AN APPLICATION OF WALL-CROSSING TO NOETHER–LEFSCHETZ LOCI

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Abstract. Consider a smooth projective 3-fold $X$ satisfying the Bogomolov–Gieseker conjecture of Bayer-Macrì-Toda (such as $\mathbb{P}^3$, the quintic threefold or an abelian threefold).

Let $L$ be a line bundle supported on a very positive surface in $X$. If $c_1(L)$ is a primitive cohomology class then we show it has very negative square.

1. Introduction

Let $(X, O(1))$ be a smooth polarised complex threefold. For the strongest results we take $O(1)$ to be primitive. Set $H := c_1(O(1))$, though we do not require it to be effective.

Weak stability conditions on the derived category $D(X)$ were introduced by Bayer-Macrì-Toda [BMT14]. Together with their Bogomolov-Gieseker Conjecture 3.1 below they constitute the main technique for producing Bridgeland stability conditions on threefolds.

We only need certain weakenings of the conjecture described in $(BG1), (BG2)$ below. They are known to hold for many threefolds [BMS16, Ko18a, Ko18b, Li19b, Li19a, MP16, Ma14, Sc14] such as $\mathbb{P}^3$ or the quintic 3-fold. We apply them to certain weak-semistable objects of $D(X)$ as we move through the space of weak stability conditions. Combined with wall-crossing techniques this proves results about line bundles on surfaces in $|O(n)|$.

Theorem 1.1. Fix any irreducible divisor $\iota: D \hookrightarrow X$ in $|O(n)|$ and any line bundle $L$ on $D$ with $c_1(L) \neq 0$ in $H^2(D, \mathbb{Q})$ and $c_1(L).H = 0$.

\begin{itemize}
  \item[(A)] If $(BG1)$ holds on $X$ and $n \geq 4$ then $L^2 \leq -\frac{2n}{3}$.
  \item[(B)] If $(BG2)$ holds on $X$ and $n \geq 10$ then $L^2 \leq -2n + 4$.
\end{itemize}

See below for consequences of (B) on $\mathbb{P}^3$, for the observation that it is sharp, and for stronger inequalities for line bundles $L = L|D$ which are restricted from $X$.

It is the classes on $D$ which are not restricted from $X$ that most interest us. One obvious source of such classes is the vanishing cycles of $D$ — the (co)homology classes of the Lagrangian two-spheres in $D$ that are contracted to nodes as we deform $D$ inside $|O(n)|$ to a nodal surface. These classes all have square $-2 > -\frac{2n}{3}$ so Theorem 1.1 tells us they can never be the class of a line bundle $L$ on $D$.

Corollary 1.2. The vanishing cycles of $D \in |O(n)|$ have empty Noether-Lefschetz loci. In fact any sum of $m$ disjoint vanishing cycles has empty Noether-Lefschetz locus when

- $X$ satisfies $(BG1)$, $n \geq 4$ and $m \leq \lfloor \frac{n-1}{3} \rfloor$, or
- $X$ satisfies $(BG2)$, $n \geq 10$ and $m \leq n - 3$.

In other words, if we look for irreducible $D \in |O(n)|$ where our vanishing class has Hodge type $(1,1)$ we should find only singular $D$ on which our cohomology class has ceased to exist (or, considered as a homology class, some part of it has vanished).

\footnote{$D$ may be singular. The results also apply to $D$ reducible, so long as $L$ is slope semistable on $D$.}
So not all classes in $H^2(D,\mathbb{Z})$ become $(1, 1)$ under some deformation inside $|\mathcal{O}(n)|$, even though those which do generate $H^2(D,\mathbb{Z})$ over $\mathbb{Z}$ by [Vo07, p19].

**Method.** To prove Theorem 1.1 we move in a space of weak stability conditions on $D(X)$, and show that if $L^2 > -2n/3$ then the Bogomolov-Gieseker inequality (BG1) implies $\iota_* L$ is unstable in certain regions. We find the wall on which it becomes unstable, where we show it is destabilised by a map from $\iota_* L$ to $T(-n)[1]$, for some line bundle $T$ with torsion $c_1(T)$. Thus by relative Serre duality for the map $\iota$,

$$\text{Hom}_X(\iota_* L, T(-n)[1]) = \text{Hom}_D(L, |T|)_D 
eq 0,$$

which means $L^* \otimes T|_D$ is effective.\(^2\) Since $L.H = 0$ this implies $L = T|_D$, so, in particular $c_1(L) = 0$ in $H^2(D,\mathbb{Q})$.

**Projective space.** There are two different ways to saturate the inequality (B) on $\mathbb{P}^3$ and hence deduce it is sharp.

Firstly, we can take $D$ to contain disjoint lines $L_1, L_2 \subset \mathbb{P}^3$. Their normal bundles inside $D$ are $\mathcal{O}_{\mathbb{P}^3}(−n+2)$, so $L := \mathcal{O}_D(L_1 - L_2)$ satisfies $L.H = 0$ and $L^2 = -2n + 4$.

Secondly, if an irreducible $D \in |\mathcal{O}_{\mathbb{P}^3}(n)|$, $n \geq 10$, contains disjoint degree $d \neq 1$ plane curves $C_1, C_2$, then (B) applied to $\mathcal{O}_D(C_1 - C_2)$ proves $n \geq d + 2$. Thus (B) is saturated if $n = d + 2$, and it is indeed easy to construct $D \supset C_1, C_2$ of any degree $n \geq d + 2$.

More generally if $D \in |\mathcal{O}_{\mathbb{P}^3}(n)|$ contains disjoint degree $d$ curves $C_1, C_2$ of genus $g_1, g_2$ then (B) applied to $\mathcal{O}_D(C_1 - C_2)$ gives $g_1 + g_2 \leq (n - 4)(d - 1)$ for $n \geq 10$.

**Line bundles restricted from $X$.** When $L = \mathcal{L}|_D$ extends to a line bundle $\mathcal{L}$ on $X$ with $\mathcal{L}.H^2 = 0$ then (A) is trivial on any $X$. In fact $L^2 = n\mathcal{L}^2.H$ is divisible by $n$ and $< 0$ by the Hodge index theorem, so

$$L^2 \leq -n.\leqno{(2)}$$

But then if (BG2) holds, (B) gives $\mathcal{L}^2.nH \leq -2n + 4$, i.e. any line bundle $\mathcal{L}$ on $X$ satisfies

$$\mathcal{L}.H^2 = 0 \implies \mathcal{L}^2.H \leq -2.\leqno{(3)}$$

This appears to be nontrivial, but not very (the Hodge index theorem already gives $\leq 1$). Plugging it back into the argument that gave (2) strengthens it to

$$L^2 \leq -2n.\leqno{(3)}$$

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\(^2\)This also shows that if $c_1(L)$ is torsion then it lifts to $X$. For $D$ smooth this follows already from the Lefschetz hyperplane theorem: $X$ is made from $D$ by attaching $(n \geq 3)$-cells, so $H^3(X, D)$ is torsion free.
2. Weak stability conditions

In this section, we review the notion of a weak stability condition on the derived category of coherent sheaves on a smooth threefold. The main references are [BMT14, BMS16].

Let \((X, \mathcal{O}(1))\) be a smooth polarised complex threefold, and \(H = c_1(\mathcal{O}(1))\). Denote the bounded derived category of coherent sheaves on \(X\) by \(D(X)\) and its Grothendieck group by \(K(X) := K(D(X))\). We define the \(\mu_H\)-slope of a coherent sheaf \(E\) on \(X\) to be

\[
\mu_H(E) := \begin{cases} 
\frac{\text{ch}_1(E).H^2}{\text{ch}_0(E)H} & \text{if } \text{ch}_0(E) \neq 0, \\
+\infty & \text{if } \text{ch}_0(E) = 0.
\end{cases}
\]

Associated to this slope every sheaf \(E\) has a Harder-Narasimhan filtration. Its graded pieces have slopes whose maximum we denote by \(\mu_H^+(E)\) and minimum by \(\mu_H^-(E)\).

