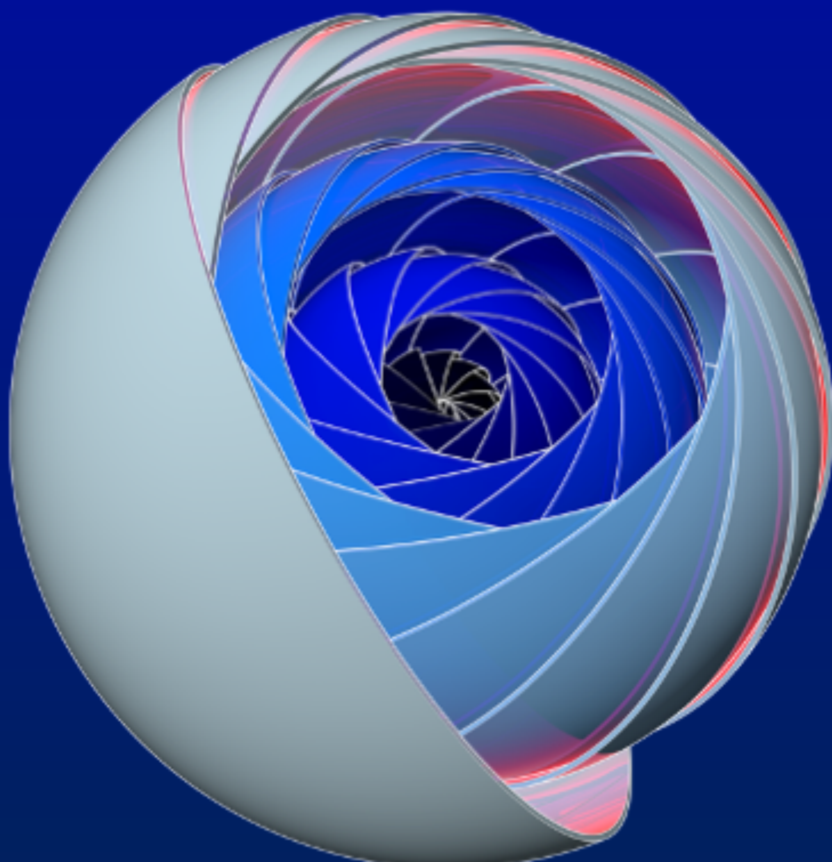
JOURNAL OF OPEN MATHEMATICAL PROBLEMS



AMR
ASSOCIATION FOR MATHEMATICAL RESEARCH

VOLUME 1 ISSUE 1
2025

JOURNAL OF OPEN MATHEMATICAL PROBLEMS

PUBLISHED BY

Association for Mathematical Research, amathr.org,
Davis, CA; Jenkintown, PA.

JOURNAL OF OPEN MATHEMATICAL PROBLEMS, 1(1):16–51, 2025
jomprob.org

Hodge and generalized Hodge conjectures, coniveau and algebraic cycles

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Received 9 Jan 2025; Revised 4 Jun 2025; Accepted 13 Jul 2025

Abstract:
This is a survey of the Hodge conjecture, with emphasis on its companion, the generalized Hodge conjecture, which involves the theory of Hodge structures, algebraic cycles and motives.

Key words and phrases:
Hodge Structures; Hodge Classes; Algebraic Cycles

1 Introduction

We present in this paper the Hodge conjecture and its much more general version, "the generalized Hodge conjecture" due to Grothendieck [Gro69]. The Hodge conjecture involves two distinct objects associated with a smooth projective algebraic variety X defined over the complex numbers. On one side, we have the algebraic variety X and its algebraic subvarieties. On the other side, we have the associated complex manifold X_{an}, and we associate to it the Betti cohomology with rational coefficients of its underlying topological space, which is related to the study of differentiable submanifolds of X_{an}. The data coming from algebraic geometry and topology are not disjoint, thanks to the beautiful comparison theorems due to Serre and Grothendieck for cohomology with complex coefficients (see Section 3.4). One geometric bridge between the two sets of data is given by the cycle class, that associates to an algebraic subvariety $Z \subset X$ the cohomology class $[Z_{\mathrm{an}}]$ of the corresponding closed analytic subset of X_{an}. The Hodge conjecture stated in Section 3 proposes a characterization of the subspace of $H^{2k}(X_{\mathrm{an}}, \mathbb{Q})$ generated over $\mathbb{Q}$ by the classes $[Z_{\mathrm{an}}]$ above with $\operatorname{codim} Z = k$: it should be the space of *Hodge classes*. This conjectural characterization involves the complex geometry of X_{an} and the notion of type of differential

*The author is supported by the ERC Synergy Grant HyperK (Grant agreement No. 854361).

DOI: 10.56994/JOMP.001.001.002

forms on a complex manifold X: one can speak of differential forms of type (p,q), hence degree $p+q$, and the space $H^{p,q}(X)$ of classes of type (p,q) is defined as the set of de Rham cohomology classes of closed (p,q)-forms. Hodge classes are the rational Betti cohomology classes of even degree $2k$ on X_{an}, which are of type (k,k).

The Hodge conjecture is elegant and fascinating but what makes it deep is the general theory of *Hodge structures* associated to smooth projective varieties, that we will present in Section 2.1. As we will see, the Hodge conjecture is part of the theory of cohomological motives, and is particularly interesting when applied to products $X \times Y$ of two algebraic varieties, having in mind to understand how their motives are related. In the theory of motives, one considers as morphisms between two algebraic varieties X and Y the correspondences between X and Y, which are cycles in the product $X \times Y$, that is, combinations with rational coefficients of closed algebraic subsets of the product $X \times Y$. We have to divide out the space of cycles by an adequate equivalence relation in order to be able to compose correspondences. In the theory of cohomological motives (over the complex numbers), the cycles will be considered modulo homological equivalence, namely via their cycle class. The importance of the Hodge conjecture appears in the following fact: *The Hodge classes on the product $X \times Y$ are naturally identified with the morphisms of Hodge structures from the cohomology of X to the cohomology of Y*. We refer to Section 2.1 for an introduction to Hodge structures, their construction and their functoriality properties in the geometric context.

