Derived Functors, Cohomology and Hypercohomology

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Contents

Abstract

In this report, we will give a brief introduction to abelian categories, injective objects, injective resolutions, acylic resolutions and hypercohomology.

1 Introduction

Definition 1. [Abelian Category] An abelian category $\mathcal{C}$ is a category with additional information:

1. For all objects A and B, $\text{Hom}(A, B)$ is an abelian group.
2. The following map:

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$$

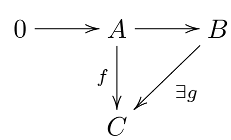
via $(f, g) \mapsto g \circ f$ is $\mathbb{Z}$-bilinear.

Remark: In this special category setting, we require functor between objects to respect direct sum. For example, for a functor $F : \mathcal{A} \to \mathcal{B}$, we have for every U and V in $\mathcal{A}$, $F(U \oplus V) = F(U) \oplus F(V)$.

Example 2. [Example of an Abelian Category] A category of modules over a ring.

2 Injective Objects and Resolution

Definition 3. [Injective Object] A object in an abelian category $\mathcal{C}$ is injective if for every injective morphisms $A \to B$ and a morphism $A \to C$, there exists a morphism $B \to C$ that makes the following diagram commute



Example 4. In an abelian category, the object $\{0\}$ is injective.

Example 5. In modules over $\mathbb{Z}$, an abelian group D is divisible if and only if D is an injective $\mathbb{Z}$ module. In particular, $\mathbb{Q}$ is an injective $\mathbb{Z}$ module.

Definition 6. [Complex] A complex $M^{\cdot}$ is a set of objects $\{M_i\}_{i\in\mathbb{Z}}$, and a set of morphisms $\{d^i : M_i \to M_{+1}\}_{i\in\mathbb{Z}}$ such that $d^i \circ d^{i-1} = 0$.

From now on, we will consider only left-bounded complexes.

Definition 7. [Morphisms of Complexes] A morphism of complexes $\phi^{\cdot} : (M^{\cdot}, d_M) \to (N^{\cdot}, d_N)$ is a collection of morphisms $\{\phi^k : M^k \to N^k\}_{k\in\mathbb{Z}}$ that makes the following diagram commute:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} & & \\ \cdots & \longrightarrow & N^i & \xrightarrow[d_N^i]{} & N^{i+1} & \longrightarrow & \cdots \end{array}$$

In other words, $\phi_{i+1} \circ d_M^i = d_N^i \circ \phi_i$

Definition 8. [The i-th degree cohomology] The i-th degree cohomology of the complex $(M^{\cdot}, d_M)$ is an object given by

$$H^i(M) := \frac{\ker d^i : M^i \to M^{i+1}}{\text{im } d^{i-1} : M^{i-1} \to M^i}$$

A morphism of complexes $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$, induces a morphism of cohomologies. In other words, we may define $H^i(\phi) : H^i(M) \to H^i(N)$ via the association $[m] \mapsto [\phi(m)]$.

Definition 9. [Homotopy] Let $\phi : (M, d_M) \to (N, d_N)$ and $\psi : (M, d_M) \to (N, d_N)$ be morphisms of two complexes. A homotopy between ϕ and ψ is a collection of morphisms $\{D^i : M^i \to N^{i-1}\}_{i\in\mathbb{N}}$ such that in the following diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{i-1} & \longrightarrow & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \cdots \\ & & \downarrow & \swarrow{\scriptstyle D^i} & \downarrow & \swarrow{\scriptstyle D^{i+1}} & \downarrow & & \\ \cdots & \longrightarrow & N^{i-1} & \xrightarrow[d_N^{i-1}]{} & N^i & \longrightarrow & N^{i+1} & \longrightarrow & \cdots \end{array}$$

We have $D^{i+1} \circ d_M^i - d_N^{i-1} \circ D^i = \phi^i - \psi^i$.

This algebraic definition of homotopy in fact may be seen in texts on differential topology or simplicial homology. We shall see why the two homotopic maps ϕ and ψ induces the same mappings $H^i(\phi)$ and $H^i(\psi)$. Suppose we have $[\omega] \in H^i(M^{\cdot})$, then $\phi^i - \psi^i(\omega)$ is exact because

$$[(\phi^i - \psi^i)(\omega)] = [(D^{i+1} \circ d_M^i - d_N^{i-1} \circ D^i)(\omega)] = [-d_N^{i-1} \circ D^i \omega] \tag{1}$$

which is 0 in cohomology. In other words, the cohomology functor is homotopy invariant.