For any \(b \in \mathbb{R}\), let \(A(b) \subset D(X)\) denote the abelian category of complexes

\[
A(b) = \{ E^{-1} \overset{d}{\to} E^0 : \mu_H^+(\ker d) \leq b, \ \mu_H^-(\coker d) > b \}.
\]

Then \(A(b)\) is the heart of a t-structure on \(D(X)\) by [BMT14, Definition 3.12], for instance. Let \(w \in \mathbb{R} \setminus \{0\}\). On \(A(b)\) we have the slope function\(^3\)

\[
N_{b,w}(E) := \begin{cases} 
\frac{w \text{ch}_2^H(E).H - \frac{w^3}{2} \text{ch}_0(E)H^3}{w^2 \text{ch}_1^H(E).H^2} & \text{if } \text{ch}_1^H(E).H^2 \neq 0, \\
+\infty & \text{if } \text{ch}_1^H(E).H^2 = 0,
\end{cases}
\]

where \(\text{ch}_1^H(E) := \text{ch}(E)e^{-bH}\). This defines a Harder-Narasimhan filtration on \(A(b)\) by [BMT14, Lemma 3.2.4]. It will be convenient to replace this with

\[
\nu_{b,w} := \sigma N_{b,\sigma} + b, \quad \text{where } \sigma := \sqrt{6(w - b^2/2)},
\]

\(^3\)This is called \(\nu_{b,w}\) in [BMT14, Equation 7], but we reserve \(\nu_{b,w}\) for its rescaling (5).
for \( w > b^2/2 \). This is because

\[
\nu_{b,w}(E) = \begin{cases} \frac{\text{ch}_2(E).H - w \cdot \text{ch}_0(E)H^3}{\text{ch}_1^H(E).H^2} & \text{if } \text{ch}_1^H(E).H^2 \neq 0, \\ +\infty & \text{if } \text{ch}_1^H(E).H^2 = 0 \end{cases}
\]

has a denominator that is linear in \( b \) and numerator linear in \( w \), so the walls of \( \nu_{b,w} \)-instability will turn out to be linear; see Proposition 4.1. Note that if \( \text{ch}_1(E).H^{n-i} = 0 \) for \( i = 0, 1, 2 \), the slope \( \nu_{b,w}(E) \) is defined by (6) to be +\( \infty \). Since (5) only rescales and adds a constant, it defines the same Harder-Narasimhan filtration as \( N_{b,\sigma} \), so it too defines a weak stability condition on \( \mathcal{A}(b) \).

**Definition 2.1.** Fix \( w > \frac{b^2}{2} \). We say \( E \in \mathcal{D}(X) \) is \( \nu_{b,w} \)-(semi)stable if and only if

- \( E[k] \in \mathcal{A}(b) \) for some \( k \in \mathbb{Z} \), and
- \( \nu_{b,w}(F)(\leq) \nu_{b,w}(E[k]) \) for all non-trivial subobjects \( F \subset E[k] \) in \( \mathcal{A}(b) \).

Here \( (\leq) \) denotes \( < \) for stability and \( \leq \) for semistability.

**Remark 2.2.** Given \((b, w) \in \mathbb{R}^2 \) with \( w > \frac{b^2}{2} \), the argument in [Br07, Proposition 5.3] describes \( \mathcal{A}(b) \). It is generated by the \( \nu_{b,w} \)-stable two-term complexes \( E = \{E^{-1} \rightarrow E^0\} \) in \( \mathcal{D}(X) \) satisfying the following conditions on the denominator and numerator of \( \nu_{b,w} \) (6):

(a) \( \text{ch}_1^H(E).H^2 \geq 0 \), and
(b) \( \text{ch}_2(E).H - w \cdot \text{ch}_0(E)H^3 \geq 0 \) if \( \text{ch}_1^H(E).H^2 = 0 \).

That is, \( \mathcal{A}(b) \) is the extension-closure of the set of these complexes.

### 3. Bogomolov-Gieseker inequality

We recall the conjectural strong Bogomolov-Gieseker inequality of [BMT14, Conjecture 1.3.1], rephrased in terms of the rescaling (5).

**Conjecture 3.1.** For \( \nu_{b,w} \)-semistable \( E \in \mathcal{A}(b) \) with \( \text{ch}_2^b(E).H = \left( w - \frac{b^2}{2} \right) \cdot \text{ch}_0(E)H^3 \),

\[
\text{ch}_1^H(E) \leq \left( \frac{w}{3} - \frac{b^2}{6} \right) \cdot \text{ch}_1^H(E).H^2.
\]

Although this conjecture is known not to hold for all classes on all threefolds [Sc17], it is possible it always holds for objects of the classes \( \text{ch}(\ell, L) \) that we consider. In Theorem 1.1 we only need the conjecture in special cases, namely

(BG1) Conjecture 3.1 holds for sheaves of class \( \text{ch}(\ell, L) \) and stability parameters \( (-\frac{n}{2}, w) \) for any \( w > \frac{n^2}{4} - \frac{3}{4} \) for fixed \( n \geq 4 \). (BG2) Conjecture 3.1 holds for both

- sheaves of class \( \text{ch}(\ell, L) \) and stability parameters \( (-\frac{n}{2}, w) \) for any \( w > \frac{n^2}{4} - \frac{3}{4} \) for fixed \( n \geq 10 \), and
- torsion-free sheaves \( F \) with \( \text{ch}_0(F) = 1, \text{ch}_1(F).H^2 = 0, \text{ch}_2(F).H \in \{-1, -2\} \), and stability parameters \( (b^*, w^*) \) with \( b^* = \text{ch}_2(F).H - \frac{1}{H^2}, w^* = (b^*)^2 + \frac{\text{ch}_2(F).H}{H^3} \).

Conjecture 3.1 is a special case of [BMS16, Conjecture 4.1], which has now been proved for
• $X$ is projective space $\mathbb{P}^3$ [Ma14], the quadric threefold [Sc14] or, more generally, any Fano threefold of Picard rank one [Li19a],
• $X$ an abelian threefold [MP16], a Calabi-Yau threefold of abelian type [BMS16], a Kummer threefold [BMS16], or a product of an abelian variety and $\mathbb{P}^n$ [Ko18a],
• $X$ with nef tangent bundle [Ko18b], and
• $X$ is a quintic threefold and $(b, w)$ are described below [Li19b].

**Theorem 3.2.** [Li19b, Theorem 2.8] Let $X$ be a smooth quintic threefold. Then Conjecture 3.1 is true for $(b, w)$ satisfying

(7) \[ w > \frac{1}{2} b^2 + \frac{1}{2} (b - \lfloor b \rfloor) (|\lfloor b \rfloor| - b + 1). \]

In particular (BG1) and (BG2) hold on $X$.

**Proof.** Using the notation $(\alpha, \beta)$ for $(w, b)$, [Li19b, Theorem 2.8] proves that (7) implies [BMS16, Conjecture 4.1]. This gives Conjecture 3.1, so we are left with checking that the parameters in (BG1), (BG2) satisfy (7).

For (BG1) we take $n \geq 4$, $b = -\frac{n}{2}$ and $w > \frac{n^2}{4} - \frac{1}{H^3}$. Then certainly $n^2 > \frac{8}{H^3} + 1$, which can be rearranged to give

(8) \[ \frac{n^2}{8} + \frac{1}{8} < \frac{n^2}{4} - \frac{1}{H^3} < w. \]

But since $b = -\frac{n}{2}$ we have

(9) \[ \frac{1}{2} b^2 + \frac{1}{2} (b - \lfloor b \rfloor) (|\lfloor b \rfloor| - b + 1) \leq \frac{n^2}{8} + \frac{1}{8} \]

which by (8) gives (7).