The original Hodge conjecture, corrected by Grothendieck in [Gro69], proposed more generally to characterize rational cohomology classes α on X_{an} that are "of geometric coniveau c", that is, are supported on a closed subset $Y \subset X$ (equivalently, their Poincaré dual is in the image of the natural morphism $H_*(Y,\mathbb{Q}) \to H_*(X,\mathbb{Q})$, see Section 4.1), where Y is closed algebraic of codimension at least c. There is an obvious restriction satisfied by such classes, namely that they are "of Hodge coniveau c", that is, the components $\alpha^{p,q}$ of the Hodge decomposition of α vanish for $p<c$ or $q<c$. Hodge believed that this would be a sufficient condition, but Grothendieck exhibited a counterexample in [Gro69], and there is indeed a stronger condition coming from the following (highly nontrivial) fact (see Corollary 4.5): the set of degree k cohomology classes on X supported on Y is a Hodge substructure of $H^k(X,\mathbb{Q})$, that is, a rational vector subspace stable under the Hodge decomposition, which consists of classes of Hodge coniveau $\geq c$. The generalized Hodge conjecture presented in Section 4 corrects the original Hodge conjecture by adding this condition. We will discuss in Section 4 the precise relation between the Hodge and generalized Hodge conjectures (see Proposition 4.8). According to this conjecture, the vanishing of the spaces $H^{p,q}(X)$ for $p+q=k$, $p<c$ or $q<c$ should imply that the Betti cohomology $H^k(X_{\mathrm{an}},\mathbb{Q})$ is supported on a codimension c closed algebraic subset Y of X. It is very interesting to note that, for this particular instance of the generalized Hodge conjecture, the assumption on X is purely algebraic, as the spaces $H^{p,q}(X)$ can be computed without any transcendental arguments, thanks to GAGA and the theory of algebraic differential forms (see Section 3.4).

One difficulty with the Hodge conjecture is that, apart from the formalism of Künneth components and Lefschetz operators described in Section 3.2, which produces by formal arguments Hodge classes on powers X^2, it is very difficult to exhibit smooth complex projective varieties with a Hodge class which is not trivially algebraic. Typically, Noether-Lefschetz type statements (see [Voi03, 6.3.1]) will say that a very general hypersurface of dimension 2 and degree ≥ 4, or dimension ≥ 3 and degree ≥ 3, has no other Hodge classes than the multiples of the restrictions $h^i_{|X}$, where h is the class of a hyperplane in projective space, so for the most natural algebraic varieties, like general hypersurfaces, the Hodge conjecture says nothing. To the contrary, the generalized Hodge conjecture predicts the existence of "interesting" suvarieties of a general hypersurface of degree d and dimension n, especially when the dimension is large compared to the degree, and this prediction remains almost completely open (see Section 6.1). Typically it is unsolved for general hypersurfaces of degree d in $\mathbb{P}^{2d}$, except for small d.

In Section 5, we will describe another set of conjectures, mostly due to Bloch and Bloch-Beilinson, relating the shape of Hodge structures of X to the size of Chow groups of cycles modulo rational equivalence on X. They basically say that the Hodge structures of a complex projective variety govern its "Chow motive". Thanks to the work of Bloch and Srinivas [BS83], these conjectures strengthen in a deep way the particular instance mentioned above of the generalized Hodge conjecture.

2 Hodge structures, Hodge classes and coniveau

2.1 The Hodge decomposition theorem

Let X be a complex manifold of dimension n, that is, a differentiable manifold equipped with the data of holomorphic coordinates $z_1,\dots,z_n$ on local charts, with the condition that the change of coordinates maps

$$z'_i = \phi_i(z_1,\dots,z_n)$$

on the overlap of two charts, are given by holomorphic functions ϕ_i. On such a manifold, the de Rham complex $d : A^*_X \to A^{*+1}_X$ of $\mathcal{C}^\infty$ differential forms with complex coefficients splits as follows. Using local holomorphic coordinates, 1-forms with complex coefficients split as

$$\alpha = \alpha^{1,0} + \alpha^{0,1}, \tag{1}$$

with

$$\alpha^{1,0} = \sum_i \alpha_i dz_i,\ \alpha^{0,1} = \sum_i \alpha_{\bar{i}} d\overline{z_i},$$

for some C^∞ functions α_i, $\alpha_{\bar{i}}$. We will denote by $A^{1,0}_X$ the space of forms $\alpha^{1,0}$ of type $(1,0)$ as above, and similarly for $A^{0,1}_X$. We thus have

$$A^1_X = A^{1,0}_X \bigoplus A^{0,1}_X, \tag{2}$$

which in turn induces a decomposition

$$A^k_X = \sum_{p+q=k} A^{p,q}_X \tag{3}$$

for any k, where forms in $A^{p,q}_X$ are said to be of type (p,q) and are locally of the form $\sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I \wedge d\overline{z_J}$.

