv}_G^\star \longleftrightarrow \mathsf{Shv}_G^{sm} : (-)^\star. \tag{7.3}$$

The right adjoint functor takes $D \in \mathsf{Shv}_G^{sm}$ to its subsheaf $D^\star$ defined by

$$D^\star(U) = \left\{ d \in D(U) : \forall s \in U,\, d_s \in D_s^{I(s)} \right\}$$

The counit is the embedding $D^\star \hookrightarrow D$ and

$$\forall s \in S : \qquad D_s^\star = D_s^{I(s)}.$$

Indeed $D_s^\star \subset D_s^{I(s)}$ by definition, and the proof of proposition 24 gives the other inclusion: the stabilizer G_{d_s} of $d_s \in D_s^{I(s)}$ is open in G and contains $I(s) = \cap_{s\in U} I(U)$, so it contains $I(U)$ for some sufficiently small affine neighborhood U of s. Shrinking U if necessary, we may assume that d_s lifts to $d \in D(U)$, and shrinking it further, that $G_d = G_{d_s}$. Then G_d contains $I(U)$, so $d \in D^\star(U)$, hence $d_s \in D_s^\star$.

7.4. **Monomorphisms and epimorphisms.** Let d be a morphism in $\mathsf{Shv}_G^\star$.

Since $\mathsf{Shv}_G^\star$ is stable under fiber products in PreShv_G, d is a monomorphism in $\mathsf{Shv}_G^\star$ if and only if d is a monomorphism in PreShv_G. Since $\mathsf{Shv}_G^\star \hookrightarrow \mathsf{Shv}_G$ has a right adjoint whose counits are monomorphisms, d is an epimorphism in $\mathsf{Shv}_G^\star$ if and only if d is an epimorphism in Shv_G (i.e. also: an epimorphism in Shv).

7.5. **Sheaves of G-sets with trivial G-action.** They belong to $\mathsf{Shv}_G^\star$, and form a strictly full subcategory which we identify with Shv. It fits into adjunctions

$$\mathrm{inc} : \mathsf{Shv} \longleftrightarrow \mathsf{Shv}_G^\star : (-)^G \tag{7.4}$$

$$(-)_G : \mathsf{Shv}_G^\star \longleftrightarrow \mathsf{Shv} : \mathrm{inc} \tag{7.5}$$

These are both restrictions of analogous adjunctions for the embedding of Shv into the larger category Shv_G, where the right and left adjoints are given by

$$D^G(U) = D(U)^G \quad \text{and} \quad D_G = a(U \mapsto G\backslash D(U)).$$

The counit $D^G \hookrightarrow D$ is a monomorphism and the unit $D \twoheadrightarrow D_G$ is an epimorphism.

7.6. **Constant sheaves.** The constant sheaf functor has left and right adjoints,

$$(-)_\eta : \mathsf{Shv}_G \longleftrightarrow \mathsf{Set}_G : (-)_S \tag{7.6}$$

$$(-)_S : \mathsf{Set}_G \longleftrightarrow \mathsf{Shv}_G : \Gamma(S, -) \tag{7.7}$$

Here $\Gamma(S, D) = D(S)$ while $(-)_\eta$ is the stalk functor at the generic point η of S. The unit $X \to \Gamma(S, X_S)$ and counit $(X_S)_\eta \to X$ are isomorphisms, so $(-)_S$ is fully faithful. Passing to the smooth subcategories, we obtain adjunctions

$$(-)_\eta : \mathsf{Shv}_G^{sm} \longleftrightarrow \mathsf{Set}_G^{sm} : (-)_S \tag{7.8}$$

$$(-)_S : \mathsf{Set}_G^{sm} \longleftrightarrow \mathsf{Shv}_G^{sm} : \Gamma(S, -)^{sm}. \tag{7.9}$$

Composing $(-)_S$ with $(-)^\star$ gives a new functor $(-)_S^\star$ fitting in an adjunction

$$(-)_\eta : \mathsf{Shv}_G^\star \longleftrightarrow \mathsf{Set}_G^{sm} : (-)_S^\star \tag{7.10}$$

with unit $D \to (D_\eta)_S^\star$. The counit is an isomorphism since $(X_S^\star)_\eta = X^{I(\eta)} = X$, so the right adjoint functor $(-)_S^\star$ is fully faithful.

Lemma 27. *For $D \in \mathsf{Shv}_G^\star$, the following conditions are equivalent:*

1. *D belongs to the essential image of $(-)_S^\star$*
2. *The unit $D \to (D_\eta)_S^\star$ is an isomorphism.*
3. *For any $s \in S$, $D_s \to D_\eta^{I(s)}$ is bijective,*
4. *For any open $U \neq \emptyset$ of S, $D(U) \to D_\eta^{I(U)}$ is bijective.*

Proof. (1) $\Leftrightarrow$ (2) by general properties of adjunctions, (2) $\Leftrightarrow$ (3) since

$$\forall s \in S : \quad ((D_\eta)_S^\star)_s = D_\eta^{I(s)}$$

and (2) $\Leftrightarrow$ (4) since for $U \neq \emptyset$,

$$(D_\eta)_S^\star(U) = \{d \in D_\eta : \forall s \in U, d \in D_\eta^{I(s)}\} = D_\eta^{I(U)}$$

by proposition 23 and smoothness of D_η. $\square$

7.7. **Et/Set sheaves.** For $D \in \mathsf{Shv}_G$, taking images of counits in Shv_G, we define

$$D_{et} = \mathrm{Im}\,(D \to (D_\eta)_S \times D_G) \quad \text{and} \quad D_{set} = \mathrm{Im}\,(D \to (D_\eta)_S)\,.$$

For $D \in \mathsf{Shv}_G^\star$, these are also images of the corresponding counits in $\mathsf{Shv}_G^\star$:

$$\begin{aligned} D_{set} &= \mathrm{Im}\,(D \to (D_\eta)_S^\star) \\ D_{et} &= \mathrm{Im}\,(D \to (D_\eta)_S^\star \times D_G) \\ &= \mathrm{Im}\,(D \to D_{set} \times D_G) \end{aligned}$$

So we have epimorphisms in $\mathsf{Shv}_G^\star$,

$$\begin{tikzcd} & & D_{set} \\ D \arrow[r, two heads] & D_{et} \arrow[ur, two heads] \arrow[dr, two heads] & \\ & & D_G \end{tikzcd}$$

We denote by Shv_G^{et} and $\mathsf{Shv}_G^{set} \subset \mathsf{Shv}_G^{et}$ the strictly full subcategories of $\mathsf{Shv}_G^\star$ where $D \simeq D_{et}$, respectively $D \simeq D_{set}$. We thus obtain new adjunctions

$$(-)_{et} : \mathsf{Shv}_G^\star \longleftrightarrow \mathsf{Shv}_G^{et} : \mathrm{inc} \tag{7.11}$$

$$(-)_{set} : \mathsf{Shv}_G^\star \longleftrightarrow \mathsf{Shv}_G^{set} : \mathrm{inc} \tag{7.12}$$

with epimorphic units $D \twoheadrightarrow D_{et}$ and $D \twoheadrightarrow D_{set}$.

7.8. **Opens.** For an open embedding $j_U : U \hookrightarrow S$, we have the usual adjunctions

$$j_U^* : \mathsf{Shv}_G(S_{Zar}) \longleftrightarrow \mathsf{Shv}_G(U_{Zar}) : j_{U*} \tag{7.13}$$

$$j_{U!} : \mathsf{Shv}_G(U_{Zar}) \longleftrightarrow \mathsf{Shv}_G(S_{Zar}) : j_U^* \tag{7.14}$$

where $j_{U!}$ takes $E \in \mathsf{Shv}_G(U_{Zar})$ to its extension

$$V \in S_{Zar} \mapsto j_{U!}(E)(V) = \begin{cases} \emptyset & \text{if } V \not\subset U, \\ E(V) & \text{if } V \subset U. \end{cases}$$

The unit $E \to j_U^* j_{U!} E$ is an isomorphism, so $j_{U!}$ is fully faithful, and the counit $j_{U!} j_U^* D \to D$ is a monomorphism. The adjunction (7.14) preserves the full subcategories Shv_G^{sm}, $\mathsf{Shv}_G^\star$, Shv, and is compatible with the stalk functor $(-)_\eta$ (if $U \neq \emptyset$), so it is compatible with all of the above constructions – details left to the reader.

7.9. **Characterization of et/set-sheaves.** For every sheaf of G-sets $D \in \mathsf{Shv}_G$, we also have the following constructions.

- The support of D is the open subset of S defined by

$$\mathrm{Supp}(D) = \{ s \in S : D_s \neq \emptyset \}\,.$$

- The set of *connected components* of D is $\pi_0(D) = G \backslash D_\eta$. For $c \in \pi_0(D)$ viewed as a G-orbit in D_η, we define $D(c)$ by the cartesian diagram

$$\begin{tikzcd} D(c) \arrow[r] \arrow[d, hook] & c_S \arrow[d, hook] \\ D \arrow[r] & (D_\eta)_S \end{tikzcd}$$

So $D(c)$ is a G-stable subsheaf of D and $D = \coprod_{c \in \pi_0(D)} D(c)$ in Shv_G.

- We denote by $\mathscr{S}_G(D)$ the set of all pairs (U,γ) where $U \neq \emptyset$ is open in S and $\gamma \subset D(U)$ is a G-orbit, and equip $\mathscr{S}_G(D)$ with the partial order
$$(U,\gamma) \leq (U',\gamma') \iff U \subset U' \quad \text{and} \quad \gamma = \gamma'|_U.$$
We denote by $\mathscr{S}_G^m(D)$ the set of maximal elements in $(\mathscr{S}_G(D), \leq)$.
- Sending $(U,\gamma) \in \mathscr{S}_G(D)$ to the image $\gamma_\eta \subset D_\eta$ of $\gamma \subset D(U)$ defines maps
$$\mathscr{S}_G^m(D) \hookrightarrow \mathscr{S}_G(D) \twoheadrightarrow \pi_0(D).$$
We denote by $\mathscr{S}_G^m(D,c) \hookrightarrow \mathscr{S}_G(D,c)$ their fibers over $c \in \pi_0(D)$. Thus
$$\mathscr{S}_G^m(D,c) = \mathscr{S}_G^m(D(c)) \quad \text{and} \quad \mathscr{S}_G(D,c) = \mathscr{S}_G(D(c)).$$
- Fix $(U,\gamma) \in \mathscr{S}_G(D)$. Since $D(U) = \Gamma(U, j_U^* D)$, the isomorphisms
$$\begin{aligned}\operatorname{Hom}_G(\gamma, D(U)) &= \operatorname{Hom}_{\mathsf{Shv}_G(U_{Zar})}(\gamma_U, j_U^* D)\\ &= \operatorname{Hom}_{\mathsf{Shv}_G(S_{Zar})}(j_{U!}(\gamma_U), D)\end{aligned}$$
map the G-equivariant embedding $\gamma \hookrightarrow D(U)$ to a morphism
$$j_{U!}(\gamma_U) \to D \quad \text{in} \quad \mathsf{Shv}_G(S_{Zar})$$
whose evaluation at $V \subset S$ is $\emptyset \to D(V)$ if $V \not\subset U$, and the composition of the embedding $\gamma \hookrightarrow D(U)$ with the restriction map $D(U) \to D(V)$ if $V \subset U$. In particular it factors through $D(c) \subset D$, where $c = \gamma_\eta \in \pi_0(D)$.
- Summing these morphisms across $\mathscr{S}_G^m(D)$, we obtain a morphism
$$\coprod_{(U,\gamma)\in\mathscr{S}_G^m(D)} j_{U!}(\gamma_U) \to D \quad \text{in} \quad \mathsf{Shv}_G(S_{Zar})$$
whose fiber over $D(c) \hookrightarrow D$ is the morphism
$$\coprod_{(U,\gamma)\in\mathscr{S}_G^m(D(c))} j_{U!}(\gamma_U) \to D(c) \quad \text{in} \quad \mathsf{Shv}_G(S_{Zar}).$$

