Aim: Fix π and let χ vary P-adically. We want to show that, generically,

$$\mathrm{ord}_{s=1/2} L(s, \pi \times \pi(\chi)) = \begin{cases} 0 & \text{if } \epsilon = +1 \\ 1 & \text{if } \epsilon = -1 \end{cases}$$

With:

- π is an irreducible automorphic representation of $GL_2(\mathbf{A}_F)$, where
- F is a totally real number field;
- K is a totally imaginary quadratic extension of F and
- χ is a Grossencharacter of K, inducing an automorphic representation $\pi(\chi)$ of $GL_2(\mathbf{A}_F)$;
- P is a prime of F.

Notations and Assumptions

- π: cuspidal, weight $(2, \cdots, 2)$, level $\mathcal{N}$, and central char. $\omega : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$.

- χ is a *ring class character*, i.e.: there exists $c \subset \mathcal{O}_F$ such that χ factors through

  $$\chi : K^\times \backslash \mathbf{A}_K^\times / K_\infty^\times \widehat{\mathcal{O}}_c^\times \simeq \mathrm{Pic}(\mathcal{O}_c) \to \mathbf{C}^\times.$$

  where $\mathcal{O}_c := \mathcal{O}_F + c\mathcal{O}_K$. The *conductor* $c(\chi)$ of χ is the largest such c.

- By CFT, we may view χ as a character on $\mathrm{Gal}(K[c]/K)$, where $K[c]$ is the *ring class field* of conductor c of K.

- $\mathcal{D}$ is the support of $\mathrm{disc}(K/F)$. Write

  $$\mathcal{D} = (P)\mathcal{D}' \quad \text{and} \quad \mathcal{N} = P^\delta \mathcal{N}'$$

  with $(\mathcal{D}', P) = 1 = (\mathcal{N}', P)$. We require:

  $$(\mathcal{N}', \mathcal{D}') = 1.$$

We need:

$$\omega \cdot \chi|_{\mathbf{A}_F^\times} = 1 \quad \text{on } \mathbf{A}_F^\times.$$

Then:

- $L(s, \pi \times \pi(\chi))$ is entire, and
- $L(s, \pi \times \pi(\chi))$ has a F.E. $s \leftrightarrow 1 - s$,
- This F.E. has a sign $\epsilon \in \{\pm 1\}$.

Moreover, ω is an unramified character of F (because χ is a ring class character of K).

(Thm) For all $n \gg 0$, there exists a ring class character χ with $c(\chi) = P^n$ and $\omega \cdot \chi|_{\mathbf{A}_F^\times} = 1$ such that

$$\text{ord}_{s=1/2} L(s, \pi \times \pi(\chi)) = \begin{cases} 0 & \text{if } \epsilon = +1 \\ 1 & \text{if } \epsilon = -1 \end{cases}$$

From now on,

$$\boxed{\text{Weonlyconsiderthe}}\, \epsilon = -1$$

case.

Thanks to the work of S. Zhang (*Gross-Zagier formulas for GL_2, II*), we now have to show:

Thm For all $n \gg 0$ and $x \in CM(P^n)$, there exists a χ with $c(\chi) = P^n$ and $\omega \cdot \chi|_{\mathbf{A}_F^\times} = 1$ such that

$$e_\chi \cdot \alpha(x) \neq 0 \quad \text{in } A \otimes \mathbf{C}.$$

Here:

- M is a Shimura curve,
  A is a quotient of $J = \mathrm{Pic}^0(M)$ and
  $\alpha : M \rightsquigarrow A$ is an F-morphism.

- CM: Points with CM by K in M and
  $CM(c) \subset M(K[c])$: those of conductor c.

- $G(n) = \mathrm{Gal}(K[P^n]/K)$ and

$$e_\chi = \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \overline{\chi}(\sigma) \cdot \sigma.$$

Formula for ϵ.