Coming back to (2), we see that for each $\mathcal{C}^\infty$ function f, we have an induced decomposition

$$df = \partial f + \overline{\partial} f,$$

which defines the first order differential operators

$$\partial : \mathcal{C}^\infty_X \to A^{1,0}_X, \overline{\partial} : \mathcal{C}^\infty_X \to A^{0,1}_X.$$

For any differential form α of type (p,q), we write locally $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I \wedge d\overline{z_J}$, hence we get

$$d\alpha = \sum_{|I|=p,|J|=q} d\alpha_{IJ} \wedge dz_I \wedge d\overline{z_J} = \sum_{|I|=p,|J|=q} \partial\alpha_{IJ} \wedge dz_I \wedge d\overline{z_J} + \sum_{|I|=p,|J|=q} \overline{\partial}\alpha_{IJ} \wedge dz_I \wedge d\overline{z_J},$$

from which we deduce that the operator d does not respect the decomposition (3) but decomposes as

$$d = \partial + \overline{\partial},$$

for some first order differential operators

$$\partial : A^{p,q}_X \to A^{p+1,q}_X, \overline{\partial} : A^{p,q}_X \to A^{p,q+1}_X.$$

The complex conjugate operators ∂ and $\overline{\partial}$ acting on A^*_X satisfy the relations

$$\partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} = -\overline{\partial}\partial.$$

The comparison of the $\overline{\partial}$-cohomology (known as Dolbeault cohomology) and the d-cohomology of complex differential forms on X (i.e. de Rham cohomology of X) gives rise to the Frölicher spectral sequence, which for general complex manifolds of dimension > 2 can be extremely complicated (see e.g. [Rol08]). However for projective complex manifolds, namely closed complex submanifolds of some projective space, or more generally compact Kähler manifolds (see Section 2.2), a miracle happens, which is called the *Hodge decomposition theorem*.

Theorem 2.1. *(Hodge [Hod41], see also [GH78, p116] or [Voi02b, 6.1.3]) Let X be a compact Kähler manifold. Then the de Rham complex cohomology groups* $H^k(X,\mathbb{C}) = \frac{\mathrm{Ker}\,(d: A^k(X)\to A^{k+1}(X))}{\mathrm{Im}\,(d: A^{k-1}(X)\to A^k(X))}$ *decompose as*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{4}$$

where $H^{p,q}(X) \subset H^k(X,\mathbb{C})$ is the set of cohomology classes of d-closed (p,q)-forms.

Furthermore, via the map which sends a closed (p,q)-form to its class as a $\overline{\partial}$-closed form, the space $H^{p,q}(X)$ is naturally isomorphic to the Dolbeault cohomology group $H^q(X,\Omega^p_X)$.

The proof of this fundamental theorem relies on the representation of cohomology classes by harmonic forms and the comparison of the Laplacians for d and $\overline{\partial}$.

An important feature of the decomposition (4) is the Hodge symmetry property. It is clear that the complex conjugate of a closed (p,q)-form is a closed (q,p)-form. It follows that complex conjugacy acting on

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{R}) \otimes \mathbb{C}$$

satisfies

$$\overline{H^{p,q}(X)} = H^{q,p}(X). \tag{5}$$

Note that the de Rham cohomology $H^k(X,\mathbb{C})$ used above is also isomorphic to Betti cohomology with complex coefficients. Betti cohomology $H^k(X,A)$ can be defined with integral or rational coefficients A and we have the change of coefficients theorem

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C}.$$

Definition 2.2. *An integral Hodge structure of weight k is the data of a lattice L, and a decomposition into complex subspaces*

$$L_{\mathbb{C}} = \oplus_{p+q=k} L^{p,q}$$

of $L_{\mathbb{C}} := L \otimes \mathbb{C}$, satisfying the Hodge symmetry condition

$$\overline{L^{p,q}} = L^{q,p}.$$

Remark 2.3. Having the Hodge decomposition on $L_{\mathbb{C}}$ allows to introduce the Hodge filtration $F^pL_{\mathbb{C}} := \oplus_{r\geq p} L^{r,k-r}$. This filtration has to satisfy the "opposite condition", namely, by Hodge symmetry, one has for any p

$$L_{\mathbb{C}} = F^pL_{\mathbb{C}} \oplus \overline{F^{k-p+1}L_{\mathbb{C}}}. \tag{6}$$

Conversely, if a decreasing filtration $F^iL_{\mathbb{C}}$ satisfies (6) for all p, it gives a Hodge decomposition by the formula

$$L^{p,q} = F^pL_{\mathbb{C}} \cap \overline{F^{k-p}L_{\mathbb{C}}}.$$

By Theorem 2.1, the Betti cohomology $H^k(X,\mathbb{Z})$ modulo torsion of a compact Kähler manifold is equiped with a Hodge structure of weight k, which is furthermore effective in the sense that $H^{p,q} = 0$ if $p < 0$ or $q < 0$. In this paper, we will mainly use the rational Hodge structures, whose definition is the same with "lattice" replaced by "finite dimensional $\mathbb{Q}$-vector space".

When looking at a Hodge structure $(L, L^{p,q})$, the first information one gets is its "coniveau", with the following

Definition 2.4. *The coniveau of a Hodge structure $(L, L^{p,q})$ with $L \neq 0$ is defined as the largest integer c, such that $L^{p,q} = 0$ for $p < c$ or $q < c$.*

If the coniveau of a weight k Hodge structure L is c, then $2c \leq k$ and its Hodge decomposition takes the form

$$L_{\mathbb{C}} = L^{k-c,c} \bigoplus \ldots \bigoplus L^{c,k-c}.$$

In particular we can shift the bidegrees of L to get an effective Hodge structure L' of weight $k-2c$

$$L'_{\mathbb{C}} = L'^{k-2c,0} \bigoplus \ldots \bigoplus L'^{0,k-2c},$$

where $L' = L$ and $L'^{p,q} = L^{p+c,q+c}$. The generalized Hodge conjecture formulated by Grothendieck and discussed in Section 4 investigates the geometric meaning of the Hodge coniveau of $H^k(X,\mathbb{Q})$.

Another information associated with a Hodge structure $(L, L^{p,q})$ of even weight $2k$ is its subgroup of Hodge classes $\mathrm{Hdg}(L)$, with the following

Definition 2.5. *The group* $\mathrm{Hdg}(L)$ *is defined as the set of elements of L which belong to $L^{k,k}$ in the Hodge decomposition of $L_{\mathbb{C}}$. If $L = H^{2k}(X,\mathbb{Q})$, where X is a compact Kähler manifold, then we will use the notation* $\mathrm{Hdg}^{2k}(X)$.

