AN APPLICATION OF WALL-CROSSING TO NOETHER–LEFSCHETZ LOCI

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Abstract. Consider a smooth projective 3-fold X satisfying the Bogomolov–Gieseker conjecture of Bayer-Macrì-Toda (such as $\mathbb{P}^3$, the quintic threefold or an abelian threefold).

Let L be a line bundle supported on a very positive surface in X. If $c_1(L)$ is a primitive cohomology class then we show it has very negative square.

1. Introduction

Let $(X, \mathcal{O}(1))$ be a smooth polarised complex threefold. For the strongest results we take $\mathcal{O}(1)$ to be primitive. Set $H := c_1(\mathcal{O}(1))$, though we do not require it to be effective.

Weak stability conditions on the derived category $\mathcal{D}(X)$ were introduced by Bayer-Macrì-Toda [BMT14]. Together with their Bogomolov-Gieseker Conjecture 3.1 below they constitute the main technique for producing Bridgeland stability conditions on threefolds.

We only need certain weakenings of the conjecture described in **(BG1)**, **(BG2)** below. They are known to hold for many threefolds [BMS16, Ko18a, Ko18b, Li19b, Li19a, MP16, Ma14, Sc14] such as $\mathbb{P}^3$ or the quintic 3-fold. We apply them to certain weak-semistable objects of $\mathcal{D}(X)$ as we move through the space of weak stability conditions. Combined with wall-crossing techniques this proves results about line bundles on surfaces in $|\mathcal{O}(n)|$.

Theorem 1.1. *Fix any irreducible divisor*[1] $\iota\colon D \hookrightarrow X$ *in* $|\mathcal{O}(n)|$ *and any line bundle* L *on* D *with* $c_1(L) \neq 0$ *in* $H^2(D, \mathbb{Q})$ *and* $c_1(L).H = 0$.

- **(A)** *If* **(BG1)** *holds on* X *and* $n \geq 4$ *then* $L^2 \;\leq\; \dfrac{-2n}{3}$.
- **(B)** *If* **(BG2)** *holds on* X *and* $n \geq 10$ *then* $L^2 \;\leq\; -2n + 4$.

See below for consequences of **(B)** on $\mathbb{P}^3$, for the observation that it is sharp, and for stronger inequalities for line bundles $L = \mathcal{L}|_D$ which are restricted from X.

It is the classes on D which are *not* restricted from X that most interest us. One obvious source of such classes is the *vanishing cycles* of D — the (co)homology classes of the Lagrangian two-spheres in D that are contracted to nodes as we deform D inside $|\mathcal{O}(n)|$ to a nodal surface. These classes all have square $-2 > \frac{-2n}{3}$ so Theorem 1.1 tells us they can never be the class of a line bundle L on D.

Corollary 1.2. *The vanishing cycles of* $D \in |\mathcal{O}(n)|$ *have empty Noether-Lefschetz loci. In fact any sum of* m *disjoint vanishing cycles has empty Noether-Lefschetz locus when*

- X *satisfies* **(BG1)**, $n \geq 4$ *and* $m \leq \lfloor \frac{n-1}{3} \rfloor$, *or*
- X *satisfies* **(BG2)**, $n \geq 10$ *and* $m \leq n - 3$.

In other words, if we look for irreducible $D \in |\mathcal{O}(n)|$ where our vanishing class has Hodge type $(1,1)$ we should find only singular D on which our cohomology class has ceased to exist (or, considered as a homology class, some part of it has vanished).

[1] D may be singular. The results also apply to D reducible, so long as L is *slope semistable* on D.

So not all classes in $H^2(D,\mathbb{Z})$ become $(1,1)$ under some deformation inside $|\mathcal{O}(n)|$, even though those which do generate $H^2(D,\mathbb{Z})$ over $\mathbb{Z}$ by [Vo07, p19].

Method. To prove Theorem 1.1 we move in a space of weak stability conditions on $\mathcal{D}(X)$, and show that if $L^2 > -2n/3$ then the Bogomolov-Gieseker inequality **(BG1)** implies $\iota_* L$ is unstable in certain regions. We find the wall on which it becomes unstable, where we show it is destabilised by a map from $\iota_* L$ to $T(-n)[1]$, for some line bundle T with torsion $c_1(T)$. Thus by relative Serre duality for the map ι,

$$\mathrm{Hom}_X(\iota_* L, T(-n)[1]) \;=\; \mathrm{Hom}_D(L, T|_D) \;\neq\; 0, \tag{1}$$

which means $L^* \otimes T|_D$ is effective.[2] Since $L.H = 0$ this implies $L = T|_D$, so, in particular $c_1(L) = 0$ in $H^2(D,\mathbb{Q})$.

Projective space. There are two different ways to saturate the inequality **(B)** on $\mathbb{P}^3$ and hence deduce it is sharp.

Firstly, we can take D to contain disjoint lines L_1, $L_2 \subset \mathbb{P}^3$. Their normal bundles inside D are $\mathcal{O}_{\mathbb{P}^1}(-n+2)$, so $L := \mathcal{O}_D(L_1 - L_2)$ satisfies $L.H = 0$ and $L^2 = -2n+4$.

Secondly, if an irreducible $D \in |\mathcal{O}_{\mathbb{P}^3}(n)|$, $n \geq 10$, contains disjoint degree $d \neq 1$ *plane curves* C_1, C_2, then **(B)** applied to $\mathcal{O}_D(C_1 - C_2)$ proves $n \geq d+2$. Thus **(B)** is saturated if $n = d+2$, and it is indeed easy to construct $D \supset C_1$, C_2 of any degree $n \geq d+2$.

More generally if $D \in |\mathcal{O}_{\mathbb{P}^3}(n)|$ contains disjoint degree d curves C_1, C_2 of genus g_1, g_2 then **(B)** applied to $\mathcal{O}_D(C_1 - C_2)$ gives $g_1 + g_2 \leq (n-4)(d-1)$ for $n \geq 10$.

Line bundles restricted from X. When $L = \mathcal{L}|_D$ extends to a line bundle $\mathcal{L}$ on X with $\mathcal{L}.H^2 = 0$ then **(A)** is trivial on *any* X. In fact $L^2 = n\mathcal{L}^2.H$ is divisible by n and < 0 by the Hodge index theorem, so

$$L^2 \;\leq\; -n. \tag{2}$$

But then if **(BG2)** holds, **(B)** gives $\mathcal{L}^2.nH \leq -2n+4$, i.e. any line bundle $\mathcal{L}$ on X satisfies

$$\mathcal{L}.H^2 \;=\; 0 \quad\Longrightarrow\quad \mathcal{L}^2.H \;\leq\; -2.$$

This appears to be nontrivial, but not very (the Hodge index theorem already gives ≤ -1). Plugging it back into the argument that gave (2) strengthens it to

$$L^2 \;\leq\; -2n. \tag{3}$$

Acknowledgements. We are most grateful to Claire Voisin. On seeing our paper she immediately saw how to produce very closely related results by more classical methods, and drafted Appendix A. We also thank Arend Bayer, Paolo Cascini, Chunyi Li, Davesh Maulik, Ivan Smith and Yukinobu Toda for useful conversations. Seven years ago Toda [To12] used very similar methods to prove the famous OSV conjecture from physics; see Section 9 for a brief discussion. The second author acknowledges support from EPSRC grant EP/R013349/1.

[2]This also shows that if $c_1(L)$ is torsion then it lifts to X. For D smooth this follows already from the Lefschetz hyperplane theorem: X is made from D by attaching $(n \geq 3)$-cells, so $H^3(X,D)$ is torsion free.

Contents

2. Weak stability conditions

In this section, we review the notion of a weak stability condition on the derived category of coherent sheaves on a smooth threefold. The main references are [BMT14, BMS16].

Let $(X, \mathcal{O}(1))$ be a smooth polarised complex threefold, and $H = c_1(\mathcal{O}(1))$. Denote the bounded derived category of coherent sheaves on X by $\mathcal{D}(X)$ and its Grothendieck group by $K(X) := K(\mathcal{D}(X))$. We define the μ_H-slope of a coherent sheaf E on X to be

$$\mu_H(E) \;:=\; \begin{cases} \frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3} & \text{if } \mathrm{ch}_0(E) \neq 0, \\ +\infty & \text{if } \mathrm{ch}_0(E) = 0. \end{cases}$$

Associated to this slope every sheaf E has a Harder-Narasimhan filtration. Its graded pieces have slopes whose maximum we denote by $\mu_H^+(E)$ and minimum by $\mu_H^-(E)$.

