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Rank 2 vector bundles and degrees of points of del Pezzo surfaces

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Abstract. We study points and 0-cycles on del Pezzo surfaces defined over a field K of characteristic 0, with emphasis on cubic surfaces. We prove that a cubic surface that admits a point defined over a field extension of K of degree coprime to 3 either has a K-point or has a point defined over a field extension of degree 4. This improves a result of Coray (who allowed also field extensions of degree 10). We also prove that 0-cycles of degree at least 18 on a cubic surface are effective and get similar results for degree 2 and degree 1 del Pezzo surfaces, improving results of Colliot-Thélène. In a different direction, we prove that the third symmetric product of a cubic hypersurface of dimension at least 2 is unirational over any field, and that, in dimension 2 and 3, it is not stably rational in general.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d over a field K. When K is not algebraically closed, it can happen that not only does X have no K-point, but all L-points of X are defined over fields extensions $L \supset K$ of degree divisible by d. The typical example is given by the generic hypersurface, where the field K is the function field $k(B)$ for some field k, the base B being the space $\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)))$. We take for X the generic fiber $\mathcal{X}_\eta \to \mathrm{Spec}(K)$ of the universal family $B \times \mathbb{P}^n \supset \mathcal{X} \stackrel{\pi}{\to} B$, where π is given by the first projection. The L-points of $\mathcal{X}_\eta$ correspond, by taking their Zariski closure in $\mathcal{X}$, to rational multisections of π. Using the fact that $\mathcal{X}$ is a projective bundle over $\mathbb{P}^n$ via the second projection, one sees that the restriction map $\mathrm{CH}(\mathcal{X}) \to \mathrm{CH}(X)$ has the same image as the restriction map $\mathrm{CH}(\mathbb{P}^n) \to \mathrm{CH}(X)$, which implies in particular that all 0-cycles of X are of degree divisible by d.

The first part of this paper is devoted to hypersurfaces of degree 3. The above argument shows that a cubic hypersurface over a field K is not in general unirational (in particular, it is not rational) since it has no K-point. This argument fails however for the third punctual

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Hilbert scheme $X^{[3]}$ of X, that always contains many K-points, namely all those obtained by intersecting X with a line in $\mathbb{P}^n$. As a consequence of Fogarty's theorem [6], the third punctual Hilbert scheme $X^{[3]}$ of a smooth variety X is smooth and birational to the symmetric product $X^{(3)}$ (see also [2], which proves that this statement is true only for the second and third punctual Hilbert schemes). We will be interested in K-points of $X^{[3]}$ corresponding to L-points of X for some field extension $K \subset L$ of degree 3, that is, after restriction to an affine open set U of X, to morphisms $\Gamma(\mathcal{O}_U) \to L$ of K-algebras. Thus the Hilbert scheme viewpoint is more natural than the viewpoint of the symmetric product, but in fact, these K-points of $X^{[3]}$ give rise to three distinct $\overline{K}$-points of X permuted by the Galois group, so the Hilbert scheme is isomorphic to the symmetric product at these points and we could work as well with the symmetric product $X^{(3)}$.

We will start by proving the following easy but useful result.

Theorem 1.1. *Let $X \subset \mathbb{P}^n_K$ be a smooth cubic hypersurface defined over a field K of characteristic* 0. *If $n \geq 3$, the variety $X^{[3]}$ is unirational.*

Remark 1.2. Our proof provides an explicit unirational parameterization. Moritz Hartlieb and Matteo Verni observed that one can also prove Theorem 1.1 using [9, Proposition 37], as any smooth cubic hypersurface of dimension at least 2 defined over a field K and having a K-point is unirational.

Theorem 1.1 immediately implies the following Corollary 1.3, which answers in particular (in the negative) the question, asked in [11, Question 6.2] and discussed in [3], whether for some cubic hypersurfaces of dimension at least 2 over a field K, all K-points of $X^{[3]}$ could be obtained by intersecting a line with X. Given a cubic hypersurface $X \subset \mathbb{P}^n$, there is a canonical 0-cycle

$$h_3 \in \mathrm{CH}_0(X) \tag{1.1}$$

defined as the intersection of a line in $\mathbb{P}^n$ with X. The effective 0-cycles $\Delta \cap X$ are all rationally equivalent to h_3, but they are clearly not Zariski-dense in $X^{[3]}$, by a dimension count. In fact, they are characterizing by collinearity.

Corollary 1.3. *Let $X \subset \mathbb{P}^n_K$ be a smooth cubic hypersurface defined over a field K of characteristic* 0. *Then the K-points of $X^{[3]}$ are Zariski-dense in $X^{[3]}$. In particular, there exist non-collinear K-points of $X^{[3]}$. Moreover, the K-points of $X^{[3]}$ which are rationally equivalent to h_3 are Zariski-dense in $X^{[3]}$.*

In contrast, as we will prove in Section 2 (see Theorem 2.5), there exist smooth cubic surfaces S over a field of characteristic 0 such that $S^{[3]}$ is not stably rational. There also exist smooth cubic threefolds X over $\mathbb{C}$ such that $X^{[3]}$ is not stably rational. This follows from the recent results proved in [5].

Coming back to the case of K-points of the cubic X itself, Cassels and Swinnerton-Dyer conjectured that if a cubic hypersurface X has L-points over a field extension $L \supset K$ of degree coprime to 3, then X has a K-point. Formulated in more geometric terms, if X has a 0-cycle of degree 1, then it has a K-point. In [4], Coray studied the possible degrees coprime to 3 of a field extension $K \subset L$ such that $X(L)$ is not empty. He proved the following.

Theorem 1.4 (see [4, Theorem 7.1]). *Let S be a smooth cubic surface over a field of characteristic* 0 *having a* 0*-cycle of degree* 1*. Then the minimal degree coprime to* 3 *of a closed point of S is* 1*,* 4 *or* 10*.*

One of our main results in this paper is that we eliminate the possibility that this minimal degree (that we will call the Coray number of S) is 10.

Theorem 1.5. *Let $S \subset \mathbb{P}^3_K$ be a smooth cubic surface defined over a field K of characteristic* 0*. If S has a* 0*-cycle of degree* 1*, then S either has a K-point or has an L-point, where L is a field extension of K of degree* 4*.*

Remark 1.6. That the Coray number of a cubic hypersurface cannot be 2 follows from the fact that if there exists an effective 0-cycle z of degree 2 on a cubic hypersurface X over a field K, then X has a K-point. When z comes from a length 2 subscheme $Z_2 \subset X$ which generates a line $\langle Z_2 \rangle$ not contained in X, this follows from the construction of the residual intersection point y of the line $\langle Z_2 \rangle$ with X. When z is supported on a line contained in X, this is still easier.

Remark 1.7. In Theorem 1.5, an equivalent conclusion is that S either has a K-point or has an effective 0-cycle of degree 4. This follows from Remark 1.6 and from the fact that an effective 0-cycle is an integral combination of classes of points defined over field extensions of K.

Remark 1.8. In [3], Colliot-Thélène established an analog of Theorem 1.4 for del Pezzo surfaces S of degree 2. Namely, he proves that if S has a 0-cycle of degree 1, then it has a closed point of degree 1, 3 or 7. It was proved by Kollár and Mella (see [3, 9]) that the minimal odd degree of a point (the Coray number of S) can be 3. It is natural to ask if one can (as in Theorem 1.5) eliminate the possibility that the Coray number of S is 7, but we were not able to do this.

The method applied to prove Theorem 1.5 will allow us to prove more general results concerning the effectivity of 0-cycles modulo rational equivalence. Such statements were first introduced in [3], which proved the following result.

Theorem 1.9 (see [3, Théorème 3.3]). *If a smooth cubic surface S over a field K of characteristic* 0 *has a K-point x, any effective* 0*-cycle z of S can be written as*

$$z = z' + \alpha x \quad \text{in } \mathrm{CH}_0(S),$$

where α is an integer and $z' = 0$ or z' is the class of an effective cycle of S of degree 1 *or* 3*. In particular,* $\mathrm{CH}_0(S)$ *is generated by classes of K-points and points defined over field extensions of K of degree* 3*.*

Another result proved in [3] concerns the effectivity of 0-cycles of geometrically rational surfaces.

Theorem 1.10 ([3, Théorèmes 3.3, 4.4, 5.1]). *Let S be a smooth del Pezzo surface of degree d_S over a field K of characteristic* 0*. If S has a K-point and*

(a) *if $d_S = 3$ (cubic surfaces), then any* 0*-cycle of degree $d \geq 3$ on S is effective;*

(b) *if $d_S = 2$, then any* 0*-cycle of degree $d \geq 6$ on S is effective;*

(c) *if $d_S = 1$, then any* 0*-cycle of degree $d \geq 21$ on S is effective.*

Remark 1.11. In [3], the bounds given are different, but these are actually the bounds that can be deduced from the results of [3] modulo a small extra argument (see the proof of Corollary 1.16 below, and also the addendum on Colliot-Thélène's website).

We will generalize these statements, mainly removing the assumption that S has a K-point, and also by improving the numerical bounds given above, for example in (c) where the assumption of having a K-point is automatic. Note that any del Pezzo surface S of degree d_S has a canonical effective 0-cycle $h_{d_S} = c_1(K_S)^2 \in \mathrm{CH}_0(S)$, which generalizes the 0-cycle h_3 of (1.1). We will first prove the following two results for smooth cubic surfaces defined over a field of characteristic 0.

Theorem 1.12. *The following statements hold.*

(a) *(Compare Theorem* 4.1*) Let S be a smooth cubic surface over a field K of characteristic* 0*. Then any effective* 0*-cycle z of S can be written as*

$$z = \pm z' + \gamma h_3 \quad \text{in } \mathrm{CH}_0(S), \tag{1.2}$$

where z' is effective of degree at most 18*.*

(b) *If S has a* 0*-cycle of degree* 1*, then S has an effective* 0*-cycle x_4 of degree* 4 *(given by Theorem* 1.5*) and any effective* 0*-cycle z on S can be written as*

$$z = \pm z' + \alpha x_4 + \beta h_3 \quad \text{in } \mathrm{CH}_0(S), \tag{1.3}$$

where z' is effective of degree at most 4*.*

Remark 1.13. As we will see in the course of the proof of Corollary 1.14, we can in fact impose the sign in front of z' in formulas (1.2) and (1.3), that is, the four formulas are true, with a positive sign or negative sign.