Lemma 28. *If D is smooth, any element of $\mathscr{S}_G(D)$ has a majorant in $\mathscr{S}_G^m(D)$.*

Proof. Fix (U,γ). By Zorn's lemma, we have to show that any chain $\mathscr{C}$ in
$$\mathscr{M} = \{(V,\theta) \in \mathscr{S}_G(D) : (U,\gamma) \leq (V,\theta)\}$$
has an upper bound in $\mathscr{M}$. Let $U' = \cup_{(V,\theta)\in\mathscr{C}} V$. Then any element of
$$\varprojlim_{(V,\theta)\in\mathscr{C}} \theta \subset \varprojlim_{(V,\theta)\in\mathscr{C}} D(V) = D(U')$$
defines a G-orbit $\gamma' \subset D(U')$ such that $(U',\gamma') \in \mathscr{M}$ dominates $\mathscr{C}$. So we have to show that the left hand side limit is not empty. For each single $(V,\theta) \in \mathscr{C}$, we may first choose some $x \in \theta$, consider the corresponding orbit map $G \twoheadrightarrow \theta$, and equip θ with the induced quotient topology. One checks that it does not depend upon x, and turns θ into a compact topological space which is Hausdorff since the stabilizer G_x is closed. Then $(V,\theta) \mapsto \theta$ is an inverse system of nonempty compact Hausdorff spaces indexed by a filtered set (namely $\mathscr{C}$), so its limit is not empty. $\square$

Proposition 29. *Suppose that $D \in \mathsf{Shv}_G^{sm}(S_{Zar})$. Then*

(1) The morphism
$$\coprod_{(U,\gamma)\in\mathscr{S}_G^m(D)} j_{U!}(\gamma_U) \to D$$
is an epimorphism.

(2) The canonical map

$$\mathscr{S}^m_G(D) \to \pi_0(D)$$

is surjective.

(3) The following conditions are equivalent:
- *(a) For any opens $\emptyset \neq V \subset U$ of S, $D(U) \to D(V)$ is injective on orbits,*
- *(b) For any open U of S and $s \in U$, $D(U) \to D_s$ is injective on orbits,*
- *(c) For any open $U \neq \emptyset$ of S, $D(U) \to D_\eta$ is injective on orbits,*
- *(d) For any $s \in S$, $D_s \to D_\eta$ is injective on orbits,*
- *(e) For any specialization $s' \rightsquigarrow s$ in S, $D_s \to D_{s'}$ is injective on orbits,*
- *(f) For every $(U, \gamma) \in \mathscr{S}_G(D)$, $j_{U!}(\gamma_U) \to D$ is a monomorphism,*
- *(g) For every $(U, \gamma) \in \mathscr{S}^m_G(D)$, $j_{U!}(\gamma_U) \to D$ is a monomorphism,*
- *(h) The product of units $D \to (D_\eta)_S \times D_G$ is a monomorphism,*
- *(i) The unit $D \to D_{et}$ is an isomorphism.*

(4) The following conditions are equivalent:
- *(a) For any opens $\emptyset \neq V \subset U$ of S, $D(U) \to D(V)$ is injective,*
- *(b) For any open U of S and $s \in U$, $D(U) \to D_s$ is injective,*
- *(c) For any open $U \neq \emptyset$ of S, $D(U) \to D_\eta$ is injective,*
- *(d) For any $s \in S$, $D_s \to D_\eta$ is injective,*
- *(e) For any specialization $s' \rightsquigarrow s$ in S, $D_s \to D_{s'}$ is injective,*
- *(f) D satisfies the conditions of (3) and $\mathscr{S}^m_G(D) \to \pi_0(D)$ is bijective,*
- *(g) The morphism $\coprod_{(U,\gamma)\in\mathscr{S}^m_G(D)} j_{U!}(\gamma_U) \to D$ is an isomorphism,*
- *(h) The unit $D \to (D_\eta)_S$ is a monomorphism.*
- *(i) The unit $D \to D_{set}$ is an isomorphism.*

Proof. (1) For $U \subset S$ and $d \in D(U)$ with G-orbit γ, pick $(U', \gamma') \in \mathscr{S}^m_G(D)$ over (U, γ). Evaluating $j_{U'!}\gamma'_{U'} \to D$ at U gives $\gamma' \hookrightarrow D(U') \to D(U)$, whose image equals γ, and so contains d: our morphism is already surjective in PreShv.

(2) Let c be a G-orbit in D_η, $d_\eta \in c$, U a small open of S where d_η lifts to $d \in D(U)$, γ the G-orbit of d, and $(U', \gamma') \in \mathscr{S}^m_G(D)$ over (U, γ). Then (U', γ') maps to $\gamma_\eta = c$ in $G\backslash D_\eta = \pi_0(D)$: our morphism $\mathscr{S}^m_G(D) \to \pi_0(D)$ is surjective.

In (3) and (4), the equivalence of conditions (a) trough (e) are easy, $(d) \Leftrightarrow (h)$ since monomorphicity is equivalent to injectivity on all stalks, and $(h) \Leftrightarrow (i)$ by definition of D_{et} and D_{set}. Also: $(3, a) \Leftrightarrow (3, f) \Rightarrow (3, g)$ and $(4, g) \Rightarrow (4, a)$ are obvious. It remains to establish the following three implications:

$(3, g) \Rightarrow (3, a)$. Let $V \subset U$ be nonempty opens of S, γ a G-orbit in $D(U)$. Pick $(U', \gamma') \in \mathscr{S}^m_G(D)$ above (U, γ). By $(3, g)$, $j_{U'!}\gamma'_{U'} \to D$ is a monomorphism. Evaluating it at V gives an injection $\gamma' \hookrightarrow D(U') \to D(V)$, which factors as

$$\gamma' \twoheadrightarrow \gamma \hookrightarrow D(U) \to D(V)$$

So $\gamma' \to \gamma$ is a bijection and the restriction map $D(U) \to D(V)$ is injective on γ.

$(4, c) \Rightarrow (4, f)$. Plainly $(4, c) \Rightarrow (3, c)$, so D satisfies all conditions of (3). Suppose that (U_1, γ_1) and (U_2, γ_2) in $\mathscr{S}^m_G(D)$ have the same image γ_η in $\pi_0(D) = G\backslash D_\eta$. Fix $x \in \gamma_\eta$ and lift it to $x_i \in \gamma_i$. By $(4, c)$ applied to $V = U_1 \cap U_2 \neq \emptyset$, $x_1|_V = x_2|_V$ in $D(V)$, so the x_i's glue to $x \in D(U)$ where $U = U_1 \cup U_2$. Let γ be the G-orbit of x. Then $(U_i, \gamma_i) \leq (U, \gamma)$ in $\mathscr{S}_G(D)$, whence $(U_1, \gamma_1) = (U, \gamma) = (U_2, \gamma_2)$ by maximality of (U_i, γ_i). Given (2), this proves $(4, f)$.

$(4,f) \Rightarrow (4,g)$. Using (1) and the decomposition $D = \coprod_{c\in\pi_0(D)} D(c)$, it is sufficient to establish that for any $c \in \pi_0(D)$, the morphism

$$\coprod_{(U,\gamma)\in\mathscr{S}_G^m(D,c)} j_{U!}(\gamma_U) \to D(c)$$

is a monomorphism. The second part of $(4,f)$ tells us that $\mathscr{S}_G^m(D,c)$ contains a single element (U,γ), and the first part of $(4,f)$ tells us that $(3,f)$ holds, which implies that $j_{U!}(\gamma_U) \to D(c)$ is indeed a monomorphism. □

Remark 30. If $D \in \mathsf{Shv}_G^\star$, we may replace $(D_\eta)_S$ by $(D_\eta)_S^\star$ in $(3,h)$ and $(4,h)$, and

$$(3,i) \iff D \in \mathsf{Shv}_G^{et}, \qquad (4,i) \iff D \in \mathsf{Shv}_G^{set}.$$

Lemma 31. *For $D \in \mathsf{Shv}_G^{et}(S_{Zar})$ and* any *open U of S,*

$$D(U) \textit{ is a smooth } G\textit{-set with trivial action of } I(U).$$

Proof. If $U = \emptyset$, $D(U) = \star$ with trivial action. If $U \neq \emptyset$ and $s \in U$, the localization $D(U) \to D_s$ is G-equivariant, injective on orbits, and D_s is smooth, with trivial action of $I(s)$. It follows that $D(U)$ is smooth, with trivial action of $I(s)$ for all $s \in U$, hence with trivial action of $I(U)$ by proposition 23. □

8. Harvest

8.1. We now assume all of the above assumptions:

S is locally henselian, geometrically unibranch, and irreducible.

We thus have equivalence of categories

$$\begin{array}{ccccccc} \mathsf{AlgSp}_{et}(S) & \xrightarrow{\alpha} & \mathsf{Shv}(S_{et}) & \xrightarrow{\beta} & \mathsf{Shv}(S_{fet}) & \xrightarrow{\gamma} & \mathsf{Shv}_G^\star(S_{Zar}) \\ A & \longmapsto & B & \longmapsto & C & \longmapsto & D \end{array} \tag{8.1}$$

where $\gamma = \gamma_3 \circ \gamma_2 \circ \gamma_1$ and $G = \pi_1(\eta, \overline{\eta})$. We set $\delta = \gamma \circ \beta \circ \alpha$. In the sequel, we may often simplify our notations to $\mathsf{AlgSp}_{et} = \mathsf{AlgSp}_{et}(S)$, $\mathsf{Shv}_G^\star = \mathsf{Shv}_G^\star(S_{Zar})$, etc...

