Generators of the pro-p Iwahori and Galois representations

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Abstract

For an odd prime p, we determine a minimal set of topological generators of the pro-p Iwahori subgroup of a split reductive group G over $\mathbb{Z}_p$. In the simple adjoint case and for any sufficiently large regular prime p, we also construct Galois extensions of $\mathbb{Q}$ with Galois group between the pro-p and the standard Iwahori subgroups of G.

1 Introduction

Let p be an odd prime, let $\mathbf{G}$ be a split reductive group over $\mathbb{Z}_p$, fix a Borel subgroup $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ of $\mathbf{G}$ with unipotent radical $\mathbf{U} \lhd \mathbf{B}$ and maximal split torus $\mathbf{T} \subset \mathbf{B}$. The Iwahori subgroup I and pro-p-Iwahori subgroup $I(1) \subset I$ of $\mathbf{G}(\mathbb{Z}_p)$ are defined [13, 3.7] by

$$I = \{g \in \mathbf{G}(\mathbb{Z}_p) : \ \mathrm{red}(g) \in \mathbf{B}(\mathbb{F}_p)\},$$
$$I(1) = \{g \in \mathbf{G}(\mathbb{Z}_p) : \ \mathrm{red}(g) \in \mathbf{U}(\mathbb{F}_p)\}.$$

where 'red' is the reduction map red: $\mathbf{G}(\mathbb{Z}_p) \to \mathbf{G}(\mathbb{F}_p)$. The subgroups I and $I(1)$ are both open subgroups of $\mathbf{G}(\mathbb{Z}_p)$. Thus $I = I(1) \rtimes T_{tors}$ and $\mathbf{T}(\mathbb{Z}_p) = T(1) \times T_{tors}$ where $T(1)$ and T_{tors} are respectively the pro$-p$ and torsion subgroups of $\mathbf{T}(\mathbb{Z}_p)$. Following [3] (who works with $\mathbf{G} = \mathbf{GL}_n$), we construct in section 2 a minimal set of topological generators for $I(1)$.

More precisely, let $M = X^*(\mathbf{T})$ be the group of characters of $\mathbf{T}$, $R \subset M$ the set of roots of $\mathbf{T}$ in $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, $\Delta \subset R$ the set of simple roots with respect to $\mathbf{B}$, $R = \coprod_{c \in \mathcal{C}} R_c$ the decomposition of R into irreducible components, $\Delta_c = \Delta \cap R_c$ the simple roots in R_c, $\alpha_{c,max}$ the highest positive root in R_c. We let $\mathcal{D} \subset \mathcal{C}$ be the set of irreducible components of type G_2 and for $d \in \mathcal{D}$, we denote by $\delta_d \in R_{d,+}$ the sum of the two simple roots in Δ_d. We denote by $M^\vee = X_*(\mathbf{T})$ the group of cocharacters of $\mathbf{T}$, by $\mathbb{Z}R^\vee$ the subgroup spanned by the coroots $R^\vee \subset M^\vee$ and we fix a set of representatives $\mathcal{S} \subset M^\vee$ for an $\mathbb{F}_p$-basis of

$$(M^\vee/\mathbb{Z}R^\vee) \otimes \mathbb{F}_p = \oplus_{s \in \mathcal{S}} \mathbb{F}_p \cdot s \otimes 1.$$

We show (see theorem 2.4.1):

Theorem. *The following elements form a minimal set of topological generators of the pro-p-Iwahori subgroup $I(1)$ of $G = \mathbf{G}(\mathbb{Q}_p)$:*

1. *The semi-simple elements $\{s(1+p) : s \in \mathcal{S}\}$ of $T(1)$,*
2. *For each $c \in \mathcal{C}$, the unipotent elements $\{x_\alpha(1) : \alpha \in \Delta_c\}$,*
3. *For each $c \in \mathcal{C}$, the unipotent element $x_{-\alpha_{c,max}}(p)$,*
4. *(If $p = 3$) For each $d \in \mathcal{D}$, the unipotent element $x_{\delta_d}(1)$.*

This result generalizes Greenberg [3] proposition 5.3, see also Schneider and Ollivier ([9], proposition 3.64, part i) for $G = SL_2$.

Let $\mathbf{T}^{ad}$ be the image of $\mathbf{T}$ in the adjoint group $\mathbf{G}^{ad}$ of $\mathbf{G}$. The action of $\mathbf{G}^{ad}$ on $\mathbf{G}$ induces an action of $\mathbf{T}^{ad}(\mathbb{Z}_p)$ on I and $I(1)$ and the latter equips the Frattini quotient $\tilde{I}(1)$ of $I(1)$ with a structure of $\mathbb{F}_p[T^{ad}_{tors}]$-module, where T^{ad}_{tors} is the torsion subgroup of $\mathbf{T}^{ad}(\mathbb{Z}_p)$ (cf. section 2.12). Any element β in $\mathbb{Z}R = M^{ad} = X^*(\mathbf{T}^{ad})$ induces a character $\beta : T^{tors}_{ad} \to \mathbb{F}_p^\times$ and we denote by $\mathbb{F}_p(\beta)$ the corresponding simple (1-dimensional) $\mathbb{F}_p[T^{ad}_{tors}]$-module. With these notations, the theorem implies that

Corollary. *The $\mathbb{F}_p[T^{ad}_{tors}]$-module $\tilde{I}(1)$ is isomorphic to*

$$\mathbb{F}_p^{\sharp\mathcal{S}} \oplus \Big(\oplus_{\alpha\in\Delta} \mathbb{F}_p(\alpha)\Big) \oplus \Big(\oplus_{c\in\mathcal{C}} \mathbb{F}_p(-\alpha_{c,max})\Big)\Big(\oplus \Big(\oplus_{d\in\mathcal{D}} \mathbb{F}_p(\delta_c)\Big) \text{ if } p = 3\Big).$$

Here $\sharp\mathcal{S}$ is the cardinality of $\mathcal{S}$. Suppose from now on in this introduction that $\mathbf{G}$ is simple and of adjoint type. Then:

Corollary *The $\mathbb{F}_p[T_{tors}]$-module $\tilde{I}(1)$ is multiplicity free unless $p = 3$ and $\mathbf{G}$ is of type A_1, B_ℓ or C_ℓ ($\ell \geq 2$), F_4 or G_2.*

Let now K be a Galois extension of $\mathbb{Q}$, Σ_p the set of primes of K lying above p. Let M be the compositum of all finite p-extensions of K which are unramified outside Σ_p, a Galois extension over $\mathbb{Q}$. Set $\Gamma = \mathrm{Gal}(M/K)$, $\Omega = \mathrm{Gal}(K/\mathbb{Q})$ and $\Pi = \mathrm{Gal}(M/\mathbb{Q})$. We say that K is p-rational if Γ is a free pro$-p$ group, see [6]. The simplest example is $K = \mathbb{Q}$, where $\Gamma = \Pi$ is also abelian and M is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Other examples of p-rational fields are $\mathbb{Q}(\mu_p)$ where p is a regular prime.

Assume K is a p-rational, totally complex, abelian extension of $\mathbb{Q}$ and $(p-1)\cdot\Omega = 0$. Then Greenberg in [3] constructs a continuous homomorphism

$$\rho_0 : \; \mathrm{Gal}(M/\mathbb{Q}) \to GL_n(\mathbb{Z}_p)$$

such that $\rho_0(\Gamma)$ is the pro-p Iwahori subgroup of $SL_n(\mathbb{Z}_p)$, assuming that there exists n distinct characters of Ω, trivial or odd, whose product is the trivial character.

In section 3, we are proving results which show the existence of p-adic Lie extensions of $\mathbb{Q}$ where the Galois group corresponds to a certain specific p-adic Lie algebra. More precisely, for p-rational fields, we construct continuous morphisms with open image $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$. We show in corollary 3.3.1 that

Corollary *Suppose that K is a p-rational totally complex, abelian extension of $\mathbb{Q}$ and $(p-1)\cdot\Omega = 0$. Assume also that if $p = 3$, our split simple adjoint group $\mathbf{G}$ is not of type A_1, B_ℓ or C_ℓ ($\ell \geq 2$), F_4 or G_2. Then there is a morphism $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$ if and only if there is morphism $\overline{\rho} : \Omega \to T_{tors}$ such that the characters $\alpha \circ \overline{\rho} : \Omega \to \mathbb{F}_p^\times$ for $\alpha \in \{\Delta \cup -\alpha_{max}\}$ are all distinct and belong to $\hat{\Omega}^{\mathcal{S}}_{odd}$.*

Here $\hat{\Omega}^{\mathcal{S}}_{odd}$ is a subset of the characters of Ω with values in $\mathbb{F}_p^\times$ (it is defined after proposition 3.2.1). Furthermore assuming $K = \mathbb{Q}(\mu_p)$ we show the existence of such a morphism $\overline{\rho} : \Omega \to T_{tors}$ provided that p is a sufficiently large regular prime (cf. section 3.2):

Corollary *There is a constant c depending only upon the type of $\mathbf{G}$ such that if $p > c$ is a regular prime, then for $K = \mathbb{Q}(\mu_p)$, M, Π and Γ as above, there is a continuous morphism $\rho : \Pi \to I$ with $\rho(\Gamma) = I(1)$.*

The constant c can be determined from lemmas 3.4.1, 3.4.2 and remark 3.4.3.

