What is an Euler System?

It's a collection of cohomology classes

$$x_Q \in H^1(Q,T)$$

satisfying certain (1) **distribution relations** and (2) **congruence relations.**

Here: K is a number field, T is a geometric p-adic representation of Gal_K and the classes are indexed by $K \subset_f Q \subset K[\infty]$ for some **fixed infinite abelian** extension $K[\infty]$ of K.

Such Euler systems are used to bound the Selmer groups of $T^\star(1)$. In the **self-dual case** $T = T^\star(1)$, one gets

$$x_K \neq \text{tors.} \Rightarrow \text{it spans } H^1_f(K,T).$$

Example: Heegner Points.

- K is a totally imaginary quadratic extension of a totally real number field F,

- $T = T_p A$ for a quotient $\pi : \mathrm{Pic}^0 X \to A$, where X is a Shimura curve X over F,

- $K[\infty]$ is the fixed field of the image of the transfert Ver : $\mathrm{Gal}_F^{ab} \to \mathrm{Gal}_K^{ab}$, so that

  $$K[\infty] = \cup_c K[c]$$

  where $K[c]$ is the ring class field with

  $$\mathrm{Gal}(K[c])/K) = \widehat{K}^\times / \widehat{F}^\times \widehat{\mathcal{O}}_c^\times$$

  where $\mathcal{O}_c = \mathcal{O}_F + c\mathcal{O}_K$.

- The x_Q's are the images by an Abel-Jacobi map of certain divisors supported on **special points** with CM by K.

Distribution and congruence relations.

For $\ell \nmid c$, the **distribution relations** compare

$$P_\ell(f_\ell)\cdot x_{K[c]} \quad \text{and} \quad \text{Cores}_{K[c\ell]/K[c]}\cdot x_{K[c\ell]}$$

in $H^1(K[c],T)$, where

$$\begin{aligned} P_\ell &= \det(f_\ell - X\cdot \mathrm{Id}|T^\star(1)), \\ f_\ell &= \text{geometric Frobenius.} \end{aligned}$$

In the classical Heegner points example,

$$P_\ell = X^2 - T_\ell X + \ell$$

and for the x_Q's that are usually used,

$$\begin{aligned} &P_\ell(f_\ell)\cdot f_\ell^{-1} x_{K[c]} \\ &\quad = \text{Cores}_{K[c\ell]/K[c]}(f_\ell^{-1}x_{K[c]} - x_{K[c\ell]}). \end{aligned}$$

The **congruence relations** compare the localisations of x_c and $x_{c\ell}$ at some place $\lambda \mid \ell$ of $K[c\ell]$ above ℓ. They can often be deduced from the distribution relations.

The general setting

We keep the CM extension K/F, the infinite abelian extension $K[\infty]/K$, but we want to consider more general T's:

- T= lattice in the p-adic etale realisation of

- M= an **irreducible** motive over F, **pure of weight** -1, with coefficients in a number field $C \subset \mathbf{C}$, with a **perfect symplectic pairing** $M \otimes M \to C(1)$.

We want to

- find out when we expect an Euler system to exist in this situation, and

- try to construct such an Euler system in some cases.

L-functions and ϵ-factors

For a place v of F, let M_v be the complex symplectic repr. of the Weil-Deligne group

$$W'_v = \begin{cases} W_v & \text{if } v \mid \infty \\ W_v \times SU_2(\mathbf{R}) & \text{if } v \nmid \infty \end{cases}$$

conjecturally attached to M. They should come from a representation of a motivic Galois group.

Each continuous character

$$\chi : \mathrm{Gal}(K[\infty]/K) \to \mathbf{C}^\times$$

yields a character of W_K (global Weil group) which induces to an *orthogonal* 2-*dimensional* representation $\mathrm{Ind}\chi$ of W_F. Write $\mathrm{Ind}_v\chi$ for its restriction to W_v or W'_v.