Definition 10. [Resolution] A complex $\{M^i\}_{i\geq 0}$ is a resolution of an object A in a category $\mathcal{C}$ if the complex $(M^{\cdot}, d_M)$ is exact and there exists an injective morphism $j : A \to M^0$ such that $\ker(d^0 : M^0 \to M^1) = j(A)$.

3 Sufficiently Many Injective Objects

Definition 11. [Sufficiently many injective objects] An abelian category $\mathcal{C}$ has sufficiently many injective objects if for each A an object in $\mathcal{C}$, there exists an injective object I and a morphism $j : A \to I$ which is injective.

Example 12. Every module over a ring R with identity can be embedded into an injective R-module.

Lemma 13. If $\mathcal{C}$ has sufficiently many injective objects, then every object of $\mathcal{C}$ has an injective resolution.

The trick to finding an exact sequence $I^{\cdot}$ is by passing through the cokernel and the application of the property of an injective object.

Proof. First we start with an object A. Since $\mathcal{C}$ has sufficiently many injective objects, there exists an injective morphism $j : A \to I^0$ for some injective object I^0. Suppose we have a sequence as follows:

$$0 \longrightarrow A \xrightarrow{j} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \xrightarrow{d^{k-1}} I^k$$

Then there exists an injective object I^{k+1} such that $\operatorname{coker} d^{k-1}$ may be embedded into I^{k+1} as follows

$$0 \longrightarrow A \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{k-1}} I^k \longrightarrow \operatorname{coker} d^{k-1} \longrightarrow I^{k+1}$$

and therefore the kernel of this map $I^k \to I^{k+1}$ is indeed the image $I^{k-1} \to I^k$. $\square$

This lemma shows us that every object has an injective resolution. In the next part, we will show that such resolution is unique up to homotopy equivalence.

Proposition 14. Let $I^{\cdot}$, $A \hookrightarrow I^0$, and $J^{\cdot}$, $B \hookrightarrow J^0$ be the respective resolutions. If $J^{\cdot}$ is an injective resolution of B, then a morphism $\phi : A \to B$ induces a morphism of complexes $\phi^{\cdot} : I^{\cdot} \to J^{\cdot}$ such that $\phi^0 \circ i = j \circ \phi$.

Proof. We want to construct a series of ϕ^k's from I^k to J^k and suppose that we have constructed morphisms from ϕ^1 to ϕ^k. The main point is that we want the property $\phi^{k+1} \circ d_I^i = d_J^i \circ \phi^i$. We will use the universal property of an injective object to construct a morphism from I^{k+1} to J^{k+1}. Therefore, we consider

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{coker} d^{k-1} & \longrightarrow & I^{k+1} \\ & & {\scriptstyle d_J^k \circ \phi^k}\downarrow & \swarrow {\scriptstyle \exists \phi^{k+1}} & \\ & & J^{k+1} & & \end{array}$$

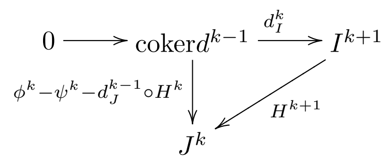
Hence we have by commutativity $\phi^{k+1} \circ d_I^k = d_J^k \circ \phi^k$. $\square$

Proposition 15. Let $I^{\cdot}$, $i : A \hookrightarrow I^0$ be a resolution of A and $J^{\cdot}$, $j : B \hookrightarrow J^0$ be an injective resolution of B. Given a morphism $\phi : A \to B$, and suppose we have two ways to extend to morphism of complexes $\phi^{\cdot} : I^{\cdot} \to J^{\cdot}$, $\psi^{\cdot} : I^{\cdot} \to J^{\cdot}$ such that $\phi^0 \circ i = j \circ \phi$ (and same for ψ), then $\phi^{\cdot}$ and $\psi^{\cdot}$ are homotopic.