Remark 2.6. We also have $\mathrm{Hdg}(L) = L \cap F^k L_{\mathbb{C}}$, where the Hodge filtration F on $L_{\mathbb{C}}$ is defined in Remark 2.3. Indeed, $L \cap F^k L_{\mathbb{C}} \subset F^k L_{\mathbb{C}} \cap \overline{F^k L_{\mathbb{C}}} = L^{k,k}$.

The Hodge conjecture discussed in Section 3 investigates the meaning of Hodge classes in the cohomology of a complex projective manifold and relates them to objects from complex geometry like closed analytic subsets, or Chern classes of holomorphic vector bundles. The same definition can be made for compact Kähler manifolds, but the results in [Voi02a], [Zuc77] indicate that no version of the Hodge conjecture can be true for compact Kähler manifolds.

2.2 Lefschetz decomposition and Hodge index theorem

A complex projective manifold, being embedded in some projective space $\mathbb{CP}^N$, admits a holomorphic line bundle L which is the restriction to X of the dual of the Hopf line bundle on $\mathbb{CP}^N$ (such a line bundle is said to be very ample). The Chern class $l := c_1(L)$ then belongs to $H^2(X,\mathbb{Z})$ and at the same time it is represented in de Rham cohomology by a positive closed $(1,1)$-form on X, namely the restriction of the Fubini-Study Kähler form on $\mathbb{CP}^N$. This is thus a degree 2 Hodge class (see Definition 2.5). A general compact Kähler manifold admits a Kähler form (i.e. a positive closed $(1,1)$-form) ω but its class cannot in general be chosen rational. The celebrated Kodaira embedding theorem [Kod54] says that a compact Kähler manifold is projective if and only if it admits a Kähler form whose de Rham cohomology class is rational.

A technical but fundamental complement to the Hodge decomposition theorem is the following

Theorem 2.7. *(Hodge, see [GH78, p122] or [Voi02b, Sections 6.2.3 and 6.3.2]) Let X be a compact Kähler manifold of dimension n and ω be a kähler form on X. Then*

(i) (hard Lefschetz theorem) For any $k \leq n$, the cup-product map

$$\smile [\omega]^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R})$$

is an isomorphism.

(ii) (Hodge-Riemann bilinear relations) For $k \leq n$, let $H^k(X,\mathbb{R})_{\mathrm{prim}} \subset H^k(X,\mathbb{R})$ be defined as the kernel of the cup-product map

$$\smile [\omega]^{n-k+1} : H^k(X,\mathbb{R}) \to H^{2n-k+2}(X,\mathbb{R}).$$

As ω is of type $(1,1)$, $H^k(X,\mathbb{C})_{\mathrm{prim}}$ has an induced Hodge decomposition into components $H^{p,q}(X)_{\mathrm{prim}}$ and the sesquilinear form h_ω on $H^k(X,\mathbb{C})$ defined by

$$h_\omega(\alpha,\beta) = i^k \int_X [\omega]^{n-k} \smile \alpha \smile \overline{\beta}$$

has the property that

(a) the Hodge decomposition is orthogonal for h_ω and

(b) the restriction of h_ω to $H^{p,q}(X)_{\mathrm{prim}}$ is definite of sign $(-1)^p$ (up to a global sign depending on k).

When X is projective, we can choose the class $[\omega]$ to be rational, and denote it by l. The results above then produce a lot of extra structure on $H^k(X,\mathbb{Q})$, namely one deduces from the hard Lefschetz theorem (now with rational coefficients) the Lefschetz decomposition

$$H^k(X,\mathbb{Q}) = \bigoplus_{k-2r\geq 0} l^r \smile H^{k-2r}(X,\mathbb{Q})_{\mathrm{prim}}, \tag{7}$$

where each $H^{k-2r}(X,\mathbb{Q})_{\mathrm{prim}} \subset H^{k-2r}(X,\mathbb{Q})$ is a *Hodge substructure*, that is, a rational vector subspace which has an induced Hodge decomposition. Furthermore, the Lefschetz intersection pairing $\langle\,,\,\rangle_l$ on $H^k(X,\mathbb{Q})$ defined by

$$\langle \alpha,\beta\rangle_l = \int_X l^{n-k} \smile \alpha \smile \beta$$

has the property that

(i) The Lefschetz decomposition is orthogonal for $\langle\,,\,\rangle_l$.

(ii) Furthermore, on each piece $l^r \smile H^{k-2r}(X,\mathbb{Q})_{\mathrm{prim}}$, the associated sesquilinear form h_l satisfies the Hodge-Riemann bilinear relations (a) and (b) described in Theorem 2.7.

A Hodge structure $(L,L^{p,q})$ of weight k equiped with a nondegenerate pairing q which is rational, symmetric if k is even, skew-symmetric if k is odd, and such that the associated sesquilinear pairing

$$h_q(\alpha,\beta) = i^k q(\alpha,\overline{\beta})$$

on $L_{\mathbb{C}}$ satisfies the Hodge-Riemann bilinear relations, is called a *polarized Hodge structure*.

2.3 The category of (polarized) Hodge structures

The Hodge structures on the cohomology of compact Kähler manifolds have some functoriality properties that we now describe. If $\phi : X \to Y$ is a morphism (holomorphic map) between compact Kähler manifolds, then the pull-back $\phi^*\alpha$ of a closed form of type (p,q) on Y is a closed form of type (p,q) on X. It follows that

$$\phi^* H^{p,q}(Y) \subset H^{p,q}(X).$$

In other words, the morphism $\phi^* : H^k(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})$ is a morphism of Hodge structures, with the following

Definition 2.8. *A morphism of Hodge structures $(L,L^{p,q})$, $(L',L'^{p',q'})$ of respective weights k and $k+2r$ is a morphism $\phi : L \to L'$ of $\mathbb{Q}$-vector spaces, such that*

$$\phi_{\mathbb{C}} : L_{\mathbb{C}} \to L'_{\mathbb{C}}$$

maps $L^{p,q}$ to $L'^{p+r,q+r}$.