For any $b \in \mathbb{R}$, let $\mathcal{A}(b) \subset \mathcal{D}(X)$ denote the abelian category of complexes

$$\mathcal{A}(b) \;=\; \big\{E^{-1} \xrightarrow{d} E^0 \;:\; \mu_H^+(\ker d) \leq b\,,\; \mu_H^-(\operatorname{coker} d) > b\big\}. \tag{4}$$

Then $\mathcal{A}(b)$ is the heart of a t-structure on $\mathcal{D}(X)$ by [BMT14, Definition 3.12], for instance. Let $w \in \mathbb{R} \setminus \{0\}$. On $\mathcal{A}(b)$ we have the slope function[3]

$$N_{b,w}(E) \;:=\; \begin{cases} \frac{w\,\mathrm{ch}_2^{bH}(E).H - \frac{1}{6}w^3\,\mathrm{ch}_0(E)H^3}{w^2\,\mathrm{ch}_1^{bH}(E).H^2} & \text{if } \mathrm{ch}_1^{bH}(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^{bH}(E).H^2 = 0, \end{cases}$$

where $\mathrm{ch}^{bH}(E) := \mathrm{ch}(E)e^{-bH}$. This defines a Harder-Narasimhan filtration on $\mathcal{A}(b)$ by [BMT14, Lemma 3.2.4]. It will be convenient to replace this with

$$\nu_{b,w} \;:=\; \sigma N_{b,\sigma} + b, \quad \text{where } \sigma := \sqrt{6(w - b^2/2)}, \tag{5}$$

[3]This is called $\nu_{b,w}$ in [BMT14, Equation 7], but we reserve $\nu_{b,w}$ for its rescaling (5).

for $w > b^2/2$. This is because

$$\nu_{b,w}(E) \;=\; \begin{cases} \frac{\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1^{bH}(E).H^2} & \text{if } \mathrm{ch}_1^{bH}(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^{bH}(E).H^2 = 0 \end{cases} \tag{6}$$

has a denominator that is linear in b and numerator linear in w, so the walls of $\nu_{b,w}$-instability will turn out to be *linear*; see Proposition 4.1. Note that if $\mathrm{ch}_i(E).H^{n-i} = 0$ for $i = 0, 1, 2$, the slope $\nu_{b,w}(E)$ is defined by (6) to be $+\infty$. Since (5) only rescales and adds a constant, it defines the same Harder-Narasimhan filtration as $N_{b,\sigma}$, so it too defines a weak stability condition on $\mathcal{A}(b)$.

Definition 2.1. Fix $w > \frac{b^2}{2}$. We say $E \in \mathcal{D}(X)$ is $\nu_{b,w}$-(semi)stable if and only if

- $E[k] \in \mathcal{A}(b)$ for some $k \in \mathbb{Z}$, and
- $\nu_{b,w}(F)\,(\leq)\,\nu_{b,w}(E[k])$ for all non-trivial subobjects $F \subset E[k]$ in $\mathcal{A}(b)$.

Here $(\leq)$ denotes $<$ for stability and $\leq$ for semistability.

Remark 2.2. Given $(b, w) \in \mathbb{R}^2$ with $w > \frac{b^2}{2}$, the argument in [Br07, Propostion 5.3] describes $\mathcal{A}(b)$. It is generated by the $\nu_{b,w}$-stable two-term complexes $E = \{E^{-1} \to E^0\}$ in $\mathcal{D}(X)$ satisfying the following conditions on the denominator and numerator of $\nu_{b,w}$ (6):

(a) $\mathrm{ch}_1^{bH}(E).H^2 \geq 0$, and
(b) $\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3 \geq 0$ if $\mathrm{ch}_1^{bH}(E).H^2 = 0$.

That is, $\mathcal{A}(b)$ is the extension-closure of the set of these complexes.

3. Bogomolov-Gieseker inequality

We recall the conjectural strong Bogomolov-Gieseker inequality of [BMT14, Conjecture 1.3.1], rephrased in terms of the rescaling (5).

Conjecture 3.1. *For $\nu_{b,w}$-semistable $E \in \mathcal{A}(b)$ with $\mathrm{ch}_2^{bH}(E).H = \left(w - \frac{b^2}{2}\right)\mathrm{ch}_0(E)H^3$,*

$$\mathrm{ch}_3^{bH}(E) \;\leq\; \left(\frac{w}{3} - \frac{b^2}{6}\right)\mathrm{ch}_1^{bH}(E).H^2.$$

Although this conjecture is known not to hold for all classes on all threefolds [Sc17], it is possible it always holds for objects of the classes $\mathrm{ch}(\iota_* L)$ that we consider. In Theorem 1.1 we only need the conjecture in special cases, namely

(BG1) Conjecture 3.1 holds for sheaves of class $\mathrm{ch}(\iota_* L)$ and stability parameters $(-\frac{n}{2}, w)$ for any $w > \frac{n^2}{4} - \frac{1}{H^3}$ for fixed $n \geq 4$.

(BG2) Conjecture 3.1 holds for both
- sheaves of class $\mathrm{ch}(\iota_* L)$ and stability parameters $(-\frac{n}{2}, w)$ for any $w > \frac{n^2}{4} - \frac{3}{H^3}$ and fixed $n \geq 10$, and
- torsion-free sheaves F with $\mathrm{ch}_0(F) = 1$, $\mathrm{ch}_1(F).H^2 = 0$, $\mathrm{ch}_2(F).H \in \{-1, -2\}$, and stability parameters (b^*, w^*) with $b^* = \mathrm{ch}_2(F).H - \frac{1}{2H^3}$, $w^* = (b^*)^2 + \frac{\mathrm{ch}_2(F).H}{H^3}$.

Conjecture 3.1 is a special case of [BMS16, Conjecture 4.1], which has now been proved for

- X is projective space $\mathbb{P}^3$ [Ma14], the quadric threefold [Sc14] or, more generally, any Fano threefold of Picard rank one [Li19a],
- X an abelian threefold [MP16], a Calabi-Yau threefold of abelian type [BMS16], a Kummer threefold [BMS16], or a product of an abelian variety and $\mathbb{P}^n$ [Ko18a],
- X with nef tangent bundle [Ko18b], and
- X is a quintic threefold and (b,w) are described below [Li19b].

Theorem 3.2. [Li19b, Theorem 2.8] *Let X be a smooth quintic threefold. Then Conjecture 3.1 is true for (b,w) satisfying*

$$w \;>\; \frac{1}{2}b^2+\frac{1}{2}\big(b-\lfloor b\rfloor\big)\big(\lfloor b\rfloor-b+1\big). \tag{7}$$

In particular **(BG1)** *and* **(BG2)** *hold on X.*

Proof. Using the notation (α,β) for (w,b), [Li19b, Theorem 2.8] proves that (7) implies [BMS16, Conjecture 4.1]. This gives Conjecture 3.1, so we are left with checking that the parameters in **(BG1)**, **(BG2)** satisfy (7).

For **(BG1)** we take $n\geq 4,\ b=-\frac{n}{2}$ and $w>\frac{n^2}{4}-\frac{1}{H^3}$. Then certainly $n^2>\frac{8}{H^3}+1$, which can be rearranged to give

$$\frac{n^2}{8}+\frac{1}{8}\;<\;\frac{n^2}{4}-\frac{1}{H^3}\;<\;w. \tag{8}$$

But since $b=-\frac{n}{2}$ we have

$$\frac{1}{2}b^2+\frac{1}{2}\big(b-\lfloor b\rfloor\big)\big(\lfloor b\rfloor-b+1\big)\;\leq\;\frac{n^2}{8}+\frac{1}{8} \tag{9}$$

which by (8) gives (7).

For **(BG2)** we take $n\geq 10,\ b=-\frac{n}{2}$ and $w>\frac{n^2}{4}-\frac{3}{H^3}$. Then certainly $n^2>\frac{24}{H^3}+1$, which can be rearranged to give

$$\frac{n^2}{8}+\frac{1}{8}\;<\;\frac{n^2}{4}-\frac{3}{H^3}\;<\;w.$$

By (9) this gives (7).