In section 2, we find a minimal set of topological generators of $I(1)$ and study the structure of $\tilde{I}(1)$ as an $\mathbb{F}_p[T^{ad}_{tors}]$-module. In section 3, assuming our group $\mathbf{G}$ to be simple and adjoint, we discuss the notion of p-rational fields and construct continuous morphisms $\rho : \Pi \to I$ with open image.

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2 Topological Generators of the pro-p Iwahori

This section is organized as follows. In sections $(2.1 - 2.3)$ we introduce the notations, then section 2.4 states our main result concerning the minimal set of topological generators of $I(1)$ (see theorem 2.4.1) with a discussion of the Iwahori factorisation in section 2.5. Its proof for $\mathbf{G}$ simple and simply connected is given in sections $(2.6 - 2.10)$, where section 2.10 deals with the case of a group of type G_2. The proof for an arbitrary split reductive group over $\mathbb{Z}_p$ is discussed in sections $(2.11 - 2.14)$. In particular, section 2.14 establishes the minimality of our set of topological generators. Finally, in section 2.15 we study the structure of the Frattini quotient $\tilde{I}(1)$ of $I(1)$ as an $\mathbb{F}_p[T^{ad}_{tors}]$-module and determine the cases when it is multiplicity free.

2.1 Let p be an odd prime, $\mathbf{G}$ be a split reductive group over $\mathbb{Z}_p$. Fix a pinning of $\mathbf{G}$ [11, XXIII 1]

$$(\mathbf{T}, M, R, \Delta, (X_\alpha)_{\alpha\in\Delta}) .$$

Thus $\mathbf{T}$ is a split maximal torus in $\mathbf{G}$, $M = X^*(\mathbf{T})$ is its group of characters,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha\in R}\mathfrak{g}_\alpha$$

is the weight decomposition for the adjoint action of $\mathbf{T}$ on $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, $\Delta \subset R$ is a basis of the root system $R \subset M$ and for each $\alpha \in \Delta$, X_α is a $\mathbb{Z}_p$-basis of $\mathfrak{g}_\alpha$.

2.2 We denote by $M^\vee = X_*(\mathbf{T})$ the group of cocharacters of $\mathbf{T}$, by $\alpha^\vee$ the coroot associated to $\alpha \in R$ and by $R^\vee \in M^\vee$ the set of all such coroots. We expand $(X_\alpha)_{\alpha\in\Delta}$ to a Chevalley system $(X_\alpha)_{\alpha\in R}$ of $\mathbf{G}$ [11, XXIII 6.2]. For $\alpha \in R$, we denote by $\mathbf{U}_\alpha \subset \mathbf{G}$ the corresponding unipotent group, by $x_\alpha : \mathbf{G}_{a,\mathbb{Z}_p} \to \mathbf{U}_\alpha$ the isomorphism given by $x_\alpha(t) = \exp(tX_\alpha)$. The height $h(\alpha) \in \mathbb{Z}$ of $\alpha \in R$ is the sum of the coefficients of α in the basis Δ of R. Thus $R_+ = h^{-1}(\mathbb{Z}_{>0})$ is the set of positive roots in R, corresponding to a Borel subgroup $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ of $\mathbf{G}$ with unipotent radical $\mathbf{U}$. We let $\mathcal{C}$ be the set of irreducible components of R, so that

$$R = \coprod_{c\in\mathcal{C}} R_c, \quad \Delta = \coprod_{c\in\mathcal{C}} \Delta_c, \quad R_+ = \coprod_{c\in\mathcal{C}} R_{c,+}$$

with R_c irreducible, $\Delta_c = \Delta \cap R_c$ is a basis of R_c and $R_{c,+} = R_+ \cap R_c$ is the corresponding set of positive roots in R_c. We denote by $\alpha_{c,max} \in R_{c,+}$ the highest root of R_c. We let $\mathcal{D} \subset \mathcal{C}$ be the set of irreducible components of type G_2 and for $d \in \mathcal{D}$, we denote by $\delta_d \in R_{d,+}$ the sum of the two simple roots in Δ_d.

2.3 Since $\mathbf{G}$ is smooth over $\mathbb{Z}_p$, the reduction map

$$\mathrm{red} : \mathbf{G}(\mathbb{Z}_p) \to \mathbf{G}(\mathbb{F}_p)$$

is surjective and its kernel $G(1)$ is a normal pro-p-subgroup of $\mathbf{G}(\mathbb{Z}_p)$. The Iwahori subgroup I and pro-p-Iwahori subgroup $I(1) \subset I$ of $\mathbf{G}(\mathbb{Z}_p)$ are defined [13, 3.7] by

$$\begin{aligned} I &= \{g \in \mathbf{G}(\mathbb{Z}_p) : \mathrm{red}(g) \in \mathbf{B}(\mathbb{F}_p)\}, \\ I(1) &= \{g \in \mathbf{G}(\mathbb{Z}_p) : \mathrm{red}(g) \in \mathbf{U}(\mathbb{F}_p)\}. \end{aligned}$$

Thus $I(1)$ is a normal pro-p-sylow subgroup of I which contains $\mathbf{U}(\mathbb{Z}_p)$ and

$$I/I(1) \simeq \mathbf{B}(\mathbb{F}_p)/\mathbf{U}(\mathbb{F}_p) \simeq \mathbf{T}(\mathbb{F}_p).$$

Since $\mathbf{T}(\mathbb{Z}_p) \twoheadrightarrow \mathbf{T}(\mathbb{F}_p)$ is split by the torsion subgroup $T_{tors} \simeq \mathbf{T}(\mathbb{F}_p)$ of $\mathbf{T}(\mathbb{Z}_p)$,

$$\mathbf{T}(\mathbb{Z}_p) = T(1) \times T_{tors} \qquad \text{and} \qquad I = I(1) \rtimes T_{tors}$$

where

$$T(1) = \mathbf{T}(\mathbb{Z}_p) \cap I(1) = \ker\left(\mathbf{T}(\mathbb{Z}_p) \to \mathbf{T}(\mathbb{F}_p)\right)$$

is the pro-p-sylow subgroup of $\mathbf{T}(\mathbb{Z}_p)$. Note that

$$\begin{aligned} T(1) &= \mathrm{Hom}\left(M, 1 + p\mathbb{Z}_p\right) = M^\vee \otimes (1 + p\mathbb{Z}_p), \\ T_{tors} &= \mathrm{Hom}\left(M, \mu_{p-1}\right) = M^\vee \otimes \mathbb{F}_p^\times. \end{aligned}$$

2.4 Let $\mathcal{S} \subset M^\vee$ be a set of representatives for an $\mathbb{F}_p$-basis of

$$(M^\vee/\mathbb{Z}R^\vee) \otimes \mathbb{F}_p = \oplus_{s\in\mathcal{S}} \mathbb{F}_p \cdot s \otimes 1.$$

Theorem 2.4.1. *The following elements form a minimal set of topological generators of the pro-p-Iwahori subgroup $I(1)$ of $G = \mathbf{G}(\mathbb{Q}_p)$:*

1. *The semi-simple elements $\{s(1 + p) : s \in \mathcal{S}\}$ of $T(1)$.*
2. *For each $c \in \mathcal{C}$, the unipotent elements $\{x_\alpha(1) : \alpha \in \Delta_c\}$.*
3. *For each $c \in \mathcal{C}$, the unipotent element $x_{-\alpha_{c,max}}(p)$.*
4. *(If $p = 3$) For each $d \in \mathcal{D}$, the unipotent element $x_{\delta_d}(1)$.*

2.5 By [11, XXII 5.9.5] and its proof, there is a canonical filtration

$$\mathbf{U} = \mathbf{U}_1 \supset \mathbf{U}_2 \supset \cdots \supset \mathbf{U}_h \supset \mathbf{U}_{h+1} = 1$$

of $\mathbf{U}$ by normal subgroups such that for $1 \leq i \leq h$, the product map (in any order)

$$\prod_{h(\alpha)=i} \mathbf{U}_\alpha \to \mathbf{U}$$

factors through $\mathbf{U}_i$ and yields an isomorphism of group schemes

$$\prod_{h(\alpha)=i} \mathbf{U}_\alpha \xrightarrow{\simeq} \overline{\mathbf{U}}_i, \quad \overline{\mathbf{U}}_i = \mathbf{U}_i/\mathbf{U}_{i+1}.$$