For (BG2) we take $n \geq 10$, $b = -\frac{n}{2}$ and $w > \frac{n^2}{4} - \frac{3}{H^3}$. Then certainly $n^2 > \frac{24}{H^3} + 1$, which can be rearranged to give

\[ \frac{n^2}{8} + \frac{1}{8} < \frac{n^2}{4} - \frac{3}{H^3} < w. \]

By (9) this gives (7).

For the second part of (BG2), use the obvious inequality $(2\epsilon - x)(\epsilon - x) + (\epsilon - 1)x > 0$ for $\epsilon \in \{1, 2\}$ and $x \in (0, 1)$. By rearranging this is equivalent to

\[ \frac{1}{2} \left( \frac{-\epsilon - x}{2} \right)^2 - \epsilon x > \frac{x}{4} \left( 1 - \frac{x}{2} \right). \]

Substituting in $\epsilon = -\text{ch}_2(F).H$, $x = \frac{1}{H^3}$ and $b^* = \text{ch}_2(F).H - \frac{1}{2H^3}$ makes this

\[ \frac{(b^*)^2}{2} + \frac{\text{ch}_2(F).H}{H^3} > \frac{1}{4H^3} \left( 1 - \frac{1}{2H^3} \right). \]

For $w^* = (b^*)^2 + \frac{\text{ch}_2(F).H}{H^3}$ this is

\[ w^* - \frac{(b^*)^2}{2} > \frac{1}{2} \left( 1 - \frac{1}{2H^3} \right) \frac{1}{2H^3} = \frac{1}{2} (b^* - \lfloor b^* \rfloor) (|b^*| - b^* + 1), \]
i.e. the inequality (7) for \((b^*, w^*)\) as required.

\[ \square \]

4. WALL AND CHAMBER STRUCTURE

In Figure 1 we plot the \((b, w)\)-plane simultaneously with the image of the projection map

\[
P : K(X) \setminus \{ E : \text{ch}_0(E) = 0 \} \to \mathbb{R}^2, 
E \mapsto \left( \frac{\text{ch}_1(E).H^2}{\text{ch}_0(E)H^3}, \frac{\text{ch}_2(E).H}{\text{ch}_0(E)H^3} \right).
\]

\[\text{Figure 1. } (b, w)\text{-plane and the projection } \Pi(E)\]

Note that for any weak stability condition \(\nu_{b,w}\), the pair \((b, w)\) is in the shaded open subset

\[
U := \left\{ (b, w) \in \mathbb{R}^2 : w > \frac{b^2}{2} \right\}.
\]

Conversely, the image \(\Pi(E)\) of \(\nu_{b,w}\)-semistable objects \(E\) with \(\text{ch}_0(E) \neq 0\) is outside \(U\),

\[
\left( \frac{\text{ch}_1(E).H^2}{\text{ch}_0(E)H^3} \right)^2 - 2 \frac{\text{ch}_2(E).H}{\text{ch}_0(E)H^3} \geq 0,
\]

by [BMS16, Theorem 3.5]. That is, the left hand side is the quotient by \((\text{ch}_0(E)H^3)^2 > 0\) of the \(H\)-discriminant

\[
\Delta_H(E) = \left( \frac{\text{ch}_1(E).H^2}{\text{ch}_0(E)H^3} \right)^2 - 2(\text{ch}_0(E)H^3)(\text{ch}_2(E).H) \geq 0
\]
which [BMS16] show satisfies the classical Bogomolov-Gieseker-type inequality $\Delta_H(E) \geq 0$ when $E$ is $\nu_{b,w}$-semistable.\footnote{[BMS16, Theorem 3.5] state (11) with $ch$ replaced by $ch^H$, but the result is still $\Delta_H(E)$. We use the stronger Bogomolov inequality $ch_1(E)^2.H - 2ch_0(E)(ch_2(E).H) \geq 0$ for $\mu_H$-semistable sheaves in (25).}

**Proposition 4.1 (Wall and chamber structure).** Fix an object $E \in D(X)$ such that the vector $(\text{ch}_0(E), \text{ch}_1(E).H^2, \text{ch}_2(E).H) \neq 0$ is non-zero. There exists a locally finite collection of lines $\{\ell_i\}_{i \in I}$ in $\mathbb{R}^2$ (called “walls”) which satisfies the following conditions:

(a) Any line $\ell_i$ passes through the point $\Pi(E)$ if $\text{ch}_0(E) \neq 0$, or has fixed slope $\frac{\text{ch}_2(E).H}{\text{ch}_1(E).H^2}$ if $\text{ch}_0(E) = 0$.

(b) The $\nu_{b,w}$-(semi)stability of $E$ is unchanged as $(b, w)$ varies within any connected component (called a “chamber”) of $U \setminus \bigcup_{i \in I} \ell_i$.

(c) For any wall $\ell_i$ there exists $k_i \in \mathbb{Z}$ and a map $f: F \to E[k_i]$ in $D(X)$ such that
   - for any $(b, w) \in \ell_i \cap U$, the objects $E[k_i]$, $F$ lies in the heart $A(b)$,
   - $E[k_i]$ is $\nu_{b,w}$-semistable with $\nu_{b,w}(E) = \nu_{b,w}(F) = \text{slope}(\ell_i)$ constant on $\ell_i \cap U$, and
   - $f$ is an injection $F \subset E[k_i]$ in $A(b)$ which strictly destablises $E[k_i]$ for $(b, w)$ in one of the two chambers adjacent to the wall $\ell_i$.

![Figure 2. The line segments $\ell_i \cap U$ are walls for $E$.](image)

**Proof.** For $E \in D(X)$ the existence of a locally finite set of walls in the $(b, w)$ plane follows from the arguments in [Br08, Proposition 9.3] or [BMS16, Proposition 12.5].

Suppose that $E$ is $\nu_{b,w}$-strictly semistable. Then there is a $k \in \mathbb{Z}$ such that $E[k] \in A(b)$ and a $\nu_{b,w}$-stable destablising object $F \subset E[k]$ in $A(b)$. The condition that $\nu_{b,w}(E[k]) = \nu_{b,w}(F)$ is

\[
\frac{w - \frac{\text{ch}_2(E[k]).H}{\text{ch}_0(E[k]).H^2}}{b - \frac{\text{ch}_1(E[k]).H^2}{\text{ch}_0(E[k]).H^2}} = \frac{w - \frac{\text{ch}_2(F).H}{\text{ch}_0(F).H^2}}{b - \frac{\text{ch}_1(F).H^2}{\text{ch}_0(F).H^2}} \quad \text{if} \quad \text{ch}_0(E[k]) \neq 0 \neq \text{ch}_0(F),
\]

which [BMS16] show satisfies the classical Bogomolov-Gieseker-type inequality $\Delta_H(E) \geq 0$ when $E$ is $\nu_{b,w}$-semistable.
implies that the Hilbert polynomial of $E$.

(13) \[
\frac{w - \frac{\text{ch}_2(E[k]).H}{\text{ch}_0(E[k]).H^2}}{b - \frac{\text{ch}_1(E[k]).H}{\text{ch}_0(E[k]).H^2}} = \frac{\text{ch}_2(F).H}{\text{ch}_1(F).H^2} \quad \text{if} \quad \text{ch}_0(E[k]) \neq 0 = \text{ch}_0(F),
\]

or

(14) \[
\frac{\text{ch}_2(E[k]).H}{\text{ch}_1(E[k]).H^2} = \frac{w - \frac{\text{ch}_2(F).H}{\text{ch}_0(F).H^2}}{b - \frac{\text{ch}_1(F).H}{\text{ch}_0(F).H^2}} \quad \text{if} \quad \text{ch}_0(E[k]) = 0 \neq \text{ch}_0(F).
\]

As we move through the $(b, w)$ plane, (12) is the equation of the straight line joining $\Pi(E)$ and $\Pi(F)$, (13) is the straight line though $\Pi(E)$ of slope $\frac{\text{ch}_2(F).H}{\text{ch}_1(F).H^2}$, and (14) is the line through $\Pi(F)$ of slope $\frac{\text{ch}_2(E[k]).H}{\text{ch}_1(E[k]).H^2}$. In each case the slopes of $E[k]$ and $F$ are constant on the wall, and satisfy strict (and opposite) inequalities on the two sides of the wall. This explains the shape of the walls of instability.