For the second part of **(BG2)**, use the obvious inequality $(2\epsilon-x)(\epsilon-x)+(\epsilon-1)x>0$ for $\epsilon\in\{1,2\}$ and $x\in(0,1)$. By rearranging this is equivalent to

$$\frac{1}{2}\left(-\epsilon-\frac{x}{2}\right)^2-\epsilon x\;>\;\frac{x}{4}\left(1-\frac{x}{2}\right).$$

Substituting in $\epsilon=-\operatorname{ch}_2(F).H,\ x=\frac{1}{H^3}$ and $b^*=\operatorname{ch}_2(F).H-\frac{1}{2H^3}$ makes this

$$\frac{(b^*)^2}{2}+\frac{\operatorname{ch}_2(F).H}{H^3}\;>\;\frac{1}{4H^3}\left(1-\frac{1}{2H^3}\right).$$

For $w^*=(b^*)^2+\frac{\operatorname{ch}_2(F).H}{H^3}$ this is

$$w^*-\frac{(b^*)^2}{2}\;>\;\frac{1}{2}\left(1-\frac{1}{2H^3}\right)\frac{1}{2H^3}\;=\;\frac{1}{2}\big(b^*-\lfloor b^*\rfloor\big)\big(\lfloor b^*\rfloor-b^*+1\big),$$

i.e. the inequality (7) for (b^*, w^*) as required. $\square$

4. Wall and chamber structure

In Figure 1 we plot the (b, w)-plane simultaneously with the image of the projection map

$$\begin{aligned} \Pi\colon\ K(X) \setminus \big\{E\colon \operatorname{ch}_0(E) = 0\big\} &\longrightarrow \mathbb{R}^2, \\ E &\longmapsto \left(\frac{\operatorname{ch}_1(E).H^2}{\operatorname{ch}_0(E)H^3}\,,\, \frac{\operatorname{ch}_2(E).H}{\operatorname{ch}_0(E)H^3}\right). \end{aligned}$$

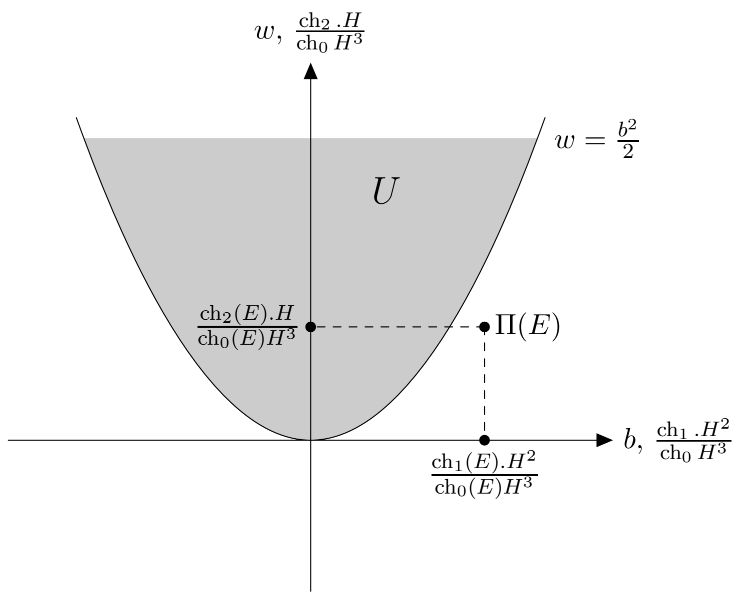

Figure 1. (b, w)-plane and the projection $\Pi(E)$

Note that for any weak stability condition $\nu_{b,w}$, the pair (b, w) is in the shaded open subset

$$U \coloneqq \left\{(b, w) \in \mathbb{R}^2 \colon w > \frac{b^2}{2}\right\}. \tag{10}$$

Conversely, the image $\Pi(E)$ of $\nu_{b,w}$-semistable objects E with $\operatorname{ch}_0(E) \neq 0$ is *outside* U,

$$\left(\frac{\operatorname{ch}_1(E).H^2}{\operatorname{ch}_0(E)H^3}\right)^2 - 2\,\frac{\operatorname{ch}_2(E).H}{\operatorname{ch}_0(E)H^3} \ \geq\ 0,$$

by [BMS16, Theorem 3.5]. That is, the left hand side is the quotient by $(\operatorname{ch}_0(E)H^3)^2 > 0$ of the H-discriminant

$$\Delta_H(E) \ =\ \big(\operatorname{ch}_1(E).H^2\big)^2 - 2(\operatorname{ch}_0(E)H^3)(\operatorname{ch}_2(E).H) \ \geq\ 0 \tag{11}$$

which [BMS16] show satisfies the classical Bogomolov-Gieseker-type inequality $\Delta_H(E) \geq 0$ when E is $\nu_{b,w}$-semistable.[4]

Proposition 4.1 (**Wall and chamber structure**). *Fix an object $E \in \mathcal{D}(X)$ such that the vector $\big(\mathrm{ch}_0(E), \mathrm{ch}_1(E).H^2, \mathrm{ch}_2(E).H\big) \neq 0$ is non-zero. There exists a locally finite collection of lines $\{\ell_i\}_{i\in I}$ in $\mathbb{R}^2$ (called "walls") which satisfies the following conditions:*

(a) *Any line ℓ_i passes through the point $\Pi(E)$ if $\mathrm{ch}_0(E) \neq 0$, or has fixed slope $\frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_1(E).H^2}$ if $\mathrm{ch}_0(E) = 0$.*

(b) *The $\nu_{b,w}$-(semi)stability of E is unchanged as (b,w) varies within any connected component (called a "chamber") of $U \setminus \bigcup_{i\in I} \ell_i$.*

(c) *For any wall ℓ_i there exists $k_i \in \mathbb{Z}$ and a map $f\colon F \to E[k_i]$ in $\mathcal{D}(X)$ such that*
   - *for any $(b,w) \in \ell_i \cap U$, the objects $E[k_i]$, F lies in the heart $\mathcal{A}(b)$,*
   - *$E[k_i]$ is $\nu_{b,w}$-semistable with $\nu_{b,w}(E) = \nu_{b,w}(F) = \mathrm{slope}\,(\ell_i)$ constant on $\ell_i \cap U$, and*
   - *f is an injection $F \subset E[k_i]$ in $\mathcal{A}(b)$ which strictly destabilises $E[k_i]$ for (b,w) in one of the two chambers adjacent to the wall ℓ_i.*

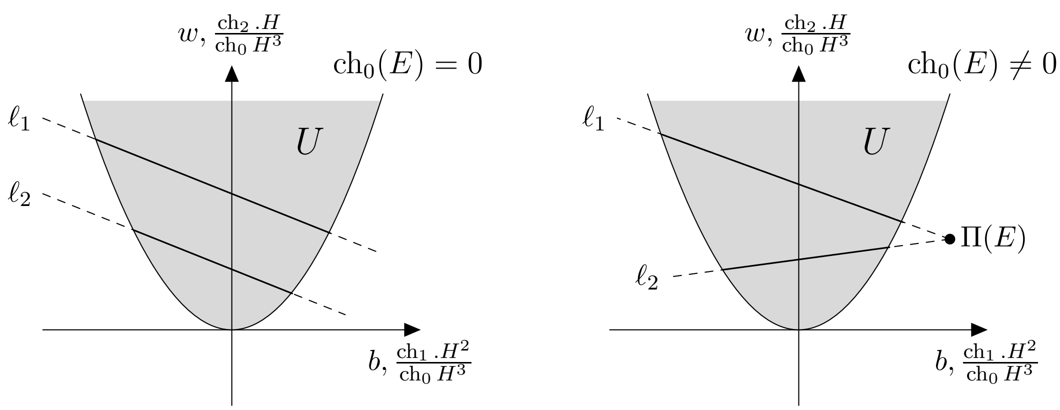

FIGURE 2. The line segments $\ell_i \cap U$ are walls for E.

Proof. For $E \in \mathcal{D}(X)$ the existence of a locally finite set of walls in the (b,w) plane follows from the arguments in [Br08, Proposition 9.3] or [BMS16, Proposition 12.5].

Suppose that E is $\nu_{b,w}$-strictly semistable. Then there is a $k \in \mathbb{Z}$ such that $E[k] \in \mathcal{A}(b)$ and a $\nu_{b,w}$-stable destabilising object $F \subset E[k]$ in $\mathcal{A}(b)$. The condition that $\nu_{b,w}(E[k]) = \nu_{b,w}(F)$ is

$$\frac{w - \frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_0(E[k])H^3}}{b - \frac{\mathrm{ch}_1(E[k]).H^2}{\mathrm{ch}_0(E[k])H^3}} = \frac{w - \frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_0(F)H^3}}{b - \frac{\mathrm{ch}_1(F).H^2}{\mathrm{ch}_0(F)H^3}} \quad \text{if} \quad \mathrm{ch}_0(E[k]) \neq 0 \neq \mathrm{ch}_0(F), \tag{12}$$

[4][BMS16, Theorem 3.5] state (11) with ch replaced by ch^{bH}, but the result is still $\Delta_H(E)$. We use the stronger Bogomolov inequality $\mathrm{ch}_1(E)^2.H - 2\,\mathrm{ch}_0(E)(\mathrm{ch}_2(E).H) \geq 0$ for μ_H-semistable sheaves in (25).

or

$$\frac{w-\frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_0(E[k])H^3}}{b-\frac{\mathrm{ch}_1(E[k]).H^2}{\mathrm{ch}_0(E[k])H^3}} \;=\; \frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_1(F).H^2} \quad \text{if} \quad \mathrm{ch}_0(E[k]) \neq 0 = \mathrm{ch}_0(F), \tag{13}$$

or

$$\frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_1(E[k]).H^2} \;=\; \frac{w-\frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_0(F)H^3}}{b-\frac{\mathrm{ch}_1(F).H^2}{\mathrm{ch}_0(F)H^3}} \quad \text{if} \quad \mathrm{ch}_0(E[k]) = 0 \neq \mathrm{ch}_0(F). \tag{14}$$

As we move through the (b,w) plane, (12) is the equation of the straight line joining $\Pi(E)$ and $\Pi(F)$, (13) is the straight line though $\Pi(E)$ of slope $\frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_1(F).H^2}$, and (14) is the line through $\Pi(F)$ of slope $\frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_1(E[k]).H^2}$. In each case the slopes of $E[k]$ and F are constant on the wall, and satisfy strict (and opposite) inequalities on the two sides of the wall. This explains the shape of the walls of instability.