By [11, XXII 5.9.6] and its proof,

$$\overline{\mathbf{U}}_i(R) = \mathbf{U}_i(R)/\mathbf{U}_{i+1}(R)$$

for every $\mathbb{Z}_p$-algebra R. It follows that the product map

$$\prod_{h(\alpha)=i} \mathbf{U}_\alpha \times \mathbf{U}_{i+1} \to \mathbf{U}_i$$

is an isomorphism of $\mathbb{Z}_p$-schemes and by induction, the product map

$$\prod_{h(\alpha)=1} \mathbf{U}_\alpha \times \prod_{h(\alpha)=2} \mathbf{U}_\alpha \times \cdots \times \prod_{h(\alpha)=h} \mathbf{U}_\alpha \to \mathbf{U}$$

is an isomorphism of $\mathbb{Z}_p$-schemes. Similarly, the product map

$$\prod_{h(\alpha)=-h} \mathbf{U}_\alpha \times \prod_{h(\alpha)=-h+1} \mathbf{U}_\alpha \times \cdots \times \prod_{h(\alpha)=-1} \mathbf{U}_\alpha \to \mathbf{U}^-$$

is an isomorphism of $\mathbb{Z}_p$-schemes, where $\mathbf{U}^-$ is the unipotent radical of the Borel subgroup $\mathbf{B}^- = \mathbf{U}^- \rtimes \mathbf{T}$ opposed to $\mathbf{B}$ with respect to $\mathbf{T}$. Then by [11, XXII 4.1.2], there is an open subscheme Ω of $\mathbf{G}$ (the "big cell") such that the product map

$$\mathbf{U}^- \times \mathbf{T} \times \mathbf{U} \to \mathbf{G}$$

is an open immersion with image Ω. Plainly, $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ is a closed subscheme of Ω. Thus by definition of I, $I \subset \Omega(\mathbb{Z}_p)$ and therefore any element of I (resp. $I(1)$) can be written uniquely as a product

$$\prod_{h(\alpha)=-h} x_\alpha(a_\alpha) \times \cdots \times \prod_{h(\alpha)=-1} x_\alpha(a_\alpha) \times t \times \prod_{h(\alpha)=1} x_\alpha(a_\alpha) \times \cdots \times \prod_{h(\alpha)=h} x_\alpha(a_\alpha)$$

where $a_\alpha \in \mathbb{Z}_p$ for $\alpha \in R_+$, $a_\alpha \in p\mathbb{Z}_p$ for $\alpha \in R_- = -R_+$ and $t \in \mathbf{T}(\mathbb{Z}_p)$ (resp. $T(1)$). This is the Iwahori decomposition of I (resp. $I(1)$). If I^+ is the group spanned by $\{x_\alpha(\mathbb{Z}_p) : \alpha \in R_+\}$ and I^- is the group spanned by $\{x_\alpha(p\mathbb{Z}_p) : \alpha \in R_-\}$, then $I^+ = \mathbf{U}(\mathbb{Z}_p)$, $I^- \subset \mathbf{U}^-(\mathbb{Z}_p)$ and every $x \in I$ (resp. $I(1)$) has a unique decomposition $x = u^- t u^+$ with $u^\pm \in I^\pm$ and $t \in \mathbf{T}(\mathbb{Z}_p)$ (resp. $t \in T(1)$).

2.6 Suppose first that $\mathbf{G}$ is semi-simple and simply connected. Then $M^\vee = \mathbb{Z}R^\vee$, thus $\mathcal{S} = \emptyset$. Moreover, everything splits according to the decomposition $R = \coprod R_c$:

$$\mathbf{G} = \prod \mathbf{G}_c, \quad \mathbf{T} = \prod \mathbf{T}_c, \quad \mathbf{B} = \prod \mathbf{B}_c, \quad I = \prod I_c \quad \text{and} \quad I(1) = \prod I_c(1).$$

To establish the theorem in this case, we may thus furthermore assume that $\mathbf{G}$ is simple. From now on until section 2.11, we therefore assume that

$$\mathbf{G} \text{ is (split) simple and simply connected.}$$

2.7 As a first step, we show that

Lemma 2.7.1. *The group generated by I^+ and I^- contains $T(1)$.*

Proof. Since $\mathbf{G}$ is simply connected,

$$\prod_{\alpha\in\Delta} \alpha^\vee : \prod_{\alpha\in\Delta} \mathbf{G}_{m,\mathbb{Z}_p} \to \mathbf{T}$$

is an isomorphism, thus

$$T_c(1) = \prod_{\alpha\in\Delta} \alpha^\vee(1 + p\mathbb{Z}_p).$$

Now for any $\alpha \in \Delta$, there is a unique morphism [11, XX 5.8]

$$f_\alpha : \mathbf{SL}(2)_{\mathbb{Z}_p} \to \mathbf{G}$$

such that for every $u, v \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p^\times$,

$$f_\alpha \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = x_\alpha(u), \quad f_\alpha \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = x_{-\alpha}(v) \quad \text{and} \quad f_\alpha \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \alpha^\vee(x).$$

Since for every $x \in 1 + p\mathbb{Z}_p$ [11, XX 2.7],

$$\begin{pmatrix} 1 & 0 \\ x^{-1}-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

in $\mathbf{SL}(2)(\mathbb{Z}_p)$, it follows that $\alpha^\vee(1 + p\mathbb{Z}_p)$ is already contained in the subgroup of $\mathbf{G}(\mathbb{Z}_p)$ generated by $x_\alpha(\mathbb{Z}_p^\times)$ and $x_{-\alpha}(p\mathbb{Z}_p)$. This proves the lemma. □

2.8 Recall from [11, XXI 2.3.5] that for any pair of non-proportional roots $\alpha \neq \pm\beta$ in R, the set of integers $k \in \mathbb{Z}$ such that $\beta + k\alpha \in R$ is an interval of length at most 3, i.e. there are integers $r \geq 1$ and $s \geq 0$ with $r + s \leq 4$ such that

$$R \cap \{\beta + \mathbb{Z}\alpha\} = \{\beta - (r-1)\alpha, \cdots, \beta + s\alpha\}.$$

The above set is called the α-chain through β and any such set is called a root chain in R. Let $\|-\| : R \to \mathbb{R}_+$ be the length function on R.

Proposition 2.8.1. *Suppose $\|\alpha\| \leq \|\beta\|$. Then for any $u, v \in \mathbf{G}_a$ the commutator*

$$[x_\beta(v) : x_\alpha(u)] = x_\beta(v)x_\alpha(u)x_\beta(-v)x_\alpha(-u)$$

is given by the following table, with (r, s) as above:

(r,s)	$[x_\beta(v) : x_\alpha(u)]$
$(-,0)$	1
$(1,1)$	$x_{\alpha+\beta}(\pm uv)$
$(1,2)$	$x_{\alpha+\beta}(\pm uv) \cdot x_{2\alpha+\beta}(\pm u^2 v)$
$(1,3)$	$x_{\alpha+\beta}(\pm uv) \cdot x_{2\alpha+\beta}(\pm u^2 v) \cdot x_{3\alpha+\beta}(\pm u^3 v) \cdot x_{3\alpha+2\beta}(\pm u^3 v^2)$
$(2,1)$	$x_{\alpha+\beta}(\pm 2uv)$
$(2,2)$	$x_{\alpha+\beta}(\pm 2uv) \cdot x_{2\alpha+\beta}(\pm 3u^2 v) \cdot x_{\alpha+2\beta}(\pm 3uv^2)$
$(3,1)$	$x_{\alpha+\beta}(\pm 3uv)$

The signs are unspecified, but only depend upon α and β.