When $\omega \cdot \chi|_{\mathbf{A}_F^\times} = 1$,

$$\begin{aligned} \epsilon &= \prod \epsilon_v = (-1)^{|S|} \\ \text{and} \quad S &= \{v; \epsilon_v \neq \eta_v \cdot \omega_v(-1)\}. \end{aligned}$$

Under our assumption that $\mathcal{N}$ and $\mathcal{D}$ are relatively prime away from P, and provided that χ is sufficently ramified at P, we find:

$$S = \left\{ v \,\middle|\, \begin{array}{l} v \mid \infty \text{ or } v \nmid P\infty \text{ is inert in } K \\ \text{and such that } v(\mathcal{N}) \equiv 1 \bmod 2 \end{array} \right\}$$

Therefore

$$\Box \epsilon = -1 \quad \Leftrightarrow \quad |S| \equiv 1 \bmod 2.$$

The Hodge "embedding".

There exists $\delta \in \mathrm{Pic}(M) \otimes \mathbf{Q}$ which has degree 1 on each geometrical connected component of M. This δ defines an element $\iota : M \rightsquigarrow J$ of $\mathrm{Mor}_F(M, J) \otimes \mathbf{Q}$ given by

$$x \mapsto (x - \delta_c)$$

where δ_c is the restriction of δ to the geometrical connected component c of $x \in M(\mathbf{C})$.

We put

$$\alpha : M \overset{\iota}{\rightsquigarrow} J \xrightarrow{\pi} A$$

so that α belongs to $\mathrm{Mor}_F(M, A) \otimes \mathbf{Q}$.

A Filtration

$$1 \subset G_2 \subset G_1 \subset G_0 \subset G(\infty).$$

- $G(\infty) = \varprojlim G(n) = \mathrm{Gal}(K[P^\infty]/K)$ and $G_0 = G(\infty)_{\mathrm{torsion}}$.

- $G(\infty)^{\mathrm{rat}} = <\mathrm{Frob}_Q;\ Q \nmid P> \subset G(\infty)$ and $G_1 = G(\infty)^{\mathrm{rat}}_{\mathrm{torsion}}$.

- $G_2 = \mathrm{rec}_K(\widehat{F}^\times) \subset G(\infty)$.

Then $G_2 \subset G_1 \subset G_0$ and

- G_0 is finite, $G(\infty) \simeq G_0 \times \mathbf{Z}_p^{[F_P, \mathbf{Q}_p]}$.

- $G_2 \simeq \mathrm{Pic}(\mathcal{O}_F)$.

- G_1/G_2 is an $\mathbf{F}_2$-vector space with basis
$$\{\sigma_q = \mathrm{Frob}_Q;\ Q^2 = q\mathcal{O}_K, q \mid \mathcal{D}'\}.$$

Fix $\chi_0 : G_0 \to \mathbf{C}^\times$ such that $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$.

Thm For all $n \gg 0$ and $x \in CM(P^n)$, there exists a χ with $c(\chi) = P^n$ and $\chi |_{G_0} = \chi_0$ such that

$$e_\chi \cdot \alpha(x) \neq 0 \quad \text{in } A \otimes \mathbf{C}.$$

Proof: Put $e_{\chi_0,n} = \sum e_\chi$ where χ runs through the above characters (i.e. those inducing χ_0 on G_0 and such that $c(\chi) = P^n$). We want:

Thm For all $n \gg 0$ and $x \in CM(P^n)$,

$$e_{\chi_0,n} \cdot \alpha(x) \neq 0 \quad \text{in } A \otimes \mathbf{C}.$$

Lemma 1 In the group ring $\mathbf{C}[G(n)]$,

$$e_{\chi_0,n} = \frac{1}{q\,|G_0|} \sum_{\sigma \in G_0} \overline{\chi}_0(\sigma)\sigma \cdot (q - \mathrm{Tr}_{Z(n)})$$

with $q = |\mathcal{O}_F/P|$ and

$$Z(n) = \mathrm{Gal}(K[P^n]/K[P^{n-1}]).$$

Lemma 2 $\exists M'$, $J' = \mathrm{Pic}^0 M'$, $\pi' : J' \twoheadrightarrow A$ and

$$x \in CM(P^n) \mapsto x' \in CM'(P^n) \quad (n \geq 2)$$

such that if $\alpha' = \pi' \circ \iota' : M' \rightsquigarrow A$,

$$(q - \mathrm{Tr}_{Z(n)})\alpha(x) = \alpha'(x') \quad \text{in } A \otimes \mathbf{C}.$$