If $\mathrm{ch}_0(E[k]) = 0 = \mathrm{ch}_0(F)$ we do not get a wall since both slopes remain constant as we move throughout the whole of U in the (b,w) plane.

Finally, if we move along a wall, the $\nu_{b,w}$-slopes of all the Jordan-Hölder factors of $E[k]$ coincide and remain constant. So long as they're finite, Remark 2.2 implies that the Jordan-Hölder factors remain in the heart $\mathcal{A}(b)$, and so $E[k]$ does too. If they're infinite the wall is vertical, and the category $\mathcal{A}(b)$ is constant, so the conclusion is the same. □

5. Large volume limit

As usual we consider a line bundle L on $D \in |\mathcal{O}(n)|$ such that $L.H = 0$. The Chern character of its push-forward is

$$\mathrm{ch}(\iota_* L) \;=\; \left(0,\, nH,\, \iota_*(c_1(L)) - \frac{n^2}{2}H^2,\, \frac{1}{2}L^2 + \frac{n^3}{6}H^3\right). \tag{15}$$

To move through the space U (10) of weak stability conditions, we begin in the large volume region $w \gg 0$. We use the fact that L is slope stable on D since it has no proper saturated subsheaves when D is irreducible. (The results of this paper also hold for reducible D if we *assume* that $\iota_* L$ is slope semistable.)

Lemma 5.1. *The sheaf $\iota_* L$ is $\nu_{b,w}$-semistable for any $b \in \mathbb{R}$ and $w \gg 0$.*

Proof. We sketch the proof, which is very similar to [Br08, Proposition 14.2]. The key point is that a sheaf $\iota_* E$ pushed forward from D has rank 0 so its $\nu_{b,w}$-slope (6),

$$\nu_{b,w}(\iota_* E) \;=\; \frac{\mathrm{ch}_2(\iota_* E).H}{\mathrm{ch}_1(\iota_* E).H^2} \;=\; \frac{\mathrm{ch}_1(E).H}{\mathrm{ch}_0(E)H^2} - \frac{n}{2} \;=\; \mu_H(E) - \frac{n}{2}\,, \tag{16}$$

is independent of $(b,w) \in \mathbb{R}^2$ and essentially reduces to the ordinary slope of E on D. Here the intersections take place on X in the second term and on D in the third term. (On reducible D the denominator $\mathrm{ch}_0(E)H^2$ would be replaced by the leading coefficient of the Hilbert polynomial of E.)

Fix a real number $b \in \mathbb{R}$. The sheaf $\iota_* L$ is in the heart $\mathcal{A}(b)$. Fix a subobject E_1 of $\iota_* L$ in $\mathcal{A}(b)$ with quotient E_2. Then the ordinary cohomology sheaves $\mathcal{H}^i$ of these objects sit in a long exact sequence

$$0 \longrightarrow \mathcal{H}^{-1}(E_2) \longrightarrow \mathcal{H}^0(E_1) \longrightarrow \iota_* L \longrightarrow \mathcal{H}^0(E_2) \longrightarrow 0.$$

In particular E_1 is a sheaf. Suppose first that $\operatorname{rank}(E_1) \neq 0$. Since $E_1 \in \mathcal{A}(b)$ we know $\mu_H^-(E_1) > b \Rightarrow \mu_H(E_1) > b \Rightarrow \operatorname{ch}_1^{bH}(E_1).H^2 > 0$. By (6) therefore, $+\infty > \nu_{b,w}(E_1) \to -\infty$ as $w \to \infty$, so E_1 does not destabilise for $w \gg 0$. As in [Br08, Proposition 14.2] one can in fact make the bound on w (so that E_1 does not destabilise) uniform in E_1.

If $\operatorname{rank}(E_1) = 0$ then $\mathcal{H}^{-1}(E_2) = 0$ because $E_2 \in \mathcal{A}(b)$ implies that $\mathcal{H}^{-1}(E_2)$ is a torsion-free sheaf. Therefore E_1 is a subsheaf of $\iota_* L$, which by (16) and the slope semistability of L cannot strictly $\nu_{b,w}$-destabilise $\iota_* L$. $\square$

6. The first wall

From now on we work in one of the situations

(i) suppose **(BG1)** holds, $n \geq 4$ and $L^2 \geq \left\lfloor \frac{-2n}{3} \right\rfloor + 1$, or

(ii) suppose **(BG2)** holds, $n \geq 10$ and $L^2 \geq -2n + 5$.

Then moving in the space U of weak stability conditions we will try to show that $c_1(L)$ is a torsion class in $H^2(D)$. This will prove Theorem 1.1.

By Proposition 4.1 the walls of instability for $\iota_* L$ are all lines of slope $-\frac{n}{2}$ in the (b, w) plane; see Figure 3. The lowest such line which intersects $\overline{U}$ is $w = -\frac{n}{2} b - \frac{n^2}{8}$, which is tangent to ∂U at $(-\frac{n}{2}, \frac{n^2}{8})$. Therefore the vertical line

$$b \;\equiv\; b_0 \;:=\; -\frac{n}{2} \tag{17}$$

intersects all the possible walls of instability of $\iota_* L$. We will move down this vertical line from the large volume region $w \gg 0$.

By (15), $\operatorname{ch}_2^{bH}(\iota_* L).H = 0 = \operatorname{ch}_0(\iota_* L)$ on the line $b = b_0$, so we can apply the Bogomolov-Gieseker Conjecture 3.1 for stability parameters $\left(-\frac{n}{2}, w\right)$. That is, if $\iota_* L$ is $\nu_{b_0,w}$-semistable then

$$\operatorname{ch}_3^{b_0 H}(\iota_* L) \;\leq\; \left(\frac{w}{3} - \frac{b_0^2}{6}\right) \operatorname{ch}_1^{b_0 H}(\iota_* L).H^2.$$

Using (15) and rearranging gives

$$w \;\geq\; w_f \;:=\; \frac{n^2}{4} + \frac{3L^2}{2nH^3}\,. \tag{18}$$

Note that case (i) gives $w_f > \frac{n^2}{4} - \frac{1}{H^3}$, while case (ii) gives $w_f > \frac{n^2}{4} - \frac{3n-6}{nH^3} > \frac{n^2}{4} - \frac{3}{H^3}$. In both cases then, $w_f > \frac{b_0^2}{2} = \frac{n^2}{8}$, so (b_0, w_f) lies inside U.

Therefore, when we move down the line $b = -\frac{n}{2}$, we find there is a point $w_0 \geq w_f$ where $\iota_* L$ is first destabilised. We next show that in fact $w_0 \in \left[w_f, \frac{n^2}{4}\right]$.