8.2. **Monomorphisms and epimorphisms.** Let $a : A_1 \to A_2$ be a morphism in AlgSp_{et} with image $d : D_1 \to D_2$ in $\mathsf{Shv}_G^\star$. Since δ is an equivalence of categories

$$\begin{array}{rcl} a \text{ is a monomorphism} & \iff & d \text{ is a monomorphism} \\ a \text{ is an epimorphism} & \iff & d \text{ is an epimorphism} \end{array}$$

We have described monomorphisms and epimorphisms of $\mathsf{Shv}_G^\star$ in section 7.4.

In AlgSp_{et}, monomorphisms are open immersions. Indeed since the embedding of AlgSp_{et} into AlgSp commutes with fiber products, a is a monomorphism in AlgSp_{et} if and only if it is a monomorphism in AlgSp. Since A_1 and A_2 are etale over S, a is an etale morphism by [6, Tag 03FV], so a is a monomorphism if and only if it is an open immersion by [6, Tag 05W5]. We thus obtain:

$$a \text{ is an open immersion} \iff d \text{ is a monomorphism of presheaves.}$$

In AlgSp_{et}, a is an epimorphism if and only if a is surjective. Indeed by [6, Tag 03MF], a is surjective if and only if there is a commutative diagram

$$\begin{tikzcd} A'_1 \arrow[r] \arrow[d] & A'_2 \arrow[d] \\ A_1 \arrow[r] & A_2 \end{tikzcd}$$

with $A'_i \in S_{et}$, $A'_1 \to A'_2$ surjective, and $A'_1 \to A_1$ (representable) surjective and etale. As observed in the proof of proposition 1, $A'_i \to A_i$ and $A'_1 \to A'_2$ are then epimorphisms in $\mathsf{Shv}((\mathsf{Sch}/S)_{et}))$, thus also in $\mathsf{Shv}((\mathsf{Sch}/S)_{fppf})$, $\mathsf{AlgSp}(S)$ and $\mathsf{AlgSp}_{et}(S)$, so a is an epimorphism. Conversely if a is an epimorphism in AlgSp_{et}, let $A'_2 \to A_2$ and $A'_1 \to A_1 \times_{A_2} A'_2$ be surjective etale morphisms with $A'_i \in S_{et}$. Then $A'_1 \to A_1$ is also surjective etale, and we want to show that $A'_1 \to A'_2$ is surjective. We have just seen that $A'_1 \to A_1 \times_{A_2} A'_2$ is an epimorphism in AlgSp_{et}; since a is an epimorphism, so is its base change $A_1 \times_{A_2} A'_2 \to A'_2$ in the topos $\mathsf{AlgSp}_{et}(S) \simeq \mathsf{Shv}(S_{et})$, thus $A'_1 \to A'_2$ is also an epimorphism in AlgSp_{et}. We are reduced to: if a morphism $a : A_1 \to A_2$ is S_{et} is an epimorphism in AlgSp_{et}, it is surjective as a morphism of schemes. This is obvious: glue two copies of A_2 along the (open!) image of a to obtain a new scheme $A_3 \in S_{et}$, equipped with two morphisms $b_1, b_2 : A_2 \to A_3$ such that $b_1 \circ a = b_2 \circ a$ in $S_{et} \subset \mathsf{AlgSp}_{et}$. Since a is an epimorphism in AlgSp_{et}, $b_1 = b_2$, therefore a is surjective. So:

$$a \text{ is surjective } \iff d \text{ is an epimorphism of sheaves.}$$

8.3. Representable etale algebraic spaces.

Lemma 32. *For $X \in S_{fet}$ with image U in S viewed as an element of AlgSp_{et},*

$$\delta(X) = j_{U!}(X(\overline{\eta})_U) \quad \textit{in} \quad \mathsf{Shv}_G.$$

Proof. Set $A = \mathrm{Hom}_S(-, X)$ and $D = \delta(A)$. Unwinding the definitions, D is the Zariski sheafification of the presheaf which takes $V \in S_{Zar}$ to the G-set with trivial action of $I(V)$ corresponding to the smooth $\pi(V)$-set defined by

$$B_V(\overline{\eta}) = \varinjlim_{(Y,y)\in \mathsf{Fet}_V(\overline{\eta})} B_V(Y) = \varinjlim_{(Y,y)\in \mathsf{Fet}_V(\overline{\eta})} \mathrm{Hom}_V(Y, X_V).$$

Since S is irreducible, so are the nonempty V's, thus for any $(Y, y) \in \mathsf{Fet}_V(\overline{\eta})$, the nonempty clopen image of the finite etale map $Y \to V$ equals V, i.e. $Y \twoheadrightarrow V$ is surjective. Hence $B_V(\overline{\eta}) = \emptyset$ when V is not contained in the image U of X in S. For $V \subset U$, $X_V \in \mathsf{Fet}_V$ and the map which takes $s \in \mathrm{Hom}_V(Y, X_V)$ to $s(y) \in X(\overline{\eta})$ induces a G-equivariant isomorphism $B_V(\overline{\eta}) \to X(\overline{\eta})$ of smooth G-sets. It follows that our presheaf is already a sheaf, namely $D = j_{U!}(X(\overline{\eta})_U)$. $\square$

Corollary 33. *There is a 2-commutative diagram*

$$\begin{tikzcd} \mathsf{Fet}_U \arrow[r, "(-)(\overline{\eta})"] \arrow[d, hook] & \mathsf{Set}^{fsm}_{\pi(U)} \arrow[r, "(-)_U"] & \mathsf{Shv}^{set}_G(U) \arrow[d, "j_{U!}"] \\ S_{fet} \arrow[rr, "\delta"] \arrow[d, hook] & & \mathsf{Shv}^{set}_G(S) \arrow[d, hook] \\ \mathsf{AlgSp}_{et}(S) \arrow[rr, "\delta"] & & \mathsf{Shv}^{\star}_G(S) \end{tikzcd}$$

Proof. For $X \in \mathsf{Fet}_U$, $Y = X(\overline{\eta})$ belongs to $\mathsf{Set}^{fsm}_{\pi(U)}$; for $Y \in \mathsf{Set}^{sm}_{\pi(U)}$, the constant G-sheaf $Z = Y_U$ belongs to $\mathsf{Shv}^{set}_G(U)$; for $Z \in \mathsf{Shv}^{set}_G(U)$, the extension $D = j_{U!}(Z)$ belongs to $\mathsf{Shv}^{set}_G(S)$. For X, Y, Z, D as indicated, we have seen that $D = \delta(X)$. This gives the 2-commutativity of the outer diagram. Since any $X \in S_{fet}$ belongs to Fet_U for some U (the image of X in S), we also have $\delta(S_{fet}) \subset \mathsf{Shv}^{set}_G(S)$. □

Theorem 34. *The functor δ induces equivalence of categories*

$$\begin{tikzcd} S_{set} \arrow[r, "\delta"] \arrow[d, hook, "\text{yon}"'] & \mathsf{Shv}^{set}_G \arrow[d, hook, "\text{inc}"] \\ S_{et} \arrow[r, "\delta"] \arrow[d, hook, "\text{yon}"'] & \mathsf{Shv}^{et}_G \arrow[d, hook, "\text{inc}"] \\ \mathsf{AlgSp}_{et} \arrow[r, "\delta"] & \mathsf{Shv}^{\star}_G \end{tikzcd}$$

where S_{set} is the strictly full subcategory of separated S-schemes in S_{et}.

Proof. Let A, B, C, D be as in 8.1. We have

$$\begin{aligned} & A \in \mathsf{AlgSp}_{et}(S) \text{ is representable (by some } X \in S_{et}) \\ \Longleftrightarrow\ & B \in \mathsf{Shv}(S_{et}) \text{ is representable (by the same } X \in S_{et}) \\ \Longleftrightarrow\ & C = \cup C_i \text{ in } \mathsf{Shv}(S_{fet}) \text{ with } C_i \text{ representable (by some } X_i \in S_{fet}) \\ \Longleftrightarrow\ & D = \cup D_i \text{ in } \mathsf{Shv}^{\star}_G(S_{Zar}) \text{ with } D_i \simeq \delta(X_i) \text{ for some } X_i \in S_{fet} \end{aligned}$$

using remark 4 and corollary 16. We have seen that δ maps S_{fet} to $\mathsf{Shv}^{set}_G \subset \mathsf{Shv}^{et}_G$. Since Shv^{et}_G is stable under arbitrary unions, we thus find that δ maps representable A's to D's in Shv^{et}_G. Suppose conversely that $D \in \mathsf{Shv}^{et}_G$. Then by proposition 29,

$$D = \cup_{(U,\gamma) \in \mathscr{S}^m_G(D)} D_{(U,\gamma)} \quad \text{in} \quad \mathsf{Shv}^{\star}_G$$

where $D_{(U,\gamma)} = j_{U!}(\gamma_U)$ with $\gamma \in \mathsf{Set}^{fsm}_{\pi(U)}$ by lemma 31, whence

$$D_{(U,\gamma)} \simeq \delta(X_{(U,\gamma)})$$

for an S-scheme $X_{(U,\gamma)} \in \mathsf{Fet}_U \subset S_{fet}$ with $X_{(U,\gamma)}(\overline{\eta}) \simeq \gamma$ as G-sets. Thus D satisfies the last displayed property, and A is representable by some $X \in S_{et}$ which is a union of the $X_{(U,\gamma)}$'s. If moreover $D \in \mathsf{Shv}^{set}_G$, then in fact

$$D = \coprod_{(U,\gamma) \in \mathscr{S}^m_G(D)} D_{(U,\gamma)} \quad \text{in} \quad \mathsf{Shv}^{\star}_G$$

so A is representable by $X = \coprod_{(U,\gamma)} X_{(U,\gamma)}$, which is indeed separated over S.