We note that there is the obvious notion of duality for Hodge structures, namely the dual of a Hodge structure $(L,L^{p,q})$ of weight k is the Hodge structure of weight $-k$ given by L^* equiped with the dual Hodge decomposition on $L^*_{\mathbb{C}}$. Observing that for a compact Kähler manifold X of dimension n, the (perfect) Poincaré pairing $\langle\,,\,\rangle_X$ between $H^k(X,\mathbb{Q})$ and $H^{2n-k}(X,\mathbb{Q})$ has the property that

$$\langle \alpha^{p,q},\beta^{p',q'}\rangle = 0 \,\text{if}\, (p',q') \neq (n-p,n-q),$$

we get that (up to a shift of bidegrees by (n,n) that is called a Tate twist) the Hodge structures on $H^k(X,\mathbb{Q})$ and $H^{2n-k}(X,\mathbb{Q})$ are dual.

This gives the covariant functoriality of Hodge structures, namely, if X, Y, ϕ are as above, the Gysin morphism

$$\phi_*: H^k(X,\mathbb{Q}) \to H^{k-2d}(Y,\mathbb{Q}),\, d := \dim X - \dim Y,$$

is a morphism of Hodge structures. Indeed, one has

$$\phi_* = PD_Y^{-1} \circ {}^t(\phi^*) \circ PD_X,$$

where the Poincaré duality isomorphisms considered here are

$$PD_X : H^k(X,\mathbb{Q}) \cong H^{2n-k}(X,\mathbb{Q})^*,\, PD_Y : H^{k-2d}(Y,\mathbb{Q}) \cong H^{2n-k}(Y,\mathbb{Q})^*.$$

A third example of a morphism of Hodge structures that can be constructed on the cohomology of a compact Kähler manifold comes from Hodge classes on X. If $\alpha \in \mathrm{Hdg}^{2l}(X)$, the cup-product map

$$\smile \alpha : H^k(X,\mathbb{Q}) \to H^{k+2l}(X,\mathbb{Q})$$

is a morphism of Hodge structures, because the wedge product of a closed form of type (p,q) and a closed form of type (l,l) is a closed form of type $(p+l,q+l)$.

A more general link between Hodge classes and morphisms of Hodge structures is given by the following

Lemma 2.9. *Let X,Y be compact Kähler manifolds. Let $\alpha \in H^{2k}(X\times Y,\mathbb{Q})$ and, for each integer $l \geq 0$, let*

$$\alpha_{*,l} : H^l(X,\mathbb{Q}) \to H^{l+2k-2n}(Y,\mathbb{Q}),\, n := \dim X$$

be defined by

$$\alpha_{*,l}(\gamma) = \mathrm{pr}_{Y*}(\alpha \smile \mathrm{pr}_X^*\gamma). \tag{8}$$

Then α is a Hodge class on $X\times Y$ if and only if the morphisms $\alpha_{*,l}$ are morphisms of Hodge structures for all l.

Remark 2.10. The collection of the morphisms $\alpha_{*,l}$ is equivalent, using Poincaré duality, to the data of the so-called Künneth components of α, obtained using the direct sum decomposition

$$H^{2k}(X\times Y,\mathbb{Q}) = \oplus_{r+s=2k} H^r(X,\mathbb{Q}) \otimes H^s(Y,\mathbb{Q}).$$

We end this section with the following important result.

Proposition 2.11. *The category of polarizable rational Hodge structures is semi-simple.*

Proof. We have to show that if $H' \subset H$ is a Hodge substructure, where H is polarizable, then there exists a Hodge substructure $H'' \subset H'$ such that $H = H' \bigoplus H''$ as Hodge structures. This is done by proving that the pairing q giving a polarization on H remains nondegenerate on H', which allows to set $H'' := H'^{\perp q}$. The nondegeneracy of $q_{|H'}$ is proved using the Hodge-Riemann bilinear relations (Theorem 2.7 (a) and (b)), which also imply that H'' is a Hodge substructure. ∎

Proposition 2.11 is completely wrong in the unpolarized setting. For example, it is not true for weight 1 unpolarized Hodge structures, that are associated to complex tori (see [Voi02b, 7.2.2]). In that case, the geometric meaning of Proposition 2.11 is the following: If T is a complex compact torus and $T' \subset T$ is a complex subtorus, then in general T is not isogenous to a product $T' \times T''$ of complex tori. However, this statement is true if T is a projective complex torus, that is, an abelian variety.

One consequence of Proposition 2.11 is

Corollary 2.12. *Let H, H' be Hodge structures of weight $2k$, with H' polarized, and let $\phi : H' \to H$ be a surjective morphism of Hodge structures. Then*

$$\phi : \mathrm{Hdg}(H') \to \mathrm{Hdg}(H)$$

is surjective.

Indeed, this follows from the fact that, thanks to Proposition 2.11, ϕ has a left inverse as morphism of Hodge structures.

3 The Hodge conjecture and standard conjectures

3.1 Cycle classes and Chern classes

Let X be a compact complex manifold, and let $j : Z \hookrightarrow X$ be a closed irreducible complex analytic subset of dimension d, that is, Z is closed, locally defined by holomorphic equations, and, away from a closed analytic subset $Z' \subset Z$ which is nowhere dense in Z, Z is a connected complex submanifold of dimension d of X. The cycle class $[Z] \in H^{2c}(X, \mathbb{Z})$, $c := \dim X - d$, has been constructed first in [BH61] (see also [GH78, p61] for a version with real coefficients). An easy construction using Hironaka's resolution of singularities [Hir64] goes as follows: there exists a resolution of singularities $\tau : \widetilde{Z} \to Z$ of Z, that is, $\widetilde{Z}$ is a complex manifold and τ is a proper holomorphic map which is an isomorphism above $Z \setminus \mathrm{Sing}\, Z$. Denoting $\tilde{j} := j \circ \tau : \widetilde{Z} \to X$, we thus have a Gysin morphism

$$\tilde{j}_* : H^0(\widetilde{Z}, \mathbb{Z}) \to H^{2c}(X, \mathbb{Z}),$$

which provides and defines a class

$$[Z] := \tilde{j}_*(1_{\widetilde{Z}}),$$

which is easily shown to be independent of the chosen resolution. If X is now a compact Kähler manifold, the morphism $\tilde{j}_*$ is a morphism of Hodge structures, hence the class $[Z]$ is an integral Hodge class on X. The subgroup of $H^{2c}(X, \mathbb{Z})$ generated by these classes is called the group of codimension c analytic cycle classes.