Proof. This is [11, XXIII 6.4]. □

Corollary 2.8.2. *If $r + s \leq 3$ and $\alpha + \beta \in R$ (i.e. $s \geq 1$), then for any $a, b \in \mathbb{Z}$, the subgroup of G generated by $x_\alpha(p^a\mathbb{Z}_p)$ and $x_\beta(p^b\mathbb{Z}_p)$ contains $x_{\alpha+\beta}(p^{a+b}\mathbb{Z}_p)$.*

Proof. This is obvious if $(r, s) = (1, 1)$ or $(2, 1)$ (using $p \neq 2$ in the latter case). For the only remaining case where $(r, s) = (1, 3)$, note that

$$[x_\beta(v) : x_\alpha(u)][x_\beta(w^2 v) : x_\alpha(uw^{-1})]^{-1} = x_{\alpha+\beta}(\pm uv(1 - w)).$$

Since $p \neq 2$, we may find $w \in \mathbb{Z}_p^\times$ with $(1 - w) \in \mathbb{Z}_p^\times$. Our claim easily follows. □

Lemma 2.8.3. *If R contains any root chain of length 3, then $\mathbf{G}$ is of type G_2.*

Proof. Suppose that the α-chain through β has length 3. By [11, XXI 3.5.4], there is a basis Δ' of R such that $\alpha \in \Delta'$ and $\beta = a\alpha + b\alpha'$ with $\alpha' \in \Delta'$, $a, b \in \mathbb{N}$. The root system R' spanned by $\Delta' = \{\alpha, \alpha'\}$ [11, XXI 3.4.6] then also contains an α-chain of length 3. By inspection of the root systems of rank 2, for instance in [11, XXIII 3], we find that R' is of type G_2. In particular, the Dynkin diagram of R contains a triple edge (linking the vertices corresponding to α and α'), which implies that actually $R = R'$ is of type G_2. □

2.9 We now establish our theorem 2.4.1 for a group $\mathbf{G}$ which is simple and simply connected, but not of type G_2.

Lemma 2.9.1. *The group I^+ is generated by $\{x_\alpha(\mathbb{Z}_p) : \alpha \in \Delta\}$.*

Proof. Let $H \subset I^+$ be the group spanned by $\{x_\alpha(\mathbb{Z}_p) : \alpha \in \Delta\}$. We show by induction on $h(\gamma) \geq 1$ that $x_\gamma(\mathbb{Z}_p) \subset H$ for every $\gamma \in R_+$. If $h(\gamma) = 1$, γ already belongs to Δ and there is nothing to prove. If $h(\gamma) > 1$, then by [1, VI.1.6 Proposition 19], there is a simple root $\alpha \in \Delta$ such that $\beta = \gamma - \alpha \in R_+$. Then $h(\beta) = h(\gamma) - 1$, thus by induction $x_\beta(\mathbb{Z}_p) \subset H$. Since also $x_\alpha(\mathbb{Z}_p) \subset H$, $x_\gamma(\mathbb{Z}_p) \subset H$ by Corollary 2.8.2. □

Lemma 2.9.2. *The group generated by I^+ and $x_{-\alpha_{max}}(p\mathbb{Z}_p)$ contains I^-.*

Proof. Let $H \subset I$ be the group spanned by I^+ and $x_{-\alpha_{max}}(p\mathbb{Z}_p)$. We show by descending induction on $h(\gamma) \geq 1$ that $x_{-\gamma}(p\mathbb{Z}_p) \subset H$ for every $\gamma \in R_+$. If $h(\gamma) = h(\alpha_{max})$, then $\gamma = \alpha_{max}$ and there is nothing to prove. If $h(\gamma) < h(\alpha_{max})$, then by [1, VI.1.6 Proposition 19], there is a pair of positive roots α, β such that $\beta = \gamma + \alpha$. Then $h(\beta) = h(\gamma) + h(\alpha) > h(\gamma)$, thus by induction $x_{-\beta}(p\mathbb{Z}_p) \subset H$. Since also $x_\alpha(\mathbb{Z}_p) \subset H$, $x_{-\gamma}(p\mathbb{Z}_p) \subset H$ by Corollary 2.8.2. □

Remark 2.9.3. *From the Hasse diagrams in [10], it seems that in the previous proof, we may always require α to be a simple root.*

Proof. (Of theorem 2.4.1 for **G** simple, simply connected, not of type G_2) By lemma 2.7.1, 2.9.1, 2.9.2 and the Iwahori decomposition of section 2.5, $I(1)$ is generated by

$$\{x_\alpha(\mathbb{Z}_p) : \alpha \in \Delta\} \cup \{x_{-\alpha_{max}}(p\mathbb{Z}_p)\}$$

thus topologically generated by

$$\{x_\alpha(1) : \alpha \in \Delta\} \cup \{x_{-\alpha_{max}}(p)\}.$$

None of these topological generators can be removed: the first ones are contained in $I^+ \subsetneq I(1)$, and all of them are needed to span the image of

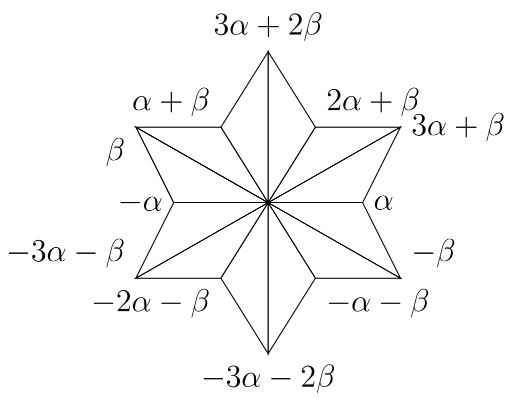
$$I(1) \twoheadrightarrow \mathbf{U}(\mathbb{F}_p) \twoheadrightarrow \overline{\mathbf{U}}_1(\mathbb{F}_p) \simeq \prod_{\alpha\in\Delta} \mathbf{U}_\alpha(\mathbb{F}_p),$$

a surjective morphism that kills $x_{-\alpha_{max}}(p)$. □

2.10 Let now **G** be simple of type G_2, thus $\Delta = \{\alpha, \beta\}$ with $\|\alpha\| < \|\beta\|$ and

$$R_+ = \{\alpha, \beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, 2\beta+3\alpha\}.$$

The whole root system looks like this:



Lemma 2.10.1. *The group generated by I^+ and $x_{-2\beta-3\alpha}(p\mathbb{Z}_p)$ contains I^-.*

Proof. Let $H \subset I(1)$ be the group generated by I^+ and $x_{-2\beta-3\alpha}(p\mathbb{Z}_p)$. Then, for every $u, v \in \mathbb{Z}_p$, H contains

$$\begin{aligned}
[x_{-2\beta-3\alpha}(pv) : x_\beta(u)] &= x_{-\beta-3\alpha}(\pm puv) \\
[x_{-2\beta-3\alpha}(pv) : x_{\beta+3\alpha}(u)] &= x_{-\beta}(\pm puv) \\
[x_{-2\beta-3\alpha}(pv) : x_{\beta+2\alpha}(u)] &= x_{-\beta-\alpha}(\pm puv) \cdot x_\alpha(\pm pu^2v) \cdot x_{\beta+3\alpha}(\pm pu^3v) \cdot x_{-\beta}(\pm p^2u^3v^2)
\end{aligned}$$

It thus contains $x_{-\beta-3\alpha}(p\mathbb{Z}_p)$, $x_{-\beta}(p\mathbb{Z}_p)$ and $x_{-\beta-\alpha}(p\mathbb{Z}_p)$, along with

$$\begin{aligned}
[x_{-\beta-3\alpha}(pv) : x_\alpha(u)] &= x_{-\beta-2\alpha}(\pm puv)\cdot x_{-\beta-\alpha}(\pm pu^2v)\cdot x_{-\beta}(\pm pu^3v)\cdot x_{-2\beta-3\alpha}(\pm p^2u^3v^2)\\
[x_{-\beta-3\alpha}(pv) : x_{\beta+2\alpha}(u)] &= x_{-\alpha}(\pm puv)\cdot x_{\beta+\alpha}(\pm pu^2v)\cdot x_{2\beta+3\alpha}(\pm pu^3v)\cdot x_{\beta}(\pm p^2u^3v^2)
\end{aligned}$$

It therefore also contains $x_{-\beta-2\alpha}(p\mathbb{Z}_p)$ and $x_{-\alpha}(p\mathbb{Z}_p)$. □