Finally, suppose that $A = \mathrm{Hom}_S(-, X)$ for some $X \in S_{set}$. We have to show that $D \in \mathsf{Shv}^{set}_G$, and we use the characterization $(4, c)$ of proposition 29. So let $U \neq \emptyset$ be open in S, and suppose that $d_1, d_2 \in D(U)$ have the same image $d_{1,\eta} = d_{2,\eta} = d$ in D_η. Let γ_i be the G-orbit of d_i, γ the G-orbit of d. We already know that $D \in \mathsf{Shv}^{et}_G$, thus $D(U) \to D_\eta$ induces G-equivariant bijections $\gamma_1 \to \gamma$ and $\gamma_2 \to \gamma$, whose inverse we denote by $\gamma \ni e \mapsto e_i \in \gamma_i$. By lemma 31, $\gamma \in \mathsf{Set}^{fsm}_{\pi(U)}$, so there is a $(Y, y) \in \mathsf{Fet}^c_U(\overline{\eta})$ with $(Y(\overline{\eta}), y) \simeq (\gamma, d)$ as pointed G-sets. Since $\delta(Y) \simeq j_{U!}(\gamma_U)$, we obtain morphisms $j_i : Y \to X$, corresponding to the morphisms $j_{U!}(\gamma_U) \simeq j_{U!}(\gamma_{i,U}) \hookrightarrow D$. Let $Z \hookrightarrow Y$ be the pull-back of the diagonal of X under $(j_1, j_2) : Y \to X \times_S X$ and set $E = \delta(Z)$. Since δ commutes with fiber products, E is the pull-back of the diagonal of D under $j_{U!}(\gamma) \to D \times D$. So

$$E(U) = \{e \in \gamma : e_1 = e_2 \text{ in } D(U)\} \quad \text{and} \quad E_\eta = \gamma.$$

Since X is separated etale over S, $Z \hookrightarrow Y$ is a clopen immersion, and since Y is finite etale over U, so is Z. Thus $E \simeq j_{U!}(Z(\overline{\eta})_U)$ and $E(U) \to E_\eta$ is a bijection. It follows that $d_1 = d_2$, which checks the characterization $(4, c)$, so $D \in \mathsf{Shv}_G^{set}$. $\square$

Remark 35. A punctual scheme S is irreducible, locally henselian and geometrically unibranch, with $\mathsf{Shv}_G^\star = \mathsf{Shv}_G^{et} = \mathsf{Shv}_G^{set}$. So we retrieve the well known fact that over such an S, every etale algebraic space is representable by a separated S-scheme.

Corollary 36. *Any $X \in S_{set}$ is a disjoint union of irreducible fet S-schemes.*

Proof. This either follows from the proof of theorem 34, or from the equivalence $\delta : S_{set} \to \mathsf{Shv}_G^{set}$ and the characterization $(4, g)$ of Shv_G^{set} in proposition 29. $\square$

8.4. **Underlying topological spaces.** Let Top be the category of topological spaces and continuous maps. For $\mathcal{S} \in \mathsf{Top}$, we denote by $\mathcal{S}_{loc}$ the strictly full subcategory of $\mathsf{Top}/\mathcal{S}$ whose objects are local homeomorphisms $\mathcal{X} \to \mathcal{S}$. It is well known that the functor $\sec : \mathcal{S}_{loc} \to \mathsf{Shv}(\mathcal{S})$ which takes $\mathcal{X}$ to its sheaf of sections is an equivalence of categories. An inverse functor maps a sheaf of sets $\mathcal{C}$ on $\mathcal{S}$ to the corresponding *espace étalé*: the underlying set is $\coprod_{s \in \mathcal{S}} \mathcal{C}_s$, and a basis of open neighborhoods of $x \in \mathcal{C}_s$ is given by the subsets $\{c_y : y \in V\}$, for V open in S containing s and $c \in \mathcal{C}(V)$ with $c_s = x$.

Likewise, if S_{loc} is the full subcategory of S_{et} whose objects are local isomorphisms $X \to S$, then the functor $\sec : S_{loc} \to \mathsf{Shv}(S_{Zar})$ which takes $X \to S$ to its sheaf of sections is an equivalence of categories. An inverse functor maps a sheaf of sets C on S_{Zar} to the object X of S_{loc} whose underlying topological space $|X|$ is as above, equipped with the sheaf of rings $\mathcal{O}_X$ which is the pull-back through $|X| \to |S|$ of $\mathcal{O}_S$. It follows that the functor $S_{loc} \to |S|_{loc}$ which forgets the structure sheaves is an equivalence of categories.

Proposition 37. *There is a 2-commutative diagram of equivalences*

$$\begin{tikzcd} S_{loc} \arrow[r, "\sec"] \arrow[d, hook, "\text{yon}"'] & \mathsf{Shv} \arrow[d, hook, "\text{inc}"] \\ \mathsf{AlgSp}_{et} \arrow[r, "\delta"] & \mathsf{Shv}_G^\star \end{tikzcd}$$

Proof. Let $f : X \to S$ be a local isomorphism, $\mathcal{X} = \sec(X)$ its sheaf of sections, $D = \delta(X)$. Since $X \in S_{loc}$, there is a Zariski covering $X = \cup X_i$ such that f induces an isomorphism from X_i to an open $U_i = f(X_i)$ of S. Then $D = \cup D_i$ with

$$D_i = \delta(X_i) = j_{U_i!}(\star_{U_i}) \quad \text{where} \quad X_i(\overline{\eta}) = \{\star\}.$$

In particular D_i has trivial action of G, and so does $D = \cup D_i$. Let U be an open of S. A section $x \in \mathcal{X}(U)$ is an S-morphism $U \to X$ which δ maps to a morphism $j_{U!}(\star_U) \to D$ corresponding to a section $d \in D(U)$. Conversely, a section $d \in D(U)$ gives a morphism $j_{U!}(\star_U) \to D$ which is the image of an S-morphism $U \to X$ which is a section $x \in \mathcal{X}(U)$. We thus obtain an isomorphism of sheaves $\mathcal{X} \to D$ on S_{Zar} which is plainly functorial in X, and this yields the desired isomorphism

$$\text{inc} \circ \sec \to \delta \circ \text{yon}$$

between functors $S_{loc} \to \mathsf{Shv}^{\star}_G$. $\square$

We have seen in 7.5 that the embedding $\mathsf{Shv} \hookrightarrow \mathsf{Shv}^{\star}_G$ has left and right adjoints

$$(-)_G, (-)^G : \mathsf{Shv}^{\star}_G \to \mathsf{Shv}$$

with epimorphic units $D \twoheadrightarrow D_G$ and monomorphic counits $D^G \hookrightarrow D$. So the Yoneda embedding $S_{loc} \to \mathsf{AlgSp}_{et}$ has matching left and right adjoints

$$(-)_{loc}, (-)^{loc} : \mathsf{AlgSp}_{et} \to S_{loc}$$

with surjective units $A \twoheadrightarrow A_{loc}$ and open immersion counits $A^{loc} \hookrightarrow A$. To describe these functors, we first record the following consequence of our assumptions.

Lemma 38. *The topological space $|A|$ associated to $A \in \mathsf{AlgSp}_{et}$ belongs to $|S|_{loc}$.*

Proof. Suppose first that $A = \mathrm{Hom}_S(-, X)$ for some $X \in S_{et}$. Then $|A| = |X|$ by [6, Tag 03BX]. By proposition 14, X has a Zariski covering $X = \cup X_i$ with $X_i \in S_{fet}$. By remark 25, we may assume that X_i is irreducible. Then by proposition 17, $X_i \to S$ is an homeomorphism onto its image. So $|X| \in |S|_{loc}$. For the general case, let $X \to A$ be a surjective etale morphism with $X \in S_{et}$. Then $R = X \times_A X$ also belongs to S_{et}, and $|A|$ is the quotient of $|X|$ by the equivalence relation $|R|$ in Top, see [6, Tag 03BX]. Since $|X|, |R| \in |S|_{loc} \simeq \mathsf{Shv}(|S|)$, there is also a quotient $Q = |X| / |R|$ in $|S|_{loc}$. Since morphisms in $|S|_{loc}$ are local homeomorphisms, hence open, $|X| \to Q$ is open. Since $|X| \to Q$ is an epimorphism in $|S|_{loc}$, it is surjective: glue two copies of Q along the open image of $|X|$ to obtain Q' in $|S|_{loc}$ with two continuous maps $Q \to Q'$ inducing the same map $|X| \to Q'$; then these two maps must be equal, so $|X| \to Q$ is indeed surjective. Since open surjective continuous maps are quotient maps in Top, Q is the quotient of $|X|$ by the equivalence relation $|X| \times_Q |X|$ in Top. Since this fiber product belongs to $|S|_{loc}$, it is equal to the corresponding fiber product in $|S|_{loc} \simeq \mathsf{Shv}(|S|)$, which is $|R|$ by general properties of topoi. Thus $|A| \simeq Q$, and $|A|$ belongs to $|S|_{loc}$. $\square$

Proposition 39. *Left and right adjoints of $S_{loc} \to \mathsf{AlgSp}_{et}$ are as follows:*

(1) $(-)_{loc} : \mathsf{AlgSp}_{et} \to S_{loc}$ *takes A to the S-scheme $A_{loc} \in S_{loc}$ with*

$$|A_{loc}| = |A|, \qquad \mathcal{O}_{A_{loc}} = \text{pull-back of } \mathcal{O}_S \text{ through } |A| \to |S|.$$

The unit $A \twoheadrightarrow A_{loc}$ is a surjective homeomorphism.

(2) $(-)^{loc} : \mathsf{AlgSp}_{et} \to S_{loc}$ *takes A to the largest open subspace A^{loc} of A which belongs to S_{loc}. The counit is the open embedding $A^{loc} \hookrightarrow A$.*

Proof. The given formula for A_{loc} defines a functor

$$(-)_{loc} : \mathsf{AlgSp}_{et} \to S_{loc}.$$

We have to show that it is left adjoint to the embedding $S_{loc} \hookrightarrow \mathsf{AlgSp}_{et}$. This amounts to constructing a functorial unit $A \to A_{loc}$ which is an isomorphism when $A \in S_{loc}$. If $A = \mathrm{Hom}_S(-, X)$ for some $f : X \to S$ in S_{et}, X_{loc} has the same underlying space as X, with structure sheaf $\mathcal{O}_{X_{loc}} = f^*\mathcal{O}_S$ (pull-back as sheaves). The structure morphism $X \to S$ gives a morphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ whose adjoint $f^*\mathcal{O}_S \to \mathcal{O}_X$ is a morphism $\mathcal{O}_{X_{loc}} \to \mathrm{Id}_*\mathcal{O}_X$, which yields our unit $\epsilon_X : X \to X_{loc}$. This construction is functorial in X and gives the identity when $X \in S_{loc}$, so it produces the desired left adjoint of $S_{loc} \hookrightarrow S_{et}$. Extending our units from S_{et} to AlgSp_{et} is then a formal consequence of proposition 1: to define $\epsilon_A : A \to A_{loc}$ in AlgSp_{et} amounts to defining a morphism between the corresponding sheaves on

S_{et}, and this we do by mapping a section $a \in A(X)$ corresponding to a morphism $a : X \to A$, to the section $a^{\sim}_{loc} \in A_{loc}(X)$ which corresponds to the morphism

$$X \xrightarrow{\epsilon_X} X_{loc} \xrightarrow{a_{loc}} A_{loc}.$$

For the right adjoint, we know that it exists, and its counits are open immersions. Thus A^{loc} is an open subspace of A belonging to S_{loc}, and by the adjunction property, any such open is contained in A^{loc}. So A^{loc} is the largest such open. □

8.5. **Connected components.** By definition, the set of connected components $\pi_0(A)$ of an algebraic space A over S is the set of connected components of the associated topological space $|A|$. For $A \in \mathsf{AlgSp}_{et}$, $|A| = |A_{loc}|$, so $\pi_0(A)$ is also the set of connected components of the etale S-scheme A_{loc}, which by remark 22, is the set of points of its generic fiber $A_{loc,\eta}$. Since $A_{loc} \to S$ is a local isomorphism, so is its generic fiber, thus $\pi_0(A)$ is also the set of sections of $A_{loc,\eta} \to \eta$. If $D = \delta(A)$, then $D_G \simeq \delta(A_{loc}) \simeq \sec(A_{loc})$ so $\pi_0(A) \simeq D_{G,\eta} \simeq G\backslash D_\eta = \pi_0(D)$.