Theorem 1.12 implies the following corollary that will be proved in Section 3.1.

Corollary 1.14. *The following statements hold.*

(a) *Let S be a smooth cubic surface over a field K of characteristic* 0*. Then any* 0*-cycle $z \in \mathrm{CH}_0(S)$ of degree $d \geq 18$ is effective.*

(b) *Let S be a smooth cubic surface over a field K of characteristic* 0*. If S has a* 0*-cycle of degree* 1*, any* 0*-cycle $z \in \mathrm{CH}_0(S)$ of degree $d \geq 8$ is effective.*

In the case of del Pezzo surfaces of degree 2, our results are as follows.

Theorem 1.15. *The following statements hold.*

(a) *(Compare Theorem* 5.3.*) Let S be a smooth degree* 2 *del Pezzo surface over a field K of characteristic* 0*. Then any effective* 0*-cycle $z \in \mathrm{CH}_0(S)$ can be written as $z = z' + \gamma h_2$, where z' is effective of degree at most* 13.

(b) *More precisely, any effective* 0*-cycle $z \in \mathrm{CH}_0(S)$ can be written as $z = z' + \gamma h_2$ in $\mathrm{CH}_0(S)$, where z' is effective of degree* 13, 12*, or at most* 7.

Corollary 1.16. *The following statements hold.*

(a) *On a smooth degree* 2 *del Pezzo surface over a field K of characteristic* 0*, any* 0*-cycle $z \in \mathrm{CH}_0(S)$ of degree $d \geq 13$ is effective.*

(b) *If S has no* 0*-cycle of odd degree, any* 0*-cycle $z \in \mathrm{CH}_0(S)$ of degree $d \geq 12$ is effective.*

Proof. (a) Let S be a del Pezzo surface of degree 2 over a field K of characteristic 0, and let $z \in \mathrm{CH}_0(S)$. As z is supported on a smooth curve $C \subset S$ in a linear system $|\mathcal{O}_S(m)|$ for m large, we can write, by applying Riemann–Roch on C, $z = z' - \gamma h_2$ in $\mathrm{CH}_0(S)$, for some large integer γ, with z' effective. We now apply (5.1) to z', which gives $z = z'' + \gamma' h_2$ in $\mathrm{CH}_0(S)$, where z'' is effective of degree at most 13. If $\deg z \geq 13$, then $\gamma' \geq 0$, so z is effective and the corollary is proved in this case.

(b) If S has no 0-cycle of odd degree, we know that $\deg z'' \leq 12$ in the above argument. It follows that if $\deg z \geq 12$, then $\gamma' \geq 0$ and z is effective. □

Our last results concern the case of a del Pezzo surfaces of degree $d_S = 1$, which will be treated by the same method in Section 6.

Theorem 1.17. *The following statements hold.*

(a) *Let S be a smooth degree* 1 *del Pezzo surface over a field K of characteristic* 0*. Then any effective* 0*-cycle $z \in \mathrm{CH}_0(S)$ can be written as $z = z' + \gamma h_1$, where z' is effective of degree at most* 15.

(b) *Under the same assumptions, any effective* 0*-cycle $z \in \mathrm{CH}_0(S)$ can be written as*

$$z = \pm z' + \gamma h_1,$$

where z' is effective of degree 15, 10, 7, 6 *or at most* 4.

The following corollary is then proved in the same way as Corollary 1.16.

Corollary 1.18. *On a smooth del Pezzo surface of degree* 1 *defined over a field K of characteristic* 0*, any* 0*-cycle $z \in \mathrm{CH}_0(S)$ of degree $d \geq 15$ is effective.*

Our main tool for the proof of Theorems 1.5, 1.12 and 1.15 is the classical Schwarzenberger construction of vector bundles of rank 2 on a smooth del Pezzo surface S of degree 3, 2, 1, respectively, starting from smooth (or *lci*) length-2 subschemes Z of S. Sections of these vector bundles then allow us to move the cycle and a section vanishing along a cycle h_3, h_2, h_1, respectively, allows us to prove effectivity results for $Z - h_3$, $Z - h_2$, $Z - h_1$, respectively. This strategy is described in Section 3 where the key Proposition 3.2 is proved.

2. Rational self-maps and the proof of Theorem 1.1

We will use the following rather standard construction (see for example [1]) that allows to construct rational self-maps of a cubic hypersurface and its higher symmetric powers. Let E be an elliptic curve over a field K and let $H \in \mathrm{Pic}(E)$ be a line bundle of degree d. Then, for any integer $m = sd + 1$, there is a morphism

$$\mu_s : E \to E, \quad x \mapsto mx - sH \in \mathrm{Pic}^1(E) = E.$$

Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface over a field K of characteristic 0. For a general element $[W] \in G := \mathrm{Grass}(n-1, H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))$, there is a rational map

$$\phi_W : X \dashrightarrow \mathbb{P}^{n-2} \tag{2.1}$$

given by the linear projection from the line $\Delta_W \subset \mathbb{P}^n$ defined by W. The generic fiber of ϕ_W is an elliptic curve over the function field of $\mathbb{P}^{n-2}$ and it carries a line bundle H of degree $d = 3$. The construction above thus gives for each s a rational self-map $\mu_{s,W} : X \dashrightarrow X$ of multiplication by $3s + 1$ over $\mathbb{P}^{n-2}$. This map induces in turn for each k a rational self-map

$$\mu^k_{s,W} : X^{(k)} \dashrightarrow X^{(k)}.$$

Using the maps $\mu^k_{s,W}$, we now construct for each s a rational map defined over K,

$$\Psi_s : G \times G \dashrightarrow X^{[3]}.$$

The construction goes as follows. For a generic $[W'] \in G = \mathrm{Grass}(n-1, H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))$, the line $\Delta_{W'}$ produces by intersection with X a subscheme of length 3 of X, hence a point $\delta_{W'}$ of $X^{[3]}$. The rational map Ψ_s is defined by

$$\Psi_s([W], [W']) = \mu^3_{s,W}(\delta_{W'}).$$

As the variety $G \times G$ is rational over K, Theorem 1.1 will now be obtained as a consequence of the following result.

Proposition 2.1. *Assume $n \geq 3$ and* char $K = 0$. *Then, for $s = -1$, the rational map*

$$\Psi_s : G \times G \dashrightarrow X^{[3]}$$

is dominant.

Remark 2.2. It is plausible that the statement is true for any $s \neq 0$. For $s = 0$, the statement does not hold because each $\mu_{s,W}$ is the identity; hence $\mu^3_{s,W}$ is also the identity, and in particular, it preserves collinearity of triples on points. The image of $\Psi_s([W], [W'])$ is then the subvariety (birational to G) of $X^{[3]}$ parameterizing collinear triples of points in X.

Remark 2.3. For $n = 2$, the statement obviously does not hold since Theorem 1.1 is wrong in this case. The conclusion of Proposition 2.1 does not hold in this case because the rational map ϕ_W of (2.1) is the constant map (and in particular does not depend on W). For any s, the map $\mu_{s,W}$ is multiplication by $3s + 1$ on the elliptic curve X, and it has in this case the property that $\mu^3_{s,W} : X^{[3]} \dashrightarrow X^{[3]}$ preserves collinearity, as one can see by considering the Abel map of the elliptic curve X.

Proof of Proposition 2.1. For $s = -1$, the morphism

$$\Phi_s: G \times X \dashrightarrow X, \quad \Phi_s([W], x) = \mu_{s,W}(x)$$

has a special form, namely, on each hyperplane section $E \subset X$, the map $\mu_{E,-1}$ is the multiplication by -2 on the elliptic curve E and maps $x \in E$ to $h_E - 2x$, where $h_E := c_1(\mathcal{O}_E(1))$, hence is geometrically realized by sending x to the residual intersection point of the projective line $\mathbb{P}(T_{E,x})$ tangent to E at x with X. It follows that $\mu_{s,W}(x)$ depends only on the tangent space at x of the curve $E_{W,x}$ passing through x. In other words, the rational map Φ_{-1} factors through the rational morphism

$$G \times X \dashrightarrow \mathbb{P}(T_X)$$

which to $([W], x)$ associates the tangent line to the fiber $E_{W,x}$ passing through x of the linear projection ϕ_W.

We observe that it suffices to prove the result when $\dim X = 2$ since any set of three points on X (or rather subscheme of length 3) is supported on a smooth cubic surface. Furthermore, we can assume that the field is algebraically closed (for example $K = \mathbb{C}$). We choose three general points x, y, z on X and have to prove that the preimage $\Psi_{-1}^{-1}(\{x, y, z\})$ is not empty. Looking at the construction of Ψ_{-1}, this preimage consists of a pencil W of elliptic plane curves on S, and a set of three *collinear* points x', y', z' on X (generating a line $\Delta_{W'}$), such that, denoting respectively E_x, E_y, E_z the fibers of the pencil passing through x, y, z, we have

$$x' \in E_x, \qquad y' \in E_y, \qquad z' \in E_z, \tag{2.2}$$

$$\mu_{E_x,-1}(x') = x, \quad \mu_{E_y,-1}(y') = y, \quad \mu_{E_z,-1}(z') = z. \tag{2.3}$$

Using the above description of the maps $\mu_{E,-1}$, we can describe differently this fiber, namely, let $C_x \subset X, C_y \subset X, C_z \subset X$ be the curve of points $x' \in X, y' \in X, z' \in X$ such that the line $\langle x', x\rangle$, $\langle y', y\rangle$, $\langle z', z\rangle$ is tangent to X at x', y', z', respectively (more rigorously, we should take the respective Zariski closures of these curves in $X \setminus \{x\}$, $X \setminus \{y\}$ and $X \setminus \{z\}$). These curves, which are ramification curves of the projection of X to $\mathbb{P}^2$ from x, y, z, respectively, are well understood. They are members of the linear system $|\mathcal{O}_X(2)|$, irreducible for general points x, y, z, and they contain respectively the points x, y, z. Furthermore, they are mobile in the sense that any point w of X can be chosen not to belong to C_x, C_y, C_z, respectively. This last point is clear since there exists a line Δ passing through w and not tangent to X at w. This line intersects X at another point $x \in X$, and w does not belong to C_x.