Then $M_v \otimes \mathrm{Ind}_v\chi$ is a *symplectic* Weil-Deligne representation. Define

$$\begin{aligned} L(M,\chi,s) &= \prod L_v(M_v \otimes \mathrm{Ind}_v\chi, s), \\ \epsilon(M,\chi,s) &= \prod \epsilon_v(M_v \otimes \mathrm{Ind}_v\chi, s, \psi_v), \end{aligned}$$

with $\psi : F\backslash\mathbf{A}_F \to \mathbf{C}^\times$ as usual.

Signs as hint for Euler Systems

• The L-function $L(M,\chi,s)$ should have a meromorphic extension to $\mathbf{C}$ and a functional equation $s \leftrightarrow -s$ with root number

$$\epsilon(M,\chi) = \epsilon(M,\chi,0) \in \{\pm 1\}.$$

Fact: $\epsilon(M,\chi) = \epsilon(M,K)$ does not depend upon χ, provided χ does not ramify where M does.

• If $\epsilon(M,K)$ equals -1, then the order of vanishing of $L(M,\chi,s)$ at $s = 0$ should be 1 for almost all χ. Accordingly, the rank of the χ component of $H^1_f(K(\chi),M)$ should be 1 for almost all χ's.

• So if $\epsilon(M,K) = -1$, there should be enough room for a non-trivial Euler system to exist, it should essentially be unique, and account (by Kolyvagin's method) for the fact that most of the $H^1_f(K(\chi),M)$'s have rank one. Thus:

Assumption 1: $\epsilon(M,K) = -1$.

Automorphicity

Assumption 2: M *is automorphic.*

There exists an admissible global parameter

$$\phi : \mathcal{L}_F \to \mathsf{Sp}_{2n}(\mathbf{C})$$

such that for all v, the local parameter

$$\phi_v : W_v' \to \mathsf{Sp}_{2n}(\mathbf{C})$$

is isomorphic to $M_v(\frac{-1}{2})$. Then:

- M pure of weight $-1 \Rightarrow \phi$ is tempered,
- M irreducible $\Rightarrow Z(\phi) = \{\pm 1\}$

where

$$\{\pm 1\} \subset Z(\phi) \subset Z(\phi_v) \subset \mathsf{Sp}_{2n}(\mathbf{C})$$

are the centralizer of ϕ and ϕ_v.

Conjectures of Langlands, Arthur, Vogan...

Let $\Pi(\phi)$ be the set of isomorphism classes of pairs (G, π) where G is an (inner) form of $SO(2n+1)$ over F, and π belongs to the global L-paquet $\Pi(G, \phi)$, i.e. **(1)** π is an irreducible unitary representation of $G(\mathbf{A}_F)$ which occurs with multiplicity $m(\pi) > 0$ in $L^2_d(G(F)\backslash G(\mathbf{A}_F))$ and **(2)** $\pi = \otimes' \pi_v$ where for each v, the representation π_v of $G_v = G(F_v)$ belongs to the local Langlands L-paquet $\Pi(G_v, \phi_v)$.

The inclusion $Z(\phi) \hookrightarrow Z(\phi_v)$ induces a map from $A = \pi_0(Z(\phi))$ to $A_v = \pi_0(Z(\phi_v))$. Let $\Delta : A \to \prod_v A_v$ be the diagonal.

Conjecture *There are natural bijections*

$$\Pi(\phi_v) \simeq A_v^\vee \quad \textit{and} \quad \Pi(\phi) \simeq \left(\prod_v A_v / \Delta(A) \right)^\vee .$$

Moreover, $m(\pi) = 1$ *for all* $(G, \pi) \in \Pi(\phi)$.

Local Signs and characters

Let X be the space of $\phi : \mathcal{L}_F \to \mathrm{Sp}(X)$. For $s \in Z(\phi_v)[2]$, put $X(s) = \{x \in X | sx = -x\}$ and

$$c_v(\chi, s) = \epsilon_v(X(s) \otimes \mathrm{Ind}_v \chi) \cdot \eta_v(-1)^{\frac{\dim X(s)}{2}}$$

where $\eta = \otimes \eta_v : \mathbf{A}_F^\times \to \mathrm{Gal}(K/F) = \{\pm 1\}$ is the reciprocity map.