If $\text{ch}_0(E[k]) = 0 = \text{ch}_0(F)$ we do not get a wall since both slopes remain constant as we move throughout the whole of $U$ in the $(b, w)$ plane.

Finally, if we move along a wall, the $\nu_{b,w}$-slopes of all the Jordan-Hölder factors of $E[k]$ coincide and remain constant. So long as they’re finite, Remark 2.2 implies that the Jordan-Hölder factors remain in the heart $\mathcal{A}(b)$, and so $E[k]$ does too. If they’re infinite the wall is vertical, and the category $\mathcal{A}(b)$ is constant, so the conclusion is the same. \(\square\)

5. Large volume limit

As usual we consider a line bundle $L$ on $D \in |\mathcal{O}(n)|$ such that $L.H = 0$. The Chern character of its push-forward is

(15) \[
\text{ch}(\iota_*L) = \left(0, nH, \iota_*(c_1(L)) - \frac{n^2}{2}H^2, \frac{1}{2}L^2 + \frac{n^3}{6}H^3\right).
\]

To move through the space $U$ (10) of weak stability conditions, we begin in the large volume region $w \gg 0$. We use the fact that $L$ is slope stable on $D$ since it has no proper saturated subsheaves when $D$ is irreducible. (The results of this paper also hold for reducible $D$ if we assume that $\iota_*L$ is slope semistable.)

Lemma 5.1. The sheaf $\iota_*L$ is $\nu_{b,w}$-semistable for any $b \in \mathbb{R}$ and $w \gg 0$.

Proof. We sketch the proof, which is very similar to [Br08, Proposition 14.2]. The key point is that a sheaf $\iota_*E$ pushed forward from $D$ has rank 0 so its $\nu_{b,w}$-slope (6),

(16) \[
\nu_{b,w}(\iota_*E) = \frac{\text{ch}_2(\iota_*E).H}{\text{ch}_1(\iota_*E).H^2} = \frac{\text{ch}_1(E).H}{\text{ch}_0(E).H^2} - \frac{n}{2} = \mu_H(E) - \frac{n}{2},
\]

is independent of $(b, w) \in \mathbb{R}^2$ and essentially reduces to the ordinary slope of $E$ on $D$. Here the intersections take place on $X$ in the second term and on $D$ in the third term. (On reducible $D$ the denominator $\text{ch}_0(E).H^2$ would be replaced by the leading coefficient of the Hilbert polynomial of $E$.)
Fix a real number $b \in \mathbb{R}$. The sheaf $\iota_*L$ is in the heart $\mathcal{A}(b)$. Fix a subobject $E_1$ of $\iota_*L$ in $\mathcal{A}(b)$ with quotient $E_2$. Then the ordinary cohomology sheaves $H^i$ of these objects sit in a long exact sequence

$$0 \rightarrow H^{-1}(E_2) \rightarrow H^0(E_1) \rightarrow \iota_*L \rightarrow H^0(E_2) \rightarrow 0.$$ 

In particular $E_1$ is a sheaf. Suppose first that $\operatorname{rank}(E_1) \neq 0$. Since $E_1 \in \mathcal{A}(b)$ we know $\mu_H(E_1) > b \Rightarrow \mu_H(E_1) > b \Rightarrow \operatorname{ch}_1(E_1).H^2 > 0$. By (6) therefore, $+\infty > \nu_{b,w}(E_1) \rightarrow -\infty$ as $w \rightarrow \infty$, so $E_1$ does not destabilise for $w \gg 0$. As in [Br08, Proposition 14.2] one can in fact make the bound on $w$ (so that $E_1$ does not destabilise) uniform in $E_1$.

If $\operatorname{rank}(E_1) = 0$ then $H^{-1}(E_2) = 0$ because $E_2 \in \mathcal{A}(b)$ implies that $H^{-1}(E_2)$ is a torsion-free sheaf. Therefore $E_1$ is a subsheaf of $\iota_*L$, which by (16) and the slope semistability of $L$ cannot strictly destabilise $\iota_*L$. $\square$

6. THE FIRST WALL

From now on we work in one of the situations

(i) suppose (BG1) holds, $n \geq 4$ and $L^2 \geq \left\lfloor \frac{-2n}{3} \right\rfloor + 1$, or

(ii) suppose (BG2) holds, $n \geq 10$ and $L^2 \geq -2n + 5$.

Then moving in the space $U$ of weak stability conditions we will try to show that $c_1(L)$ is a torsion class in $H^2(D)$. This will prove Theorem 1.1.

By Proposition 4.1 the walls of instability for $\iota_*L$ are all lines of slope $-\frac{n}{2}$ in the $(b, w)$ plane; see Figure 3. The lowest such line which intersects $\mathcal{U}$ is $w = -\frac{n}{2}b - \frac{n^2}{8}$, which is tangent to $\partial U$ at $\left( -\frac{n}{2}, \frac{n^2}{8} \right)$. Therefore the vertical line

$$b \equiv b_0 := -\frac{n}{2}$$

intersects all the possible walls of instability of $\iota_*L$. We will move down this vertical line from the large volume region $w \gg 0$.

By (15), $\operatorname{ch}_3^H(\iota_*L).H = 0 = \operatorname{ch}_0(\iota_*L)$ on the line $b = b_0$, so we can apply the Bogomolov-Gieseker Conjecture 3.1 for stability parameters $(-\frac{n}{2}, w)$. That is, if $\iota_*L$ is $\nu_{b_0,w}$-semistable then

$$\operatorname{ch}_3^H(\iota_*L) \leq \left( \frac{w}{3} - \frac{b_0^2}{6} \right) \operatorname{ch}_1(\iota_*L).H^2.$$ 

Using (15) and rearranging gives

$$w \geq w_f := \frac{n^2}{4} + \frac{3L^2}{2nH^3}.$$ 

Note that case (i) gives $w_f > \frac{n^2}{4} - \frac{1}{16}$, while case (ii) gives $w_f > \frac{n^2}{4} - \frac{3n-6}{nH^2} > \frac{n^2}{4} - \frac{3}{16}$. In both cases then, $w_f > \frac{b_0^2}{2} = \frac{n^2}{8}$, so $(b_0, w_f)$ lies inside $U$.

Therefore, when we move down the line $b = -\frac{n}{2}$, we find there is a point $w_0 \geq w_f$ where $\iota_*L$ is first destabilised. We next show that in fact $w_0 \in [w_f, \frac{2n}{3}]$. 


Proposition 6.1. There is a wall of slope $-\frac{n}{2}$ for $\iota_* L$ that bounds the large volume chamber $w \gg 0$. It passes through a point $(b_0, w_0)$, where $w_0 \in [w_f, \frac{n^2}{4}]$. In the destabilising sequence $F_1 \hookrightarrow \iota_* L \rightarrow F_2$ in $A(b_0)$, we have $\dim \text{supp} \mathcal{H}^0(F_2) \leq 1$, the object $F_1$ is a rank one sheaf with $\text{ch}_1(F_1).H^2 = 0$ and, in cases (i), (ii),

(i) $\text{ch}_2(F_1).H = 0$,
(ii) $\text{ch}_2(F_1).H \in \{0, -1, -2\}$.