For $(x', y', z') \in C_x \times C_y \times C_z$, we then choose any elliptic plane curve $E_{x',x}$ passing through x' and x, and similarly $E_{y',y}$ passing through y' and y, $E_{z',z}$ passing through z' and z. Equations (2.2) and (2.3) are then automatically satisfied since the line $\langle x', x\rangle$ is tangent to any such E_x at x' and similarly for y and z. We need now to impose the following conditions.

(1) The three points x', y', z' are collinear (producing the line $\Delta_{W'}$).

(2) The three elliptic curves $E_{x',x}$, $E_{y',y}$, $E_{z',z}$ generate a pencil (producing the pencil W).

Condition (1) has to be satisfied on $C_x \times C_y \times C_z$. Given x', y', z', condition (2) has to be satisfied on the product $\mathbb{P}^1_{x',x} \times \mathbb{P}^1_{y',y} \times \mathbb{P}^1_{z',z}$, where the line $\mathbb{P}^1_{x',x}$ parameterizes planes in $\mathbb{P}^3$ containing the line $\langle x', x\rangle$ and so on.

That the set of triples $(x', y', z') \in C_x \times C_y \times C_z$ of *distinct* points satisfying conditions (1) and (2) is not empty follows now from the following.

Lemma 2.4. *The following statements hold.*

(1) *For a general choice of points* x, y, z, *condition* (1) *is satisfied along a non-empty curve* $D \subset C_x \times C_y \times C_z$ *with the property that, for a general triple* $(x', y', z') \in D$, *we have* $x \neq x'$, $y \neq y'$, $z \neq z'$ *and the three lines* $\langle x', x\rangle$, $\langle y', y\rangle$, $\langle z', z\rangle$ *are mutually non-intersecting.*

(2) *For a general point* $(x', y', z') \in D$, *condition* (2) *is satisfied along a curve*

$$D'_{x',y',z'} \subset \mathbb{P}^1_{x',x} \times \mathbb{P}^1_{y',y} \times \mathbb{P}^1_{z',z}$$

which is isomorphic to $\mathbb{P}^1$.

Proof. (1) The set D cannot contain a surface. Indeed, it would dominate otherwise one of the 3 products $C_x \times C_y$, $C_x \times C_z$, $C_y \times C_z$ under the corresponding projections. Assume it dominates $C_x \times C_y$. Then, for any $(x', y') \in C_x \times C_y$, the third intersection point z'' of the line $\langle x', y'\rangle$ with X must belong to the curve C_z. Using the mobility of the curve C_z with z as explained above, this is not possible if z is chosen generically. Next, the set D has expected codimension at most 2 in $C_x \times C_y \times C_z$. Indeed, on each curve $C_\bullet$, where $\bullet = x, y, z$, we have the inclusion

$$\mathcal{O}_{C_\bullet}(-1) \hookrightarrow W_4 \otimes \mathcal{O}_{C_\bullet},$$

where $W_4 = H^0(X, \mathcal{O}_X(1))^*$. We combine these three inclusion maps to construct a morphism of vector bundles

$$\phi\colon \mathrm{pr}_x^* \mathcal{O}_{C_x}(-1) \oplus \mathrm{pr}_y^* \mathcal{O}_{C_y}(-1) \oplus \mathrm{pr}_z^* \mathcal{O}_{C_z}(-1) \to W_4 \otimes \mathcal{O}_{C_x \times C_y \times C_z},$$

of vector bundles of respective ranks 3 and 4. It is clear that the locus D of collinear triples (x', y', z') is contained in the locus D_2 where ϕ has rank at most 2, but we have to remove from it the sublocus D_2' where $x' = y'$ or $x' = z'$, or $y' = z'$. It is a standard fact that the rank locus D_2 has expected codimension at most 2; hence its codimension is exactly 2 by the previous assertion. The class of D_2 in $\mathrm{CH}^2(C_x \times C_y \times C_z)$ is in fact computed following [7]. This class is nothing but the Segre class $s_2(\mathcal{F})$ in $\mathrm{CH}^2(C_x \times C_y \times C_z)$, where

$$\mathcal{F} := \mathrm{pr}_x^* \mathcal{O}_{C_x}(-1) \oplus \mathrm{pr}_y^* \mathcal{O}_{C_y}(-1) \oplus \mathrm{pr}_z^* \mathcal{O}_{C_z}(-1).$$

This follows indeed from the fact that the rank at most 2 locus of ϕ is also the image in $C_x \times C_y \times C_z$, under the projection map $\pi\colon \mathbb{P}(\mathcal{F}) \to C_x \times C_y \times C_z$, of the locus $\widetilde{D}_2 \subset \mathbb{P}(\mathcal{F})$ defined by the 4 sections of $\mathcal{F}^*$ or sections of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ given by ϕ. Using the fact that D_2 has the right dimension, this is saying that

$$[D_2] = \pi_*(c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))^4) = s_2(\mathcal{F}) \quad \text{in } \mathrm{CH}^2(C_x \times C_y \times C_z).$$

The degree of D_2 is computed using the Whitney formula for the total Segre class, which gives

$$s(\mathcal{F}) = \mathrm{pr}_x^*(1 + c_1(\mathcal{O}_{C_x}(1)))\, \mathrm{pr}_y^*(1 + c_1(\mathcal{O}_{C_y}(1)))\, \mathrm{pr}_z^*(1 + c_1(\mathcal{O}_{C_z}(1))) \\ \text{in } \mathrm{CH}(C_x \times C_y \times C_z),$$

so that

$$s_2(\mathcal{F}) = \mathrm{pr}_x^*(c_1(\mathcal{O}_{C_x}(1)))\, \mathrm{pr}_y^*(c_1(\mathcal{O}_{C_y}(1))) + \mathrm{pr}_x^*(c_1(\mathcal{O}_{C_x}(1)))\, \mathrm{pr}_z^*(c_1(\mathcal{O}_{C_z}(1))) \\ + \mathrm{pr}_y^*(c_1(\mathcal{O}_{C_y}(1)))\, \mathrm{pr}_z^*(c_1(\mathcal{O}_{C_z}(1))) \quad \text{in } \mathrm{CH}(C_x \times C_y \times C_z).$$

In order to prove the non-emptiness of $D_2 \setminus D_2'$, it suffices to show that

$$\deg_{H_x} D_2 > \deg_{H_x} D_2',$$

where the degree is computed via the line bundle $H_x := \mathrm{pr}_x^* \mathcal{O}_{C_x}(1))$ on $C_x \times C_y \times C_z$. The curves C_x, C_y, C_z being defined by quadratic equations in X, we have

$$\deg_{H_x} D_2' = \deg(C_y \cdot C_z) \deg_{H_x}(C_x) = 12 \cdot 6, \quad \deg_{H_x} D_2 = 6 \cdot 6 \cdot 6,$$

from which we conclude that $\deg_{H_x} D_2 > \deg_{H_x} D_2'$ and $D_2 \setminus D_2'$ is non-empty.

By a similar counting argument, for a general element (x', y', z') of $D_2 \setminus D_2'$, we prove that the three lines

$$\langle x', x\rangle,\ \langle y', y\rangle,\ \langle z', z\rangle \tag{2.4}$$

are mutually non-intersecting.

(2) This will follow from the last statement. Indeed, this statement says that the three points $\delta_{x'x}, \delta_{y'y}, \delta_{z'z}$ which parameterize respectively the three lines (2.4) are three points in general position in the Grassmannian of lines in $\mathbb{P}^3$. The Grassmannian $G(2,4)$ is a quadric Q in $\mathbb{P}^5$, and two lines in $\mathbb{P}^3$ are non-intersecting if and only if the corresponding points in $G(2,4) = Q$ have the property that the line they generate is not contained in Q. There is thus a single orbit under the action of PGL(4) on $G(2,4)^{[3]}$ of triples parameterizing three lines mutually non-intersecting, as follows from the similar statement for the action of the orthogonal group O(6) on Q. It thus suffices to prove the result when the three lines (2.4) in $\mathbb{P}^3$ are defined by equations

$$X_0 = X_1 = 0, \quad X_2 = X_3 = 0, \quad X_0 - X_2 = 0, \quad X_1 - X_3 = 0.$$

An easy computation shows that the set of triples of planes $p_{x',x}, p_{y',y}, p_{z',z} \in (\mathbb{P}^3)^*$ such that the corresponding plane $P_{x',x}$ contains $\langle x', x\rangle$ and so on, and such that the three linear forms $p_{x',x}, p_{y',y}, p_{z',z}$ generate a pencil of planes (i.e. are collinear), is a copy of $\mathbb{P}^1$ diagonally embedded in $\mathbb{P}^1_{x',x} \times \mathbb{P}^1_{y',y} \times \mathbb{P}^1_{z',z}$. Indeed, one writes

$$p_{x',x} = u_0X_0 + u_1X_1, \quad p_{y,y'} = v_2X_2 + v_3X_3, \quad p_{z,z'} = w(X_0 - X_2) + w'(X_1 - X_3)$$

for adequate choice of homogeneous coordinates on $\mathbb{P}^3$. The fact that the three planes generate a pencil then provides equations $w = au_0 = -bv_2$, $w' = au_1 = -bv_3$ for some nonzero coefficients a, b. Hence the equations provide $u_1/v_3 = u_0/v_2 = -b/a$, $w = au_0$, $w' = au_1$, which proves the result. □

This concludes the proof of Proposition 2.1. □

We finish this section with the proof of the following result, which immediately follows from [5], and is in contrast with Theorem 1.1.