Proof. Define $H^1 : I^1 \to J^0$ by the injective property of J^0 via

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{coker} i & \xrightarrow{d_I^0} & I^1 \\ & & {\scriptstyle \phi^0 - \psi^0}\downarrow & \swarrow {\scriptstyle H^1} & \\ & & J^0 & & \end{array}$$

Suppose we have morphisms $H^1, \dots H^k$. Then we define H^{k+1} by



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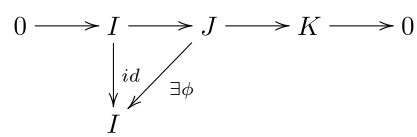
4 Derived Functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be two abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor. Suppose that $\mathcal{C}$ has sufficiently many injective objects. We will define $R^i F$, the derived functor. Before we proceed, we start off with a few lemmas:

Lemma 16. Let $0 \to I \to J \to K \to 0$ be an exact sequence. If I is injective, then the exact sequence splits.

Proof. Since I is injective, we may have a map $\phi : J \to I$ such that the following diagram commutes



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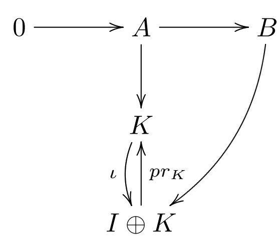
Lemma 17. If $J = I \oplus K$ and J is injective, so is K.

Proof. For every embedding $A \hookrightarrow B$ and a map $A \to K$, we have a map $B \to K$ by following the diagram below:



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Lemma 18. Given an exact sequence in $\mathcal{C}$

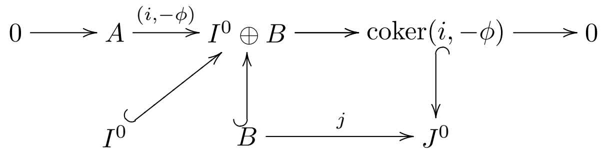
$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

there exists injective resolutions $A \hookrightarrow I^\cdot$, $B \hookrightarrow J^\cdot$ and $C \hookrightarrow K^\cdot$ such that the following sequence

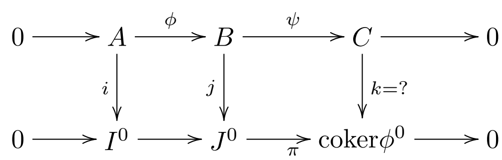
$$0 \longrightarrow I^\cdot \xrightarrow{\phi^\cdot} J^\cdot \xrightarrow{\psi^\cdot} K^\cdot \longrightarrow 0.$$

is exact and $\phi^0 \circ i = j \circ \phi$ and $\psi^0 \circ j = k \circ \psi$.

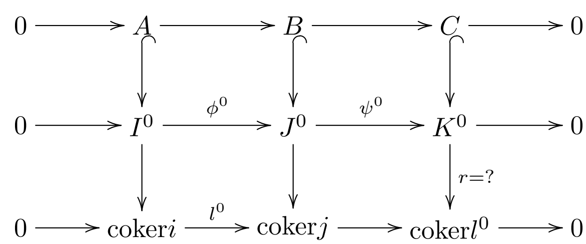
Proof. First we construct the first batch (I^0, J^0, K^0). Given that we have sufficiently injective objects, we may have the following embedding $i : A \to I^0$ for some injective object. We consider the following commutative diagram



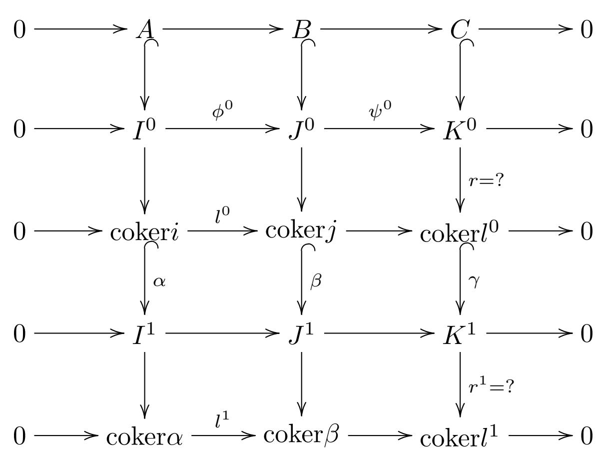
And we define $j : B \to J^0$ and $\phi^0 : I^0 \to J^0$ by following the arrows. Then j is injective because i is, and ϕ^0 is injective because ϕ is. We may thus have the following diagram



To define k, let $c \in C$. By surjectivity of ψ, there exists b such that $\psi(b) = c$. Therefore, we define $k(c) = \pi \circ j(b)$. This is well-defined, independant of the choice of b because of exactness at B. The mapping k turns out to be injective (i.e. 5-lemma). Then we proceed to construct the next set.