Proof. By Proposition 4.1 and (18), $\iota_* L$ is $\nu_{b_0, w_0}$-destabilised by a sequence $F_1 \hookrightarrow \iota_* L \rightarrow F_2$ in $A(b_0)$ for $b_0 = -\frac{n}{2}$ and some $w_0 \geq w_f$. The corresponding wall is denoted by $\ell$ in Figure 3. It has equation $w = -\frac{n}{2}b + x$, where

$$x = w_0 - \frac{n^2}{4} \geq w_f - \frac{n^2}{4} = \frac{3L^2}{2nH^3}$$

satisfies

$$x > \begin{cases} -\frac{1}{H^3} & \text{in case (i)}, \\ -\frac{3}{H^3} & \text{in case (ii)}. \end{cases}$$
Let $b_2 < b_1$ be the values of $b$ at the intersection points of $\ell$ and the boundary $w = \frac{\ell^2}{2}$ of $U$, 
\[ b_1 = \sqrt{\frac{n^2}{4} + 2x - \frac{n}{2}}, \quad b_2 = -\sqrt{\frac{n^2}{4} + 2x - \frac{n}{2}}. \]
We claim that 
\begin{equation}
(20) \quad b_1 > -\frac{1}{2H^3} \quad \text{and} \quad b_2 + n < \frac{1}{2H^3}.
\end{equation}
Both are equivalent to $\sqrt{\frac{n^2}{4} + 2x} > \frac{n}{2} - \frac{1}{2H^3}$, and therefore to $2x > \frac{1}{4(H^3)^2} - \frac{n}{2H^3}$, which is 
\[ n > -4H^3x + \frac{1}{2H^3} = -\frac{6L^2}{n} + \frac{1}{2H^3}. \]

For (i) this follows from $L^2 \geq \lceil -\frac{2n}{3} \rceil + 1$ and the inequality $n > -\frac{6}{n} \left( \lceil -\frac{2n}{3} \rceil + 1 \right) + \frac{1}{2H^3}$ that holds for all $n \geq 4$. For (ii) it follows from $L^2 \geq -2n+5$ and the inequality $n > 12 - \frac{30}{n} + \frac{1}{2H^3}$ that holds for all $n \geq 10$.

Taking cohomology from the destabilising sequence $F_1 \hookrightarrow \iota_*L \twoheadrightarrow F_2$ gives the long exact sequence of coherent sheaves 
\begin{equation}
(21) \quad 0 \longrightarrow \mathcal{H}^{-1}(F_2) \longrightarrow \mathcal{H}^0(F_1) \longrightarrow \iota_*L \longrightarrow \mathcal{H}^0(F_2) \longrightarrow 0.
\end{equation}
In particular, the destabilising subobject $F_1$ is a coherent sheaf. As we saw in the proof of Proposition 4.1, if it had rank 0 then its slope would be constant throughout $U$, like that of $\iota_*L$, so we would not have a wall. Thus $\text{ch}_0(F_1) > 0$ so (21) gives 
\[ \text{ch}_0(\mathcal{H}^{-1}(F_2)) = \text{ch}_0(F_1) > 0. \]
As in Proposition 4.1, $\Pi(F_1)$ and $\Pi(F_2)$ lie on the line $\ell$. All along $\ell \cap U$ (i.e. for $b \in (b_2, b_1)$) the objects $F_1$ and $F_2$ lie in the heart $\mathcal{A}(b)$ and (semi)destabilise $\iota_*L$. Therefore by the definition (4) of $\mathcal{A}(b)$ and the inequalities (20), 
\begin{equation}
(22) \quad \mu_+^H(\mathcal{H}^{-1}(F_2)) \leq b_2 < -n + \frac{1}{2H^3} \quad \text{and} \quad \mu_-^H(F_1) \geq b_1 > -\frac{1}{2H^3}.
\end{equation}
Thus dividing $(\text{ch}_1(\iota_*L) - \text{ch}_1(\mathcal{H}^0(F_2)).H^2 = (\text{ch}_1(F_1) - \text{ch}_1(\mathcal{H}^{-1}(F_2))).H^2$ by $\text{ch}_0(F_1)H^3$ gives 
\begin{equation}
(23) \quad \frac{n}{\text{ch}_0(F_1)} - \frac{\text{ch}_1(\mathcal{H}^0(F_2)).H^2}{\text{ch}_0(F_1)H^3} = \mu_+^H(F_1) - \mu_-^H(\mathcal{H}^{-1}(F_2)) \\geq \mu_-^H(F_1) - \mu_-^H(\mathcal{H}^{-1}(F_2)) \geq b_1 - b_2 > n - \frac{1}{H^3}.
\end{equation}
Since $\mathcal{H}^0(F_2)$ has rank zero, $\text{ch}_1(\mathcal{H}^0(F_2)).H^2 \geq 0$ so $\frac{n}{\text{ch}_0(F_1)} \geq (23)$. Thus the inequalities imply $\text{ch}_0(F_1) = 1$ and $\text{ch}_1(\mathcal{H}^0(F_2)).H^2 = 0$. In particular, $\mathcal{H}^0(F_2)$ is supported in dimension $\leq 1$.

Hence $\mu_+^H(F_1) = \frac{\text{ch}_1(F_1).H^2}{H^3}$ is an integer multiple of $\frac{1}{H^2}$, so the inequality (22) implies that $\mu_+^H(F_1) \geq 0$. Similarly (22) gives $\mu_-^H(\mathcal{H}^{-1}(F_2)) \leq -n$ while (23) gives $\mu_-^H(\mathcal{H}^{-1}(F_2)) \geq -n$. 

The upshot is that $\mu_H(F_1) = 0$ and $\mu_H(\mathcal{H}^{-1}(F_2)) = -n$. Hence $\text{ch}_1(F_1).H^2 = 0$ and $\Pi(F_1)$ lies on the $w$-axis. But it also lies on the wall $\ell$ given by $w = -\frac{n^2}{2} b + x$, so

$$x = \frac{\text{ch}_2(F_1).H}{\text{ch}_0(F_1)H^3} = \frac{\text{ch}_2(F_1).H}{H^3}. \quad (24)$$

Since the sheaf $F_1$ is $\nu_{b_0,w_0}$-semistable, $\Pi(F_1)$ lies outside $U$ by (11). Thus $x \leq 0$ which is $w_0 \leq \frac{n^2}{4}$, as claimed. Combining this with (19) and (24) gives, finally,

$$0 \geq \text{ch}_2(F_1).H > \begin{cases} -1 & \text{in case (i),} \\ -3 & \text{in case (ii).} \end{cases}$$

**Proposition 6.2.** Under the assumptions of Proposition 6.1, the destabilising subobject $F_1$ of $i_*L$ satisfies $\text{ch}_2(F_1).H = 0$. That is, $x = 0$, $w_0 = \frac{n^2}{4}$, and the wall bounding the large volume chamber is the line of slope $-\frac{n^2}{2}$ through the origin.

Proposition 6.1 proves this in case (i). We will prove Proposition 6.2 in case (ii) in Section 8 by applying the Bogomolov-Gieseker conjecture 3.1 to $F_1$ and $F_2$. This gives upper bounds for $\text{ch}_3(F_1)$ and $\text{ch}_3(F_2)$ respectively. In turn the latter gives a lower bound for $\text{ch}_3(F_1)$. If we work only at $(b_0, w_0)$, as in [To12], the bounds are not optimal, but by working at more general points of the $(b, w)$-plane we get stronger bounds which together force $\text{ch}_2(F_1).H = 0$.