We thus want:

Thm For all $n \gg 0$ and $x' \in CM'(P^n)$,

$$\sum_{\sigma \in G_0} \overline{\chi}_0(\sigma) \cdot \sigma\alpha'(x') \neq 0 \quad \text{in } A \otimes \mathbf{C}.$$

Put $\chi = \chi_0$, $M = M'$, $J = J'$, $\alpha = \alpha' \ldots$

Put $C_i = \mathbf{Z}[\chi(G_i)]$ so that

$$\mathbf{Z} \subset C_2 \subset C_1 \subset C_0 \subset \mathbf{C}$$

We want:

Thm For all but finitely many $x \in CM(P^\infty)$,

$$(*) \qquad \sum_{\sigma \in G_0} \overline{\chi}(\sigma)\sigma\alpha(x) \neq \text{tors. in } A \otimes_{\mathbf{Z}} C_0$$

Proposition 3 $(*) \Leftrightarrow$

$$\Leftrightarrow \quad \sum_{\sigma \in G_0/G_2} \overline{\chi}(\sigma)\sigma\alpha(x) \neq \text{tors. in } A \otimes_{C_2} C_0$$

$$\Leftrightarrow \quad \sum_{\sigma \in G_0/G_1} \overline{\chi}(\sigma)\sigma\alpha_1(x_1) \neq \text{tors. in } A_1 \otimes_{C_1} C_0$$

Action of G_2.

The center $\widehat{F}^\times$ of $G(\mathbf{A}_f)$ acts on M, J, and A through its quotient

$$\widehat{F}^\times / F^\times \widehat{\mathcal{O}}_F^\times \simeq \mathrm{Pic}(\mathcal{O}_F) \quad (\simeq G_2)$$

For a CM point $x = [g] \in T(\mathbf{Q}) \backslash G(\mathbf{A}_f)/H$,

Galois action: $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$ acts by

$$\sigma \cdot x = [\lambda g]$$

where $\sigma = \mathrm{rec}_K(\lambda)$ with $\lambda \in \widehat{K}^\times$.

Automorphic action: $\theta \in \mathrm{Pic}(\mathcal{O}_F)$ acts by

$$\theta(x) = [g\lambda]$$

where $\theta = [\lambda]$ with $\lambda \in \widehat{F}^\times$.

For $\sigma \in G_2 \leftrightarrow \theta \in \mathrm{Pic}(\mathcal{O}_F)$, $\lambda \in \widehat{F}$ and

$$\sigma \cdot x = [\lambda g] = [g\lambda] = \theta(x).$$

Action of G_1/G_2.

Recall that $G_1/G_2 = \{\sigma_D;\ D \mid \mathcal{D}'\}$ where

$$\sigma_D = \prod_{q|D} \sigma_q, \quad \sigma_q = \mathrm{Frob}_Q, \quad Q^2 = q\mathcal{O}_K.$$

Lemma 4 $\exists\{\deg_D : M_1 \to M;\ D \mid \mathcal{D}'\}$ and

$$x \in CM(P^n) \mapsto x_1 \in CM_1(P^n)$$

such that for all $D \mid \mathcal{D}$ and $x \in CM(P^n)$,

$$\sigma_D \cdot x = \deg_D(x_1) \quad \text{(in } M.)$$

Take $A_1 = A \otimes_{C_2} C_1$ and

$$\alpha_1 : M_1 \overset{\iota_1}{\rightsquigarrow} J_1 \overset{\pi_1}{\twoheadrightarrow} A_1$$

where $\pi_1 : J_1 \twoheadrightarrow A_1$ is defined by

$$\begin{array}{ccccccc} J_1 & \to & J^{\{D|\mathcal{D}\}} & \xrightarrow{\pi} & A^{\{D|\mathcal{D}\}} & \to & A_1 \\ x & \mapsto & (\deg_D(x)) & & (x_D) & \mapsto & \sum \overline{\chi}(\sigma_D) x_D \end{array}$$

Fix a set $\mathcal{R} \subset G_0$ of representatives for G_0/G_1. We now want:

Thm For all but finitely many $x \in CM_1(P^\infty)$

$$\sum_{\sigma \in \mathcal{R}} \overline{\chi}(\sigma) \cdot \sigma\alpha_1(x) \neq \text{torsion} \quad \text{in } A_1 \otimes_{C_1} C_0.$$

Lemma 5 For any abelian variety B,

$$B(K[P^\infty])_{\text{torsion}} \quad \text{is finite.}$$

Let $\mathcal{E} \subset CM_1(P^\infty)$ be an infinite subset of counterexample to our theorem. Then:

$$x \mapsto \sum_{\sigma \in \mathcal{R}} \overline{\chi}(\sigma) \cdot \sigma\alpha_1(x) \in A_1 \otimes_{C_1} C_0$$

takes finitely many values on $\mathcal{E}$, and we want to get a contradiction out of this.

Replace $M_1, J_1, A_1 \ldots$ by $M, J, A \ldots$

First Proof: using Andre-Oort (Edixhoven + Yafaev)

Proposition 6 For any infinite subset $\mathcal{E}$ of $CM(P^\infty)$, the Zariski closure of $\delta(\mathcal{E})$ in $M^{\mathcal{R}}/\mathbf{C}$ contains a connected component.

If $\Phi \circ \delta(\mathcal{E})$ is finite, Φ is constant on some geometrical connected component of $M^{\mathcal{R}}$. Then $\alpha : M \to A$ should be constant on some geometrical connected component of M. Being defined over F, α should then be constant on M (because M is connected as an F-curve), and π would be trivial, a contradiction.

Sketch of the proof of Proposition 6.

If some component of the Zariski closure of

$$\{(x, \sigma x);\, x \in \mathcal{E}\} \subset M^2$$

is a curve, then

- This curve is a connected component of some Hecke correspondance $\mathcal{T}_{\mathcal{M}}$, and

- σ belongs to the subgroup

  $$< \mathrm{Frob}_Q; Q \mid \mathcal{M}; Q \nmid P >$$

  of $G(\infty)^{\mathrm{rat}}$.

- If also $\sigma \in G_0 = G(\infty)_{\mathrm{torsion}}$, we obtain

  $$\sigma \in G(\infty)^{\mathrm{rat}}_{\mathrm{torsion}} = G_1.$$

Second Proof: using Ratner's Theorem

1- We want: for all $n \gg 0$ and $x \in CM(P^n)$,

$$\sum_{\sigma \in \mathcal{R}} \overline{\chi}(\sigma) \cdot \sigma\alpha(x) = \Phi \circ \delta(x)$$

has a large Galois orbit. This orbit equals

$$G(\infty) \cdot \Phi \circ \delta(x) = \Phi \circ \delta(G(\infty) \cdot x).$$

2- Reducing everything at some place v of $K[P^\infty]$ and using $\text{red} \circ \Phi = \Phi \circ \text{red}$, we want

$$\text{RED}(G(\infty) \cdot x) \subset M^{\mathcal{R}}(k)$$

to be large when $n \gg 0$, where

$$\text{RED} = \text{red} \circ \delta : CM(P^\infty) \to M^{\mathcal{R}}(k)$$

maps x to $(\text{red}(\sigma x))_{\sigma \in \mathcal{R}}$ and k is the residue field of v.

3- Choose v above some prime ℓ of F which is inert in K and not in $\mathcal{N}$. Then k is finite and

$$\text{red}(CM) \subset M^{\text{ss}}(k) = \{\text{supersingular points}\}.$$

Is it true that

$$\mathsf{RED} : CM(P^\infty) \to M^{\mathsf{ss}}(k)^{\mathcal{R}}$$

is surjective? The answer is no.