We may also work this out as follows, without passing through A_{loc}. Since A is etale over S, $|A|$ is locally connected: this follows from remark 22 and [6, Tag 03BT] (and holds whenever S locally has a finite number of irreducible components). We may then redefine $\pi_0(A)$ as the set of minimal nonempty clopen subsets of $|A|$, and $|A|$ is the disjoint union of these connected components. By [6, Tag 03BZ], opens of $|A|$ correspond bijectively with opens of A, which as seen above, are just the subobjects of A in AlgSp_{et}. Thus $\pi_0(A)$ is also the set of minimal nontrivial complemented elements in the poset of all subobjects of A in AlgSp_{et}, and A is the disjoint union of these subobjects. Passing this categorical description through the equivalence δ, we obtain another set $\pi'_0(D)$ of connected components of $D = \delta(A)$, and we must check that it matches the set $\pi_0(D)$ from section 7.9. Since

$$D = \textstyle\coprod_{c \in \pi_0(D)} D(c) \quad \text{in} \quad \mathsf{Shv}^\star_G$$

we have to show that any nontrivial complemented G-stable subsheaf D_1 of D contained in $D(c)$ equals $D(c)$. Since D_1 is complemented in D, it is also complemented in $D(c)$: there is a G-stable subsheaf D_2 with $D(c) = D_1 \coprod D_2$. But then $c = D_{1,\eta} \coprod D_{2,\eta}$, and this forces one of the $D_{i,\eta}$'s to be empty, which can only occur if D_i itself is the empty sheaf. So $D_1 = D(c)$, and indeed $\pi_0(D) = \pi'_0(D)$.

8.6. **Base change.** Let S' be another irreducible, locally henselian and geometrically unibranch scheme, and let $f : S' \to S$ be a morphism inducing an isomorphism on generic points. For instance, we could take any of the following:

(1) An open immersion $j_U : U \hookrightarrow S$ with $U \neq \emptyset$,
(2) A monomorphism $S(s) \hookrightarrow S$ with $S(s) = \mathrm{Spec}(\mathcal{O}_{S,s})$ for some $s \in S$,
(3) The closed immersion $S_{red} \hookrightarrow S$,
(4) The generic point map $\iota : \eta \to S$.

Proposition 40. *There is a 2-commutative diagram*

$$\begin{tikzcd}
\mathsf{AlgSp}_{et}(S) \arrow[r, "\alpha"] \arrow[d, "f^*"'] & \mathsf{Shv}(S_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[d, "f^*"] & \mathsf{Shv}^\star_G(S_{Zar}) \arrow[r, hook, "\mathrm{inc}"] \arrow[d, "f^*"] & \mathsf{Shv}_G(S_{Zar}) \arrow[d, "f^*"] \\
\mathsf{AlgSp}_{et}(S') \arrow[r, "\alpha"] & \mathsf{Shv}(S'_{et}) \arrow[r, "\gamma\circ\beta"] & \mathsf{Shv}^\star_G(S'_{Zar}) \arrow[r, hook, "\mathrm{inc}"] & \mathsf{Shv}_G(S'_{Zar})
\end{tikzcd}$$

Proof. The 2-commutativity of the first square was established in proposition 5. For the second and third square, we first construct a base change morphism

$$\begin{tikzcd} \mathsf{Shv}\,(S_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[d, "f^*"'] & \mathsf{Shv}_G(S_{Zar}) \arrow[d, "f^*"] \arrow[dl, phantom, "bc"] \\ \mathsf{Shv}\,(S'_{et}) \arrow[r, "\gamma\circ\beta"'] & \mathsf{Shv}_G(S'_{Zar}) \end{tikzcd} \quad \text{adjoint to} \quad \begin{tikzcd} \mathsf{Shv}\,(S_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[d, "f^*"'] \arrow[dr, phantom, "bc'"] & \mathsf{Shv}_G(S_{Zar}) \\ \mathsf{Shv}\,(S'_{et}) \arrow[r, "\gamma\circ\beta"'] & \mathsf{Shv}_G(S'_{Zar}) \arrow[u, "f_*"'] \end{tikzcd}$$

So let $B \in \mathsf{Shv}(S_{et})$, $B' = f^*B$ and let C and C' be their images under $\gamma_2 \circ \gamma_1 \circ \beta$, viewed as presheaves of G-sets, so that γ_3 is the sheafification functor a. We have to construct a morphism of sheaves $bc' : a(C) \to f_*a(C')$, or equivalently a morphism of presheaves $C \to f_*a(C')$; it is sufficient to construct a morphism of presheaves $C \to f_*C'$. Now for any open U of S with preimage $U' = f^{-1}(U)$ in S', we have

$$C(U) = \varinjlim_{(X,x)\in\mathsf{Fet}_U(\overline{\eta})} B(X)$$
$$f_*C'(U) = \varinjlim_{(X',x')\in\mathsf{Fet}_{U'}(\overline{\eta})} B'(X')$$

The unit $B \to f_*f^*B = f_*B'$ induces a morphism

$$\begin{tikzcd} \mathsf{Fet}_U(\overline{\eta}) \arrow[drr, "B(-)"] \arrow[d, "-\times_U U'"'] & & \\ & & \mathsf{Set} \\ \mathsf{Fet}_{U'}(\overline{\eta}) \arrow[urr, "B'(-)"'] & & \end{tikzcd}$$

whose colimit gives a G-equivariant map $C(U) \to f_*C'(U)$, which yields the desired morphism $C \to f_*C'$. As a consequence, we obtain a factorization

$$\begin{tikzcd} \mathsf{Shv}\,(S_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[d, "f^*"'] & \mathsf{Shv}^{\star}_G(S_{Zar}) \arrow[r, hook, "\mathrm{inc}"] \arrow[d, "f^*"] \arrow[dl, phantom, "bc"] & \mathsf{Shv}_G(S_{Zar}) \arrow[d, "f^*"] \\ \mathsf{Shv}\,(S'_{et}) \arrow[r, "\gamma\circ\beta"'] & \mathsf{Shv}^{\star}_G(S'_{Zar}) \arrow[r, hook, "\mathrm{inc}"] & \mathsf{Shv}_G(S'_{Zar}) \end{tikzcd}$$

and we now have to show that the natural transformation of the first square is an isomorphism $bc : f^* \circ (\gamma \circ \beta) \to (\gamma \circ \beta) \circ f^*$. Since all functors in sight are left adjoints or equivalences, they commute with all colimits. Since any etale sheaf B on S is a colimit of representable sheaves, we may, by proposition 14, restrict our attention to $B = \mathrm{Hom}_S(-, X)$ with X in S_{fet}, say with image U in S. Then $f^*B = \mathrm{Hom}_{S'}(-, X')$ with $X' = X \times_S S'$ in S'_{fet} with image $U' = f^{-1}(U)$ in S'. Note that $X'(\overline{\eta}) \simeq X(\overline{\eta})$ as G-sets. By lemma 32, we have

$$\gamma \circ \beta(B) = j_{U!}(X(\overline{\eta})_U) \quad \text{and} \quad \gamma \circ \beta(f^*B) = j_{U'!}(X'(\overline{\eta})_{U'}).$$

So we are reduced to showing that the following diagram is 2-commutative:

$$\begin{tikzcd}
 & \mathsf{Shv}_G(U_{Zar}) \arrow[r, "j_{U!}"] \arrow[d, "f^*"] & \mathsf{Shv}_G(S_{Zar}) \arrow[d, "f^*"] \\
\mathsf{Set}_G \arrow[ur, "(-)_U"] \arrow[dr, "(-)_{U'}"'] & & \\
 & \mathsf{Shv}_G(U'_{Zar}) \arrow[r, "j_{U'!}"] & \mathsf{Shv}_G(S'_{Zar})
\end{tikzcd}$$

This follows from the obvious 2-commutativity of the right adjoint diagram:

$$\begin{tikzcd}
 & \mathsf{Shv}_G(U_{Zar}) \arrow[dl, "\Gamma(U,-)"'] & \mathsf{Shv}_G(S_{Zar}) \arrow[l, "j_U^*"'] \\
\mathsf{Set}_G & & \\
 & \mathsf{Shv}_G(U'_{Zar}) \arrow[ul, "\Gamma(U',-)"] \arrow[uu, "f_*"] & \mathsf{Shv}_G(S'_{Zar}) \arrow[l, "j_{U'}^*"'] \arrow[uu, "f_*"]
\end{tikzcd}$$

This proves the proposition. $\square$

Corollary 41. *There is a right adjoint dual* 2*-commutative diagram*

$$\begin{tikzcd}
\mathsf{AlgSp}_{et}(S) \arrow[r, "\alpha"] & \mathsf{Shv}(S_{et}) \arrow[r, "\gamma\circ\beta"] & \mathsf{Shv}^\star_G(S_{Zar}) \\
\mathsf{AlgSp}_{et}(S') \arrow[r, "\alpha"] \arrow[u, "f^{et}_*"] & \mathsf{Shv}(S'_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[u, "f_*"] & \mathsf{Shv}^\star_G(S'_{Zar}) \arrow[u, "f^{et}_*"]
\end{tikzcd}$$

Remark 42. As in remark 6, while f^* on $\mathsf{Shv}^\star_G$ is the restriction of the eponymous functor on Shv_G, their respective right adjoints f^{et}_* and f_* are related by

$$\begin{tikzcd}
\mathsf{Shv}^\star_G(S_{Zar}) & \mathsf{Shv}_G(S_{Zar}) \arrow[l, "(-)^{sm,\star}"'] \\
\mathsf{Shv}^\star_G(S'_{Zar}) \arrow[r, hook, "\mathrm{inc}"] \arrow[u, "f^{et}_*"] & \mathsf{Shv}_G(S'_{Zar}) \arrow[u, "f_*"]
\end{tikzcd}$$

When f is quasi-compact, f_* takes smooth sheaves to smooth sheaves, and we may thus replace the right column by $f_* : \mathsf{Shv}^{sm}_G(S'_{Zar}) \to \mathsf{Shv}^{sm}_G(S_{Zar})$.