Another method to construct Hodge classes is due to Chern and uses Chern classes of holomorphic vector bundles E on X. The topological Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ depend only on the underlying topological complex vector bundle. Following [BT82], the Chern classes (in real de Rham cohomology) can be represented by choosing a complex connection ∇ on E, with curvature operator

$$R_\nabla \in \Gamma(X, \mathcal{A}_X^2 \otimes \mathcal{E}nd\, E).$$

Then the closed differential forms $\alpha_i := \sigma_i(\frac{R_\nabla}{2\iota\pi})$ represent the classes $c_i(E)$, where σ_i is the polynomial functions on matrices of size (r,r) which to a matrix associates the i-th symmetric function of its eigenvalues. The holomorphic structure of E and the data of a Hermitian metric h on E determine a complex connection ∇ on E (the Chern connection) having the property that the curvature operator R_∇ is of type $(1,1)$, that is, belongs to $\Gamma(X, \mathcal{A}_X^{1,1} \otimes \mathcal{E}nd\, E)$. It then follows that the corresponding representative α_i is a (real) closed form of type (i,i), hence the classes $c_i(E)$ are Hodge classes.

When X is a projective complex manifold, passing to $\mathbb{Q}$-coefficients, the $\mathbb{Q}$-vector spaces generated by classes $c_i(E)$ and by cycle classes $[Z]$ are equal. To see that cycle classes are combinations of Chern classes, one first constructs the extension of the theory of Chern classes to analytic coherent sheaves (see [BS58]). The existence (in the projective setting) of a finite locally free resolution

$$0 \to \mathcal{E}^n \to \ldots \to \mathcal{E}^0 \to \mathcal{F} \to 0$$

for any coherent sheaf $\mathcal{F}$, provides by the Whitney formula the equality

$$c(\mathcal{F}) = \prod_i c(\mathcal{E}^i)^{\varepsilon_i},\ \varepsilon_i := (-1)^i, \tag{9}$$

where $c(\mathcal{E}) := 1 + c_1(\mathcal{E}) + \ldots + c_n(\mathcal{E}) \in H^*(X, \mathbb{Q})$ is the total Chern class of any coherent sheaf $\mathcal{E}$. We use finally the Grothendieck-Riemann-Roch formula

$$c_c(\mathcal{O}_Z) = (-1)^{c-1}(c-1)![Z] \text{ in } H^{2c}(X, \mathbb{Z}), \tag{10}$$

valid for any codimension c closed analytic subset Z. Formula (10) expresses cycle classes with rational coefficients as Chern classes of coherent sheaves, and formula (9) shows that Chern classes of coherent sheaves do not provide more classes than Chern classes of algebraic vector bundles.

In the other direction, Chern classes of vector bundles can be expressed as combinations of cycle classes using the fact that for any holomorphic vector bundle E on X, an adequate twist $E \otimes L$, where L is a very ample line bundle on X, is generated by global sections, hence is the pull-back of a tautological vector bundle on a Grassmannian via a holomorphic morphism $\phi : X \to G(k,n)$. Finally, one uses the fact that the whole integral cohomology of any Grassmannian is generated by cycle classes, as shows the theory of Schubert varieties.

These comparisons and arguments do not work in the general compact Kähler setting, as shown in [Voi02a], where examples of coherent sheaves without locally free resolutions on compact Kähler manifolds are exhibited. The $\mathbb{Q}$-vector space generated by Chern classes of coherent sheaves can be strictly larger than the one generated by Chern classes of vector bundles, and also than the one generated by analytic cycles classes as in the example of [Zuc77].

The statement of the Hodge conjecture is the following

Conjecture 3.1. *(Hodge conjecture) let X be a smooth projective complex manifold. Then for any c, the $\mathbb{Q}$-vector space $H^{2c}(X, \mathbb{Q})_{\mathrm{alg}} \subset H^{2c}(X, \mathbb{Q})$ of codimension c cycle classes of X is equal to the $\mathbb{Q}$-vector space $\mathrm{Hdg}^{2c}(X) \subset H^{2c}(X, \mathbb{Q})$ of degree $2c$ Hodge classes of X.*

Remark 3.2. By Chow's theorem [Cho49], closed analytic subsets of a projective complex variety X are also closed algebraic. The cycle classes will thus be called "algebraic classes".

It was known since Atiyah-Hirzebruch [AH62] that the similar statement with integral coefficients fails, and quite different examples have been exhibited by Kollár in [Kol90]. In [Voi02a], it is shown that the extension of the Hodge conjecture to compact Kähler manifolds, replacing "cycles classes" by "Chern classes of coherent sheaves", is wrong.

The Hodge conjecture is obvious for Hodge classes of degree 0 (the class of a point) and degree $2n$, $n = \dim X$, (the class of a point). The only other cases where it is known in general follow from

Theorem 3.3. *(Lefschetz theorem on $(1,1)$-classes) Let X be a compact Kähler manifold and $\alpha \in H^2(X,\mathbb{Z})$ be an integral Hodge class, that is, its image $\alpha_{\mathbb{C}}$ in $H^2(X,\mathbb{C})$ is of type $(1,1)$. Then*

(i) there exists a holomorphic line bundle L on X such that $c_1(L) = \alpha$.