Theorem 2.5. *The following statements hold.*

(a) *Let $X \subset \mathbb{P}^4$ be a very general cubic threefold over $\mathbb{C}$. Then $X^{[3]}$ is not stably rational.*

(b) *Let $\mathcal{S} \to (\mathbb{P}^4)^* := B$ be the universal hyperplane section of X, and let $S_\eta \to B_\eta$ be its generic fiber over B, which is a smooth cubic surface over the field $\mathbb{C}(B)$. Then $S_\eta^{[3]}$ is not stably rational.*

Proof. Using the natural inclusion $\mathcal{S} \subset B \times X$, the relative Hilbert scheme $\mathcal{S}^{[3/B]}$ maps naturally to $X^{[3]}$ and is generically a projective bundle over $X^{[3]}$, with fiber, over a general point $z \in X^{[3]}$ parameterizing the length-3 subscheme $Z \subset X$, the projective line

$$\mathbb{P}(H^0(X, \mathcal{I}_Z(1))).$$

The stable rationality of $S_\eta^{[3]}$ over $\mathbb{C}(B)$ would imply the stable rationality over $\mathbb{C}$ of $\mathcal{S}^{[3/B]}$ which is birational to a projective bundle over $X^{[3]}$ by the construction above, and thus the stable rationality of $X^{[3]}$ over $\mathbb{C}$. We now claim that the results of [5] imply that $X^{[3]}$ is not stably rational over $\mathbb{C}$, which proves both (a) and (b) by the argument above. To prove this, we recall that the stable rationality of $X^{[3]}$ implies that $X^{[3]}$ has trivial universal CH_0 group, so we just have to show that this is not the case. To see this, we use the natural incidence correspondence $I \subset X^{[3]} \times X$ defined as follows: recalling that $X^{[3]}$ is the Hilbert scheme parameterizing length-3 subschemes $Z \subset X$, it carries a universal object which is described by a flat morphism $p: I \to X^{[3]}$ of degree 3. The morphism $I \to X$ is the inclusion as a subscheme of X of length 3 on each fiber of p. Moreover, there is an inclusion $i: \mathbb{P}(\Omega_X) \hookrightarrow X^{[3]}$, which to $x \in X$, $0 \neq \eta \in \Omega_{X,x}$ associates the subscheme of length 3 of X that is supported on x and has as Zariski tangent space at x the hyperplane defined by η. We have the obvious relation

$$I_* \circ i_*(x) = 3x \quad \text{in } \mathrm{CH}_0(X). \tag{2.5}$$

Assume by contradiction that $X^{[3]}$ has trivial universal CH_0 group, that is, any 0-cycle of $X^{[3]}$ of degree 0 defined over a field containing $\mathbb{C}$ is trivial (the interesting case being the function field of $X^{[3]}$ itself). By the construction described above of the fat points, there is, after choosing a rational section $\sigma: X \dashrightarrow \mathbb{P}(\Omega_X)$ of the structural morphism $\mathbb{P}(\Omega_X) \to X$, a rational map $i \circ \sigma: X \dashrightarrow X^{[3]}$, and thus a point $\gamma_{X^{[3]}}$ of $X^{[3]}$ over the function field M of X. We apply the CH_0-universal triviality to the difference $\gamma_{X^{[3]}} - z_0$, where $z_0 = i(z_0')$ for some point of X defined over $\mathbb{C}$. We thus get that

$$\gamma_{X^{[3]}} - z_0 = 0 \quad \text{in } \mathrm{CH}_0(X_M^{[3]}). \tag{2.6}$$

Applying (2.5), we get $I_*(\gamma_{X^{[3]}}) = 3\delta_X$ in $\mathrm{CH}_0(X_M)$, where $\delta_X \in X(M)$ is the generic point of X. It thus follows from (2.6) that the cycle $z = \delta_X - z_0' \in \mathrm{CH}_0(X_M)$ satisfies $3z = 0$ in $\mathrm{CH}_0(X_M)$. On the other hand, as X admits a unirational parameterization of degree 2, we also know that z satisfies $2z = 0$ in $\mathrm{CH}_0(X_M)$. Thus $z = 0$ and X has trivial universal CH_0 group. This contradicts [5], which proves that X does not have trivial universal CH_0 group (more precisely, it is proved in [5] that the minimal class of the intermediate Jacobian of X is not algebraic and this prevents X having trivial universal CH_0 group using [13]). □

3. Zero-cycles and rank 2 vector bundles on del Pezzo surfaces

Let S be a smooth del Pezzo surface over a field K of characteristic 0. We will denote the ample line bundle K_S^{-1} by $\mathcal{O}_S(1)$. We denote by d_S the canonical degree of S, namely $d_S := \deg c_1(K_S)^2$. For all $l \geq 0$, we have, by Riemann–Roch,

$$h^0(S, \mathcal{O}_S(l)) = 1 + \frac{d_S}{2}(l^2 + l). \tag{3.1}$$

Let d be a positive integer and let $x \in S^{[d]}(K)$ be a K-point, parameterizing a reduced (or *lci*) subscheme $Z_x \subset S$ of length d, which is defined over K. Suppose l is a nonnegative integer such that

$$h^0(S, \mathcal{O}_S(l)) < d, \tag{3.2}$$

that is, by (3.1),

$$1 + \frac{d_S}{2}(l^2 + l) < d.$$

The strict inequality in (3.2) implies that the restriction map

$$H^0(S, \mathcal{O}_S(l)) \to H^0(Z_x, \mathcal{O}_{Z_x}(l))$$

is not surjective, hence that $H^1(S, \mathcal{I}_{Z_x}(l)) \neq 0$. As is standard (see [10, 12]), we use Serre's duality $H^1(S, \mathcal{I}_{Z_x}(l))^* \cong \mathrm{Ext}^1(\mathcal{I}_{Z_x}(l), K_S)$, and as $K_S = \mathcal{O}_S(-1)$, it follows that the K-vector space $\mathrm{Ext}^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S)$ is nontrivial. Any element $e \in \mathrm{Ext}^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S)$ provides us with a rank 2 coherent sheaf E constructed as an extension

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_{Z_x}(l+1) \to 0. \tag{3.3}$$

We now have the following lemma.

Lemma 3.1. *Assume x corresponds to a L-point of S defined over a field extension $K \subset L$ of degree d. Then*

(a) *for any nonzero extension class $e \in \mathrm{Ext}^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S)$, the coherent sheaf E constructed from e is locally free;*

(b) *E has a section σ whose zero-set is exactly Z_x;*

(c) *we have $h^0(S, E) = 1 + h^0(S, \mathcal{I}_{Z_x}(l+1))$.*

Proof. As is well known (see [8, p. 726] or [12]), the coherent sheaf E is locally free away from Z_x and it is locally free on S if and only if the extension class e is nonzero at each point z of Z_x (over the algebraic closure of K). Passing to the algebraic closure of K, we see that the set S_{nlf} of points of $S_{\overline{K}}$ where E is not locally free is contained in $Z_{x,\overline{K}}$ and, as E is defined over K, S_{nlf} is invariant under $\mathrm{Gal}(\overline{K}/K)$. As Z_x is an L-point of S, $\mathrm{Gal}(\overline{K}/K)$ acts transitively on the set of points in $Z_{x,\overline{K}}$. Thus S_{nlf} is either empty or the whole of $Z_{x,\overline{K}}$. In the second case, the extension class $e \in \mathrm{Ext}^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S)$ vanishes in

$$H^0(S, \mathcal{E}xt^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S)).$$

However, as follows from the vanishing of $H^1(S, \mathcal{O}_S(-l-1))$, the natural map

$$\mathrm{Ext}^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S) \to H^0(S, \mathcal{E}xt^1(\mathcal{I}_{Z_x}(l+1), \mathcal{O}_S))$$

is injective, so in the second case, the extension class e is identically 0, which contradicts our assumption. This proves (a).

(b) The section σ being given by the morphism $\mathcal{O}_S \to E$ on the left in (3.3), (b) follows from (a) and the exact sequence (3.3).

(c) This follows from the exact sequence (3.3) and the vanishing $H^1(S, \mathcal{O}_S) = 0$. □

Using rank 2 vector bundles as in Lemma 3.1 will be our main tool in this paper, as they will be used to show that some 0-cycles are effective. As a sample result, let us prove the following statement, which will be systematically used for the proof of Theorems 1.5 and 1.12.

Proposition 3.2. *Let S be a smooth cubic surface over a field K of characteristic* 0, *and let $l \geq 0$, $d \geq 0$ be integers satisfying inequality* (3.2)*, namely*

$$h^0(S, \mathcal{O}_S(l)) < d. \tag{3.4}$$

Assume there is an effective cycle $z_d \in \mathrm{CH}_0(S)$ of degree d and let $s \geq 1$ be an integer. Then, if S has an effective 0*-cycle z_s of degree s, and*

$$h^0(S, \mathcal{O}_S(l+1)) - d = 1 + \frac{3}{2}((l+1)^2 + l + 1) - d \geq 2s, \tag{3.5}$$

the cycle $z_d - z_s \in \mathrm{CH}_0(S)$ is effective. In particular, S has an effective 0*-cycle $z \in \mathrm{CH}_0(S)$ of degree $d - s$.*

Proof. First of all, we note that the existence of an effective 0-cycle z_d of degree d defined over K implies the existence of a length-d subscheme $Z \subset S$ which is curvilinear, hence *lci*, of Chow class z_d. Indeed, the fibers of the Hilbert–Chow morphism $S^{[d]} \to S^{(d)}$ over K-points are rational over K, and the subset of the fiber parameterizing curvilinear (hence *lci*) subschemes is open and Zariski-dense in this fiber. The same remark applies to z_s.

Our assumption thus gives us a subscheme $Z_d \subset S$ of length d which is *lci*. Using inequality (3.4), we can perform the construction of the rank 2 coherent sheaf E as above. By Lemma 3.1, it has a section σ which vanishes exactly along Z_d. Furthermore, thanks to (3.5), we get $h^0(S, \mathcal{I}_{Z_d}(l+1)) \geq 2s$; hence $h^0(S, E) \geq 2s + 1$ by Lemma 3.1 (c). For a subscheme $Z_s \subset S$ of length s, there is by (3.5) a nonzero section σ' of E vanishing along Z_s. Assuming the coherent sheaf E is locally free and the zero-set $V(\sigma')$ is of dimension 0, then the residual cycle Z' of Z_s in $V(\sigma')$ is effective, defined over K and of degree $d - s$; more precisely, it is of class $c_2(E) - z_s = z_d - z_s$, so the proof is finished in this case. Unfortunately, as we want to prove the result for a specific subscheme $Z \subset S$, it might be that E is not locally free and that any section of E vanishing along any subscheme Z_s of length s defined over K vanishes along a curve in S (see Example 3.5).