**Lemma 6.3.** Under the assumptions of Proposition 6.1, $\dim \text{supp} \mathcal{H}^0(F_2) = 0$ and

$$\text{ch}_1(\mathcal{H}^{-1}(F_2)) = -nH \quad \text{in } H^2(X, \mathbb{Q}).$$

**Proof.** By Proposition 6.1, $F_2$ has rank 1 and lies in $\mathcal{A}(b_0) (4)$, so $\mathcal{H}^{-1}(F_2)$ is a torsion-free rank one sheaf. Therefore it is $\mu_H$-semistable and the classical Bogomolov inequality says

$$\text{ch}_1(\mathcal{H}^{-1}(F_2))^2.H - 2 \text{ch}_2(\mathcal{H}^{-1}(F_2)).H \geq 0. \quad (25)$$

From the exact sequence (21) we calculate $\text{ch}_1(\mathcal{H}^{-1}(F_2)) = \text{ch}_1(F_1) - \text{ch}_1(i_*L) + \text{ch}_1(\mathcal{H}^0(F_2))$. Taking $i = 2$ and intersecting with $H$, Proposition 6.2 kills the first term while (15) and $L.H = 0$ calculate the second, yielding

$$\text{ch}_2(\mathcal{H}^{-1}(F_2)).H = \frac{n^2H^3}{2} + \text{ch}_2(\mathcal{H}^0(F_2)).H. \quad (26)$$

Taking $i = 1$ and intersecting with $H^2$, Proposition 6.1 kills the first and third terms, giving

$$\text{ch}_1(\mathcal{H}^{-1}(F_2)).H^2 = -nH^3.$$ 

So by the Hodge index theorem

$$n^2H^3 = \frac{(\text{ch}_1(\mathcal{H}^{-1}(F_2)).H^2)^2}{H^3} \geq \text{ch}_1(\mathcal{H}^{-1}(F_2))^2.H, \quad (27)$$

with equality if and only if $\text{ch}_1(\mathcal{H}^{-1}(F_2))$ is a multiple of $H$ in $H^2(X, \mathbb{Q})$.

Combining (25), (26) and (27) gives

$$-2 \text{ch}_2(\mathcal{H}^0(F_2)).H \geq 0. \quad (28)$$
But Proposition 6.1 also showed that $\mathcal{H}^0(F_2)$ is supported in dimension $\leq 1$, so (28) shows it must have 0-dimensional support and (28, 27) are equalities. Thus $\text{ch}_1(\mathcal{H}^{-1}(F_2))$ is a multiple of $H$ in $H^2(X, \mathbb{Q})$.

To determine the multiple we calculate from the sequence (21) that $\text{ch}_1(\mathcal{H}^{-1}(F_2)).H^2 = \text{ch}_1(F).H^2 - \text{ch}_1(\iota_*L).H^2$. The former is zero by Proposition 6.1 and the second is $nH^3$. □

So $\mathcal{H}^0(F_2)$ is supported in dimension 0 and is a quotient of $\iota_*L$ by (21). Thus there is a 0-dimensional subscheme $Z \subset D \subset X$ such that (21) simplifies to

$$0 \to \mathcal{H}^{-1}(F_2) \to F_1 \to \iota_* (L \otimes I_Z) \to 0,$$

where $\mathcal{H}^{-1}(F_2)$ and $F_1$ are rank 1 torsion free sheaves. By Lemma 6.3 there is a dim $\leq 1$ subscheme $C \subset X$ such that

$$\mathcal{H}^{-1}(F_2) \cong T(-n) \otimes I_C$$

for some line bundle $T$ with $c_1(T) = 0 \in H^2(X, \mathbb{Q})$. Rotating the exact triangle (29), we get a short exact sequence in $\mathcal{A}(b_0)$:

$$0 \to F_1 \to \iota_* (L \otimes I_Z) \to T(-n) \otimes I_C[1] \to 0.$$

In fact any rank zero sheaf such as $\iota_* (L \otimes I_Z)$ lies in the heart $\mathcal{A}(b_0)$. Since $T(-n)$ is a line bundle, it is a $\mu_H$-semistable sheaf of the same slope as $\mathcal{H}^{-1}(F_2)$, and thus its shift by [1] lies in $\mathcal{A}(b_0)$ because $F_2$ does. By the same reasoning,

$$0 \to T(-n) \otimes O_C \to T(-n) \otimes I_C[1] \to T(-n)[1] \to 0$$

is also a short exact sequence in $\mathcal{A}(b_0)$.

7. Proof of main Theorem

We are now ready to prove Theorem 1.1. We compose the $\mathcal{A}(b_0)$-surjections (the third arrows) of (31) and (32) to give

$$\iota_* (L \otimes I_Z) \to T(-n)[1].$$

Since this is a surjection in $\mathcal{A}(b_0)$, it is a nonzero element of

$$\text{Ext}^1(\iota_* (L \otimes I_Z), T(-n)) \cong \text{Ext}^1(\iota_* L, T(-n)) \cong \text{Hom}(L, T|_D).$$

(The first isomorphism follows from $\text{Ext}^{<3}(O_Z, T(-n)) = 0$, by dim $Z = 0$, and the second from relative Serre duality for $\iota$.) Thus $L^* \otimes T|_D$ is effective. Since $L.H = 0$ this implies $L = T|_D$. In particular, $c_1(L) = 0$ in $H^2(D, \mathbb{Q})$. □

Remark 7.1. In fact, calculating $\text{ch}_2(F_1).H$ from (29) and (30) gives $-H.C$, which by Proposition 6.2 is zero. Therefore both $C$ and $Z$ are 0-dimensional and the $\nu_{b,w}$ slopes of $T(-n) \otimes I_C$ and $\iota_* (L \otimes I_Z)$ are the same as those of $T(-n)$ and $\iota_* L$ respectively. Thus the map $\iota_* L \to T(-n)[1]$ produced in (33) also destabilises in $\mathcal{A}(b)$ on the first wall. That is,

$$0 \to O(-n) \to O \to O_D \to 0$$

– tensored with $T$ and rotated – gives the destabilising short exact sequence in $\mathcal{A}(b)$. 

8. Destabilising objects in case (ii)

What remains is to prove Proposition 6.2 in case (ii). So we assume (BG2) holds, 
\(n \geq 10\) and \(L^2 \geq -2n + 5\). By Proposition 6.1, in \(A(b)\) there is a a destabilising sequence 
\(F_1 \leftrightarrow \iota_* L \rightarrow F_2\) for \(\iota_* L\) along the wall \(\ell\) with equation 
\[w = \frac{n}{2} b + \frac{\text{ch}_2(F_1).H}{H^3}.\]
Moreover rank \(F_1 = 1 = -\text{rank} F_2\), and, by Proposition 6.1, 
\[\text{ch}_1(F_1).H^2 = 0 \quad \text{and} \quad \text{ch}_2(F_1).H \in \{0, -1, -2\}.\]
We will assume that \(\text{ch}_2(F_1).H \neq 0\) and apply the Bogomolov-Gieseker inequality to 
\(F_1\) and \(F_2\) to get a contradiction.

It will be convenient to work with \(b = b_1 := -\frac{1}{H^3}\) because then, by (6), 
\[\nu_{b_1,w}(E) = \frac{\text{ch}_2(E).H - w \text{ch}_0(E)H^3}{\text{ch}_1(E)H^2 + \text{ch}_0(E)}\]
has a denominator \(D_1(E) := \text{ch}_1(E)H^2 + \text{ch}_0(E)\) which
- is integral and \(\geq 0\) for \(E \in A(b_1)\),
- is additive on K-theory classes: \(D_1(E_1 + E_2) = D_1(E_1) + D_1(E_2)\), and
- takes the minimal nonzero value 1 on \(F_1\).
This means that in \(A(b_1)\) the object \(F_1\) can only be destabilised by objects with denominator \(D_1 = 0\).\(^5\) Such objects have \(\nu = +\infty\) so, in particular, \(F_1\) can never be semi-destabilised: it is either stable or strictly unstable, and has no walls of instability. Since it is semistable 
on \(\ell\), and this intersects \(b = b_1\) at the point 
\[w_1 = \frac{n}{2H^3} + \frac{\text{ch}_2(F_1).H}{H^3}\]
which defines a stability condition in \(U\) by 
\[w_1 - \frac{b_1^2}{2} = \frac{n}{2H^3} - \frac{1}{2(H^3)^2} + \frac{\text{ch}_2(F_1).H}{H^3} \geq \frac{nH^3 - 1}{2(H^3)^2} > 0,\]
we conclude the following.