Corollary 43. *For $A \in \mathsf{AlgSp}_{et}(S)$ with pull-back $A_{red} \in \mathsf{AlgSp}_{et}(S_{red})$,*

$$A \textit{ is representable } \iff A_{red} \textit{ is representable.}$$

Moreover, $A_{red}(X \times_S S_{red}) = A(X)$ for any proetale morphism $X \to S$.

Proof. Apply the proposition to $f : S_{red} \to S$ and note that the pull-back functor on Zariski sheaves $\mathsf{Shv}^\star_G(S_{Zar}) \to \mathsf{Shv}^\star_G(S_{red,Zar})$ is the identity of $\mathsf{Shv}^\star_G(|S|)$. For the second assertion, the proetale case follows from the etale case by [6, Tag 0468], which itself follows from the fact that f^* is here an equivalence. $\square$

8.7. **Generic fiber.** Applying proposition 40 to $\iota : \eta \to S$, we obtain

$$\begin{tikzcd} \mathsf{AlgSp}_{et}(S) \arrow[r, "\alpha"] \arrow[d, "\iota^*"'] & \mathsf{Shv}(S_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[d, "\iota^*"] & \mathsf{Shv}^\star_G(S_{Zar}) \arrow[d, "\iota^*"] \arrow[dr, bend left, "(-)_\eta"] & \\ \mathsf{AlgSp}_{et}(\eta) \arrow[r, "\alpha"] & \mathsf{Shv}(\eta_{et}) \arrow[r, "\gamma\circ\beta"] & \mathsf{Shv}^\star_G(\eta_{Zar}) \arrow[r, "\Gamma(\eta,-)"] & \mathsf{Set}^{sm}_G \end{tikzcd}$$

where we have added the obviously 2-commutative triangle at the end, in which $\Gamma(\eta, -)$ is an equivalence. Passing to right adjoints, we obtain

$$\begin{tikzcd} \mathsf{AlgSp}_{et}(S) \arrow[r, "\alpha"] & \mathsf{Shv}(S_{et}) \arrow[r, "\gamma\circ\beta"] & \mathsf{Shv}^\star_G(S_{Zar}) & \\ \mathsf{AlgSp}_{et}(\eta) \arrow[r, "\alpha"] \arrow[u, "\iota^{et}_*"] & \mathsf{Shv}(\eta_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[u, "\iota_*"] & \mathsf{Shv}^\star_G(\eta_{Zar}) \arrow[r, "\Gamma(\eta,-)"] \arrow[u, "\iota^{et}_*"] & \mathsf{Set}^{sm}_G \arrow[ul, bend right, "(-)^\star_S"'] \end{tikzcd}$$

where the last adjunction is taken from (7.10), whose counits are isomorphisms. It follows that our right adjoints are fully faithful. The composite bottom equivalence is easily computed: it takes $\mathcal{A} \in \mathsf{AlgSp}_{et}(\eta)$ to the G-set $\mathcal{A}(\overline{\eta})$. We thus find that

$$\delta(\iota^{et}_*\mathcal{A}) \simeq \mathcal{A}(\overline{\eta})^\star_S.$$

Proposition 44. *The right adjoint ι^{et}_* factors through the strictly full subcategory of algebraic spaces which are representable by separated etale S-schemes.*

Proof. For $\mathcal{A} \in \mathsf{AlgSp}_{et}(\eta)$, set $A = \iota^{et}_*\mathcal{A}$ and $D = \mathcal{A}(\overline{\eta})^\star_S$, so that $D \simeq \delta(A)$, and D plainly belongs to $\mathsf{Shv}^{set}_G(S)$: we conclude by theorem 34. Concretely:

$$D = \coprod_{c\in\pi_0(D)} D(c) \quad \text{with} \quad D(c) \simeq j_{U(c)!}(\gamma(c)_{U(c)})$$

where $(U(c), \gamma(c))$ is the unique element of $\mathscr{S}^m_G(D)$ above $c \in G\backslash\mathcal{A}(\overline{\eta})$. By lemma 27, $U(c)$ is the largest open U of S such that c is fixed pointwise by $I(U)$, namely

$$U(c) = \{s \in S : c \subset \mathcal{A}(\overline{\eta})^{I(s)}\}.$$

Also, $\gamma(c) = c$ in $D(U(c)) = \mathcal{A}(\overline{\eta})^{I(U(c))}$. Let $X(c)$ be a connected finite etale $U(c)$-scheme with $X(c)(\overline{\eta}) \simeq c$ as $\pi(U(c))$-sets. Then $D(c) \simeq \delta(X(c))$, so

$$A \simeq \mathrm{Hom}_S(-, X) \quad \text{with } X = \coprod_{c\in\pi_0(A)} X(c).$$

Since all $X(c)$'s are separated over S, so is X. □

8.8. **The functor $S[-]^{et}$.** The previous proposition tells us that there is a functor

$$S[-]^{et} : \mathsf{AlgSp}_{et}(\eta) \to S_{set}$$

whose composition with the Yoneda embedding

$$S_{set} \hookrightarrow S_{et} \hookrightarrow \mathsf{AlgSp}_{et}(S)$$

is right adjoint to the generic fiber functor: for $A \in \mathsf{AlgSp}_{et}(S)$ and $\mathcal{A} \in \mathsf{AlgSp}_{et}(\eta)$,

$$\mathrm{Hom}_{\mathsf{AlgSp}(\eta)}(A_\eta, \mathcal{A}) \simeq \mathrm{Hom}_{\mathsf{AlgSp}(S)}(A, S[\mathcal{A}]^{et})$$

where $A_\eta = \iota^*A$ is the generic fiber of A. The counit $S[\mathcal{A}]^{et}_\eta \to \mathcal{A}$ is an isomorphism, so $S[-]^{et}$ is fully faithful, and δ maps the unit $A \to S[A_\eta]^{et}$ to the unit $D \to (D_\eta)^\star_S$, where $D = \delta(A)$. For an irreducible $\mathcal{A}$, $S[\mathcal{A}]^{et}$ belongs to the subcategory Fet^c_U of S_{set}, where U is the largest open of S such that $I(U)$ acts trivially on $\mathcal{A}(\overline{\eta})$.

Proposition 45. *If S is normal, then $S[\mathcal{A}]^{et}$ is the largest open $S[\mathcal{A}]^{\circ}$ of the normalization $S[\mathcal{A}]$ of S along $\mathcal{A} \to \eta \to S$ which is etale over S.*

Proof. By corollary 36, we may write $S[\mathcal{A}]^{et} = \coprod X_i$ with irreducible X_i's in S_{fet}. Then $\mathcal{A} = S[\mathcal{A}]^{et}_{\eta} = \coprod X_{i,\eta}$, so $S[\mathcal{A}] = \coprod S[X_{i,\eta}]$ where $S[X_{i,\eta}]$ is the normalization of S along $X_{i,\eta} \to \eta \to S$, and $S[\mathcal{A}]^{\circ} = \coprod S[X_{i,\eta}]^{\circ}$ where $S[X_{i,\eta}]^{\circ}$ is the largest open of $S[X_{i,\eta}]$ which is etale over S. Let U_i be the image of X_i in S. Since $X_i \to U_i$ is finite etale and U_i is normal, X_i is normal and isomorphic to the normalisation $U_i[X_{i,\eta}]$ of U_i along $X_{i,\eta} \to \eta \to U_i$. Since $U_i[X_{i,\eta}] \simeq X_i$ is etale, the open $U_i[X_{i,\eta}]$ of $S[X_{i,\eta}]$ is contained in $S[X_{i,\eta}]^{\circ}$. We thus obtain an open embedding

$$S[\mathcal{A}]^{et} \simeq \coprod U_i[X_{i,\eta}] \hookrightarrow \coprod S[X_{i,\eta}]^{\circ} = S[\mathcal{A}]^{\circ}$$

extending $\mathrm{Id} : \mathcal{A} \to \mathcal{A}$. Conversely by the universal property of $S[\mathcal{A}]^{et}$, there is a unique S-morphism $S[\mathcal{A}]^{\circ} \to S[\mathcal{A}]^{et}$ extending $\mathrm{Id} : \mathcal{A} \to \mathcal{A}$. Composing it with our open embedding, we obtain an S-morphism $S[\mathcal{A}]^{\circ} \to S[\mathcal{A}]^{\circ}$ extending $\mathrm{Id} : \mathcal{A} \to \mathcal{A}$. Since $S[\mathcal{A}]^{\circ}$ is separated etale over S, there is a unique such morphism, namely the identity of $S[\mathcal{A}]^{\circ}$. Our embedding is therefore surjective, and $S[\mathcal{A}]^{et} \simeq S[\mathcal{A}]^{\circ}$. □

8.9. **The $(-)_{set}$ and $(-)_{et}$ functors.** For $A \in \mathsf{AlgSp}_{et}$, define

$$A_{set} = \mathrm{Im}\left(A \to S[A_{\eta}]^{et}\right) \quad \text{and} \quad A_{et} = \mathrm{Im}\left(A \to S[A_{\eta}]^{et} \times A_{loc}\right).$$

By the results of section 8.2,

- A_{set} is open in $S[A_{\eta}]^{et}$ and thus belongs to S_{set},
- A_{et} is open in $S[A_{\eta}]^{et} \times A_{loc}$ and thus belongs to S_{et},

and we have a diagram of surjective morphisms in AlgSp_{et}

$$\begin{tikzcd} & & A_{set} \\ A \arrow[r, two heads] & A_{et} \arrow[ur, two heads] \arrow[dr, two heads] & \\ & & A_{loc} \end{tikzcd}$$

which δ maps to the analogous diagram from section 7.7,

$$\begin{tikzcd} & & D_{set} \\ D \arrow[r, two heads] & D_{et} \arrow[ur, two heads] \arrow[dr, two heads] & \\ & & D_G \end{tikzcd}$$