Lemma *This yields a character*

$$c_v(\chi) : A_v \to \{\pm 1\}$$

which is $\equiv 1$ *if* K_v *splits or* ϕ_v *is unramified.*

By construction, the root number

$$\epsilon(M, \chi) = \prod_v c_v(\chi, -1) = (-1)^{\# S(\chi)}$$

where $S(\chi) = \{v | c_v(\chi, -1) = -1\}$.

Modification of the c_v's at ∞, I

For χ's which do not ramify where M does,

$$c_v(\chi) = c_v \quad \text{and} \quad S(\chi) = S$$

with

$$(\prod c_v)(-1) = (-1)^{\# S} = \epsilon(M, K) = -1$$

so we need to modify some c_v.

Assumption 3: *for $v \mid \infty$, ϕ_v is discrete.*

Which means that

$$\phi_v = \oplus_{i=1}^{n} \mathrm{Ind}(z/\overline{z})^{a_{i,v}}$$

with $a_{i,v} \in \frac{1}{2}\mathbf{Z} - \mathbf{Z}$ and $a_{1,v} > \cdots > a_{n,v} > 0$.

Then $A_v = Z(\phi_v) = \oplus \mathbf{Z}/2\mathbf{Z} \cdot \epsilon_i$ and $c_v(\epsilon_i) = -1$ corresponds to a representation of

$$\begin{cases} SO(n+1, n) & \text{if } n \equiv 0 \bmod 2, \\ SO(n+2, n-1) & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

Modification of the c_v's at ∞, II

For $j = 0, \cdots, n$ define

$$c_v(j)(\epsilon_i) = (-1)^{i+1+n+\delta(i,j)}$$

which corresponds to

$$\begin{aligned} \{\pi_v^0\} &= \Pi(SO(2n+1), \phi_v) \\ \{\pi_v^1, \cdots, \pi_v^n\} &= \Pi(SO(2n-1,2), \phi_v) \end{aligned}$$

Fix $v_0 \mid \infty$ and define

$$c_v^j = \begin{cases} c_v & \text{if } v \nmid \infty \\ c_v(j) & \text{if } v = v_0 \\ c_v(0) & \text{if } v \mid \infty, v \neq v_0 \end{cases}$$

so that

$$(\prod c_v^j)(-1) = (-1)^{\delta_{0,j} + \frac{[F:\mathbf{Q}] \cdot n(3n+5)}{2}}$$

Assumption 4: $2 \mid [F : \mathbf{Q}]$ *if* $n \equiv 2, 3 \bmod 4$.

Then $(\prod c_v^j)(-1) = 1$ for $j = 1, \cdots, n$.

The Automorphic Representations

These characters (for $j = 1, \cdots, n$)

$$(\prod_v c_v^j) \in (\prod_v A_v/\Delta A)^\vee \simeq \Pi(\phi)$$

correspond to automorphic representations

$$\pi^j = \otimes'_v \pi_v^j = \pi_f \otimes \pi_\infty^j$$

• on the *same group* $G_0 = SO(V, \phi)$, where (V, ϕ) is a quadratic space of dimension $2n+1$ with Witt invariants

$$\mathbf{Witt}_v(V, \phi) = c_v^j(-1)$$

and signature at $v \mid \infty$

$$\mathrm{sign}_v(V, \phi) = \begin{cases} (2n-1, 2) & \text{if } v = v_0, \\ (2n+1, 0) & \text{if } v \neq v_0; \end{cases}$$

• with the *same finite part* π_f,
• with infinite parts running through a single L-paquet for $G_{0,\infty} = SO(V \otimes \mathbf{R}, \phi \otimes \mathbf{R})$.

$$\{\pi_\infty^j | j = 1, \cdots, n\} = \Pi(\phi_\infty, G_{0,\infty}).$$

The Shimura Varieties

Lemma *$\exists$ a K-hermitian F-hyperplane (W, ψ) of (V, ϕ): for some anisotropic line L of V,*

$$(V, \phi) = (W, \mathrm{Tr}\psi) \perp (L, \phi|L).$$

Moreover, any two such (W, ψ)'s are conjugated by an element of $SO(V, \phi)$.