The filtration $(\mathbf{U}_i)_{i\geq 1}$ of $\mathbf{U}$ in section 2.5 induces a filtration

$$I^+ = I_1^+ \supset \cdots \supset I_5^+ \supset I_6^+ = 1$$

of $I^+ = \mathbf{U}(\mathbb{Z}_p)$ by normal subgroups $I_i^+ = \mathbf{U}_i(\mathbb{Z}_p)$ whose graded pieces

$$\overline{I}_i^+ = \overline{\mathbf{U}}_i(\mathbb{Z}_p) = I_i^+/I_{i+1}^+$$

are free $\mathbb{Z}_p$-modules, namely

$$\begin{gathered}
\overline{I}_1^+ = \mathbb{Z}_p\cdot\overline{x}_\alpha \oplus \mathbb{Z}_p\cdot\overline{x}_\beta, \qquad \overline{I}_2^+ = \mathbb{Z}_p\cdot\overline{x}_{\alpha+\beta}\\
\overline{I}_3^+ = \mathbb{Z}_p\cdot\overline{x}_{2\alpha+\beta}, \qquad \overline{I}_4^+ = \mathbb{Z}_p\cdot\overline{x}_{3\alpha+\beta}, \qquad \overline{I}_5^+ = \mathbb{Z}_p\cdot\overline{x}_{3\alpha+2\beta}
\end{gathered}$$

where $\overline{x}_\gamma$ is the image of $x_\gamma(1)$. The commutator defines $\mathbb{Z}_p$-linear pairings

$$[-,-]_{i,j} : \overline{I}_i^+ \times \overline{I}_j^+ \to \overline{I}_{i+j}^+$$

with $[y,x]_{j,i} = -[x,y]_{i,j}$, $[x,x]_{i,i} = 0$ and, by Proposition 2.8.1,

$$\begin{gathered}
[\overline{x}_\beta, \overline{x}_\alpha] = \pm\overline{x}_{\alpha+\beta}, \quad [\overline{x}_{\alpha+\beta}, \overline{x}_\alpha] = \pm 2\overline{x}_{2\alpha+\beta}, \quad [\overline{x}_{2\alpha+\beta}, \overline{x}_\alpha] = \pm 3\overline{x}_{3\alpha+\beta},\\
[\overline{x}_{\alpha+\beta}, \overline{x}_{2\alpha+\beta}] = \pm x_{3\alpha+2\beta} \quad \text{and} \quad [\overline{x}_\beta, \overline{x}_{3\alpha+\beta}] = \pm x_{2\alpha+2\beta}
\end{gathered}$$

Let H be the subgroup of I^+ generated by $x_\alpha(\mathbb{Z}_p)$ and $x_\beta(\mathbb{Z}_p)$ and denote by H_i its image in $I^+/I_{i+1}^+ = G_i$. Then $H_1 = G_1$, H_2 contains $[\overline{x}_\beta, \overline{x}_\alpha] = \pm\overline{x}_{\alpha+\beta}$ thus $H_2 = G_2$, H_3 contains $[\overline{x}_{\alpha+\beta}, \overline{x}_\alpha] = \pm 2\overline{x}_{2\alpha+\beta}$ thus $H_3 = G_3$ since $p \neq 2$, H_4 contains $[\overline{x}_{2\alpha+\beta}, \overline{x}_\alpha] = \pm 3\overline{x}_{3\alpha+\beta}$ thus $H_4 = G_4$ if $p \neq 3$, in which case actually $H = H_5 = G_5 = I^+$ since H always contains $[\overline{x}_{\alpha+\beta}, \overline{x}_{2\alpha+\beta}] = \pm x_{3\alpha+2\beta}$.

If $p = 3$, let us also consider the exact sequence

$$0 \to J_4 \to G_4 \to \overline{I}_1^+ \to 0$$

The group $J_4 = I_2^+/I_5^+$ is commutative, and in fact again a free $\mathbb{Z}_3$-module:

$$J_4 = (\mathbf{U}_2/\mathbf{U}_5)(\mathbb{Z}_p) = \mathbb{Z}_3\tilde{x}_{\alpha+\beta} \oplus \mathbb{Z}_3\tilde{x}_{2\alpha+\beta} \oplus \mathbb{Z}_3\overline{x}_{3\alpha+\beta}$$

where $\tilde{x}_\gamma$ is the image of $x_\gamma(1)$. The action by conjugation of $\overline{I}_1^+$ on J_4 is given by

$$\overline{x}_\alpha \mapsto \begin{pmatrix} 1 & & \\ \pm 2 & 1 & \\ \pm 3 & \pm 3 & 1 \end{pmatrix} \quad \overline{x}_\beta \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

in the indicated basis of J_4. The $\mathbb{Z}_3$-submodule $H_4' = H_4 \cap J_4$ of J_4 satisfies

$$H_4' + \mathbb{Z}_3\overline{x}_{3\alpha+\beta} = J_4 \quad \text{and} \quad 3\overline{x}_{3\alpha+\beta} \in H_4'.$$

Naming signs $\epsilon_i \in \{\pm 1\}$ in formula $(1,3)$ of proposition 2.8.1, we find that H_4' contains

$$\epsilon_1 uv \cdot \tilde{x}_{\alpha+\beta} + \epsilon_2 u^2 v \cdot \tilde{x}_{2\alpha+\beta} + \epsilon_3 u^3 v \cdot \overline{x}_{3\alpha+\beta}$$

for every $u, v \in \mathbb{Z}_3$. Adding these for $v = 1$ and $u = \pm 1$, we obtain

$$\tilde{x}_{2\alpha+\beta} \in H_4'.$$

It follows that H_4' actually contains the following $\mathbb{Z}_3$-submodule of J_4:

$$J_4' = \{a \cdot \tilde{x}_{\alpha+\beta} + b \cdot \tilde{x}_{2\alpha+\beta} + c \cdot \overline{x}_{3\alpha+\beta} : a, b, c \in \mathbb{Z}_3,\ \epsilon_1 a \equiv \epsilon_3 c \bmod 3\}.$$

Now observe that J_4' is a normal subgroup of G_4, and the induced exact sequence

$$0 \to J_4/J'_4 \to G_4/J_4' \to \overline{I}_1^+ \to 0$$

is an *abelian* extension of $\overline{I}_1^+ \simeq \mathbb{Z}_3^2$ by $J_4/J_4' \simeq \mathbb{F}_3$. Since H_4/J_4' is topologically generated by two elements and surjects onto $\overline{I}_1^+$, it actually defines a splitting:

$$G_4/J_4' = H_4/J_4' \oplus J_4/J_4'.$$

Thus $H_4' = J_4'$, H_4 is a normal subgroup of G_4, H is a normal subgroup of I^+ and

$$I^+/H \simeq G_4/H_4 \simeq J_4/J_4' \simeq \mathbb{F}_3$$

is generated by the class of $x_{\alpha+\beta}(1)$ or $x_{3\alpha+\beta}(1)$. We have shown:

Lemma 2.10.2. *The group I^+ is spanned by $x_\alpha(\mathbb{Z}_p)$ and $x_\beta(\mathbb{Z}_p)$ plus $x_{\alpha+\beta}(1)$ if $p = 3$.*

Proof. (Of theorem 2.4.1 for $\mathbf{G}$ simple of type G_2) By lemma 2.7.1, 2.10.1, 2.10.2 and the Iwahori decomposition of section 2.5, the pro-p-Iwahori $I(1)$ is generated by $x_\alpha(\mathbb{Z}_p)$, $x_\beta(\mathbb{Z}_p)$, $x_{-2\beta-3\alpha}(p\mathbb{Z}_p)$, along with $x_{\alpha+\beta}(1)$ if $p = 3$. It is therefore topologically generated by $x_\alpha(1)$, $x_\beta(1)$, $x_{-2\beta-3\alpha}(p)$, along with $x_{\alpha+\beta}(1)$ if $p = 3$. The surjective reduction morphism $I(1) \twoheadrightarrow \mathbf{U}(\mathbb{F}_p) \twoheadrightarrow \overline{\mathbf{U}}_1(\mathbb{F}_p)$ shows that the first two generators can not be removed. The third one also can not, since all the others belong to the closed subgroup $I_+ \subsetneq I(1)$. Finally, suppose that $p = 3$ and consider the extension

$$1 \to \mathbf{U}_2/\mathbf{U}_5 \to \mathbf{U}/\mathbf{U}_5 \to \mathbf{U}/\mathbf{U}_1 \to 1$$

With notations as above, the reduction of

$$J_4' \subset J_4 = \mathbf{U}_2(\mathbb{Z}_3)/\mathbf{U}_5(\mathbb{Z}_3) = (\mathbf{U}_2/\mathbf{U}_5)(\mathbb{Z}_3)$$

is a normal subgroup Y of $X = (\mathbf{U}/\mathbf{U}_5)(\mathbb{F}_3)$ with quotient $X/Y \simeq \mathbb{F}_3^3$. The surjective reduction morphism

$$I(1) \twoheadrightarrow \mathbf{U}(\mathbb{F}_3) \twoheadrightarrow \mathbf{U}(\mathbb{F}_3)/\mathbf{U}_5(\mathbb{F}_3) = X \twoheadrightarrow X/Y$$

then kills $x_{-2\beta-3\alpha}(p)$. The fourth topological generator $x_{\alpha+\beta}(1)$ of $I(1)$ thus also can not be removed, since the first two certainly do not span $X/Y \simeq \mathbb{F}_3^3$. □