(ii) If X is projective, there exists a divisor D, namely an integral combination $\sum_i n_i D_i$ with $D_i \subset X$ analytic hypersurfaces, such that $\alpha = [D] := \sum_i n_i [D_i]$.

Proof. Statement (ii) follows from (i), as discussed above. The proof of (i) follows from a sheaf cohomology argument. Let $\mathcal{O}_X$ and $\mathcal{O}_X^*$ be respectively the sheaves of holomorphic functions and invertible holomorphic functions on X. Then one has the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1,$$

a short exact sequence of sheaves that says that an invertible holomorphic function is locally the exponential of a holomorphic function. This induces a long exact sequence of sheaf cohomology groups

$$\ldots H^1(X,\mathcal{O}_X^*) \stackrel{\delta}{\to} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)\ldots \tag{11}$$

One notes that the group of (isomorphism classes of) holomorphic line bundles on X (where the group structure is given by the tensor product) is isomorphic to $H^1(X,\mathcal{O}_X^*)$. Indeed, this follows from local trivializations of holomorphic line bundles, with transition matrices given by invertible holomorphic functions. Then one shows that the connecting map δ is nothing but the first Chern class (in fact this can be taken as a definition of the first Chern class). Finally, the last map in (11) can be identified to the map

$$H^2(X,\mathbb{Z}) \ni \alpha \mapsto \alpha_{\mathbb{C}}^{0,2} \in H^{0,2}(X) = H^2(X,\mathcal{O}_X).$$

This concludes the proof since by assumption $\alpha_{\mathbb{C}}$ is of type $(1,1)$, hence its $(0,2)$-component is 0. □

Remark 3.4. The argument given above also shows that on any affine complex variety X^0, namely the complement of a hyperplane section H in a projective complex manifold X, any integral degree 2 cohomology class can be written as $c_1(M^0)$ for some holomorphic line bundle M^0 on X^0, (and a similar result is true for higher even degree rational cohomology, see [CG75]). Indeed, on the affine variety X^0, the cohomology groups $H^i(X^0,\mathcal{O}_{X^0})$ vanish for $i > 0$, hence in particular $H^2(X^0,\mathcal{O}_{X^0}) = 0$. A strategy for an analytic approach to the Hodge conjecture would be to start from *holomorphic* vector bundles on X^0 and to describe which of them extend, at least as coherent sheaves, to X, with the hope that the Hodge condition on Chern classes is the only obstruction. This strategy is described in [CG75].

Remark 3.5. It is to be noted that the proof of Theorem 3.3 is of a transcendental nature, since it uses the exponential exact sequence that has no analogue in algebraic geometry. However, when X is projective, we get at the end algebraic vector bundles and algebraic cycles since holomorphic vector bundles on X are algebraic by [Ser56] and closed analytic subsets of X are algebraic by [Cho49].

Theorem 3.3 implies now the following other and last case where the Hodge conjecture is known.

Corollary 3.6. *Let X be a projective complex manifold of dimension n and let $\alpha \in \mathrm{Hdg}^{2n-2}(X,\mathbb{Q})$. Then there exists a 1-cycle $Z = \sum_i n_i Z_i$ with $\mathbb{Q}$-coefficients n_i, such that*

$$\alpha = [Z] := \sum_i n_i [Z_i] \text{ in } H^{2n-2}(X,\mathbb{Q}).$$

Proof. We choose an ample line bundle L on X, set $l := c_1(L)$, and use the Lefschetz isomorphism

$$\smile l^{n-2} : H^2(X,\mathbb{Q}) \to H^{2n-2}(X,\mathbb{Q}).$$

This isomorphism is an isomorphism of Hodge structures, so there exists a Hodge class $\beta \in \mathrm{Hdg}^2(X)$ such that

$$\alpha = l^{n-2} \smile \beta.$$

By Theorem 3.3, (ii), $\beta = [D] = \sum_i n_i D_i$ for some divisor of X with rational coefficients. As L is very ample, the class l is the class of any hyperplane section H_i of X. Then for general choices of H_i

$$\alpha = \sum_i n_i [H_1 \cap \ldots \cap H_{n-2} \cap D_i] \in H^{2n-2}(X,\mathbb{Q}),$$

and each $Z_i = H_1 \cap \ldots \cap H_{n-2} \cap D_i$ is a closed algebraic curve in X. □

Remark 3.7. A big difference with the divisor case is that Corollary 3.6 is not true in general with integral coefficients, as show examples constructed by Kollár [Kol90].

3.2 Standard conjectures and some consequences

The natural question concerning the Hodge conjecture is: are there so many examples of Hodge classes that are not trivially algebraic? The answer is yes and no. We refer to [BKU24], [KO21] for results concerning the sparsity of Hodge classes, except in the divisor case. In the opposite direction, we will spell-out in this section many examples of Hodge classes on powers of any given projective variety, all formally constructed by application of Lemma 2.9, and for which the Hodge conjecture is still open. The Hodge conjecture applied to these Hodge classes gives rise to the so-called standard conjectures (see [Kle68]) and have strong consequences on the theory of cohomological versus numerical motives. In Section 3.4, we will exhibit an arithmetic property satisfied by these standard Hodge classes, and that is conjecturally satisfied by all Hodge classes, as a consequence of the Hodge conjecture.