**Lemma 8.1.** The destabilising sheaf \(F_1\) is \(\nu_{b_1,w}\)-stable for any \(w > \frac{b_1^2}{2}\). \(\square\)

Similarly if we work with \(b = b_2 := -n + \frac{1}{H^3}\) then the denominator of \(\nu_{b_2,w}\) is 
\[D_2(E) := \text{ch}_1(E(n)).H^2 - \text{ch}_0(E)\].
This has the same properties as \(D_1(E)\), except the third is replaced now by \(D_1(F_2) = 1\) 
being minimal. Again \(\ell\) intersects \(b = b_2\) in a point 
\[w_2 = \frac{n^2}{2} - \frac{n}{2H^3} + \frac{\text{ch}_2(F_1).H}{H^3}\]
\(^5\)This argument is familiar from the analogous fact that rank 1 sheaves can only be destabilised by rank 0 torsion sheaves when working with slope (for which the denominator is rank).
inside the space $U$ of stability conditions, by

$$w_2 - \frac{b^2}{2} = \frac{n}{2H^3} - \frac{1}{2(H^3)^2} + \frac{\text{ch}_2(F_1).H}{H^3} \geq \frac{nH^3 - 1}{2(H^3)^2} > 0.$$ 

So the same argument as for Lemma 8.1 gives the following.

**Lemma 8.2.** The destabilising quotient $F_2$ is $\nu_{b_2,w}$-stable for any $w > \frac{b^2}{2}$. $\square$

**Proposition 8.3.** $\text{ch}_3(F_1) \leq \frac{2}{3} \text{ch}_2(F_1).H \left( \text{ch}_2(F_1).H - \frac{1}{2H^3} \right)$.

**Proof.** Recall the line $\{b = b_1\} \cap U$ used in Lemma 8.1. Its base on $w = \frac{b^2}{2}$ is the point $(-\frac{1}{H^3}, \frac{1}{2(H^3)^2})$. Let $\ell_2$ denote the line connecting this point to $\Pi(F_1) = (0, \frac{\text{ch}_2(F_1).H}{H^3})$.

$$w = \left( \text{ch}_2(F_1).H - \frac{1}{2H^3} \right) b + \frac{\text{ch}_2(F_1).H}{H^3}.$$

![Figure 4. The first wall for the sheaf $F_1$](image)

By the description of the walls of instability (Proposition 4.1), the $w \downarrow \frac{b^2}{2}$ limit of Lemma 8.1 therefore shows that $F_1$ is $\nu_{b,w}$-semistable for any $(b, w) \in \ell_2 \cap U$; see Figure 4.

To apply the Bogomolov-Gieseker Conjecture 3.1 to $F_1$ on $\ell_2$ we need to find a point of $\ell_2 \cap U$ satisfying $\text{ch}_2^{bH}(F_1).H = \left( w - \frac{b^2}{2} \right) \text{ch}_0(F_1)H^3$, i.e.

$$\frac{\text{ch}_2(F_1).H}{H^3} + \frac{b^2}{2} = w - \frac{b^2}{2}.$$
This intersects \( \ell_2 \) (34) at the point \((b^*, w^*)\), where
\[
b^* = \text{ch}_2(F_1).H - \frac{1}{2H^3} \quad \text{and} \quad w^* = \left(\text{ch}_2(F_1).H\right)^2 + \frac{1}{4(4H^3)^2}.
\]
\( \nu_{b^*, w^*} \) is a stability condition since \( w^* - \frac{(b^*)^2}{2} = \frac{1}{2}\left(\text{ch}_2(F_1).H + \frac{1}{2H^3}\right)^2 > 0 \), so by (BG2) we may apply Conjecture 3.1 to give
\[
\text{ch}_3(F_1) - b^* \text{ch}_2(F_1).H - \frac{(b^*)^3H^3}{6} \leq \frac{1}{3} \left(w^* - \frac{(b^*)^2}{2}\right)(-b^*H^3)
\]
\[
= \frac{1}{3} \left(\frac{\text{ch}_2(F_1).H}{H^3} + \frac{(b^*)^2}{2}\right)(-b^*H^3).
\]
Simplifying gives
\[
\text{ch}_3(F_1) \leq \frac{2}{3} b^* \text{ch}_2(F_1).H. \tag{35}
\]

**Proposition 8.4.** \( \text{ch}_3(F_2(n)) \leq \frac{2}{3} \text{ch}_2(F_2(n)).H \left(\text{ch}_2(F_2(n)).H + \frac{1}{2H^3}\right) \).

**Proof.** By Lemma 8.2, \( F_2 \in \mathcal{A}(b_2) \) is \( \nu_{b_2,w} \)-semistable for \( w \gg 0 \). Thus \( F_2(n) \in \mathcal{A}(b_2 + n) = \mathcal{A}(-b_2) \) is \( \nu_{-b_2, w} \)-semistable for \( w \gg 0 \). Therefore, by [BMT14, Lemma 5.1.3(b)] the shifted derived dual \( F_2(n)^{\vee}[1] \) lies in an exact triangle
\[
F \hookrightarrow F_2(n)^{\vee}[1] \rightarrow Q[-1],
\]
with \( Q \) a zero-dimensional sheaf and \( F \) a \( \nu_{b_2,w} \)-semistable object of \( \mathcal{A}(b_1) \) for \( w \gg 0 \). Since rank \( F = 1 \) it is a torsion-free sheaf by [BMS16, Lemma 2.7]. We also have \( \text{ch}_1(F).H^2 = \text{ch}_1(F_2(n)).H^2 = 0 \). Thus \( F \) has all the properties of \( F_1 \) used in Lemma 8.1 and Proposition 8.3, so the latter gives
\[
\text{ch}_3(F) \leq \frac{2}{3} \text{ch}_2(F).H \left(\text{ch}_2(F).H - \frac{1}{2H^3}\right).
\]
Since \( \text{ch}_3(F_2(n)).H = -\text{ch}_2(F).H \) and \( \text{ch}_3(F_2(n)) = \text{ch}_3(F_2(n)^{\vee}[1]) = \text{ch}_3(F) - \text{ch}_3(Q) \leq \text{ch}_3(F) \) the claim follows. \( \square \)

**Proof of Proposition 6.2.** Set \( c := \text{ch}_2(F_1).H \in \{0, -1, -2\} \), so by Proposition 8.3,
\[
\text{ch}_3(F_1) \leq \frac{2c}{3} \left(c - \frac{1}{2H^3}\right). \tag{35}
\]
Using \( \text{ch}_0(F_1) = 1, \text{ch}_1(F_1).H^2 = 0 \) and the exact triangle \( F_1 \rightarrow \iota_*L \rightarrow F_2 \) we compute
\[
\text{ch}_1(F_2(n)).H^2 = 0, \quad \text{ch}_2(F_2(n)).H = -c \quad \text{and} \quad \text{ch}_3(F_2(n)) = -nc - \text{ch}_3(F_1) + \frac{L^2}{2}.
\]
The inequality of Proposition 8.4 therefore becomes
\[
-nc - \text{ch}_3(F_1) + \frac{L^2}{2} \leq -\frac{2c}{3} \left(-c + \frac{1}{2H^3}\right).
\]
Combined with (35) and our assumption $L^2 > -2n + 4$ this gives

$$-n(c + 1) + 2 < -nc + \frac{L^2}{2} \leq \chi_3(F_1) - \frac{2c}{3} \left( -c + \frac{1}{2H^3} \right) \leq \frac{4c}{3} \left( c - \frac{1}{2H^3} \right).$$

If $c = -1$ this gives the contradiction $2 < \frac{4}{3} + \frac{2}{2H^3}$. If $c = -2$ we get $n + 2 < \frac{16}{3} + \frac{4}{2H^3} < 7$ but $n \geq 10$. So $c = 0$.