We need to define l. We define $l([i^0] \in \text{coker}i) = [\phi^0(i^0)] \in \text{coker}j$, which is well-defined and injective. Also, we define r in a similar manner as before. From the above diagram, we may embed coker$i$ into I^1, and similar methods give J^1 and K^1. It remains to show that the last vertical line starting from C is a complex. We may continue on with



$\square$

The importance of taking an injective resolution is that it gives rise to long exact sequences which helps to compute cohomology.

Theorem 19. [Existence] Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor. For every object M of $\mathcal{C}$, there exists objects $R^iF(M)$ in $\mathcal{C}'$ such that

1. $R^0F(M) = F(M)$,
2. Every short exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

   Induces a long exact sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\phi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \longrightarrow \cdots$$

Proof. Let A be an object in $\mathcal{C}$, choose an injective resolution $A \hookrightarrow I^\cdot$ and define $R^iF(M) = H^i(F(I^\cdot))$ of the complex:

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots$$

To show property 2, given any exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, by the lemma 18, there exists a sequence of complexes $0 \to I^\cdot \to J^\cdot \to K^\cdot \to 0$ with the properties in the lemma. Since short exact sequence of injective objects

$$0 \longrightarrow I^l \longrightarrow J^l \longrightarrow K^l \longrightarrow 0$$

splits, applying the functor to the sequence obtains another split exact sequence

$$0 \longrightarrow F(I^l) \longrightarrow F(J^l) \longrightarrow F(K^l) \longrightarrow 0.$$

It is a property that split exact sequences gives rise to long exact sequence in cohomology. □

Theorem 20. [Uniqueness] The object $R^iF(M)$ is determined up to isomorphism.

Proof. Suppose we have two choices of injective resolutions for A, such as $I^\cdot$ and $J^\cdot$, by a previous proposition, there exists homomorphisms $\phi : I^\cdot \to J^\cdot$ and $\psi : J^\cdot \to I^\cdot$ and homotopies $D_I^\cdot : I^\cdot \to I^{\cdot-1}$, $D_J^\cdot : J^\cdot \to J^{\cdot-1}$ between $\psi^\cdot \circ \phi^\cdot$ and id, and $\phi^\cdot \circ \psi^\cdot$ and id. Applying the functor gives morphisms $F(\phi)$, $F(\psi)$ and the corresponding homotopies $F(D_I^\cdot)$, $F(D_J^\cdot)$. Hence the morphisms $H^i(F(\phi))$ and $H^i(F(\psi))$ are inverses of each other. □

Corollary 21. If I is injective, $R^iF(I)$ is 0.

Proof. For the injective object, choose an injective resolution $0 \to I \to I \to 0$. Then $R^iF(I)$ is zero by uniqueness. □

5 Acyclic Objects

In practice, injective objects are difficult to work with, and one would like to work with acyclic objects instead.

Definition 22. M is acyclic for the functor F if $R^iF(M) = 0$ for all $i > 0$.

Proposition 23. Let $M^\cdot$ be an acyclic resolution of A with M^i's all F-acyclic. Then $R^iF(A) = H^iF(M^\cdot)$.

Remark: This means that acyclic objects work just as well as injective objects.

Proof. The proof is by induction on i. We have the exact sequence

$$0 \longrightarrow A \xrightarrow{d^0} M^0 \longrightarrow B \longrightarrow 0 \tag{2}$$

where B is the cokernel of the map $A \to M^0$. Moreover, we have the following resolution

$$0 \longrightarrow B \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow \cdots.$$

Since (2) is exact, previous theorem gives us a long exact sequence of derived objects such as this:

$$0 \longrightarrow F(A) \longrightarrow F(M^0) \longrightarrow F(B) \longrightarrow R^1F(A) \longrightarrow R^1F(M^0) \longrightarrow R^1F(B) \longrightarrow \cdots$$