2.11 We now return to an arbitrary split reductive group $\mathbf{G}$ over $\mathbb{Z}_p$. Let

$$\mathbf{G}^{sc} \twoheadrightarrow \mathbf{G}^{der} \hookrightarrow \mathbf{G} \twoheadrightarrow \mathbf{G}^{ad}$$

be the simply connected cover $\mathbf{G}^{sc}$ of the derived group $\mathbf{G}^{der}$ of $\mathbf{G}$, and the adjoint group $\pi : \mathbf{G} \twoheadrightarrow \mathbf{G}^{ad}$ of $\mathbf{G}$. Then

$$\left(\mathbf{T}^{ad}, M^{ad}, R^{ad}, \Delta^{ad}, \left(X_\alpha^{ad}\right)_{\alpha\in\Delta^{ad}}\right) = \left(\pi(\mathbf{T}), \mathbb{Z}R, R, \Delta, (\pi(X_\alpha))_{\alpha\in\Delta}\right)$$

is a pinning of $\mathbf{G}^{ad}$ and this construction yields a bijection between pinnings of $\mathbf{G}$ and pinnings of $\mathbf{G}^{ad}$. Applying this to $\mathbf{G}^{sc}$ or $\mathbf{G}^{der}$, we obtain pinnings

$$\left(\mathbf{T}^{sc}, M^{sc}, R^{sc}, \Delta^{sc}, (X_\alpha^{sc})_{\alpha\in\Delta^{sc}}\right) \quad \text{and} \quad \left(\mathbf{T}^{der}, M^{der}, R^{der}, \Delta^{der}, \left(X_\alpha^{der}\right)_{\alpha\in\Delta^{sc}}\right)$$

for $\mathbf{G}^{sc}$ and $\mathbf{G}^{der}$: all of the above constructions then apply to $\mathbf{G}^{ad}$, $\mathbf{G}^{sc}$ or $\mathbf{G}^{der}$, and we will denote with a subscript *ad*, *sc* or *der* for the corresponding objects. For instance, we have a sequence of Iwahori (resp. pro-p-Iwahori) subgroups

$$I^{sc} \to I^{der} \hookrightarrow I \to I^{ad} \quad \text{and} \quad I^{sc}(1) \to I^{der}(1) \hookrightarrow I(1) \to I^{ad}(1).$$

2.12 The action of $\mathbf{G}$ on itself by conjugation factors through a morphism

$$\mathrm{Ad} : \mathbf{G}^{ad} \to \mathrm{Aut}(\mathbf{G}).$$

For $b \in \mathbf{B}^{ad}(\mathbb{F}_p)$, $\mathrm{Ad}(b)(\mathbf{B}_{\mathbb{F}_p}) = \mathbf{B}_{\mathbb{F}_p}$ and $\mathrm{Ad}(b)(\mathbf{U}_{\mathbb{F}_p}) = \mathbf{U}_{\mathbb{F}_p}$. We thus obtain an action of the Iwahori subgroup I^{ad} of $G^{ad} = \mathbf{G}^{ad}(\mathbb{Q}_p)$ on I or $I(1)$. Similar consideration of course apply to $\mathbf{G}^{sc}$ and $\mathbf{G}^{der}$, and the sequence

$$I^{sc}(1) \to I^{der}(1) \hookrightarrow I(1) \to I^{ad}(1)$$

is equivariant for these actions of $I^{ad} = I^{ad}(1) \rtimes T^{ad}_{tors}$.

2.13 Let J be the image of $I^{sc}(1) \to I(1)$, so that J is a normal subgroup of I. From the compatible Iwahori decompositions for $I(1)$ and $I^{sc}(1)$ in section 2.5, we see that $T(1) \hookrightarrow I(1)$ induces a T^{ad}-equivariant isomorphism

$$T(1)/T(1) \cap J \to I(1)/J.$$

Since the inverse image of $\mathbf{T}(\mathbb{Z}_p)$ in $\mathbf{G}^{sc}(\mathbb{Z}_p)$ equals $\mathbf{T}^{sc}(\mathbb{Z}_p)$ and since also

$$T^{sc}(1) = \mathbf{T}^{sc}(\mathbb{Z}_p) \cap I^{sc}(1),$$

we see that $T(1) \cap J$ is the image of $T^{sc}(1) \to T(1)$. Also, the kernel of $I^{sc}(1) \to I(1)$ equals $Z \cap I^{sc}(1)$ where

$$Z = \ker(\mathbf{G}^{sc} \to \mathbf{G})(\mathbb{Z}_p) = \ker(\mathbf{T}^{sc} \to \mathbf{T})(\mathbb{Z}_p).$$

Therefore $Z \cap I^{sc}(1)$ is the kernel of $T^{sc}(1) \to T(1)$, which is trivial since Z is finite and $T^{sc}(1) \simeq \mathrm{Hom}(M^{sc}, 1 + p\mathbb{Z}_p)$ has no torsion. We thus obtain exact sequences

$$\begin{array}{ccccccccc} 1 & \to & T^{sc}(1) & \to & T(1) & \to & Q & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 1 & \to & I^{sc}(1) & \to & I(1) & \to & Q & \to & 0 \end{array}$$

where the cokernel Q is the finitely generated $\mathbb{Z}_p$-module

$$Q = (M^\vee/\mathbb{Z}R^\vee) \otimes (1 + p\mathbb{Z}_p).$$

Remark 2.13.1. *If $\mathbf{G}$ is simple, then $M^\vee/\mathbb{Z}R^\vee$ is a finite group of order c, with $c \mid \ell+1$ if $\mathbf{G}$ is of type A_ℓ, $c \mid 3$ if $\mathbf{G}$ is of type E_6 and $c \mid 4$ in all other cases. Thus $Q = 0$ and $I^{sc}(1) = I(1)$ unless $\mathbf{G}$ is of type A_ℓ with $p \mid c \mid \ell+1$ or $p = 3$ and $\mathbf{G}$ is adjoint of type E_6. In these exceptional cases, $M^\vee/\mathbb{Z}R^\vee$ is cyclic, thus $Q \simeq \mathbb{F}_p$.*

2.14 It follows that $I(1)$ is generated by $I^{sc}(1)$ and $s(1+p\mathbb{Z}_p)$ for $s \in \mathcal{S}$, thus topologically generated by $I^{sc}(1)$ and $s(1+p)$ for $s \in \mathcal{S}$. In view of the results already established in the simply connected case, this shows that the elements listed in $(1-4)$ of Theorem 2.4.1 indeed form a set of topological generators for $I(1)$.

None of the semi-simple elements in (1) can be removed: they are all needed to generate the above abelian quotient Q of $I(1)$ which indeed kills the unipotent generators in $(2-4)$. Likewise, none of the unipotent elements in (2) can be removed: they are all needed to generate the abelian quotient

$$I(1) \twoheadrightarrow \mathbf{U}(\mathbb{F}_p) \twoheadrightarrow \overline{\mathbf{U}}_1(\mathbb{F}_p) \simeq \prod_{\alpha\in\Delta} \mathbf{U}_\alpha(\mathbb{F}_p)$$

which kills the other generators in (1), (3) and (4). One checks easily using the Iwahori decomposition of $I(1)$ and the product decomposition $\mathbf{U}^- = \prod_{c\in\mathcal{C}} \mathbf{U}_c^-$ that none of the unipotent elements in (3) can be removed. Finally if $p = 3$ and $d \in \mathcal{D}$, the central isogeny $\mathbf{G}^{sc} \to \mathbf{G}^{ad}$ induces an isomorphism $\mathbf{G}_d^{sc} \to \mathbf{G}_d^{ad}$ between the simple (simply connected *and* adjoint) components corresponding to d, thus also an isomorphism between the corresponding pro-p-Iwahori's $I_d^{sc}(1) \to I_d^{ad}(1)$. In particular, the projection $I(1) \to I^{ad}(1) \twoheadrightarrow I_d^{ad}(1)$ is surjective. Composing it with the projection $I_d^{ad}(1) \twoheadrightarrow \mathbb{F}_3^3$ constructed in section 2.10, we obtain an abelian quotient $I(1) \twoheadrightarrow \mathbb{F}_3^3$ that kills all of our generators except $x_\alpha(1)$, $x_\beta(1)$ and $x_{\alpha+\beta}(1)$ where $\Delta_d = \{\alpha, \beta\}$. In particular, the generator $x_{\alpha+\beta}(1)$ from (4) is also necessary. This finishes the proof of Theorem 2.4.1.