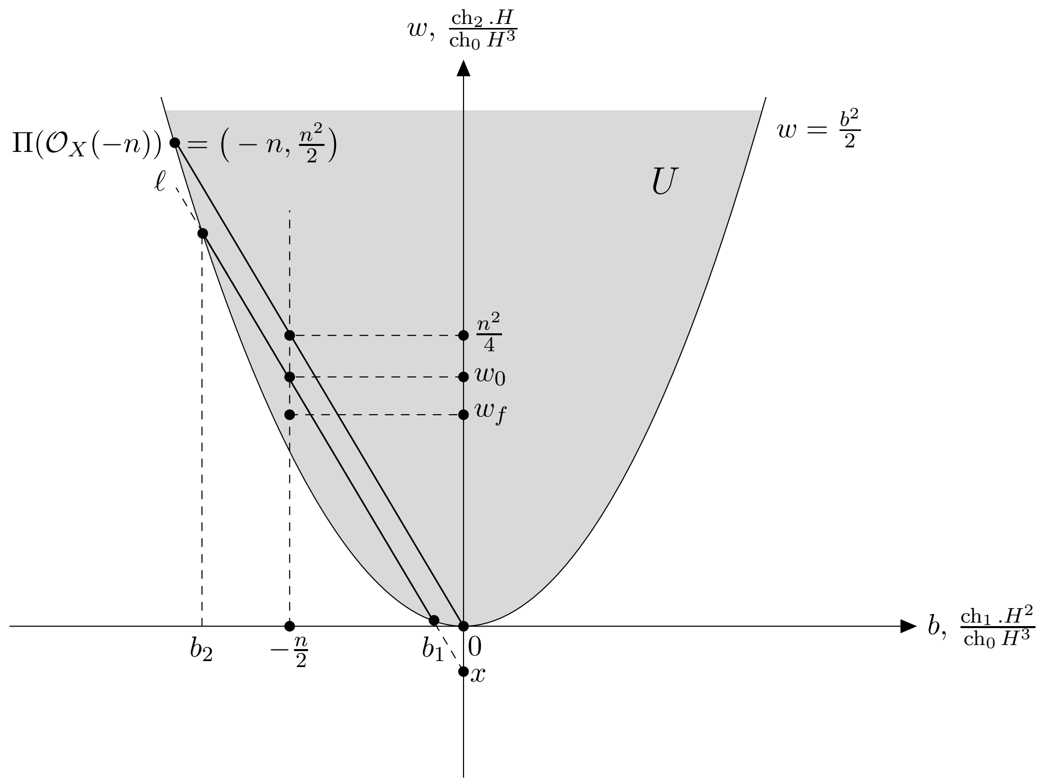

FIGURE 3. Walls for $\iota_* L$

Proposition 6.1. *There is a wall of slope $-\frac{n}{2}$ for $\iota_* L$ that bounds the large volume chamber $w \gg 0$. It passes through a point (b_0, w_0), where $w_0 \in \big[w_f, \frac{n^2}{4}\big]$.*

In the destabilising sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ in $\mathcal{A}(b_0)$, we have $\dim \operatorname{supp} \mathcal{H}^0(F_2) \leq 1$, the object F_1 is a rank one sheaf with $\mathrm{ch}_1(F_1).H^2 = 0$ and, in cases (i), (ii),

(i) $\mathrm{ch}_2(F_1).H = 0$,
(ii) $\mathrm{ch}_2(F_1).H \in \{0, -1, -2\}$.

Proof. By Proposition 4.1 and (18), $\iota_* L$ is ν_{b_0,w_0}-destabilised by a sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ in $\mathcal{A}(b_0)$ for $b_0 = -\frac{n}{2}$ and some $w_0 \geq w_f$. The corresponding wall is denoted by ℓ in Figure 3. It has equation $w = -\frac{n}{2} b + x$, where

$$x \ = \ w_0 - \frac{n^2}{4} \ \geq \ w_f - \frac{n^2}{4} \ = \ \frac{3L^2}{2nH^3}$$

satisfies

$$x \ > \ \begin{cases} -\frac{1}{H^3} & \text{in case (i)}, \\ -\frac{3}{H^3} & \text{in case (ii)}. \end{cases} \tag{19}$$

Let $b_2 < b_1$ be the values of b at the intersection points of ℓ and the boundary $w = \frac{b^2}{2}$ of U,

$$b_1 \;=\; \sqrt{\frac{n^2}{4}+2x}-\frac{n}{2}\,, \qquad b_2 \;=\; -\sqrt{\frac{n^2}{4}+2x}-\frac{n}{2}\,.$$

We claim that

$$b_1 \;>\; -\frac{1}{2H^3} \quad\text{and}\quad b_2+n \;<\; \frac{1}{2H^3}\,. \tag{20}$$

Both are equivalent to $\sqrt{\frac{n^2}{4}+2x} > \frac{n}{2}-\frac{1}{2H^3}$, and therefore to $2x > \frac{1}{4(H^3)^2}-\frac{n}{2H^3}$, which is

$$n \;>\; -4H^3x+\frac{1}{2H^3} \;=\; -\frac{6L^2}{n}+\frac{1}{2H^3}\,.$$

For (i) this follows from $L^2 \geq \lfloor -\frac{2n}{3}\rfloor+1$ and the inequality $n > -\frac{6}{n}\big(\lfloor -\frac{2n}{3}\rfloor+1\big)+\frac{1}{2H^3}$ that holds for all $n \geq 4$. For (ii) it follows from $L^2 \geq -2n+5$ and the inequality $n > 12-\frac{30}{n}+\frac{1}{2H^3}$ that holds for all $n \geq 10$.

Taking cohomology from the destabilising sequence $F_1 \hookrightarrow \iota_*L \twoheadrightarrow F_2$ gives the long exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{H}^{-1}(F_2) \longrightarrow \mathcal{H}^0(F_1) \longrightarrow \iota_*L \longrightarrow \mathcal{H}^0(F_2) \longrightarrow 0. \tag{21}$$

In particular, the destabilising subobject F_1 is a coherent sheaf. As we saw in the proof of Proposition 4.1, if it had rank 0 then its slope would be constant throughout U, like that of ι_*L, so we would not have a wall. Thus $\mathrm{ch}_0(F_1) > 0$ so (21) gives

$$\mathrm{ch}_0(\mathcal{H}^{-1}(F_2)) \;=\; \mathrm{ch}_0(F_1) \;>\; 0.$$

As in Proposition 4.1, $\Pi(F_1)$ and $\Pi(F_2)$ lie on the line ℓ. All along $\ell \cap U$ (i.e. for $b \in (b_2, b_1)$) the objects F_1 and F_2 lie in the heart $\mathcal{A}(b)$ and (semi)destabilise ι_*L. Therefore by the definition (4) of $\mathcal{A}(b)$ and the inequalities (20),

$$\mu_H^+(\mathcal{H}^{-1}(F_2)) \;\leq\; b_2 \;<\; -n+\frac{1}{2H^3} \quad\text{and}\quad \mu_H^-(F_1) \;\geq\; b_1 \;>\; \frac{-1}{2H^3}\,. \tag{22}$$

Thus dividing $(\mathrm{ch}_1(\iota_*L)-\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2 = (\mathrm{ch}_1(F_1)-\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2$ by $\mathrm{ch}_0(F_1)H^3$ gives

$$\begin{aligned} \frac{n}{\mathrm{ch}_0(F_1)}-\frac{\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2}{\mathrm{ch}_0(F_1)H^3} \;&=\; \mu_H(F_1)-\mu_H(\mathcal{H}^{-1}(F_2)) \\ &\geq\; \mu_H^-(F_1)-\mu_H^+(\mathcal{H}^{-1}(F_2)) \;\geq\; b_1-b_2 \;>\; n-\frac{1}{H^3}\,. \end{aligned} \tag{23}$$

Since $\mathcal{H}^0(F_2)$ has rank zero, $\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2 \geq 0$ so $\frac{n}{\mathrm{ch}_0(F_1)} \geq$ (23). Thus the inequalities imply $\mathrm{ch}_0(F_1) = 1$ and $\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2 = 0$. In particular, $\mathcal{H}^0(F_2)$ is supported in dimension ≤ 1.

Hence $\mu_H(F_1) = \frac{\mathrm{ch}_1(F_1).H^2}{H^3}$ is an integer multiple of $\frac{1}{H^3}$, so the inequality (22) implies that $\mu_H(F_1) \geq 0$. Similarly (22) gives $\mu_H(\mathcal{H}^{-1}(F_2)) \leq -n$ while (23) gives $\mu_H(\mathcal{H}^{-1}(F_2)) \geq -n$.

The upshot is that $\mu_H(F_1)=0$ and $\mu_H(\mathcal{H}^{-1}(F_2))=-n$. Hence $\mathrm{ch}_1(F_1).H^2=0$ and $\Pi(F_1)$ lies on the w-axis. But it also lies on the wall ℓ given by $w=\frac{-n}{2}b+x$, so

$$x \;=\; \frac{\mathrm{ch}_2(F_1).H}{\mathrm{ch}_0(F_1)H^3} \;=\; \frac{\mathrm{ch}_2(F_1).H}{H^3}\,. \tag{24}$$

Since the sheaf F_1 is ν_{b_0,w_0}-semistable, $\Pi(F_1)$ lies outside U by (11). Thus $x\le 0$ which is $w_0\le\frac{n^2}{4}$, as claimed. Combining this with (19) and (24) gives, finally,

$$0 \;\ge\; \mathrm{ch}_2(F_1).H \;>\; \begin{cases} -1 & \text{in case (i)}, \\ -3 & \text{in case (ii)}. \end{cases} \qquad \square$$

Proposition 6.2. *Under the assumptions of Proposition 6.1, the destabilising subobject F_1 of ι_*L satisfies $\mathrm{ch}_2(F_1).H=0$. That is, $x=0$, $w_0=\frac{n^2}{4}$, and the wall bounding the large volume chamber is the line of slope $-\frac{n}{2}$ through the origin.*

Proposition 6.1 proves this in case (i). We will prove Proposition 6.2 in case (ii) in Section 8 by applying the Bogomolov-Gieseker conjecture 3.1 to F_1 and F_2. This gives upper bounds for $\mathrm{ch}_3(F_1)$ and $\mathrm{ch}_3(F_2)$ respectively. In turn the latter gives a lower bound for $\mathrm{ch}_3(F_1)$. If we work only at (b_0,w_0), as in [To12], the bounds are not optimal, but by working at more general points of the (b,w)-plane we get stronger bounds which together force $\mathrm{ch}_2(F_1).H=0$.