This construction defines functors $A \mapsto A_{set}$ and $A \mapsto A_{et}$ which are left adjoint to the Yoneda embeddings $S_{set} \hookrightarrow \mathsf{AlgSp}_{et}$ and $S_{et} \hookrightarrow \mathsf{AlgSp}_{et}$, with units $A \twoheadrightarrow A_{set}$ and $A \twoheadrightarrow A_{et}$, the latter inducing an homeomorphism on the underlying topological spaces. We have obtained 2-commutative diagrams of adjunctions

$$\begin{tikzcd} \mathsf{AlgSp}_{et} \arrow[r, shift left, "(-)_{et}"] \arrow[d, "\delta"'] & S_{et} \arrow[l, hook', shift left, "\mathrm{yon}"] \arrow[r, shift left, "(-)_{set}"] \arrow[d, "\delta"'] & S_{set} \arrow[l, hook', shift left, "\mathrm{inc}"] \arrow[r, shift left, "(-)_{\eta}"] \arrow[d, "\delta"'] & \eta_{et} \arrow[l, shift left, "S[-]^{et}"] \arrow[d, "(-)(\overline{\eta})"] \\ \mathsf{Shv}^{\star}_G \arrow[r, shift left, "(-)_{et}"] & \mathsf{Shv}^{et}_G \arrow[l, hook', shift left, "\mathrm{inc}"] \arrow[r, shift left, "(-)_{set}"] & \mathsf{Shv}^{set}_G \arrow[l, hook', shift left, "\mathrm{inc}"] \arrow[r, shift left, "(-)_{\eta}"] & \mathsf{Set}^{sm}_G \arrow[l, shift left, "(-)^{\star}_S"] \end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{AlgSp}_{et} \arrow[r, shift left, "(-)_{et}"] \arrow[d, "\delta"'] & S_{et} \arrow[l, hook', shift left, "\mathrm{yon}"] \arrow[r, shift left, "(-)_{loc}"] \arrow[d, "\delta"'] & S_{loc} \arrow[l, hook', shift left, "\mathrm{inc}"] \arrow[r, "|-|", "\sim"'] \arrow[d, "\delta"'] & |S|_{loc} \arrow[d, "\mathrm{sec}"] \\
\mathsf{Shv}^{\star}_{G} \arrow[r, shift left, "(-)_{et}"] & \mathsf{Shv}^{et}_{G} \arrow[l, hook', shift left, "\mathrm{inc}"] \arrow[r, shift left, "(-)_{G}"] & \mathsf{Shv} \arrow[l, hook', shift left, "\mathrm{inc}"] \arrow[r, equal] & \mathsf{Shv}
\end{tikzcd}$$

8.10. **Stalks.** Since all morphisms in Fet_S are finite etale, we may define

$$S\{\overline{\eta}\} = \varprojlim_{(X,x)\in\mathsf{Fet}_S(\overline{\eta})} X = \varprojlim_{(X,x)\in\mathsf{Fet}^c_S(\overline{\eta})} X = \varprojlim_{(X,x)\in\mathsf{Fet}^g_S(\overline{\eta})} X.$$

Here Fet^c_S and $\mathsf{Fet}^g_S \subset \mathsf{Fet}^c_S$ are the strictly full subcategories of X's in Fet_S which are respectively connected and Galois over S: the corresponding strictly full subcategories $\mathsf{Fet}^c_S(\overline{\eta})$ and $\mathsf{Fet}^g_S(\overline{\eta})$ are initial in $\mathsf{Fet}_S(\overline{\eta})$. So $S\{\overline{\eta}\}$ is a connected proetale cover of S with Galois group $\pi(S) = G/I(S)$. By remark 25 and [3, 8.2.9], $S\{\overline{\eta}\}$ is irreducible; by proposition 17, $S\{\overline{\eta}\} \to S$ is an homeomorphism. Let $E(S)$ be the category of finite subextensions L of the Galois extension $k(\eta, S) = k(\eta,\overline{\eta})^{I(S)}$ of $k(\eta)$, where $k(\eta,\overline{\eta})$ is the separable closure of $k(\eta)$ in $k(\overline{\eta})$. Mapping $(X,x) \in \mathsf{Fet}^c_S(\overline{\eta})$ to

$$L(X,x) = \text{image of } x^\sharp : \Gamma(X_\eta, \mathcal{O}_{X_\eta}) \to k(\overline{\eta})$$

defines a G-equivariant equivalence of categories $\mathsf{Fet}^c_S(\overline{\eta}) \to E(S)^{\mathrm{opp}}$. An inverse functor takes $L \in E(S)$ to the finite etale S-scheme $S[L]^{et} = S[\mathrm{Spec}(L)]^{et}$ equipped with the $\overline{\eta}$-valued point given by $\overline{\eta} \to \mathrm{Spec}(L) \simeq S[L]^{et}_\eta \to S[L]^{et}$. Thus also

$$S\{\overline{\eta}\} = \varprojlim_{L\in E(S)} S[L]^{et}.$$

For $s \in S$ and $U \ni s$ open in S, the base change functors $\mathsf{Fet}_S \to \mathsf{Fet}_U \to \mathsf{Fet}_{S(s)}$ induce G-equivariant proetale S-morphisms $S(s)\{\overline{\eta}\} \to U\{\overline{\eta}\} \to S\{\overline{\eta}\}$, and

$$S(s)\{\overline{\eta}\} = \varprojlim_{s\in U\subset S} U\{\overline{\eta}\} = \varprojlim_{(X,x)\in S_{fet}(\overline{\eta},s)} X.$$

Here $S(s) = \mathrm{Spec}(\mathcal{O}_{S,s})$ while $S_{fet}(\overline{\eta}, s) = \cup_{s\in U} \mathsf{Fet}_U(\overline{\eta})$ is the category of all pairs (X,x) where $X \in S_{fet}$ is such that its image U in S contains s, and $x \in X(\overline{\eta})$. For any such pair, we denote by $\tilde{x} : S(s)\{\overline{\eta}\} \to X$ the corresponding S-morphism. Since $S(s)\{\overline{\eta}\} \to S(s)$ is an homeomorphism, $S(s)\{\overline{\eta}\}$ is a local scheme. Let $\tilde{s} \to S(s)\{\overline{\eta}\}$ be a geometric point over the closed point $\overline{s}$ of $S(s)\{\overline{\eta}\}$. This yields a geometric point of S over s, and the corresponding strict henselization of $\mathcal{O}_{S,s}$ is given by

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,\tilde{s}}) = \varprojlim_{(Y,y)\in S_{et}(\tilde{s})} Y = \varprojlim_{(Y,y)\in S_{fet}(\tilde{s})} Y.$$

Here $S_{et}(\tilde{s})$ is the category of pairs (Y,y) with $Y \in S_{et}$ and $y \in Y(\tilde{s})$, and $S_{fet}(\tilde{s})$ is the strictly full subcategory where $Y \in S_{fet}$, which is initial by proposition 14. Mapping (X,x) to (Y,y) where $Y = X \in S_{fet}$ and $y \in Y(\tilde{s})$ is the composition

$$\tilde{s} \longrightarrow S(s)\{\overline{\eta}\} \xrightarrow{\tilde{x}} X$$

defines an equivalence of categories $S_{fet}(\overline{\eta}, s) \to S_{fet}(\tilde{s})$. We thus obtain

$$S(s)\{\overline{\eta}\} \simeq \mathrm{Spec}(\mathcal{O}^{sh}_{S,\tilde{s}}).$$

A posteriori, we find that we can essentially take $\tilde{s} \to S(s)\{\overline{\eta}\}$ to be the closed immersion $\overline{s} \hookrightarrow S(s)\{\overline{\eta}\}$ of the closed point $\overline{s}$ of $S(s)\{\overline{\eta}\}$, whose residue field $k(\overline{s})$ is indeed a separable closure of $k(s)$. With these conventions:

Proposition 46. *For any $s \in S$, there is a 2-commutative diagram*

$$\begin{tikzcd} \mathsf{AlgSp}_{et}(S) \arrow[r, "\alpha"] \arrow[dr, bend right, "(-)(\overline{s})"'] & \mathsf{Shv}(S_{et}) \arrow[r, "\gamma\circ\beta"] \arrow[d, "(-)_{\overline{s}}"] & \mathsf{Shv}^{\star}_{G}(S_{Zar}) \arrow[dl, bend left, "(-)_s"] \\ & \mathsf{Set}^{sm}_{G} & \end{tikzcd}$$

Proof. Proposition 7 gives commutativity of the first triangle, except that the target category was $\mathsf{Set}_{\Gamma(\overline{s})}$, with $\Gamma(s) = \mathrm{Gal}(k(\overline{s})/k(s))$. Since $S(s)\{\overline{\eta}\} = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$,

$$\Gamma(s) = \mathrm{Aut}(\mathcal{O}^{sh}_{S,\overline{s}}/\mathcal{O}_{S,s}) = \mathrm{Aut}(S(s)\{\overline{\eta}\}/S(s)) = \pi(S(s)) = G/I(s).$$

Since the third functor lands in the strictly full subcategory Set^{sm}_{G} of Set_G, it remains to establish the 2-commutativity of the second triangle. Unwinding the definitions, we find that for $B \in \mathsf{Shv}(S_{et})$, $C = \gamma_2 \circ \gamma_1 \circ \beta(B)$ and $D = \gamma_3(C)$,

$$B_{\overline{s}} = \varinjlim_{(Y,y)\in S_{et}(\overline{s})} B(Y) = \varinjlim_{(Y,y)\in S_{fet}(\overline{s})} B(Y)$$

$$D_s = C_s = \varinjlim_{s\in U} C(U) = \varinjlim_{s\in U} \varinjlim_{(X,x)\in \mathsf{Fet}_U(\overline{\eta})} B(X) = \varinjlim_{(X,x)\in S_{fet}(\overline{\eta},s)} B(X)$$

The equivalence $S_{fet}(\overline{\eta}, s) \to S_{fet}(\overline{s})$ considered above between the indexing categories of these colimits then yields the desired functorial isomorphism. □

For a specialization $s' \rightsquigarrow s$ in S, the base change functor $\mathsf{Fet}_{S(s)} \to \mathsf{Fet}_{S(s')}$ induces a G-equivariant S-morphism

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}'}) = S(s')\{\overline{\eta}\} \longrightarrow S(s)\{\overline{\eta}\} = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}).$$

The localization morphism between Zariski stalk functors

$$\mathsf{loc} : (-)_s \to (-)_{s'}$$

corresponds to the localization morphism between etale stalk functors

$$\mathsf{loc} : (-)_{\overline{s}} \to (-)_{\overline{s}'}$$

induced by the functor $S_{et}(\overline{s}) \to S_{et}(\overline{s}')$ mapping (Y, y) to (Y, y'), with y' given by

$$\overline{s}' \hookrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}'}) \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \xrightarrow{\tilde{y}} Y$$

where $\tilde{y}$ is the canonical map $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \varprojlim_{(Y,y)\in S_{et}(\overline{s})} Y \to Y$. On $\mathsf{AlgSp}_{et}(S)$, it corresponds to the localization morphism between geometric section functors

$$\mathsf{loc} : (-)(\overline{s}) \to (-)(\overline{s}')$$

whose evaluation at $A \in \mathsf{AlgSp}_{et}(S)$ is given by

$$A(\overline{s}) \xleftarrow{\simeq} A\left(\mathcal{O}^{sh}_{S,\overline{s}}\right) \longrightarrow A\left(\mathcal{O}^{sh}_{S,\overline{s}'}\right) \xrightarrow{\simeq} A(\overline{s}')$$

The morphism $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}'}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is a proetale cover of its image, which is the inverse image of $S(s')$ in $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. It follows that the middle map factors as

$$A\left(\mathcal{O}^{sh}_{S,\overline{s}}\right) \longrightarrow A\left(\mathcal{O}^{sh}_{S,\overline{s}} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s'}\right) \hookrightarrow A\left(\mathcal{O}^{sh}_{S,\overline{s}'}\right).$$

In particular by proposition 29, A is representable if and only if for all $s \in S$, $A\left(\mathcal{O}^{sh}_{S,\overline{s}}\right) \to A\left(\mathcal{O}^{sh}_{S,\overline{s}} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,\eta}\right)$ is injective on G-orbits.