2.15 The action of $I^{ad} = I^{ad}(1) \rtimes T^{ad}_{tors}$ on $I(1)$ induces an $\mathbb{F}_p$-linear action of

$$T^{ad}_{tors} = \mathrm{Hom}\left(M^{ad}, \mu_{p-1}\right) = \mathrm{Hom}\left(\mathbb{Z}R, \mathbb{F}_p^\times\right)$$

on the Frattini quotient $\tilde{I}(1)$ of $I(1)$. Our minimal set of topological generators of $I(1)$ reduces to an eigenbasis of $\tilde{I}(1)$, i.e. an $\mathbb{F}_p$-basis of $\tilde{I}(1)$ made of eigenvectors for the action of T^{ad}_{tors}. We denote by $\mathbb{F}_p(\alpha)$ the 1-dimensional representation of T^{ad}_{tors} on $\mathbb{F}_p$ defined by $\alpha \in \mathbb{Z}R$. We thus obtain:

Corollary 2.15.1. *The $\mathbb{F}_p[T^{ad}_{tors}]$-module $\tilde{I}(1)$ is isomorphic to*

$$\mathbb{F}_p^{\sharp\mathcal{S}} \oplus \left(\oplus_{\alpha\in\Delta} \mathbb{F}_p(\alpha)\right) \oplus \left(\oplus_{c\in\mathcal{C}} \mathbb{F}_p(-\alpha_{c,max})\right)\left(\oplus \left(\oplus_{d\in\mathcal{D}} \mathbb{F}_p(\delta_c)\right) \text{ if } p=3\right).$$

Here $\sharp\mathcal{S}$ denotes the cardinality of the set $\mathcal{S}$. The map $\alpha \mapsto \mathbb{F}_p(\alpha)$ yields a bijection between $\mathbb{Z}R/(p-1)\mathbb{Z}R$ and the isomorphism classes of simple $\mathbb{F}_p[T^{ad}_{tors}]$-modules. In particular some of the simple modules in the previous corollary may happen to be isomorphic. For instance if $\mathbf{G}$ is simple of type B_ℓ and $p = 3$, then $-\alpha_{max} \equiv \alpha \bmod 2$ where $\alpha \in \Delta$ is a long simple root. An inspection of the tables in [1] yields the following:

Corollary 2.15.2. *If $\mathbf{G}$ is simple, the $\mathbb{F}_p[T^{ad}_{tors}]$-module $\tilde{I}(1)$ is multiplicity free unless $p = 3$ and $\mathbf{G}$ is of type A_1, B_ℓ or C_ℓ ($\ell \geq 2$), F_4 or G_2.*

In the next section we use this result to construct Galois representations landing in I^{ad} with image containing $I^{ad}(1)$.

3 The Construction of Galois Representations

Let $\mathbf{G}$ be a split simple adjoint group over $\mathbb{Z}_p$ and let $I(1)$ and $I = I(1) \rtimes T_{tors}$ be the corresponding Iwahori groups, as defined in the previous section. We want here to construct Galois representations of a certain type with values in I with image containing $I(1)$. After a short review of p-rational fields in section 3.1, we establish a criterion for the existence of our representations in sections 3.2 and 3.3 and finally give some examples in section 3.4.

3.1 Let K be a number field, $r_2(K)$ the number of complex primes of K, Σ_p the set of primes of K lying above p, M the compositum of all finite p-extensions of K which are unramified outside Σ_p, M^{ab} the maximal abelian extension of K contained in M, and L the compositum of all cyclic extensions of K of degree p which are contained in M or M^{ab}. If we let Γ denote $\mathrm{Gal}(M/K)$, then Γ is a pro-p group, $\Gamma^{ab} \cong \mathrm{Gal}(M^{ab}/K)$ is the maximal abelian quotient of Γ, and $\tilde{\Gamma} \cong \Gamma^{ab}/p\Gamma^{ab} \cong \mathrm{Gal}(L/K)$ is the Frattini quotient of Γ.

Definition *A number field K is p-rational if the following equivalent conditions are satisfied:*

(1) *$rank_{\mathbb{Z}_p}(\Gamma^{ab}) = r_2(K) + 1$ and Γ^{ab} is torsion-free as a $\mathbb{Z}_p$-module,*
(2) *Γ is a free pro-p group with $r_2(K) + 1$ generators,*
(3) *Γ is a free pro-p group.*

The equivalence of (1), (2) and (3) follows from [6], see also proposition 3.1 and the discussion before remark 3.2 of [3]. There is a considerable literature concerning p-rational fields, including [8], [4].

Examples:

(1) Suppose that K is a quadratic field and that either $p \geq 5$ or $p = 3$ and is unramified in $K/\mathbb{Q}$. If K is real, then K is p-rational if and only if p does not divide the class number of K and the fundamental unit of K is not a p-th power in the completions K_v of K at the places v above p. On the other hand, if K is complex and p does not divide the class number of K, then K is a p-rational field (cf. proposition 4.1 of [3]). However, there are p-rational complex K's for which p divides the class number (cf. chapter 2, section 1, p. 25 of [7]). For similar results, see also [2] and [5] if K is complex.

(2) Let $K = \mathbb{Q}(\mu_p)$. If p is a regular prime, then K is a p-rational field (cf. [12], see also [3], proposition 4.9 for a shorter proof).

3.2 Suppose that K is Galois over $\mathbb{Q}$ and p-rational with $p \nmid [K : \mathbb{Q}]$.

Since K is Galois over $\mathbb{Q}$, so is M and we have an exact sequence

$$1 \to \Gamma \to \Pi \to \Omega \to 1 \tag{3.2.1}$$

where $\Omega = \mathrm{Gal}(K/\mathbb{Q})$ and $\Pi = \mathrm{Gal}(M/\mathbb{Q})$. Conjugation in Π then induces an action of Ω on the Frattini quotient $\tilde{\Gamma} = \mathrm{Gal}(L/K)$ of Γ. Any continuous morphism $\rho : \Pi \to I$ maps Γ to $I(1)$ and induces a morphism $\overline{\rho} : \Omega \to I/I(1) = T_{tors}$ and a $\overline{\rho}$-equivariant morphism $\tilde{\rho} : \tilde{\Gamma} \to \tilde{I}(1)$. If $\rho(\Gamma) = I(1)$, then $\tilde{\rho}$ is also surjective. Suppose conversely that we are given the finite data

$$\overline{\rho} : \Omega \to T_{tors} \quad \text{and} \quad \tilde{\rho} : \tilde{\Gamma} \twoheadrightarrow \tilde{I}(1).$$

Then as Ω has order prime to p, the Schur-Zassenhaus theorem ([14], proposition 2.3.3) implies that the exact sequence 3.2.1 splits. The choice of a splitting $\Pi \simeq \Gamma \rtimes \Omega$ yields a non-canonical action of Ω on Γ which lifts the canonical action of Ω on the Frattini quotient $\tilde{\Gamma}$. By [3], proposition 2.3, $\tilde{\rho}$ lifts to a continuous Ω-equivariant surjective morphism $\rho' : \Gamma \twoheadrightarrow I(1)$, which plainly gives a continuous morphism

$$\rho = (\rho', \overline{\rho}) : \Pi \simeq \Gamma \rtimes \Omega \to I = I(1) \rtimes T_{tors}$$

inducing $\overline{\rho} : \Omega \to T_{tors}$ and $\tilde{\rho} : \tilde{\Gamma} \twoheadrightarrow \tilde{I}(1)$. Thus:

Proposition 3.2.1. *Under the above assumptions on K, there is a continuous morphism $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$ if and only if there is a morphism $\overline{\rho} : \Omega \to T_{tors}$ such that the induced $\mathbb{F}_p[\Omega]$-module $\overline{\rho}^* \tilde{I}(1)$ is a quotient of $\tilde{\Gamma}$.*

The Frattini quotient $\tilde{I}(1)$ is an $\mathbb{F}_p[T_{tors}]$-module and by the map $\overline{\rho}$, we can consider $\tilde{I}(1)$ as an $\mathbb{F}_p[\Omega]$-module which we denote by $\overline{\rho}^* \tilde{I}(1)$.

3.3 Suppose now that

A(K): K is a totally complex abelian (thus CM) Galois extension of $\mathbb{Q}$ which is p-rational of degree $[K : \mathbb{Q}] \mid p - 1$.

Let $\hat{\Omega}$ be the group of characters of Ω with values in $\mathbb{F}_p^\times$, $\hat{\Omega}_{odd} \subset \hat{\Omega}$ the subset of odd characters (those taking the value -1 on complex conjugation), and $\chi_0 \in \hat{\Omega}$ the trivial character. Then by [3] proposition 3.3,

$$\tilde{\Gamma} = \oplus_{\chi \in \hat{\Omega}_{odd} \cup \{\chi_0\}} \mathbb{F}_p(\chi)$$

as an $\mathbb{F}_p[\Omega]$-module. In particular, $\tilde{\Gamma}$ is multiplicity free. Suppose therefore also that the $\mathbb{F}_p[T_{tors}]$-module $\tilde{I}(1)$ is multiplicity free, i.e. by corollary 2.15.2,

B(G): If $p = 3$, then $\mathbf{G}$ is not of type A_1, B_ℓ or C_ℓ $(\ell \geq 2)$, F_4 or G_2.