Fix an embedding $\tau : K \to \mathbf{C}$ above v_0. Put

$$G = \mathrm{Res}_{F/\mathbf{Q}} SO(V, \phi), \quad H = \mathrm{Res}_{F/\mathbf{Q}} U(W, \psi)$$

and let $\mathcal{X}$ (resp. $\mathcal{Y}$) be the set of all oriented negative $\mathbf{R}$-planes (resp. $\mathbf{C}$-lines) in $(V, \phi)_\tau$ (resp. $(W, \psi)_\tau$).

We obtain a morphism of Shimura data

$$(H, \mathcal{Y}) \hookrightarrow (G, \mathcal{X})$$

with reflex fields $\tau(K) \supset \tau(F)$ and dimension $n-1 \leq 2n-1$, which is well-defined up to $G(\mathbf{Q})$.

Remark: $(H, \mathcal{Y})$ is a *twist* of the classical PEL type Shimura datum considered by Kottwitz, Harris-Taylor...

Realisation of M, I

Let L be a compact open subgroup of $G(\mathbf{A}_f)$ such that $\pi_f^L \neq 0$. This is an irreducible complex representation of the Hecke algebra $\mathcal{H}_L$ of $G(\mathbf{A}_f)$ relative to L, which also acts on the cohomology of $\mathrm{Sh}_L = \mathrm{Sh}_L(G, \mathcal{X})$. Suppose that

Assumption 5: $\forall i, v \mid \infty,\ a_{i,v} = n + \frac{1}{2} - i.$

Then for the Betti realisation of

$$H^\star(\mathrm{Sh}_L)[\pi_f] = \mathrm{Hom}_{\mathcal{H}_L}(\pi_F^L, H^\star(\mathrm{Sh}_L))$$

we find using Matsushima's formula, Arthur's conjecture and the computations of Vogan-Zuckerman that

$$\begin{aligned} H_B^\star(\mathrm{Sh}_L)[\pi_f] &= \oplus_{\pi_\infty} m(\pi_f \otimes \pi_\infty) H^\star(\mathcal{G}, K_\infty, \pi_\infty) \\ &= \oplus_{j=1}^n H^\star(\mathcal{G}, K_\infty, \pi_\infty^j) \\ &= \oplus_{j=1}^n H^{2n-j,j-1} \oplus H^{j-1,2n-j} \\ &= M_{v_0}(-n) \end{aligned}$$

This suggests that

$$H^\star(\mathrm{Sh}_L)[\pi_f] \simeq H^{2n-1}(\mathrm{Sh}_L)[\pi_f] \simeq M(-n)$$

Realisation of M, II

If the number field $C \subset \mathbf{C}$ is large enough, the representation ρ of $\mathcal{H}_L$ on π_f^L is defined over C. For $\lambda \mid p$ of C, we thus obtain a λ-adic representation $H^\star_\lambda(\mathrm{Sh}_L)[\pi_f]$.

Conjecture (Blasius-Rogawski) *Suppose that $L = L^v L_v$ with $L_v \subset G_0(F_v)$ hyperspecial. Let*

$$\mathbf{H}_v(T) \in \mathcal{H}_v[T] \text{ with } \mathcal{H}_v = \mathcal{H}_{\mathbf{Q}}(G_0(F_v), L_v)$$

be the Hecke polynomial with specialisation $H_v(T) \in C[T]$ through $\rho : \mathcal{H}_v \to C = \mathrm{End}_C(\pi_v^{L_v})$. Then Sh_L has good reduction at v and

$$H_v(f_v) = 0 \quad \text{on} \quad H^\star_\lambda(\mathrm{Sh}_L)[\pi_f].$$

Fact $H_v(T) = \det(T \cdot Id - q_v^n f_v | M_v)$.