9. Henselian valuation rings

We now apply the above results to the case where S is the spectrum of a valuation ring $\mathcal{O}$ with fraction field K, maximal ideal m, residue field $k(m) = \mathcal{O}/m$, and value group $\Gamma = K^\times/\mathcal{O}^\times$, whose group structure will be denoted additively. So if $v : K^\times \twoheadrightarrow \Gamma$ is the quotient map, $v(xy) = v(x) + v(y)$ and the formula

$$v(x) \geq v(y) \iff x \in \mathcal{O}y$$

turns Γ into a totally ordered commutative group. We extend v to $K \to \Gamma \cup \{\infty\}$ by $v(0) = \infty$, so that $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ and $m = \{x \in K : v(x) > 0\}$.

The set of $\mathcal{O}$-submodules of K is totally ordered by inclusion: if I_1 and I_2 are $\mathcal{O}$-submodules of K such that $I_1 \not\subset I_2$, then for any $x_1 \in I_1 \setminus I_2$ and $x_2 \in I_2$, $x_1 \notin \mathcal{O}x_2$, so $v(x_1) < v(x_2)$, hence $x_2 \in mx_1 \subset \mathcal{O}x_1 \subset I_1$, whence $I_2 \subset I_1$.

In particular, $S = \mathrm{Spec}(\mathcal{O})$ is totally ordered by inclusion. This totally ordered set is not entirely random: it has a smallest element 0, a largest element m, and any subset $\mathcal{S} \neq \emptyset$ of S has an inf and a sup in S, respectively given by

$$\inf(\mathcal{S}) = \cap_{q \in \mathcal{S}} q \qquad \text{and} \qquad \sup(S) = \cup_{q \in \mathcal{S}} q.$$

Thus all nonempty closed subsets of S are irreducible, of the form $V(p) = [p, m]$ for a unique $p \in S$, using standard notations for intervals in posets. Accordingly, any open $U \neq S$ of S is of the form $[0, p[$ for a unique $p \in S$. For $p \subset q$ in S, p is a prime ideal of $\mathcal{O}_q$ and $\mathcal{O}(p, q) = \mathcal{O}_q/p$ is a valuation ring with spectrum $[p, q]$, fraction field $k(p) = \mathcal{O}_p/p$, residue field $k(q) = \mathcal{O}_q/q$, and value group $\Gamma(p)/\Gamma(q)$, where $\Gamma(p) = v(\mathcal{O}_p^\times)$ and $\Gamma(q) = v(\mathcal{O}_q^\times)$ are convex subgroups of Γ. If $\mathcal{O}$ is henselian, then so are all $\mathcal{O}(p, q)$'s; in particular, $\mathcal{O}$ and all $\mathcal{O}(p, q)$'s are locally henselian.

Proposition 47. *For an open $U \neq \emptyset$ of S, the following conditions are equivalent:*

(1) U is a local scheme.
(2) U is affine.
(3) U is quasi-compact.
(4) U is special, i.e. $U = D(f)$ for some nonzero $f \in \mathcal{O}$.
(5) $U = [0, p]$ for some $p \in \mathrm{Spec}(\mathcal{O})$.

Proof. Plainly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (1) with $U = \mathrm{Spec}(\mathcal{O}_p)$. For (3) $\Rightarrow$ (4): If U is quasi-compact, it is covered by finitely many special opens $D(f_i) = \mathrm{Spec}(\mathcal{O}_{f_i})$ for nonzero f_i's in $\mathcal{O}$, and so $U = D(f)$ for any $f \in \{f_i\}$ with $v(f) = \min\{v(f_i)\}$. For (4) $\Rightarrow$ (5): If $U = D(f)$, then $p = \cup_{q \in U} q$ belongs to $D(f) = U$, so $U = [0, p]$. $\square$

Definition 48. A prime p of $\mathcal{O}$ is *special* if $[0, p]$ is open in $\mathrm{Spec}(\mathcal{O})$.

The map $p \mapsto [0, p]$ is an increasing bijection from special primes of $\mathcal{O}$ to special opens of $\mathrm{Spec}(\mathcal{O})$. A prime $p \neq m$ is special if and only if $\{r : p \subsetneq r\}$ has a minimal element q; then $[0, p] = [0, q[= D(f)$ for any $f \in q \setminus p$ and the valuation ring $\mathcal{O}(p, q)$ has height 1. The constructible partitions of $\mathrm{Spec}(\mathcal{O})$ are given by

$$\mathrm{Spec}(\mathcal{O}) = [0, p_1] \cup]p_1, p_2] \cup \cdots \cup]p_{n-1}, p_n]$$

for finite sequences $p_1 \subsetneq p_2 \subsetneq \cdots \subsetneq p_n = m$ of special primes of $\mathcal{O}$.

Given the simple structure of S, a Zariski sheaf on S is uniquely characterized by its restriction to nonempty special opens, its sections on such opens match the stalk at the corresponding special primes, and the restriction maps between these spaces of sections match the localization maps associated to the corresponding specializations among special points. In other words, the category of Zariski sheaves of sets D on S is equivalent to the category of functors $\mathscr{D} : \mathrm{Sp}(\mathcal{O})^\circ \to \mathsf{Set}$, where $\mathrm{Sp}(\mathcal{O})$ is the totally ordered set of special points in $\mathrm{Spec}(\mathcal{O})$, viewed as a category:

$$\mathscr{D}(p) = D([0,p]) = D_p \qquad \text{and} \qquad D(U) = \varprojlim\nolimits_{p\in \mathrm{Sp}(\mathcal{O})\cap U} \mathscr{D}(p).$$

In particular for all $q \in \mathrm{Spec}(\mathcal{O})$,

$$D([0,q[) = \varprojlim\nolimits_{p\in\mathrm{Sp}(\mathcal{O}), p<q} \mathscr{D}(p) \quad \text{and} \quad D_q = \varinjlim\nolimits_{p\in\mathrm{Sp}(\mathcal{O}), p\geq q} \mathscr{D}(p).$$

Similar considerations apply to sheaves of G-sets.

Suppose now that $\mathcal{O}$ is henselian and fix a geometric point $\overline{\eta} \to S$ over the generic point η of S. Let $K^{sep} = k(\eta, \overline{\eta})$ be the separable closure of $k(\eta) = K$ in $k(\overline{\eta})$ and set $G = \mathrm{Gal}(K^{sep}/K)$. For $s \in U \subset S$, let $K(U) \subset K(s) \subset K^{sep}$ be the fixed fields of $I(U) \supset I(s)$, and let $E(U) \subset E(s)$ be the finite extensions of K in $K(U)$ and $K(s)$. So $E(\eta)$ is the set of all finite extensions of K in K^{sep}, and

$$\begin{array}{rclrcl} K(s) & = & \cup_{s\in U} K(U), & K(U) & = & \cap_{s\in U} K(s), \\ E(s) & = & \cup_{s\in U} E(U), & E(U) & = & \cap_{s\in U} E(s), \end{array}$$

by proposition 23. For any integral K-algebra L, let $S[L]$ be the normalization of S in $\mathrm{Spec}(L) \hookrightarrow S$, i.e. $S[L] = \mathrm{Spec}(\mathcal{O}[L])$ where $\mathcal{O}[L]$ is the integral closure of $\mathcal{O}$ in L. If L is field, then $S[L] \to S$ is an homeomorphism by proposition 17, and for $s \in S$, we denote by $s_L \in S[L]$ the unique point above s. For $L \in E(\eta)$, $S[L] \to S$ is etale at s_L if and only if $L \in E(s)$, and $S[L]^{et} = U[L]$ where U is the largest open of S such that $L \in E(U)$. In particular for a special $s \in \mathrm{Sp}(\mathcal{O})$,

$$I(s) = I([0,s]), \qquad K(s) = K([0,s]) \quad \text{and} \quad E(s) = E([0,s]).$$

With notations as in section 8.10, we have

$$S\{\overline{\eta}\} = \varprojlim\nolimits_{L\in E(S)} S[L] = S[K(S)]$$

$$S(s)\{\overline{\eta}\} = \varprojlim\nolimits_{s\in U} U\{\overline{\eta}\} = \varprojlim\nolimits_{L\in E(s)} S(s)[L] = S(s)[K(s)] = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$$

where $\overline{s} = s_{K(s)}$ is the closed point of $S(s)\{\overline{\eta}\}$.

Summary. The category of etale algebraic spaces A over S and the category of etale sheaves B on S are equivalent to the category of presheaves of smooth G-sets $\mathscr{D}$ on $\mathrm{Sp}(\mathcal{O})$ such that for all special prime s of $\mathcal{O}$, $I(s)$ acts trivially on $\mathscr{D}(s)$, with

$$\mathscr{D}(s) = A(\overline{s}) = A(\mathcal{O}^{sh}_{S,\overline{s}}) = B_{\overline{s}} = \varinjlim\nolimits_{L\in E(s)} B(S[L]^{et}) = \varinjlim\nolimits_{L\in E(s)} B(S(s)[L]).$$

The representable (resp. representable and separated) objects correspond to those $\mathscr{D}'s$ such that for every $s' \subset s$ in $\mathrm{Sp}(\mathcal{O})$, the localization map $\mathscr{D}(s) \to \mathscr{D}(s')$ is injective on G-orbits (resp. injective). Under these equivalences,

- $A \mapsto A_\eta$ corresponds to $\mathscr{D} \mapsto \mathscr{D}_\eta = \varinjlim\nolimits_{s\in\mathrm{Sp}(\mathcal{O})} \mathscr{D}(s)$,
- $A \mapsto A_{set}$ to $\mathscr{D} \mapsto \mathscr{D}_{set}$, with $\mathscr{D}_{set}(s) = \mathrm{Im}(\mathscr{D}(s) \to \mathscr{D}_\eta)$,
- $A \mapsto A_{loc}$ to $\mathscr{D} \mapsto \mathscr{D}_{loc}$, with $\mathscr{D}_{loc}(s) = G\backslash\mathscr{D}(s)$,
- $A \mapsto A_{et}$ to $\mathscr{D} \mapsto \mathscr{D}_{et}$, with $\mathscr{D}_{et}(s) = \mathrm{Im}(\mathscr{D}(s) \to \mathscr{D}_\eta \times G\backslash\mathscr{D}(s))$.

References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, volume Vol. 269 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964.
[3] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.
[4] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.
[5] Laurent Moret-Bailly. A henselian preparation theorem. *Israel J. Math.*, 257(2):519–531, 2023.
[6] The Stacks project authors. The stacks project, 2022.

Sorbonne Université, Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France.