Lemma 6.3. *Under the assumptions of Proposition 6.1,* $\dim\operatorname{supp}\mathcal{H}^0(F_2)=0$ *and*

$$\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)) \;=\; -nH \quad \text{in } H^2(X,\mathbb{Q}).$$

Proof. By Proposition 6.1, F_2 has rank 1 and lies in $\mathcal{A}(b_0)$ (4), so $\mathcal{H}^{-1}(F_2)$ is a torsion-free rank one sheaf. Therefore it is μ_H-semistable and the classical Bogomolov inequality says

$$\mathrm{ch}_1(\mathcal{H}^{-1}(F_2))^2.H-2\,\mathrm{ch}_2(\mathcal{H}^{-1}(F_2)).H \;\ge\; 0. \tag{25}$$

From the exact sequence (21) we calculate $\mathrm{ch}_i(\mathcal{H}^{-1}(F_2))=\mathrm{ch}_i(F_1)-\mathrm{ch}_i(\iota_*L)+\mathrm{ch}_i(\mathcal{H}^0(F_2))$. Taking $i=2$ and intersecting with H, Proposition 6.2 kills the first term while (15) and $L.H=0$ calculate the second, yielding

$$\mathrm{ch}_2(\mathcal{H}^{-1}(F_2)).H \;=\; \frac{n^2H^3}{2}+\mathrm{ch}_2(\mathcal{H}^0(F_2)).H. \tag{26}$$

Taking $i=1$ and intersecting with H^2, Proposition 6.1 kills the first and third terms, giving

$$\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2 \;=\; -nH^3.$$

So by the Hodge index theorem

$$n^2H^3 \;=\; \frac{\left(\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2\right)^2}{H^3} \;\ge\; \mathrm{ch}_1(\mathcal{H}^{-1}(F_2))^2.H, \tag{27}$$

with equality if and only if $\mathrm{ch}_1(\mathcal{H}^{-1}(F_2))$ *is a multiple of H in $H^2(X,\mathbb{Q})$.*

Combining (25), (26) and (27) gives

$$-\,2\,\mathrm{ch}_2(\mathcal{H}^0(F_2)).H \;\ge\; 0. \tag{28}$$

But Proposition 6.1 also showed that $\mathcal{H}^0(F_2)$ is supported in dimension ≤ 1, so (28) shows it must have 0-dimensional support and (28, 27) are equalities. Thus $\mathrm{ch}_1(\mathcal{H}^{-1}(F_2))$ is a multiple of H in $H^2(X,\mathbb{Q})$.

To determine the multiple we calculate from the sequence (21) that $\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2 = \mathrm{ch}_1(F).H^2 - \mathrm{ch}_1(\iota_* L).H^2$. The former is zero by Proposition 6.1 and the second is nH^3. $\square$

So $\mathcal{H}^0(F_2)$ is supported in dimension 0 and is a quotient of $\iota_* L$ by (21). Thus there is a 0-dimensional subscheme $Z \subset D \subset X$ such that (21) simplifies to

$$0 \longrightarrow \mathcal{H}^{-1}(F_2) \longrightarrow F_1 \longrightarrow \iota_*(L \otimes I_Z) \longrightarrow 0, \tag{29}$$

where $\mathcal{H}^{-1}(F_2)$ and F_1 are rank 1 torsion free sheaves. By Lemma 6.3 there is a dim ≤ 1 subscheme $C \subset X$ such that

$$\mathcal{H}^{-1}(F_2) \;\cong\; T(-n) \otimes I_C \tag{30}$$

for some line bundle T with $c_1(T) = 0 \in H^2(X,\mathbb{Q})$. Rotating the exact triangle (29), we get a short exact sequence in $\mathcal{A}(b_0)$:

$$0 \longrightarrow F_1 \longrightarrow \iota_*(L \otimes I_Z) \longrightarrow T(-n) \otimes I_C[1] \longrightarrow 0. \tag{31}$$

In fact any rank zero sheaf such as $\iota_*(L \otimes I_Z)$ lies in the heart $\mathcal{A}(b_0)$. Since $T(-n)$ is a line bundle, it is a μ_H-semistable sheaf of the same slope as $\mathcal{H}^{-1}(F_2)$, and thus its shift by [1] lies in $\mathcal{A}(b_0)$ because F_2 does. By the same reasoning,

$$0 \longrightarrow T(-n) \otimes \mathcal{O}_C \to T(-n) \otimes I_C[1] \longrightarrow T(-n)[1] \longrightarrow 0 \tag{32}$$

is also a short exact sequence in $\mathcal{A}(b_0)$.

7. Proof of main Theorem

We are now ready to prove Theorem 1.1. We compose the $\mathcal{A}(b_0)$-surjections (the third arrows) of (31) and (32) to give

$$\iota_*(L \otimes I_Z) \longrightarrow T(-n)[1].$$

Since this is a surjection in $\mathcal{A}(b_0)$, it is a nonzero element of

$$\mathrm{Ext}^1(\iota_*(L \otimes I_Z), T(-n)) \;\cong\; \mathrm{Ext}^1(\iota_* L, T(-n)) \;\cong\; \mathrm{Hom}(L, T|_D). \tag{33}$$

(The first isomorphism follows from $\mathrm{Ext}^{<3}(\mathcal{O}_Z, T(-n)) = 0$, by $\dim Z = 0$, and the second from relative Serre duality for ι.) Thus $L^* \otimes T|_D$ is effective. Since $L.H = 0$ this implies $L = T|_D$. In particular, $c_1(L) = 0$ in $H^2(D,\mathbb{Q})$. $\square$

Remark 7.1. In fact, calculating $\mathrm{ch}_2(F_1).H$ from (29) and (30) gives $-H.C$, which by Proposition 6.2 is zero. Therefore both C and Z are 0-dimensional and the $\nu_{b,w}$ slopes of $T(-n) \otimes I_C$ and $\iota_*(L \otimes I_Z)$ are the same as those of $T(-n)$ and $\iota_* L$ respectively. Thus the map $\iota_* L \to T(-n)[1]$ produced in (33) also destabilises in $\mathcal{A}(b)$ on the first wall. That is,

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

– tensored with T and rotated – gives the destabilising short exact sequence in $\mathcal{A}(b)$.

8. Destabilising objects in case (ii)

What remains is to prove Proposition 6.2 in case (ii). So we assume **(BG2)** holds, $n \geq 10$ and $L^2 \geq -2n+5$. By Proposition 6.1, in $\mathcal{A}(b)$ there is a a destabilising sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ for $\iota_* L$ along the wall ℓ with equation

$$w \;=\; -\frac{n}{2}b + \frac{\operatorname{ch}_2(F_1).H}{H^3}\,.$$

Moreover $\operatorname{rank} F_1 = 1 = -\operatorname{rank} F_2$, and, by Proposition 6.1,

$$\operatorname{ch}_1(F_1).H^2 = 0 \quad \text{and} \quad \operatorname{ch}_2(F_1).H \in \{0,-1,-2\}.$$

We will assume that $\operatorname{ch}_2(F_1).H \neq 0$ and apply the Bogomolov-Gieseker inequality to F_1 and F_2 to get a contradiction.

It will be convenient to work with $b = b_1 := -\frac{1}{H^3}$ because then, by (6),

$$\nu_{b_1,w}(E) \;=\; \frac{\operatorname{ch}_2(E).H - w\operatorname{ch}_0(E)H^3}{\operatorname{ch}_1(E).H^2 + \operatorname{ch}_0(E)}$$

has a denominator $D_1(E) := \operatorname{ch}_1(E).H^2 + \operatorname{ch}_0(E)$ which

- is integral and ≥ 0 for $E \in \mathcal{A}(b_1)$,
- is additive on K-theory classes: $D_1(E_1+E_2) = D_1(E_1) + D_1(E_2)$, and
- takes the minimal nonzero value 1 on F_1.

This means that in $\mathcal{A}(b_1)$ the object F_1 can only be destabilised by objects with denominator $D_1 = 0$.[5] Such objects have $\nu = +\infty$ so, in particular, F_1 can never be semi-destabilised: it is either stable or strictly unstable, and has no walls of instability. Since it is semistable on ℓ, and this intersects $b = b_1$ at the point

$$w_1 \;=\; \frac{n}{2H^3} + \frac{\operatorname{ch}_2(F_1).H}{H^3}$$

which defines a stability condition in U by

$$w_1 - \frac{b_1^2}{2} \;=\; \frac{n}{2H^3} - \frac{1}{2(H^3)^2} + \frac{\operatorname{ch}_2(F_1).H}{H^3} \;\geq\; \frac{nH^3-1}{2(H^3)^2} \;>\; 0,$$

we conclude the following.