However, we can circumvent this problem by making both subschemes $Z_d \subset S$, $Z_s \subset S$ generic. Let $B := S^{[d]} \times S^{[s]}$ and let S_η be the generic fiber (defined over $K(B)$) of the projection $\pi \colon S \times B \to B$. Then, denoting by $[Z] \in S^{[k]}(K)$ the point parameterizing the subscheme $Z \subset S$ of length k defined over K, the subscheme $Z_d \subset S$ is the specialization at any point $([Z_d], [Z_s]) \in (S^{[d]} \times S^{[s]})(K)$ of the pull-back $\mathrm{pr}_1^* \mathcal{Z}_d \subset S^{[d]} \times S^{[s]} \times S$ of the universal subscheme $\mathcal{Z}_d \subset S^{[d]} \times S$. We will denote the generic fiber of $\mathrm{pr}_1^* \mathcal{Z}_d$ over $\operatorname{Spec} K(B)$ by $Z_{d,\eta} \subset S_\eta$. Using the subscheme $Z_{d,\eta} \subset S_\eta$, we now perform the construction of the rank 2 coherent sheaf E_η over S_η. Lemma 3.1 applies in this situation, so E_η is locally free, and thanks to (3.5), we get

$$h^0(S_\eta, \mathcal{I}_{Z_{d,\eta}}(l+1)) \geq 2s;$$

hence $h^0(S_\eta, E_\eta) \geq 2s + 1$ by Lemma 3.1 (c).

Finally, the pull-back $\mathrm{pr}_2^* \mathcal{Z}_s$ to $S^{[d]} \times S^{[s]}$ of the universal subscheme $\mathcal{Z}_s \subset S^{[s]} \times S$ parameterized by $S^{[s]}$ has a generic fiber $Z_{s,\eta}$ which is a subscheme of length s of S_η. Using inequality (3.5), there exists a nonzero section σ' of E_η vanishing along the corresponding subscheme $Z_{s,\eta} \subset S_\eta$. Lemma 3.3 proved below tells that, under our numerical assumptions, there exists a section σ' as above vanishing along $Z_{s,\eta}$ and with zero-locus $Z'_{d,\eta}$ of codimension 2. It then follows that the cycle $Z'_{d,\eta} - Z_{s,\eta} \in \mathrm{CH}_0(S_\eta)$ is effective. Note that we have the equality of cycles $V(\sigma') = Z'_{d,\eta} = c_2(E_\eta) = Z_{d,\eta}$ in $\mathrm{CH}_0(S_\eta)$; hence the Fulton specialization (see [7, Section 10.3]) of $Z'_{d,\eta}$ to the fiber S of π over the point $([Z_d],[Z_s]) \in S^{[d]} \times S^{[s]}(K)$ equals $z_d \in \mathrm{CH}_0(S)$. The Fulton specialization of the class of $Z_{s,\eta}$ is z_s. The Fulton specialization of an effective cycle in $\mathrm{CH}_0(S)$ is effective (see [3, Lemme 2.10]); hence we conclude that $z_d - z_s$ is effective, proving Proposition 3.2. □

We reduced above the proof of Proposition 3.2 to the case of the generic subscheme $Z_\eta \subset S_\eta$ of length d, and we can even assume without loss of generality that the field K is $\mathbb{C}$.

Lemma 3.3. *Let S be a smooth cubic surface over $\mathbb{C}$, and let d, l, s be three integers satisfying the two inequalities*

$$h^0(S, \mathcal{O}_S(l)) < d,$$

$$1 + \frac{3}{2}((l+1)^2 + l + 1) - d = h^0(S, \mathcal{O}_S(l+1)) - d \geq 2s.$$

Then, for a general subscheme $Z_d \subset S^{[d]}$ of length d, a general vector bundle E constructed from an extension class $e \in \mathrm{Ext}^1(\mathcal{I}_{Z_d}(l+1), \mathcal{O}_S)$, and general set of s points $x_1, \dots, x_s \in S$, there exists a section σ' of E vanishing at the points x_i and whose zero-set is of dimension 0.

Proof. The only part of the statement that is not proved either in Lemma 3.1 or in the beginning of the proof of Proposition 3.2 is the fact that the vanishing locus of σ' has codimension 2. We argue by contradiction. We note first that we can assume that Z_d is very general, since the considered property is Zariski open. We observe then that, by very generality of Z_d and countability of $\mathrm{Pic}(S)$, for any line bundle M on S, we have

$$H^0(S, \mathcal{I}_{Z_d}(M)) = 0 \quad \text{if } h^0(S, M) \leq d. \tag{3.6}$$

We now choose $x_1, \dots, x_s$ generically and assume by contradiction that any section $\sigma' \in H^0(S, E \otimes \mathcal{I}_{Z_s}) \subset H^0(S, E)$ vanishes along a (possibly reducible) curve $C \subset S$. We can choose C to be maximal with this property. We first prove the result assuming that the generic such curve C is not disjoint from the set $\{x_1, \dots, x_s\}$. As the set of points $\{x_1, \dots, x_s\}$ is unordered, and more precisely the Galois group of the function field of the base parameterizing the data of $\{x_1, \dots, x_s\}$, C as above acts as the full symmetric group $\mathfrak{S}_s$ on $\{x_1, \dots, x_s\}$, we conclude that the curve C passes in fact through all the points x_i, $i = 1, \dots, s$. We can assume by countability of $\mathrm{Pic}(S)$ that the class of the curve C does not depend on the points x_i, so we have $C \in |H|$ for some $H \in \mathrm{Pic}(S)$ independent of the points x_i and H satisfies

$$H^0(S, E \otimes H^{-1}) \neq 0. \tag{3.7}$$

We now use the exact sequence (3.3) and conclude from (3.7) that

$$H^0(S, \mathcal{I}_{Z_d}(l+1) \otimes H^{-1}) \neq 0.$$

By (3.6), this implies

$$h^0(S, \mathcal{O}_S(l+1) \otimes H^{-1}) \geq d + 1. \tag{3.8}$$

However, we also have

$$h^0(S,H) \geq s+1, \tag{3.9}$$

since there exists a member C of $|H|$ passing through a general set of s points in S. This provides us with a contradiction for $s \geq 2$. Indeed, if $s \geq 2$, we conclude from (3.9) that the degree of the curves C in $|H|$ is at least 3 and this contradicts the following

Claim 3.4. *For any curve $D \subset S$ such that $h^0(S, E(-D)) \neq 0$, we have* $\deg D \leq 2$.

Proof. Indeed, if $\deg D \geq 3$, the rank of the restriction map

$$H^0(S, \mathcal{O}_S(l+1)) \to H^0(D, \mathcal{O}_D(l+1))$$

is at least $3(l+1)$. Hence

$$h^0(S, \mathcal{O}_S(l+1)) \geq h^0(S, \mathcal{O}_S(l+1)(-D)) + 3(l+1).$$

As $h^0(S, E(-D)) \neq 0$, $h^0(S, \mathcal{I}_{Z_d}(l+1)(-D)) \neq 0$ by the exact sequence (3.3). By (3.6), we thus conclude that

$$h^0(S, \mathcal{O}_S(l+1)(-D)) > d \geq h^0(S, \mathcal{O}_S(l)) + 1.$$

This provides a contradiction since $h^0(S, \mathcal{O}_S(l+1)) - h^0(S, \mathcal{O}_S(l)) = 3(l+1)$. □

We next deal with the case $s = 1$. By the case $s \geq 2$ which is already treated, we can assume that

$$h^0(S, \mathcal{O}_S(l+1)) = d+2 \quad \text{or} \quad h^0(S, \mathcal{O}_S(l+1)) = d+3.$$

In this case, the inequality $h^0(S, \mathcal{O}_S(l+1) \otimes H^{-1}) \geq d+1$ of (3.8) gives us

$$h^0(S, \mathcal{O}_S(l+1) \otimes H^{-1}) \geq h^0(S, \mathcal{O}_S(l+1)) - 2,$$

which is impossible since $l \geq 0$; hence the curve C which is mobile imposes at least 3 conditions on the linear system $|\mathcal{O}_S(1)|$.

In order to conclude the proof, we need to study the case where the curve C is disjoint from $\{x_1, \dots, x_s\}$. By assumption, there exist for a generic set Z_s of points x_i, $i = 1, \dots, s$, a curve C and a section σ' of $E(-C)$ which vanishes at all the points x_i, while the curve C does not pass through any of them. It follows that $\sigma' \in H^0(S, E(-C) \otimes \mathcal{I}_{Z_s})$. However, we have

$$h^0(S, E(-C)) \leq h^0(S, \mathcal{I}_{Z_d}(l+1)(-C)). \tag{3.10}$$

We can obviously assume that

$$h^0(S, \mathcal{I}_{Z_d}(l+1)) = 2s \quad \text{or} \quad h^0(S, \mathcal{I}_{Z_d}(l+1)) = 2s+1. \tag{3.11}$$

It follows from (3.10), (3.11) and (3.6) that

$$h^0(S, E(-C)) \leq 2s + 1 - (l+2) < 2s, \tag{3.12}$$

because the curve C imposes at least $l+2$ conditions on $H^0(S,\mathcal{O}_S(l+1))$, hence also on $H^0(S,\mathcal{I}_{Z_d}(l+1))$. We know that there exists a nonzero section of $E(-C)$ vanishing along a generic set of s points in S. We claim that this implies that, for a generic set of s points in S, there exists a curve C' passing through at least one of these points and such that any section of $E(-C)$ vanishing at these s points vanishes along C'. Indeed, this is done by a dimension count: we consider the universal vanishing locus $\Gamma\subset\mathbb{P}(H^0(S,E(-C)))\times S^{[s]}$ and its Zariski open set $\Gamma_f\subset\Gamma$ consisting of pairs $(\sigma,\{x_1,\dots,x_s\})$, where the x_i are all distinct and the 0-locus of σ is 0-dimensional near all x_i. Using (3.12), we have $\dim\Gamma_f\le 2s-1$; hence Γ_f cannot dominate $S^{[s]}$ by the second projection, which proves the claim.

By the claim, we are now reduced to the previous situation where the curve of vanishing of the section σ' contains at least one of the points x_i. The lemma is thus fully proved. □

Let us give an example illustrating the difficulty for a nongeneric choice of $Z\subset S$.