For $\mathcal{S}$ as in section 2.4, we define

$$\hat{\Omega}_{odd}^{\mathcal{S}} = \begin{cases} \hat{\Omega}_{odd} \cup \chi_0, & \text{if } \mathcal{S} = \emptyset \\ \hat{\Omega}_{odd}, & \text{if } \mathcal{S} \neq \emptyset. \end{cases}$$

Note that $\mathcal{S} = \emptyset$ unless $\mathbf{G}$ if of type A_ℓ with $p \mid \ell + 1$ or $\mathbf{G}$ is of type E_6 with $p = 3$, in which both cases $\mathcal{S}$ is a singleton. We thus obtain:

Corollary 3.3.1. *Under the assumptions **A(K)** on K and **B(G)** on $\mathbf{G}$, there is a morphism $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$ if and only if there is morphism $\overline{\rho} : \Omega \to T_{tors}$ such that the characters $\alpha \circ \overline{\rho} : \Omega \to \mathbb{F}_p^\times$ for $\alpha \in \Delta \cup \{-\alpha_{max}\}$ are all distinct and belong to $\hat{\Omega}_{odd}^{\mathcal{S}}$.*

3.4 Some examples. Write $\Delta = \{\alpha_1, ..., \alpha_\ell\}$ and $\alpha_{max} = n_1\alpha_1 + \cdots + n_\ell\alpha_\ell$ using the conventions of the tables in [1]. In this part we suppose that p is a regular (odd) prime and take $K = \mathbb{Q}(\mu_p)$, so that K is p-rational and $\Omega = \mathbb{Z}/(p-1)\mathbb{Z}$.

Lemma 3.4.1. *Suppose* $\mathbf{G}$ *is of type* A_ℓ, B_ℓ, C_ℓ *or* D_ℓ *and* $p \geq 2l+3$ *(resp.* $p \geq 2l+5$*) if* $p \equiv 1 \mod 4$ *(resp.* $p \equiv 3 \mod 4$*). Then we can find distinct characters* $\phi_1, ..., \phi_{\ell+1} \in \hat{\Omega}_{odd} \cup \chi_0$ *such that* $\phi_1^{n_1}\phi_2^{n_2}\cdots\phi_\ell^{n_\ell}\phi_{\ell+1} = \chi_0$*. Furthermore, if* $\mathbf{G}$ *is of type* A_ℓ *and* ℓ *is odd, then one can even choose the characters* $\phi_1, ..., \phi_{\ell+1}$ *to be inside* $\hat{\Omega}_{odd}$*.*

Proof. Since Ω is (canonically) isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$, $\sharp\hat{\Omega}_{odd} = \frac{p-1}{2}$ and there are exactly $[\frac{p-1}{4}]$ pairs of characters $\{\chi, \chi^{-1}\}$ with $\chi \neq \chi^{-1}$ in $\hat{\Omega}_{odd}$. The condition on p is equivalent to $\ell \leq 2[\frac{p-1}{4}]-1$.

If $\mathbf{G}$ is of type A_ℓ, then $\alpha_{max} = \alpha_1 + \cdots + \alpha_\ell$. If ℓ is even and $\frac{\ell}{2} \leq [\frac{p-1}{4}]$, then we can pick $\frac{\ell}{2}$ distinct pairs of odd characters $\{\chi, \chi^{-1}\}$ as above for $\{\phi_1, \cdots, \phi_\ell\}$ and set $\phi_{\ell+1} = \chi_0$. If ℓ is odd and $\frac{\ell+1}{2} \leq [\frac{p-1}{4}]$, then we can choose $\frac{\ell+1}{2}$ distinct such pairs for the whole set $\{\phi_1, \cdots, \phi_{\ell+1}\}$.

If $\mathbf{G}$ is of type D_ℓ (with $\ell \geq 4$), then $\alpha_{max} = \alpha_1 + 2\alpha_2 + ... + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$. Now if ℓ is odd we can pick $\frac{\ell+1}{2}$ such pairs $\{\chi, \chi^{-1}\}$, one for $\{\phi_{\ell-1}, \phi_\ell\}$, another pair for $\{\phi_1, \phi_{\ell+1}\}$ and $\frac{\ell-3}{2}$ such pairs for $\{\phi_2, ..., \phi_{\ell-2}\}$. If ℓ is even, we let ϕ_2 be the trivial character, and we can choose $\frac{\ell}{2}$ such pairs of characters $\{\chi, \chi^{-1}\}$, one pair for $\{\phi_1, \phi_{\ell-1}\}$, another pair for $\{\phi_\ell, \phi_{\ell+1}\}$ and $\frac{\ell-4}{2}$ such pairs for $\{\phi_3, ..., \phi_{\ell-2}\}$. So the inequality that we will need is $4 \leq \ell \leq 2[\frac{p-1}{4}] - 1$.

If $\mathbf{G}$ is of type B_ℓ (with $\ell \geq 2$), then $\alpha_{max} = \alpha_1 + 2\alpha_2 + ... + 2\alpha_\ell$. If ℓ is odd then we pick $\frac{\ell+1}{2}$ pairs of characters $\{\chi, \chi^{-1}\}$; one pair for $\{\phi_1, \phi_{\ell+1}\}$ and $\frac{\ell-1}{2}$ such pairs for $\{\phi_2, ..., \phi_\ell\}$. If ℓ is even then we need $\frac{\ell}{2}$ pairs of $\{\chi, \chi^{-1}\}$; one pair for $\{\phi_1, \phi_{\ell+1}\}$ and $\frac{\ell-2}{2}$ such pairs for $\{\phi_3, ..., \phi_\ell\}$ and we let ϕ_2 be the trivial character. So in this case we need $3 \leq \ell \leq 2[\frac{p-1}{4}] - 1$.

The remaining C_ℓ case is analogous. □

Lemma 3.4.2. *Suppose* $\mathbf{G}$ *is of type* E_6, E_7, E_8, F_4 *or* G_2 *and* $p \geq \sum_{i=1}^{\ell}(2i-1)n_i + 2\ell$*. Then we can find distinct characters* $\phi_1, ..., \phi_{\ell+1} \in \hat{\Omega}_{odd}$ *such that* $\phi_1^{n_1}\phi_2^{n_2}\cdots\phi_\ell^{n_\ell}\phi_{\ell+1} = \chi_0$*.*

Proof. The choice of a generator ξ of $\mathbb{F}_p^\times$ yields an isomorphism $\mathbb{Z}/(p-1)\mathbb{Z} \simeq \hat{\Omega}$, mapping i to χ_i and $1 + 2\mathbb{Z}/(p-1)\mathbb{Z}$ to $\hat{\Omega}_{odd}$. Set $\phi_i = \chi_{2i-1} \in \hat{\Omega}_{odd}$ for $i = 1, \cdots, \ell$ and $\phi_{\ell+1} = \chi_{-r}$ where $r = \sum_{i=1}^{\ell} n_i \cdot (2i-1)$. The tables in [1] show that $h = \sum_{i=1}^{\ell} n_i$ is odd, thus also $\phi_{\ell+1} \in \hat{\Omega}_{odd}$ and plainly $\phi_1^{n_1}\cdots\phi_\ell^{n_\ell}\phi_{\ell+1} = 1$. If $p \geq \sum_{i=1}^{\ell}(2i-1)n_i + 2\ell$, the elements $\{2i-1, -\sum_{i=1}^{\ell} n_i \cdot (2i-1); i \in [1, \ell]\}$ are all distinct modulo $p-1$, which proves the lemma. □

Remark 3.4.3. *For* $\mathbf{G}$ *of type* E_6, E_7, E_8, F_4 *or* G_2*, the tables in [1] show that the constant* $\sum_{i=1}^{\ell}(2i-1)n_i + 2\ell$ *of lemma 3.4.2 is* $79, 127, 247, 53, 13$ *respectively.*

Corollary 3.4.4. *There is a constant* c *depending only upon the type of* $\mathbf{G}$ *such that if* $p > c$ *is a regular prime, then for* $K = \mathbb{Q}(\mu_p)$*,* M*,* Π *and* Γ *as above, there is a continuous morphism* $\rho : \Pi \to I$ *with* $\rho(\Gamma) = I(1)$*.*

In conclusion, we have determined a minimal set of topological generators of the pro-p Iwahori subgroup of a split reductive groups over $\mathbb{Z}_p$ (theorem 2.4.1) and used it to study the structure of the Frattini quotient $\tilde{I}(1)$ as an $\mathbb{F}_p[T^{ad}_{tors}]$-module (corollary 2.15.1). Then we have used corollary 2.15.1 to determine when $\tilde{I}(1)$ is multiplicity free (see corollary 2.15.2). Furthermore in proposition 3.2.1 and corollary 3.3.1, assuming p-rationality, we have shown that we can construct Galois representations if and only if we can find a suitable list of distinct characters in Ω, the existence of which is discussed in section 3.4 under the assumption $K = \mathbb{Q}(\mu_p)$, for any sufficiently large regular prime p (see corollary 3.4.4).

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