Lemma 8.1. *The destabilising sheaf F_1 is $\nu_{b_1,w}$-stable for any $w > \frac{b_1^2}{2}$.* □

Similarly if we work with $b = b_2 := -n + \frac{1}{H^3}$ then the denominator of $\nu_{b_2,w}$ is

$$D_2(E) \;:=\; \operatorname{ch}_1(E(n)).H^2 - \operatorname{ch}_0(E).$$

This has the same properties as $D_1(E)$, except the third is replaced now by $D_1(F_2) = 1$ being minimal. Again ℓ intersects $b = b_2$ in a point

$$w_2 \;=\; \frac{n^2}{2} - \frac{n}{2H^3} + \frac{\operatorname{ch}_2(F_1).H}{H^3}$$

[5] This argument is familiar from the analogous fact that rank 1 sheaves can only be destabilised by rank 0 torsion sheaves when working with slope (for which the denominator is rank).

inside the space U of stability conditions, by

$$w_2 - \frac{b_2^2}{2} \;=\; \frac{n}{2H^3} - \frac{1}{2(H^3)^2} + \frac{\mathrm{ch}_2(F_1).H}{H^3} \;\geq\; \frac{nH^3-1}{2(H^3)^2} \;>\; 0.$$

So the same argument as for Lemma 8.1 gives the following.

Lemma 8.2. *The destabilising quotient F_2 is $\nu_{b_2,w}$-stable for any $w > \frac{b_2^2}{2}$.* □

Proposition 8.3. $\mathrm{ch}_3(F_1) \;\leq\; \dfrac{2}{3}\,\mathrm{ch}_2(F_1).H\left(\mathrm{ch}_2(F_1).H - \dfrac{1}{2H^3}\right).$

Proof. Recall the line $\{b = b_1\} \cap U$ used in Lemma 8.1. Its base on $w = \frac{b^2}{2}$ is the point $\left(-\frac{1}{H^3}, \frac{1}{2(H^3)^2}\right)$. Let ℓ_2 denote the line connecting this point to $\Pi(F_1) = \left(0, \frac{\mathrm{ch}_2(F_1).H}{H^3}\right)$,

$$w \;=\; \left(\mathrm{ch}_2(F_1).H - \frac{1}{2H^3}\right)b + \frac{\mathrm{ch}_2(F_1).H}{H^3}\,. \tag{34}$$

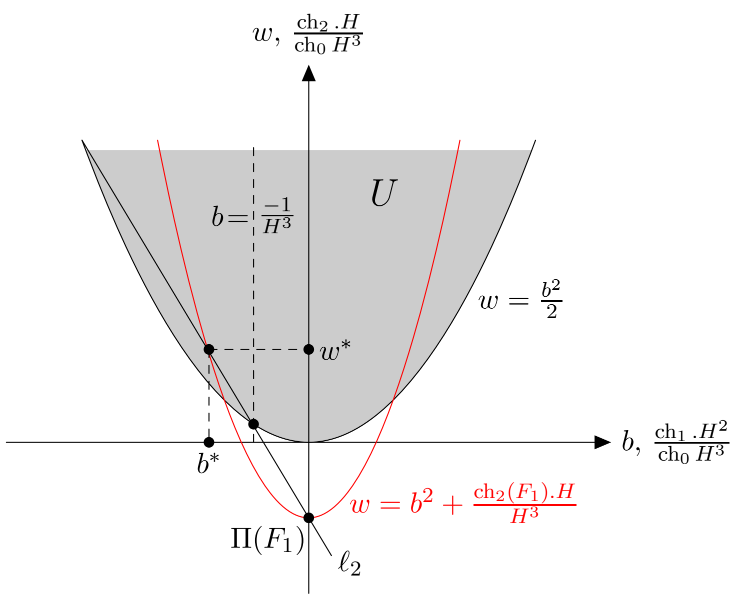

FIGURE 4. The first wall for the sheaf F_1

By the description of the walls of instability (Proposition 4.1), the $w \downarrow \frac{b_1^2}{2}$ limit of Lemma 8.1 therefore shows that F_1 is $\nu_{b,w}$-semistable for any $(b,w) \in \ell_2 \cap U$; see Figure 4.

To apply the Bogomolov-Gieseker Conjecture 3.1 to F_1 on ℓ_2 we need to find a point of $\ell_2 \cap U$ satisfying $\mathrm{ch}_2^{bH}(F_1).H = \left(w - \frac{b^2}{2}\right)\mathrm{ch}_0(F_1)H^3$, i.e.

$$\frac{\mathrm{ch}_2(F_1).H}{H^3} + \frac{b^2}{2} \;=\; w - \frac{b^2}{2}\,.$$

This intersects ℓ_2 (34) at the point (b^*, w^*), where

$$b^* \;=\; \mathrm{ch}_2(F_1).H - \frac{1}{2H^3} \quad \text{and} \quad w^* \;=\; \big(\mathrm{ch}_2(F_1).H\big)^2 + \frac{1}{4(H^3)^2}\,.$$

ν_{b^*,w^*} is a stability condition since $w^* - \frac{(b^*)^2}{2} = \frac{1}{2}\big(\mathrm{ch}_2(F_1).H + \frac{1}{2H^3}\big)^2 > 0$, so by **(BG2)** we may apply Conjecture 3.1 to give

$$\begin{aligned} \mathrm{ch}_3(F_1) - b^*\,\mathrm{ch}_2(F_1).H - \frac{(b^*)^3H^3}{6} \;&\leq\; \frac{1}{3}\left(w^* - \frac{(b^*)^2}{2}\right)(-b^*H^3) \\ &=\; \frac{1}{3}\left(\frac{\mathrm{ch}_2(F_1).H}{H^3} + \frac{(b^*)^2}{2}\right)(-b^*H^3). \end{aligned}$$

Simplifying gives

$$\mathrm{ch}_3(F_1) \;\leq\; \frac{2}{3}\,b^*\,\mathrm{ch}_2(F_1).H. \qquad \square$$

Proposition 8.4. $\mathrm{ch}_3(F_2(n)) \;\leq\; \frac{2}{3}\,\mathrm{ch}_2(F_2(n)).H\left(\mathrm{ch}_2(F_2(n)).H + \frac{1}{2H^3}\right).$

Proof. By Lemma 8.2, $F_2 \in \mathcal{A}(b_2)$ is $\nu_{b_2,w}$-semistable for $w \gg 0$. Thus $F_2(n) \in \mathcal{A}(b_2+n) = \mathcal{A}(-b_1)$ is $\nu_{-b_1,w}$-semistable for $w \gg 0$. Therefore, by [BMT14, Lemma 5.1.3(b)] the shifted derived dual $F_2(n)^\vee[1]$ lies in an exact triangle

$$F \;\hookrightarrow\; F_2(n)^\vee[1] \;\twoheadrightarrow\; Q[-1],$$

with Q a zero-dimensional sheaf and F a $\nu_{b_1,w}$-semistable object of $\mathcal{A}(b_1)$ for $w \gg 0$. Since $\mathrm{rank}\,F = 1$ it is a torsion-free sheaf by [BMS16, Lemma 2.7]. We also have $\mathrm{ch}_1(F).H^2 = \mathrm{ch}_1(F_2(n)).H^2 = 0$. Thus F has all the properties of F_1 used in Lemma 8.1 and Proposition 8.3, so the latter gives

$$\mathrm{ch}_3(F) \;\leq\; \frac{2}{3}\,\mathrm{ch}_2(F).H\left(\mathrm{ch}_2(F).H - \frac{1}{2H^3}\right).$$

Since $\mathrm{ch}_2(F_2(n)).H = -\,\mathrm{ch}_2(F).H$ and $\mathrm{ch}_3(F_2(n)) = \mathrm{ch}_3(F_2(n)^\vee[1]) = \mathrm{ch}_3(F) - \mathrm{ch}_3(Q) \leq \mathrm{ch}_3(F)$ the claim follows. $\square$

Proof of Proposition 6.2. Set $c := \mathrm{ch}_2(F_1).H \in \{0,-1,-2\}$, so by Proposition 8.3,

$$\mathrm{ch}_3(F_1) \;\leq\; \frac{2c}{3}\left(c - \frac{1}{2H^3}\right). \tag{35}$$

Using $\mathrm{ch}_0(F_1) = 1,\ \mathrm{ch}_1(F_1).H^2 = 0$ and the exact triangle $F_1 \to \iota_*L \to F_2$ we compute

$$\mathrm{ch}_1(F_2(n)).H^2 \;=\; 0, \quad \mathrm{ch}_2(F_2(n)).H \;=\; -c \quad \text{and} \quad \mathrm{ch}_3(F_2(n)) \;=\; -nc - \mathrm{ch}_3(F_1) + \frac{L^2}{2}\,.$$