Since M^0 is acyclic, therefore $R^iF(M) = 0$ for all $i \geq 1$. Hence for all $i \geq 1$ we have $R^{i+1}F(A) = R^iF(B))$ and $R^1F(A) = \text{coker}(F(M^0) \to F(B)$. By induction hypothesis, suppose $R^kF(A) = H^k(F(M^\cdot))$ for all $k = 1, \ldots i$ (and any object A), therefore, $R^{i+1}F(A) = R^iF(B) = H^i(F(M^{\cdot+1})) = H^{i+1}(F(M^\cdot))$. □

6 Hypercohomology

In this section we will only give a brief exposition of hypercohomology. The full treatment can be found in [Voi]. We will use the setting for our discussion. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a left exact functor. We assume that $\mathcal{A}$ has sufficiently many injective objects. We will define for every left bounded complex $M^\cdot$, a derived functor $R^iF(M^\cdot)$.

Proposition 24. For each left bounded complex $M^\cdot$ in $\mathcal{A}$, there is a complex $I^\cdot$ in $\mathcal{A}$ such that

1. $I^\cdot$ is left bounded.
2. I^k is injective in $\mathcal{A}$.
3. $\phi : M^\cdot \to I^\cdot$ is a quasi-isomorhism.
4. For each k, $\phi^k : M^k \to I^k$ is injective.

We will prove this proposition later. First we need a result:

Proposition 25. For each left bounded complex $M^\cdot$ in $\mathcal{A}$, there exists a double complex $(I^{k,l}, (D_1, D_2))$ with

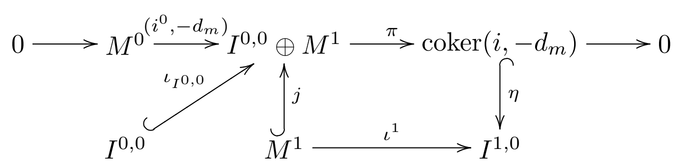
1. $I^{k,l}$ is injective.
2. $(I^{k,\cdot}, D_2)$ is a resolution of M^k.
3. The inclusion $(M^k, d_M) \hookrightarrow (I^{\cdot,0}, D_1)$

Proof of proposition 24. Given the double complex $(I^{k,l}, (D_1, D_2))$, we may let $(I^\cdot, D)$ to be

$$I^k = \bigoplus_{p+q=k} I^{p,q}, \qquad D = D_1 + (-1)^p D_2.$$

□

Proof of proposition 25. We will construct the first line $(I^{\cdot,0}, D_1)$. Already, we have the following injection $M^0 \hookrightarrow I^{0,0}$. We will construct $I^{1,0}$. In the following diagram:



Where η is the inclusion by the condition that $\mathcal{A}$ has sufficiently many injective objects. We thus let $\iota^1 : M^1 \to I^{1,0}$ by $\iota^1 = \eta \circ \pi \circ j$ and $D_1 : I^{0,0} \to I^{1,0}$ by $D_1 = \eta \circ \pi \circ \iota_{I^{0,0}}$. Suppose we have constructed the sequence up till $I^{k,0}$ as follows:

$$\begin{array}{ccccccc}
M^{k-1} & \longrightarrow & M^k & \longrightarrow & M^{k+1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \iota^{k-1}} & & \downarrow{\scriptstyle \iota^k} & & & & \\
I^{k-1,0} & \xrightarrow{D_1} & I^{k,0} & & & &
\end{array}$$

We construct $I^{k+1,0}$ by the following injection $\mathrm{coker}((\iota^k, -d_M) : M^k \to \mathrm{coker} D_1 \oplus M^{k+1}) \hookrightarrow I^{k+1,0}$ and note that $i^k : M^k \to I^{k,0}$ is an injection. For the last assertion, we refer the readers to [Voi]. It uses the following result:

Lemma 26. Let $(I^{\cdot}, D)$ be the simple complex associated to the double complex $(I^{p,q}, D_1, D_2)$ and suppose for each p, the complex $(I^{p,\cdot}, D_2)$ is a resolution of M^p via the injection $\iota^p : M^p \hookrightarrow I^{p,0}$. Then the morphism of complexes $\iota^{\cdot} : M^{\cdot} \to I^{\cdot}$ induces isomorphism of cohomologies $H^p(M^{\cdot}, d_M) \cong H^p(I^{\cdot}, D)$.

□

7 Bibliography

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