9. Curve counting

The results of this paper are a special case of the results in [FT19], which in turn builds on [GST14]. Consider 2-dimensional torsion sheaves of the form $\iota_* (L \otimes I_C)$, where $C \subset D \subset X$ is a subscheme of dimension $\leq 1$ and $I_C \subset O_D$ is its ideal sheaf. We take $D \in |O(n)|$, $L.H = 0$ and $n$ is sufficiently large as in this paper; the main difference in [FT19] is that we allow nonempty $C$.

We show the moduli space of slope semistable sheaves in the class of $\iota_* (L \otimes I_C)$ is isomorphic to the product of $\text{Pic}^{\text{tors}}(X)$ – the line bundles on $X$ with torsion $c_1$ – and the moduli space of Joyce-Song pairs

$$\mathcal{O}(-n) \xrightarrow{\delta} \mathcal{I}_C.$$

Here $\mathcal{I}_C \subset O_X$ is an ideal sheaf on $X$ and $s \in H^0(O(n))$ is a nonzero section with zero divisor $D \supset C$. The correspondence takes the cokernel of (36) and tensors it with a line bundle $T$ with $c_1(T) = 0$ to get a sheaf of the form $\iota_* (L \otimes I_C)$.

For $n \gg 0$ the moduli space of pairs (36) is a projective bundle over the moduli space of ideal sheaves $\mathcal{I}_C$. The fibre $\mathbb{P}(H^0(\mathcal{I}_C(n)))$ has Euler characteristic $\chi(\mathcal{I}_C(n))$. If $X$ is a Calabi-Yau 3-fold with $H^1(O_X) = 0$ this gives the relation

$$\#(\text{2-dimensional sheaves}) = \#H^2(X, \mathcal{Z})_{\text{tors}} \cdot \chi(\mathcal{I}_C(n)) \cdot \#(\text{ideal sheaves}).$$

The first term is a DT invariant counting Gieseker stable sheaves of the same topological type as $\iota_* (L \otimes I_C)$. The next two terms are topological constants. The final term is the DT invariant counting ideal sheaves of the topological type of $\mathcal{I}_C$.

The set of all of these DT invariants counting ideal sheaves is equivalent, by the MNOP conjecture [MNOP] (proved for most Calabi-Yau 3-folds in [PP17]), to the set of Gromov-Witten invariants of $X$. The upshot is that the Gromov-Witten invariants of $X$ are governed by counts of 2-dimensional sheaves. In turn the generating series of the latter are conjectured by physicists to be mock modular forms due to $S$-duality.

Both this paper and [FT19] use very similar methods to those employed so impressively by Toda [To12] to prove the famous OSV conjecture on Calabi-Yau threefolds $X$ with $\text{Pic} X = \mathbb{Z}$ satisfying the Bogomolov-Gieseker conjecture. Toda also considers slope stable sheaves of dimension two and follows them down the wall $b = b_0$ (17), using the Bogomolov-Gieseker inequality to find the first wall of instability $\ell$. One difference between the papers is that in Proposition 6.1 we analyse the destabilising objects $F_1, F_2$ along the wall $\ell$, and use the fact that they lie in $\mathcal{A}(b)$ at its endpoints $\ell \cap \partial U$ to constrain $\chi(F_1)$. Toda works

\footnote{We show slope semistability is equivalent to Gieseker stability for $n \gg 0$.}
only on \( b = b_0 \) and uses different arguments to analyse \( \text{ch}(F_i) \). A similar comment applies to the work in Section 8 to prove Proposition 6.2, as described in the discussion below Proposition 6.2.

The main difference between our work and Toda’s is that we consider subtly different Chern characters. In [FT19] we consider two dimensional sheaves with \( \text{ch}_1 = nH \) and

\[
\text{ch}_2 = -\beta - \frac{n^2}{2}H^2
\]

for \( n \gg 0 \) and some curve class \( \beta \) (in this paper ultimately \( \beta = 0 \in H^4(X, \mathbb{Q}) \)). Toda considers \( \text{ch}_1 = nH \) and \( \text{ch}_2 = -\beta \), for fixed \( \beta \) and \( n \gg 0 \). To apply his methods to our class would require a bound like \( \beta.H \geq \frac{1}{2}n^2H^3 \), while his paper works in the opposite regime \( \beta.H < \epsilon n^{-2} \). As a result he manages to express counts of 2-dimensional sheaves in terms of both ideal sheaves and stable pairs, whereas for us the stable pairs are absent and the results rather different.

**Appendix A. The case of \( \mathbb{P}^3 \)**

*By Claire Voisin*

When \( X = \mathbb{P}^3 \) we can prove a very similar result to \((B)\) by more classical methods.

**Theorem A.1.** Let \( D \) be a smooth surface of degree \( n \geq 4 \) in \( \mathbb{P}^3 \). Any nontrivial line bundle \( L \) on \( D \) with \( c_1(L).H = 0 \) satisfies \( L^2 \leq -\frac{4}{2}n^2 + 5 \).

**Proof.** The \( K3 \) case \( n = 4 \) is trivial: Riemann-Roch gives \( h^0(L) + h^0(L^{-1}) = h^1(L) + 2 + \frac{L^2}{2} \) so if \( L \) is nontrivial with \( L.H = 0 \) this gives \( 0 = h^1(L) + 2 + \frac{L^2}{2} \) and so \( L^2 \leq -4 \).

So we can take \( n \geq 5 \). By Riemann-Roch,

\[
(37) \quad h^0(L) + h^0(K_D \otimes L^{-1}) \geq \chi(L) = \chi(O_D) + \frac{1}{2}L^2 - \frac{1}{2}K_D.L.
\]

We assume for a contradiction that \( L \) is nontrivial and \( L^2 \geq -2n + 6 \). Using \( K_D = O_D(n - 4) \), \( L.H = 0 \) and \( h^1(O_D) = 0 \), (37) gives

\[
(38) \quad h^0(L^{-1}(n - 4)) \geq h^0(O_D(n - 4)) - (n - 4).
\]

Let \( C := H \cap D \) be a smooth plane section and \( L_C := L|_C \). Then the exact sequences \( 0 \to L^{-1}(i) - 1 \to L^{-1}(i) \to L^{-1}_C(i) \to 0 \) give

\[
(39) \quad h^0(L^{-1}(i)) - h^0(L^{-1}(i - 1)) \leq h^0(L^{-1}_C(i)).
\]

Since \( h^0(L^{-1}) = 0 \), summing over \( 1 \leq i \leq n - 4 \) gives

\[
(40) \quad h^0(L^{-1}(n - 4)) \leq \sum_{i=1}^{n-4} h^0(L^{-1}_C(i)).
\]
Replacing $L^{-1}$ by $\mathcal{O}_D$ gives equality in (39) for $1 \leq i \leq n-4$ by Kodaira vanishing, and so

\begin{equation}
\tag{41}
h^0(\mathcal{O}_D(n-4)) - 1 = \sum_{i=1}^{n-4} h^0(\mathcal{O}_C(i)).
\end{equation}

Comparing (38), (40) and (41) shows that $h^0(L_C^{-1}(i)) \geq h^0(\mathcal{O}_C(i))$ for some $1 \leq i \leq n-4$. Since $\deg L_C = 0$ this implies $L_C = \mathcal{O}_C$ by [Ha86, Theorem 2.1, 2(b)].

By standard methods, this now implies the contradiction $L = \mathcal{O}_D$. For instance, consider the blow up $\pi: \hat{D} \to D$ of $D$ in the baselocus of a pencil of $C$s, giving a fibration $p: \hat{D} \to \mathbb{P}^1$. Then $\pi^*L$ is trivial on the fibres, so is the pullback from $\mathbb{P}^1$ of the line bundle $p_*(\pi^*L) \cong \mathcal{O}_{\mathbb{P}^1}(d)$. Restricting $\pi^*L$ to (the proper transform of) another plane section (a multisection of $p$) and using $L.H = 0$ shows that $d = 0$. \hfill \Box

References


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