The inequality of Proposition 8.4 therefore becomes

$$-nc - \mathrm{ch}_3(F_1) + \frac{L^2}{2} \;\leq\; -\frac{2c}{3}\left(-c + \frac{1}{2H^3}\right).$$

Combined with (35) and our assumption $L^2 > -2n+4$ this gives

$$-n(c+1)+2 \;<\; -nc+\frac{L^2}{2} \;\le\; \mathrm{ch}_3(F_1)-\frac{2c}{3}\left(-c+\frac{1}{2H^3}\right) \;\le\; \frac{4c}{3}\left(c-\frac{1}{2H^3}\right).$$

If $c=-1$ this gives the contradiction $2<\frac{4}{3}+\frac{2}{3H^3}$. If $c=-2$ we get $n+2<\frac{16}{3}+\frac{4}{3H^3}<7$ but $n\ge 10$. So $c=0$. □

9. Curve counting

The results of this paper are a special case of the results in [FT19], which in turn builds on [GST14]. Consider 2-dimensional torsion sheaves of the form $\iota_*(L\otimes I_C)$, where $C\subset D\subset X$ is a subscheme of dimension ≤ 1 and $I_C\subset\mathcal{O}_D$ is its ideal sheaf. We take $D\in|\mathcal{O}(n)|$, $L.H=0$ and n is sufficiently large as in this paper; the main difference in [FT19] is that we allow nonempty C.

We show the moduli space of slope semistable sheaves in the class of $\iota_*(L\otimes I_C)$ is isomorphic to the product of $\mathrm{Pic}_{\mathrm{tors}}(X)$ – the line bundles on X with torsion c_1 – and the moduli space of Joyce-Song pairs

$$\mathcal{O}(-n)\xrightarrow{\;s\;}\mathcal{I}_C. \tag{36}$$

Here $\mathcal{I}_C\subset\mathcal{O}_X$ is an ideal sheaf on X and $s\in H^0(\mathcal{O}(n))$ is a nonzero section with zero divisor $D\supset C$. The correspondence takes the cokernel of (36) and tensors it with a line bundle T with $c_1(T)=0$ to get a sheaf of the form $\iota_*(L\otimes I_C)$.

For $n\gg 0$ the moduli space of pairs (36) is a projective bundle over the moduli space of ideal sheaves $\mathcal{I}_C$. The fibre $\mathbb{P}\big(H^0(\mathcal{I}_C(n))\big)$ has Euler characteristic $\chi(\mathcal{I}_C(n))$. If X is a Calabi-Yau 3-fold with $H^1(\mathcal{O}_X)=0$ this gives the relation

$$\#(\text{2-dimensional sheaves}) \;=\; \#H^2(X,Z)_{\mathrm{tors}}\cdot\chi(\mathcal{I}_C(n))\cdot\#(\text{ideal sheaves}).$$

The first term is a DT invariant counting Gieseker stable sheaves[6] of the same topological type as $\iota_*(L\otimes I_C)$. The next two terms are topological constants. The final term is the DT invariant counting ideal sheaves of the topological type of $\mathcal{I}_C$.

The set of all of these DT invariants counting ideal sheaves is equivalent, by the MNOP conjecture [MNOP] (proved for most Calabi-Yau 3-folds in [PP17]), to the set of Gromov-Witten invariants of X. The upshot is that the Gromov-Witten invariants of X are governed by counts of 2-dimensional sheaves. In turn the generating series of the latter are conjectured by physicists to be mock modular forms due to *S-duality*.

Both this paper and [FT19] use very similar methods to those employed so impressively by Toda [To12] to prove the famous OSV conjecture on Calabi-Yau threefolds X with $\mathrm{Pic}\,X=\mathbb{Z}$ satisfying the Bogomolov-Gieseker conjecture. Toda also considers slope stable sheaves of dimension two and follows them down the wall $b=b_0$ (17), using the Bogomolov-Gieseker inequality to find the first wall of instability ℓ. One difference between the papers is that in Proposition 6.1 we analyse the destabilising objects F_1, F_2 along the wall ℓ, and use the fact that they lie in $\mathcal{A}(b)$ at its endpoints $\ell\cap\partial U$ to constrain $\mathrm{ch}(F_i)$. Toda works

[6] We show slope semistability is equivalent to Gieseker stability for $n\gg 0$.

only on $b = b_0$ and uses different arguments to analyse $\mathrm{ch}(F_i)$. A similar comment applies to the work in Section 8 to prove Proposition 6.2, as described in the discussion below Proposition 6.2.

The main difference between our work and Toda's is that we consider subtly different Chern characters. In [FT19] we consider two dimensional sheaves with $\mathrm{ch}_1 = nH$ and

$$\mathrm{ch}_2 \;=\; -\beta - \frac{n^2}{2}H^2$$

for $n \gg 0$ and some curve class β (in this paper ultimately $\beta = 0 \in H^4(X, \mathbb{Q})$). Toda considers $\mathrm{ch}_1 = nH$ and

$$\mathrm{ch}_2 \;=\; -\beta,$$

for fixed β and $n \gg 0$. To apply his methods to our class would require a bound like $\beta.H \geq \frac{1}{2}n^2H^3$, while his paper works in the opposite regime $\beta.H < \epsilon n^2$. As a result he manages to express counts of 2-dimensional sheaves in terms of both ideal sheaves and stable pairs, whereas for us the stable pairs are absent and the results rather different.

Appendix A. The case of $\mathbb{P}^3$

By Claire Voisin

When $X = \mathbb{P}^3$ we can prove a very similar result to **(B)** by more classical methods.

Theorem A.1. *Let D be a smooth surface of degree $n \geq 4$ in $\mathbb{P}^3$. Any nontrivial line bundle L on D with $c_1(L).H = 0$ satisfies $L^2 \leq -2n + 5$.*

Proof. The $K3$ case $n = 4$ is trivial: Riemann-Roch gives $h^0(L) + h^0(L^{-1}) = h^1(L) + 2 + \frac{L^2}{2}$ so if L is nontrivial with $L.H = 0$ this gives $0 = h^1(L) + 2 + \frac{L^2}{2}$ and so $L^2 \leq -4$.

So we can take $n \geq 5$. By Riemann-Roch,

$$h^0(L) + h^0(K_D \otimes L^{-1}) \;\geq\; \chi(L) \;=\; \chi(\mathcal{O}_D) + \frac{1}{2}L^2 - \frac{1}{2}K_D.L. \tag{37}$$

We assume for a contradiction that L is nontrivial and $L^2 \geq -2n + 6$. Using $K_D = \mathcal{O}_D(n-4)$, $L.H = 0$ and $h^1(\mathcal{O}_D) = 0$, (37) gives

$$h^0(L^{-1}(n-4)) \;\geq\; h^0(\mathcal{O}_D(n-4)) - (n-4). \tag{38}$$

Let $C := H \cap D$ be a smooth plane section and $L_C := L|_C$. Then the exact sequences $0 \to L^{-1}(i-1) \to L^{-1}(i) \to L_C^{-1}(i) \to 0$ give

$$h^0\big(L^{-1}(i)\big) - h^0\big(L^{-1}(i-1)\big) \;\leq\; h^0\big(L_C^{-1}(i)\big). \tag{39}$$

Since $h^0(L^{-1}) = 0$, summing over $1 \leq i \leq n-4$ gives

$$h^0\big(L^{-1}(n-4)\big) \;\leq\; \sum_{i=1}^{n-4} h^0\big(L_C^{-1}(i)\big). \tag{40}$$

Replacing L^{-1} by $\mathcal{O}_D$ gives equality in (39) for $1 \le i \le n-4$ by Kodaira vanishing, and so

$$h^0\big(\mathcal{O}_D(n-4)\big) - 1 \;=\; \sum_{i=1}^{n-4} h^0\big(\mathcal{O}_C(i)\big). \tag{41}$$

Comparing (38), (40) and (41) shows that $h^0\big(L_C^{-1}(i)\big) \ge h^0\big(\mathcal{O}_C(i)\big)$ for some $1 \le i \le n-4$. Since $\deg L_C = 0$ this implies $L_C = \mathcal{O}_C$ by [Ha86, Theorem 2.1, 2(b)].

By standard methods, this now implies the contradiction $L = \mathcal{O}_D$. For instance, consider the blow up $\pi\colon \widehat{D} \to D$ of D in the baselocus of a pencil of Cs, giving a fibration $p\colon \widehat{D} \to \mathbb{P}^1$. Then π^*L is trivial on the fibres, so is the pullback from $\mathbb{P}^1$ of the line bundle $p_*(\pi^*L) \cong \mathcal{O}_{\mathbb{P}^1}(d)$. Restricting π^*L to (the proper transform of) another plane section (a multisection of p) and using $L.H = 0$ shows that $d = 0$. $\square$

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