This suggest again that

$$M = H^\star(\mathrm{Sh}_L, n)[\pi_f] = H^{2n-1}(\mathrm{Sh}_L, n)[\pi_f].$$

Cycles (Definition)

For $g \in G(\mathbf{A}_f)$, let $\mathcal{Z}_L(g)$ be the image of

$$gL \times \mathcal{Y} \subset G(\mathbf{A}_f) \times \mathcal{X}$$

inside

$$\mathrm{Sh}_L(\mathbf{C}) = G(\mathbf{Q})\backslash \left(G(\mathbf{A}_f)/L \times \mathcal{X}\right).$$

This yields a collection $\mathcal{Z}_L$ of cycles of codimension n in $\mathrm{Sh}_L = \mathrm{Sh}_L(G, \mathcal{X})$, and

$$\begin{aligned} \mathcal{Z}_L &\simeq H(\mathbf{Q})\backslash G(\mathbf{A}_f)/L \\ &\simeq H(\mathbf{Q})H^1(\mathbf{A}_f)\backslash G(\mathbf{A}_f)/L \end{aligned}$$

where

$$\begin{array}{lrcl} & H^1 & = & \ker\left(\det : H \to T^1\right) \\ \text{with} & T^1 & = & \ker\left(N : T \to Z\right) \\ \text{for} & T & = & \mathrm{Res}_{K/\mathbf{Q}}\mathbf{G}_{m,K} \\ \text{and} & Z & = & \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_{m,F}. \end{array}$$

Cycles (Fields of Definition)

Left multiplication by $H(\mathbf{A}_f)$ on

$$\mathcal{Z}_L = H(\mathbf{Q}) \cdot H^1(\mathbf{A}_f) \backslash G(\mathbf{A}_f)/L$$

descends to an action of

$$\begin{aligned} H(\mathbf{A}_f)/H^1(\mathbf{A}_f) & \overset{\text{det}}{\simeq} T^1(\mathbf{A}_f) \\ & \overset{\text{Hilbert 90}}{\simeq} T(\mathbf{A}_f)/Z(\mathbf{A}_f) \\ & \overset{\text{rec}_K}{\simeq} \text{Gal}(K[\infty]/K) \end{aligned}$$

Proposition: *The cycles $\mathcal{Z}_L(g) \subset \text{Sh}_L(G, \mathcal{X})$ are defined over the abelian extension $K[\infty]$ of K, with the Galois action given as above.*

Schwartz space

Let $\mathcal{S}$ be the space of locally constant and compactly supported functions

$$s : H^1(\mathbf{A}_f)\backslash G(\mathbf{A}_f) \to \mathbf{Z}.$$

The group $T^1(\mathbf{A}_f) \times G(\mathbf{A}_f)$ acts on $\mathcal{S}$.

Definition *The* field of definition *of $s \in \mathcal{S}$ is the subfield of $K[\infty]$ which is fixed by the image in* Gal$(K[\infty]/K)$ *of the stabilizer of s in* $T^1(\mathbf{A}_f)$.

For a compact open subgroup L of $G(\mathbf{A}_f)$ and a finite extension $K \subset_f Q \subset K[\infty]$, we put

$$\mathcal{S}_L(Q) = \left\{ s \in \mathcal{S} \;\middle|\; \begin{array}{l} s \text{ is defined over } Q \text{ and} \\ s \in \Gamma(L, \mathcal{S}) \end{array} \right\}.$$

Distribution

For each $g \in G(\mathbf{A}_f)$, put

$$c_L(g) = \frac{\left|\mathcal{D}_L(g) \cap T^1(\mathbf{Q})\right|}{\left|\mathcal{C}_L(g) \cap T^1(\mathbf{Q})\right|} \in \mathbf{Z}\mu_K^{-1}$$

where $\mu_K = |\mu(K)|$ and

$$\begin{aligned} \mathcal{D}_L(g) &= \det\left(gLg^{-1} \cap H(\mathbf{A}_f)\right) \\ \mathcal{C}_L(g) &= gLg^{-1} \cap Z(H(\mathbf{A}_f)) \end{aligned}$$

Then c_L factors through

$$c_L : H(\mathbf{A}_f)\backslash G(\mathbf{A}_f)/\mathcal{N}_{G(\mathbf{A}_f)}L \to \mathbf{Z}\mu_K^{-1}.$$

Lemma *The map $g \mapsto \mathcal{Z}_L(g) \otimes c_L(g)$ extends to an $\mathcal{H}_L[T^1(\mathbf{A}_f)]$-equivariant morphism*

$$\mu_L : \mathcal{S}_L(Q) \to \mathsf{CH}^n_L(Q) \otimes \mathbf{Z}\mu_K^{-1}$$

where $\mathsf{CH}^n_L(Q)$ = group of cycles of codimension n on $\mathsf{Sh}_L \times Q$ modulo linear equivalence.