Example 3.5. Assume that S contains a line Δ with residual conic $C\in|\mathcal{O}_S(1)(-\Delta)|$. Consider the vector bundle $E=\mathcal{O}_S(\Delta)\oplus\mathcal{O}_S(C)$ on S. If we take a general section σ, its zero-locus Z is the intersection $\Delta\cap C$ of the line $\Delta\subset S$ with one conic in the pencil, hence consists of 2 points, and we have an extension $0\to\mathcal{O}_S\to E\to\mathcal{I}_Z(1)\to 0$. The vector bundle thus has 3 sections, and a general section has a length-2 vanishing locus, but a section vanishing at a general point vanishes along a conic.

3.1. Another useful effectivity result. We will use in the next sections the following result for 0-cycles on a surface. A similar statement already appears in [3] where it is used to simplify Coray's proof of Theorem 1.4.

Lemma 3.6. *Let S be a smooth projective surface over a field K of characteristic* 0 *and let H be a very ample line bundle on S. Then if $Z\subset S$ is a subscheme of length d, of class $z\in\mathrm{CH}_0(S)$, and*

$$d\le h^0(S,H)-2, \tag{3.13}$$

the 0*-cycle $c_1(H)^2-z\in\mathrm{CH}_0(S)$ defined over K is rationally equivalent to an effective* 0*-cycle on S.*

Proof. Consider the smooth projective variety $B=S^{[d]}$, which is defined over K, with function field $K(B)$. Let $\eta=\operatorname{Spec}K(B)$ be its generic point. The universal subscheme

$$\mathcal{Z}_d\subset B\times S$$

has for generic fiber a subscheme $Z_\eta\subset S_\eta$ of length d, which specializes to $Z\subset S$ at the point $[Z]\in S^{[d]}(K)$ parameterizing Z.

Claim 3.7. *The effectivity statement of Lemma* 3.6 *is true for the generic subscheme Z_η.*

Proof. Indeed, we have $h^0(S_\eta,\mathcal{I}_{Z_\eta}(H))\ge 2$ by (3.13). It thus suffices to show that there are two curves C_1,C_2 in $|\mathcal{I}_{Z_\eta}(H)|$ which intersect properly, since then $C_1\cap C_2$ is a 0-dimensional subscheme containing Z, so the class of $C_1\cap C_2-Z$ is effective. The existence of C_1, C_2 as above follows once the linear system $|\mathcal{I}_{Z_\eta}(H)|$ has no fixed component. Assume by contradiction there is a fixed component C. We discuss as before the two cases where C intersects Z_η or is disjoint from Z_η. In the first case, C contains Z_η by a symmetric group argument, and so we conclude by (3.6) that $h^0(S, \mathcal{O}_S(C)) \geq d+1$. As we can obviously assume that $d = h^0(S,H) - 2$, and we have $h^0(S, H(-C)) \geq 2$, we get a contradiction in this case, since H is very ample.

In the second case, the curve C does not intersect Z_η, so we get

$$h^0(S, H(-C) \otimes \mathcal{I}_{Z_\eta}) \neq 0.$$

This implies by (3.6) that $h^0(S, H(-C)) \geq d+1$, and as $h^0(S,H) = d+2$ and H is very ample, this also provides a contradiction, which proves the claim. □

The result then follows from Claim 3.7, using Fulton's specialization from $\mathrm{CH}_0(S_\eta)$ to $\mathrm{CH}_0(S)$ at the point $[Z] \in B(K)$, which, as noted in [3], preserves effectivity. □

For the applications, we will need the following variant, which is proved exactly in the same way.

Lemma 3.8. *Let S be a smooth projective surface and $L = \mathcal{O}_S(1)$ a line bundle on S, which is assumed to be ample. Let d, l be integers such that $\mathcal{O}_S(l+1)$ is generated by its sections and*

$$h^0(S, \mathcal{O}_S(l)) \geq d+1, \tag{3.14}$$

$$h^0(S, \mathcal{O}_S(l+1)) - d \geq 2. \tag{3.15}$$

Then if $z_d \in \mathrm{CH}_0(S)$ is effective of degree d, the cycle

$$z' := l(l+1)c_1(L)^2 - z_d \in \mathrm{CH}_0(S)$$

is effective.

Proof. The proof works as before. Inequality (3.14) guarantees that the generic subscheme $Z_{d,\eta}$ of S of length d is supported on a curve C_1 which is a member of $|\mathcal{O}_{S_\eta}(l)|$. Inequality (3.15) guarantees by the proof of Lemma 3.6 that the linear system $|\mathcal{I}_{Z_{d,\eta}}(l+1)|$ has no fixed component, hence has a member C_2 which intersects C_1 properly. Hence the generic subscheme $Z_{d,\eta}$ is supported on the complete intersection of the curves C_1 and C_2, so the cycle $l(l+1)c_1(L)^2 - Z_{d,\eta} \in \mathrm{CH}_0(S_\eta)$ is effective. Lemma 3.8 then follows by Fulton's specialization. □

Lemmas 3.6 and 3.8 will be used in next section to deduce Corollary 1.14 from Theorem 1.12.

4. Zero-cycles on smooth cubic surfaces

We prove in this section Theorems 1.5 and 1.12, and Corollary 1.14. We start with the proof of Theorem 1.12 (a), which is the following statement.

Theorem 4.1. *Let S be a smooth cubic surface over a field K of characteristic* 0. *Then any effective* 0*-cycle z of S can be written as*

$$z = \pm z' + \gamma h_3 \quad \textit{in } \mathrm{CH}_0(S), \tag{4.1}$$

where z' is effective of degree at most 18.

Let us first prove the following proposition.

Proposition 4.2. *Let S be a smooth cubic surface over a field K of characteristic* 0. *Let $z \in \mathrm{CH}_0(S)$ be effective of degree d. Then, if $d \geq 20$, the cycle z can be written as*

$$z = \pm z' + \gamma h_3 \quad \textit{in } \mathrm{CH}_0(S),$$

where γ is an integer and z' is effective of degree less than 20.

Proof. We argue by induction on $d \geq 20$. Given an effective 0-cycle $z \in \mathrm{CH}_0(S)$ of degree d, we introduce the nonnegative integer l such that

$$h^0(S, \mathcal{O}_S(l)) < d \leq h^0(S, \mathcal{O}_S(l+1)) \tag{4.2}$$

We first assume that $d \leq h^0(S, \mathcal{O}_S(l+1)) - 2$. By Lemma 3.6, the cycle $(l+1)^2 h_3 - z$ is effective; hence if $\deg((l+1)^2 h_3 - z) < \deg z$, we can replace z by the effective cycle

$$z' = (l+1)^2 h_3 - z,$$

to which we can apply the inductive argument. In conclusion, if $d \leq h^0(S, \mathcal{O}_S(l+1)) - 2$, we can assume that

$$\deg z \leq \frac{1}{2} \deg((l+1)^2 h_3) = \frac{3}{2}(l+1)^2. \tag{4.3}$$

By the strict left inequality in (4.2), we can apply the vector bundle construction of Section 3. Using (4.3), we get

$$h^0(S, \mathcal{O}_S(l+1)) - \deg z \geq 1 + \frac{3(l+1)}{2}.$$

Thus Proposition 3.2 (b) (with $s = 3$) and Corollary 1.3 tell that $z - h_3$ is effective if

$$1 + \frac{3(l+1)}{2} \geq 6,$$

that is, $l \geq 3$. In conclusion, we proved the induction step when $l \geq 3$, hence when

$$d \geq 20 = h^0(S, \mathcal{O}_S(3)) + 1,$$

assuming that we do not have

$$d = h^0(S, \mathcal{O}_S(l+1)) \quad \text{or} \quad d = h^0(S, \mathcal{O}_S(l+1)) - 1. \tag{4.4}$$

The missing cases (4.4) are now treated as follows. We consider, instead of the effective cycle z, the effective cycle $z' = z + h_3$. Then, in both cases, we have

$$\deg z' > h^0(S, \mathcal{O}_S(l+1)).$$

We also have

$$h^0(S, \mathcal{O}_S(l+2)) - \deg z' = h^0(S, \mathcal{O}_S(l+2)) - \deg z - 3. \tag{4.5}$$

As $\deg z \leq h^0(S, \mathcal{O}_S(l+1))$, (4.5) gives $h^0(S, \mathcal{O}_S(l+2)) - \deg z' \geq 3(l+2) - 3$; hence if $l \geq 3$, we get $h^0(S, \mathcal{O}_S(l+2)) - \deg z' \geq 12$. Proposition 3.2 thus applies with $s = 6$ once $l \geq 3$ and this says that the cycle $z' - 2h_3 = z + h_3 - 2h_3 = z - h_3 \in \mathrm{CH}_0(S)$ is effective. This concludes the proof of the induction step in the case (4.4), since for $l \leq 2$, (4.4) implies $d \leq 19$. □

We now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Using Proposition 4.2, we get that any effective 0-cycle of degree greater than 19 on S can be written as $z = z' \pm \lambda h_3$ in $\mathrm{CH}_0(S)$, where z' is effective of degree at most 19. To complete the proof of Theorem 4.1, it thus suffices to study the case of an effective cycle z of degree $d = 19$. Let $z' := z + 3h_3 \in \mathrm{CH}_0(S)$. This is an effective 0-cycle of degree $28 = h^0(S, \mathcal{O}_S(4)) - 3$. It follows from Lemma 3.6 that the cycle

$$z'' := 16h_3 - z' = 13h_3 - z$$

is effective of degree $20 = h^0(S, \mathcal{O}_S(3)) + 1$. We now apply Proposition 4.2 to z'', and we conclude that the cycle $z'' - h_3 = 12h_3 - z$ is effective of degree 17 on S. Theorem 4.1 is proved. □

We will next prove the following result.

Proposition 4.3. *The following statements hold.*

(a) *Let S be a smooth cubic surface over a field K of characteristic* 0 *which has an effective* 0*-cycle of degree at most* 17 *coprime to* 3*. Then S has a point of degree* 1 *or* 4*.*

(b) *Under the same assumptions, denoting by x_4 the class of an effective cycle of degree* 4*, any effective* 0*-cycle z of degree at most* 17 *on S can be written as $z = \pm z' + \alpha x_4 + \beta h_3$ in $\mathrm{CH}_0(S)$, where z' is effective of degree at most* 4*.*

Before proving Proposition 4.3, let us explain how it implies both Theorem 1.5 and Theorem 1.12 (b).