4- Let $c : M \to \mathcal{M}$ be the Stein factorisation of $M \to \mathsf{Spec}(\mathcal{O}_{F_v})$, put $C = c^{\mathcal{R}}$

$$C : M^{\mathsf{ss}}(k)^{\mathcal{R}} \to \mathcal{M}(k)^{\mathcal{R}}.$$

Then $C \circ \mathsf{RED} : CM(P^\infty) \to \mathcal{M}(k)^R$ is not surjective, but:

Thm For any $x \in CM(P^n)$ with $n \gg 0$,

$$\mathsf{RED}(G(\infty) \cdot x) = C^{-1} \left(C \circ \mathsf{RED} \left(G(\infty) \cdot x \right) \right).$$

5- Carayol gives an adelic description of $M^{\mathrm{ss}}(k)$, $\mathcal{M}(k)$ and $c: M^{\mathrm{ss}}(k) \to \mathcal{M}(k)$. The strong approximation theorem at P then provides a P-adic description of the fibers of $c: M^{\mathrm{ss}}(k) \to \mathcal{M}(k)$. On the other hand, one checks that $CM(P^\infty)$ may be covered by finitely many "strong P-isogeny classes", and the latter each have a P-adic parametrisation by $B_P^1 \simeq SL_2(F_P)$.

Altogether, we find that for a strong P-isogeny class $\mathcal{H} \subset CM(P^\infty)$ and $z = C \circ \mathrm{RED}(\mathcal{H})$, the map

$$\mathcal{H} \to C^{-1}(z) \subset M^{\mathrm{ss}}(k)^{\mathcal{R}}$$

looks like

$$SL_2(F_P) \xrightarrow{\Delta} \prod_{\sigma \in \mathcal{R}} \Gamma_\sigma \backslash SL_2(F_P)/V$$

where Δ is the diagonal, V is a compact open subgroup of $SL_2(F_P)$ and the Γ_σ's are *non-commensurable* cocompact lattices in $SL_2(F_P)$.

6- Let us already show that this map is *surjective*. Put $\mathcal{G} = SL_2(F_P)$ and $\Gamma = \prod_{\sigma\in\mathcal{R}} \Gamma_\sigma$.

Ratner's theorem (Orbit closure) *There exists a closed subgroup Σ of $\mathcal{G}^{\mathcal{R}}$ such that*

$$\overline{\Gamma \cdot \Delta(\mathcal{G})} = \Gamma \cdot \Sigma \quad \textit{in} \quad \mathcal{G}^{\mathcal{R}}.$$

We may furthermore assume that $\Delta(\mathcal{G}) \subset \Sigma$: then Σ is essentially a "product of diagonals". As $\Gamma \cdot \Sigma$ is closed, $\Gamma \cap \Sigma$ is a cocompact lattice in Σ. The non commensurability of the Γ_σ's then implies $\Sigma = \mathcal{G}^{\mathcal{R}}$, and $\Gamma \cdot \Delta(\mathcal{G})$ is dense in $\mathcal{G}^{\mathcal{R}}$. A fortiori, $\Gamma \cdot \Delta(\mathcal{G}) \cdot V^{\mathcal{R}} = \mathcal{G}^{\mathcal{R}}$.

7- To obtain our theorem, we need a better decomposition of $CM(P^\infty)$: a finite union of

$$\cup_{n\geq 0} G(\infty) \cdot U(\kappa_n)$$

where $t \mapsto U(t)$ is "one parameter unipotent familly of CM points", κ is a compact open subgroup of $\mathcal{O}_F^\times$ and $\kappa_n = \pi^{-n}\kappa$, with π a local uniformiser at P.

Another theorem of M. Ratner tells us that

$$\frac{1}{\lambda(\kappa_n)} \int_{\kappa_n} f \circ \Delta(\sigma U(t)) dt \to \int f d\mu_\sigma$$

for any continuous function f on $\Gamma \backslash \mathcal{G}^{\mathcal{R}}$. We show that for almost all $\sigma \in G(\infty)$, $\mu_\sigma = \mu$ is the unique $\mathcal{G}^{\mathcal{R}}$ invariant measure on $\Gamma \backslash \mathcal{G}^{\mathcal{R}}$. Fubini's theorem then allows us to analyse the asymptotic behavior of

$$\int_{G(\infty)} f \circ \mathsf{RED}(g \cdot x) dg$$

for $x \in CM(P^n)$ with $n \to \infty$ and f any function on the finite set $M^{\mathsf{ss}}(k)^{\mathcal{R}}$.