Distribution relations

Theorem *For any* $K \subset_f Q \subset K[\infty]$ *and* $s \in \mathcal{S}_L(Q)$*, for all but finitely many* ℓ*'s such that* ℓ *is inert in* K*, unramified in* Q*, and* $L = L^\ell L_\ell$ *with* L_ℓ *hyperspecial, there exists*

$$s(\ell) \in \mathcal{S}_L(Q[\ell]) \quad Q(\ell) = Q \cdot K[\ell]$$

for which

$$\mathbf{H}_\ell(f_\ell) \cdot s = \mathrm{Tr}_{Q[\ell]/Q}(s(\ell)).$$

What comes into the proof:

- Computation of $\mathbf{H}_\ell$: computation of Kazhdan-Luztig polynomials, using a formula of Brylinski.

- Description of the action of a $U(n)$ on the Bruhat-Tits buldings of a split $SO(2n+1)$.

Etale cohomology

Fix a prime $p \nmid \mu_K$. The Hochschild-Serre spectral sequence and the cycle map in p-adic (continuous) etale cohomology together yield the Abel-Jacobi map:

$$\mathrm{CH}^n_L(Q)_0 \xrightarrow{AJ} H^1\left(Q, H^{2n-1}_{et}\left(\overline{\mathrm{Sh}}_L, \mathbf{Z}_p(n)\right)\right)$$

where $\mathrm{CH}^n_L(Q)_0$ is the kernel of the cycle map

$$\mathrm{cyc} : \mathrm{CH}^n_L(Q) \to H^0\left(Q, H^{2n}_{et}\left(\overline{\mathrm{Sh}}_L, \mathbf{Z}_p(n)\right)\right).$$

We would like to obtain classes in

$$H^1\left(Q, H^{2n-1}_{et}\left(\overline{\mathrm{Sh}}_L, \mathbf{Z}_p(n)\right)\right)$$

from our cycles $\mathcal{Z}_L(g)$, and thus need to trivialize the distribution

$$\mu^{et}_L : \mathcal{S}_L \to H^{2n}_{et}(\overline{\mathrm{Sh}}_L, \mathbf{Z}_p(n))$$

obtained as $\mu^{et}_L = \mathrm{cyc} \circ \mu_L$.

Strategies

(1) Work with $\mathcal{S}_L^0 = \ker \mu_L^{et}$ instead of $\mathcal{S}_L$. But: one then needs to construct elements of $\mathcal{S}_L^0$.

(2) Apply some functor $\mathcal{F}$ that kills $H_{et}^{2n}(\overline{\mathsf{Sh}}_L)$, such as $\mathrm{Hom}_{\mathcal{H}_L}(\pi_f^L, \bullet)$. But: one then needs to construct elements of $\mathcal{F}(\mathcal{S}_L)$.

(3) Find some $\mathcal{H}_L$-equivariant section of

$$\begin{array}{ccc} ? & & \mu_L^{et}(\mathcal{S}_L) \\ & \swarrow & \cap \\ H_{et}^{2n}(\mathsf{Sh}_L, \mathbf{Z}_p(n)) & \rightarrow & H_{et}^{2n}(\overline{\mathsf{Sh}}_L, \mathbf{Z}_p(n)) \end{array}$$

or even of the cycle map

$$\mathsf{CH}^n(F) \rightarrow H_{et}^{2n}(\overline{\mathsf{Sh}}_L, \mathbf{Z}_p(n))$$

This works for $n = 1$, because H_{et}^{2n} is then very simple. In general, I don't know what $\mu_L^{et}(\mathcal{S}_L)$ looks like.