Proof of Theorems 1.5 *and* 1.12 (b). Let S be a smooth cubic surface over a field K of characteristic 0, which has an effective 0-cycle of degree coprime to 3. By Theorem 4.1, any effective 0-cycle z can be written as in (4.1), that is,

$$z = \pm z' + \gamma h_3 \quad \text{in } \mathrm{CH}_0(S), \tag{4.6}$$

where z' is effective of degree at most 18. By assumption, S has an effective 0-cycle z of degree coprime to 3, for which the cycle z' in (4.6) has degree at most 17. By Proposition 4.3 (a), we then conclude that S has a K-point or a point x_4 of degree 4, which proves Theorem 1.5.

We next prove Theorem 1.12 (b). Starting from any effective 0-cycle $z \in \mathrm{CH}_0(S)$, we write the decomposition (4.6). If $\deg z' \leq 17$, then we apply Proposition 4.3 (b) to z' and thus we get a decomposition for z' which takes the form

$$z' = \pm z'' + \alpha x_4 + \beta h_3 \quad \text{in } \mathrm{CH}_0(S), \tag{4.7}$$

where z'' is effective of degree at most 4. Combining (4.7) with (4.6), we get (1.3) for z, so the conclusion of Theorem 1.12 (b) is proved in this case.

In order to conclude the proof, it only remains to study the case where $\deg z' = 18$. In this case, we denote by x_4 the class of an effective 0-cycle of degree 4 that exists on S by Theorem 1.5 which is now proved. We observe that the cycle $z'' = z' + 2x_4 + h_3$ is effective of degree $29 = h^0(S, \mathcal{O}_S(4)) - 2$. Thus, by Lemma 3.6, the cycle

$$z''' = 16h_3 - z'' = 15h_3 - 2x_4 - z'$$

is effective of degree $48 - 29 = 19$. If we now look at the proof of Theorem 4.1, we see that it proves more precisely that effective cycles w of degree 19 on S can be written as

$$w = -w' + 12h_3 \quad \text{in } \mathrm{CH}_0(S),$$

where w' is effective of degree 17. We can then apply Proposition 4.3 (b) to $w = z'''$, and get formula (1.3) for z also in this case, so Theorem 1.12 (b) is fully proved. □

We now prove Proposition 4.3.

Proof of Proposition 4.3 (a). Given a smooth cubic surface S having a 0-cycle of degree 1 over a field K of characteristic 0, we know by Coray's Theorem 1.4 that S has a point of degree 1, 4 or 10, so we only have to study effective 0-cycles of degree 10 on S. Let $z \in \mathrm{CH}_0(S)$ be such a cycle. Then, if $z' = z + 2h_3$, we have $\deg z' = 16 = h^0(S, \mathcal{O}_S(3)) - 3$; hence, by Lemma 3.6, the cycle

$$z'' := 9h_3 - z' = 7h_3 - z$$

is effective of degree $11 = h^0(S, \mathcal{O}_S(2)) + 1$. We can thus apply the vector bundle method of Section 3 with $l = 2$. As furthermore we have $h^0(S, \mathcal{O}_S(3)) - \deg z' = 8$, we can apply Proposition 3.2 with $s = 3$ and conclude that the cycle $z'' - h_3$ is effective. As

$$\deg z'' = 8 = h^0(S, \mathcal{O}_S(2)) - 2,$$

we get an effective 0-cycle of degree 4 by applying Lemma 3.6. □

Proof of Proposition 4.3 (b). Let $z \in \mathrm{CH}_0(S)$ be effective of degree $d \leq 17$. We observe that $h^0(S, \mathcal{O}_S(3)) \geq \deg z + 2$, so by Lemma 3.6, $z' = 9h_3 - z$ is effective. As

$$\deg z' = 27 - \deg z,$$

we conclude that it suffices to prove the statement for effective 0-cycles of degree at most 13. If z has degree d with $11 \leq d \leq 13$, we have $d > h^0(S, \mathcal{O}_S(2))$, so the vector bundle strategy of Section 3 works with $l = 2$, and as we have $h^0(S, \mathcal{O}_S(3)) = 19 \geq d + 6$, Proposition 3.2 applies with $s = 3$ and shows that $z - h_3$ is effective. We are thus reduced to the case of a 0-cycle z of degree at most 10. The case of degree $d = 10$ has been treated in the course of the proof of Theorem 4.3 (a), so we only have to treat $d \leq 9$. When z is effective of degree

$d = 9$, we consider the cycle $z' = z + 2x_4$ which has degree $17 = h^0(S, \mathcal{O}_S(3)) - 2$. By Lemma 3.6, the cycle $z'' = 9h_3 - z'$ is effective and has degree 10, and we proved in the proof of Theorem 4.3 (a) that any effective 0-cycle z'' of degree 10 can be written as a sum

$$z'' = w + 2h_3 \quad \text{in } \mathrm{CH}_0(S), \tag{4.8}$$

where w is effective of degree 4.

It thus remains to study effective 0-cycles z of degree $d \leq 8$. When $6 \leq d \leq 8$, the cycle $4h_3 - z$ is effective of degree at most 6 by Lemma 3.6, so we are reduced to the case of an effective cycle z of degree at most 6. In the case where $d = 5$, then $z + h_3$ has degree $8 = h^0(S, \mathcal{O}_S(2)) - 2$, so $4h_3 - (z + h_3)$ is effective of degree 4, proving the result. Finally, when $d = 6$, $z + x_4$ has degree 10, and we can use again (4.8). The proof is thus finished. □

Proof of Corollary 1.14. (a) Let S be a smooth cubic surface over a field K of characteristic 0. First of all, we prove that, as mentioned in Remark 1.13, we can in fact impose in formula (1.3) the sign $\pm$ to be positive (or negative). Indeed, we claim that if a 0-cycle z is effective of degree at most 18, then we can write $z = \lambda h_3 - z'$, where z' is effective of degree at most 18. To see this, suppose first $\deg z = 18$. As $h^0(S, \mathcal{O}_S(3)) = 19$, we get by Lemma 3.8 that the cycle $z' := 12h_3 - z$, which is also of degree 18, is effective, so the claim is proved in this case. Next, if z' is an effective 0-cycle of degree 17, Lemma 3.6 implies that $z' := 9h_3 - z$ is effective. Furthermore, if $\deg z \geq 9$, then $\deg z' \leq 18$. Finally, if $\deg z \leq 8$, then $4h_3 - z$ is effective of degree at most 12, again by application of Lemma 3.6. So the claim is proved. We then deduce from Theorem 4.1 that any effective cycle $z \in \mathrm{CH}_0(S)$ can be written as

$$z = z' + \alpha h_3 \quad \text{in } \mathrm{CH}_0(S), \tag{4.9}$$

for some integer α, where z' is effective of degree at most 18. Finally, (4.9) holds as well for any 0-cycle on S, effective or not, since an easy argument involving Riemann–Roch on curves in S shows that, for $z \in \mathrm{CH}_0(S)$, there exists a $\gamma \in \mathbb{Z}$ such that $z + \gamma h_3$ is effective. Let now $z \in \mathrm{CH}_0(S)$ be a 0-cycle with $\deg z \geq 18$. We write z as in (4.9) where z' is effective of degree at most 18. As $\deg z \geq 18$ and $\deg z' \leq 18$, we have $\gamma \geq 0$; hence γh_3 is effective and z is effective.

(b) Let S be as above and admitting a 0-cycle of degree 1. Let $z \in \mathrm{CH}_0(S)$ be a 0-cycle of degree at least 8. We first prove that, in Theorem 1.12, the sign in front of z' can be chosen positive, that is, we can write

$$z = z'' + \gamma h_3 + \delta x_4 \quad \text{in } \mathrm{CH}_0(S), \tag{4.10}$$

where z'' is effective of degree at most 4. To see this, we start from formula (1.3) and notice that the cycles z'' and x_4 can be (at least generically as in the proof of Lemma 3.6) chosen supported on a smooth curve $C \in |\mathcal{O}_S(2)|$, since $h^0(S, \mathcal{O}_S(2)) = 10$. As $g(C) = 4$, the 0-cycle $z'' := 4h_3 - x_4 - z'$ of degree 4 supported on C is effective by Riemann–Roch, so we can replace z' by $-z''$ in formula (1.3), getting (4.10) if the original sign was negative.

Assume now that $\deg z \geq 8$. Then, as $\deg z'' \leq 4$, we have $\deg(\gamma h_3 + \delta x_4) \geq 4$. Any 0-cycle of the form $\gamma h_3 + \delta x_4$ which is of degree at least 4 is effective, for the same reason as before, since we can arrange (at least for a generic choice of x_4) the class h_3 and x_4 to be supported on a smooth curve $C \subset S$ of genus 4. This implies by Riemann–Roch that z is effective for a generic choice of x_4. Using as before the specialization from the generic case, we get that any cycle z as above is effective. □

5. Zero-cycles on degree 2 del Pezzo surfaces

We prove in this section Theorem 1.15 concerning degree 2 del Pezzo surfaces. As in the cubic surface case, the key tool is the rank 2 vector bundle construction from Section 3. In that section however, Proposition 3.2 had been fully established only in the cubic surface case. Let us first prove the analogous result for degree 2 del Pezzo surface.

Proposition 5.1. *Let S be a del Pezzo surface of degree 2 over a field of characteristic 0. Let d, l, s be three nonnegative integers with $l \geq 1$, and let $z_d \in \mathrm{CH}_0(S)$, $z_s \in \mathrm{CH}_0(S)$ be two effective 0-cycles on S of respective degrees d and s. Assume the following inequalities are satisfied:*

$$h^0(S, \mathcal{O}_S(l)) = 1 + l^2 + l < d,$$
$$h^0(S, \mathcal{O}_S(l+1)) - d = 1 + (l+1)^2 + l + 1 - d \geq 2s.$$

Then the cycle $z_d - z_s$ is effective.

Proof. The assumptions are exactly the same as in Proposition 3.2 (only the Riemann–Roch formula has changed). The strategy of the proof is of course the same and the only point that needs to be checked more precisely is the analog of Lemma 3.3, the proof of which was specific to the cubic surface case. In other words, we reduced the proof to showing the following lemma.

Lemma 5.2. *Let S be a smooth degree 2 del Pezzo surface over $\mathbb{C}$, and let d, $l \geq 1$, s be three nonnegative integers satisfying the two inequalities*

$$h^0(S, \mathcal{O}_S(l)) < d,$$
$$1 + (l+1)^2 + l + 1 - d = h^0(S, \mathcal{O}_S(l+1)) - d \geq 2s.$$

Then, for a general subscheme $Z_d \subset S$ of length d, a general vector bundle E constructed from an extension class $e \in \mathrm{Ext}^1(\mathcal{I}_{Z_d}(d+1), \mathcal{O}_S)$, and general set of s points $x_1, \dots, x_s \in S$, there exists a section σ' of E vanishing at all points x_i, and whose zero-set is of dimension 0.

Proof. Mutatis mutandis, the proof is the same as that of Lemma 3.3. □

Proposition 5.1 is now proved. □

Let us now prove Theorem 1.15 (a), which is the following statement.

Theorem 5.3. *Let S be a smooth del Pezzo surface of degree 2 over a field K of characteristic 0. Then any effective 0-cycle $z \in \mathrm{CH}_0(S)$ can be written as*

$$z = z' + \gamma h_2 \quad \text{in } \mathrm{CH}_0(S), \tag{5.1}$$

where z' is effective of degree at most 13.

Proof. Let $z \in \mathrm{CH}_0(S)$ be an effective 0-cycle of degree d and let l be such that

$$h^0(S, \mathcal{O}_S(l)) = 1 + l^2 + l < d.$$

Assume first that $d \leq h^0(S, \mathcal{O}_S(l+1)) - 2$. Then $(l+1)^2 h_2 - z$ is effective thanks to Lemma 3.6, which applies since $l \geq 1$ and $\mathcal{O}_S(2)$ is very ample. Replacing z by $(l+1)^2 h_2 - z$ if necessary, we can thus assume $d \leq 2(l+1)^2 - d$. Thus, from now on, we can assume that

$$d \leq (l+1)^2. \tag{5.2}$$

Using (5.2), we get

$$h^0(S, \mathcal{O}_S(l+1)) - d \geq l + 2,$$

so once $l \geq 2$, we get $h^0(S, \mathcal{O}_S(l+1)) - d \geq 4$. We can thus apply Proposition 5.1 and thus we proved that $z - h_2$ is effective when $\deg z \geq 1 + h^0(S, \mathcal{O}_S(2)) = 8$, unless we are in one of the following cases:

$$\deg z = h^0(S, \mathcal{O}_S(l+1)) \quad \text{or} \quad \deg z = h^0(S, \mathcal{O}_S(l+1)) - 1. \tag{5.3}$$

We now treat the missing cases (5.3). Let $z' := z + h_2$ and $d' := \deg z' = d + 2$. Then

$$d' = h^0(S, \mathcal{O}_S(l+1)) + 2, \quad \text{resp. } d' = h^0(S, \mathcal{O}_S(l+1)) + 1.$$

Furthermore,

$$h^0(S, \mathcal{O}_S(l+2)) - d' = 2(l+2) - 2, \quad \text{resp. } h^0(S, \mathcal{O}_S(l+2)) - d' = 2(l+2) - 1.$$

We can thus apply Proposition 5.1 to z' with $s = 4$ once $2(l+2) - 2 \geq 8$, that is, $l \geq 3$. So we proved that, in the cases (5.3),

$$z' - 2h_2 = z - h_2$$

is effective if $l \geq 3$. When $l = 2$ and (5.3) holds, we have $d = 13$ or $d = 12$. Putting everything together, we have proved by induction that any effective cycle $z \in \mathrm{CH}_0(S)$ can be written as

$$z = \pm z' + \gamma h_2 \quad \text{in } \mathrm{CH}_0(S), \tag{5.4}$$

where z' is effective of degree at most 13. However, there is an involution of S which follows from the existence of the degree 2 morphism $S \to \mathbb{P}^2$ given by the anticanonical system. This involution ι has the property that, for any $z \in \mathrm{CH}_0(S)$,

$$z + \iota(z) = (\deg z) h_2 \quad \text{in } \mathrm{CH}_0(S). \tag{5.5}$$

Using (5.5), we can arrange to make the sign in (5.4) positive, so Theorem 5.3 is proved. □

We finally prove Theorem 1.15 (b).

Proof of Theorem 1.15 (b). By Theorem 5.3, it suffices to study the case of an effective 0-cycle z of degree $d \leq 11$. When $10 \leq d \leq 11$, we argue as follows. In this case,

$$d \leq h^0(S, \mathcal{O}_S(3)) - 2$$

so that, by Lemma 3.6, $z' := 9h_2 - z_d$ is effective, with $\deg z' < \deg z$. We are thus reduced to the case where $d = 8, 9$. As $h^0(S, \mathcal{O}_S(2)) = 7$, we can apply in these cases the vector bundle method of Section 3 with $l = 2$. As we have $h^0(S, \mathcal{O}_S(3)) = 13$, we get in both cases $h^0(S, \mathcal{O}_S(3)) - d \geq 4$; hence Proposition 5.1 tells that, in both cases, $z - h_2$ is effective of degree at most 7. □

6. Zero-cycles on del Pezzo surfaces of degree 1

In the case of a del Pezzo surface S of degree $d_S = 1$, using again the notation

$$-K_S =: \mathcal{O}_S(1),$$

we have

$$h^0(S, \mathcal{O}_S(l)) = 1 + \frac{1}{2}(l^2 + l)$$

for $l \geq 0$. Note that S has a point, defined as the base-locus of the linear system $|-K_S|$. As usual, we will denote its class by $h_1 \in \mathrm{CH}_0(S)$. We note that, in this case, the line bundle $\mathcal{O}_S(3)$ is very ample, as one sees by looking at its restrictions to the elliptic curves in the pencil $|\mathcal{O}_S(1)|$. The line bundle $\mathcal{O}_S(2)$ is generated by sections and induces a degree 2 morphism onto a surface of degree 2 in $\mathbb{P}^3$, which has to be a singular quadric.

The analog of Proposition 3.2 is now as follows.

Proposition 6.1. *Let S be a del Pezzo surface of degree 1 defined over a field K of characteristic 0. Let $d, l \geq 1$, s be three integers such that*

$$h^0(S, \mathcal{O}_S(l)) = 1 + \frac{1}{2}(l^2 + l) < d,$$
$$h^0(S, \mathcal{O}_S(l+1)) \geq d + 2s.$$

Then, for any effective 0-cycle $z_d \in \mathrm{CH}_0(S)$, the cycle $z_d - sh_1$ is effective.

The proof is similar to the proof of Proposition 3.2 and we will not repeat the details here. We will now prove Theorem 1.17 (b), which is the following statement.

Theorem 6.2. *Let S be a smooth degree 1 del Pezzo surface over a field K of characteristic 0. Then any effective 0-cycle $z \in \mathrm{CH}_0(S)$ can be written as*

$$z = \pm z' + \gamma h_1,$$

where z' is effective of degree 15, 10, 7, 6 or at most 4.

Proof. Let $z_d \in \mathrm{CH}_0(S)$ be an effective 0-cycle of degree $d \geq 5$. We choose l such that

$$h^0(S, \mathcal{O}_S(l)) < d \leq h^0(S, \mathcal{O}_S(l+1)).$$

Note that, in particular $l \geq 2$; hence $\mathcal{O}_S(l+1)$ is very ample.

If $d \leq h^0(S, \mathcal{O}_S(l+1)) - 2$, we get by Lemma 3.6 that $(l+1)^2 h_1 - z_d$ is effective, so we can assume up to replacing z_d by $(l+1)^2 h_1 - z_d$ that

$$d \leq \frac{1}{2}(l+1)^2. \tag{6.1}$$

It follows from (6.1) that

$$h^0(S, \mathcal{O}_S(l+1)) - d \geq 1 + \frac{1}{2}(l+1).$$

As $l \geq 2$, we have $1 + \frac{1}{2}(l+1) \geq 2$; hence Proposition 6.1 applies with $s = 1$ and tells that $z_d - h_1$ is effective.

We next have to consider the cases where

(1) $d = h^0(S, \mathcal{O}_S(l+1))$,

(2) $d = h^0(S, \mathcal{O}_S(l+1)) - 1 = \frac{(l+1)(l+2)}{2}$.

In case (1), we replace z_d by $z' = z_d + h_1$, and in case (2), we replace z_d by $z' = z_d + 2h_1$. In both cases, we have $\deg z' = h^0(S, \mathcal{O}_S(l+1)) + 1$. Furthermore, we have

$$h^0(S, \mathcal{O}_S(l+2)) - \deg z' = l + 1.$$

- In case (1), if $l \geq 3$, we get by Proposition 6.1 that $z' - 2h_1$ is effective; hence $z_d - h_1$ is effective.
- In case (2), if $l \geq 5$, we get by Proposition 6.1 that $z' - 3h_1$ is effective; hence $z_d - h_1$ is effective.

In conclusion, when $d \geq 5$, the only cases where we cannot conclude that $\pm z_d - h_1$ is effective of degree strictly smaller than d is when we are in case (2) with $2 \leq l \leq 4$ or in case (1) with $l = 2$. Case (2) with $l = 4, 3, 2$ provides respectively $d = 15, 10, 6$. Case (1) with $l = 2$ provides $d = 7$. So Theorem 1.17 (b) is proved. □

Proof of Theorem 1.17 (a). In view of Theorem 6.2, it suffices to prove the following.

Lemma 6.3. *Let S be a del Pezzo surface of degree* 1 *defined over a field K of characteristic* 0. *Then, for any effective cycle z of degree $d \leq 15$, we can write $z = \lambda h_1 - z'$ in* $\mathrm{CH}_0(S)$*, where $\lambda \in \mathbb{Z}$ and z' is effective of degree at most* 15.

Proof. If $\deg z = 15$, using the fact that

$$h^0(S, \mathcal{O}_S(5)) = 16$$

and Lemma 3.8, then we find that $z' := 30h_1 - z$ is effective. As $\deg z' = 15$, the lemma is proved in this case. If $10 \leq \deg z \leq 14$, Lemma 3.6 tells that $z' := 25h_1 - z$ is effective. As $\deg z' \leq 15$, the lemma is proved in this case. If $1 \leq \deg z \leq 9$, Lemma 3.6 tells that $z' := 16h_1 - z$ is effective; hence the lemma is fully proved. □

Theorem 1.17 (a) is now proved. □

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