3.2.1 Künneth components of the diagonal

Let X be a smooth projective variety of dimension n. The cohomology class

$$\delta_X = [\Delta_X] \in H^{2n}(X \times X,\mathbb{Q})$$

of the diagonal $\Delta_X \subset X \times X$ of X acts as the identity on $H^k(X,\mathbb{Q})$ for any k, via the formalism described in (8). The Künneth decomposition theorem combined with Poincaré duality says that

$$H^*(X \times X,\mathbb{Q}) \cong \mathrm{End}(H^*(X,\mathbb{Q})) \tag{12}$$
$$H^{2n}(X \times X,\mathbb{Q}) \cong \mathrm{End}_0(H^*(X,\mathbb{Q})), \tag{13}$$

where in (13), the subscript 0 means “degree preserving endomorphisms". Using (13), we can write

$$\delta_X = \delta_0 + \delta_1 + \ldots + \delta_{2n} \text{ in } H^{2n}(X \times X,\mathbb{Q}), \tag{14}$$

where $\delta_i \in \mathrm{End}_0(H^*(X,\mathbb{Q}))$ acts as the identity on $H^i(X,\mathbb{Q})$ and as 0 on $H^j(X,\mathbb{Q})$ for $j \neq i$. By Lemma 2.9, each δ_i provides a rational Hodge class on $X \times X$. By definition the class $\delta_X = \sum_i \delta_i = [\Delta_X]$ is algebraic but it is not known in general if each δ_i is algebraic. This problem is the *Künneth standard conjecture* and is stated in [Kle68]. The only general results are

Proposition 3.8. *(See [Kle72]) Let X be smooth projective of dimension n. Then the Künneth components δ_i of X are algebraic when $i = 0, 1, 2n-1, 2n$.*

Proof. For δ_0, δ_{2n}, this is obvious since they are respectively the classes of $\mathrm{pt} \times X$ and $X \times \mathrm{pt}$, where pt is any point in X. For δ_1, by the Lefschetz theorem on hyperplane sections, if we consider a smooth curve $j : C \hookrightarrow X$ which is a complete intersection of hyperplane sections of X, then $j_* : H^1(C,\mathbb{Q}) \to H^{2n-1}(X,\mathbb{Q})$ is surjective, and the class $\delta_1 \in H^{2n-1}(X,\mathbb{Q}) \otimes H^1(X,\mathbb{Q})$ belongs to

$$\mathrm{Im}\,(j_* \otimes \mathrm{Id} : H^1(C,\mathbb{Q}) \otimes H^1(X,\mathbb{Q}) \to H^{2n-1}(X,\mathbb{Q}) \otimes H^1(X,\mathbb{Q})).$$

The morphism $j_* \otimes \mathrm{Id} = (j,\mathrm{Id})_*$ is a morphism of polarized Hodge structures, hence we can apply Corollary 2.12 to conclude that

$$\delta_1 = (j,\mathrm{Id})_*(\beta)$$

for some degree 2 Hodge class

$$\beta \in H^1(C,\mathbb{Q}) \otimes H^1(X,\mathbb{Q}) \subset H^2(C \times X,\mathbb{Q}).$$

The Hodge conjecture being known for degree 2 Hodge classes, we conclude that β is algebraic, hence δ_1 is algebraic. Similarly for δ_{2n-1}. □

Corollary 3.9. *The Künneth standard conjecture is true for smooth complex projective surfaces.*

Indeed, for a surface, the only Künneth components of the diagonal are δ_0, δ_1, δ_2, δ_3, δ_4. We know that δ_0, δ_1, δ_3, δ_4 are algebraic by Proposition 3.8, and that $\sum_i \delta_i$ is algebraic, so δ_2 is also algebraic.

Remark 3.10. The arguments above are given a sophisticated version in [Mur90], where a Chow-Künneth decomposition of the diagonal is given.

3.2.2 Lefschetz inverse isomorphisms

Let X be a smooth projective variety over $\mathbb{C}$ (or complex projective manifold) and let L be a very ample line bundle on X. Recalling the hard Lefschetz isomorphism of Theorem 2.7(i), that we apply to $[\omega] = l := c_1(L)$, we get inverse isomorphisms

$$\gamma_k := (l^{n-k})^{-1} : H^{2n-k}(X,\mathbb{Q}) \to H^k(X,\mathbb{Q}) \tag{15}$$

for $k \leq n = \dim X$. Note that γ_k is a morphism of Hodge structures, hence provides by Lemma 2.9 a Hodge class of degree $2k$ on $X \times X$. The *Lefschetz standard conjecture* for degree k cohomology is the following statement.

Conjecture 3.11. *For each $k \leq n$, there exists a codimension k cycle Z in $X \times X$, such that the class $[Z] \in \mathrm{Hdg}^{2k}(X \times X)$ satisfies*

$$[Z]_* = (l^{n-k})^{-1} : H^{2n-k}(X,\mathbb{Q}) \to H^k(X,\mathbb{Q}). \tag{16}$$

Conjecture 3.11 and the Künneth standard conjecture are known for abelian varieties, that is, projective complex tori (see [Lie68]). To prove the Künneth standard conjecture, we observe that there are plenty of interesting cycles in $A \times A$, where A is a projective complex torus. Namely, A being also an abelian group, the multiplication by i maps

$$\mu_i : A \to A$$
$$a \mapsto ia$$

are holomorphic. One has

$$\mu_i^* = i^k \mathrm{Id} : H^k(A,\mathbb{Q}) \to H^k(A,\mathbb{Q}). \tag{17}$$

Formula (17) and the definition of the Künneth projectors δ_k of (14) provide the following formula

$$[\Gamma_i] = \sum_k i^k \delta_k \text{ in } H^{2g}(A \times A,\mathbb{Q}) \tag{18}$$

where $g = \dim A$ and $\Gamma_i \subset A \times A$ is the graph of μ_i. Using (18) for several values of i, a Vandermonde determinant argument tells us that the Künneth components of the diagonal can be computed as combinations with rational coefficients of classes $[\Gamma_i]$ for various i's, hence they are cycle classes on $A \times A$. Note that this argument works as well for any complex torus and does not use the algebraicity of A. Finally we can construct algebraic cycles first in A by taking successive powers θ^i of an ample divisor class θ, and then on $A \times A$ using the pull-back under the sum map

$$\mu : A \times A \to A,$$
$$(a,b) \mapsto a+b.$$

It is not hard to prove the Lefschetz standard conjecture for A using the classes $\mu^*\theta^i$.

Conjecture 3.11 has been proved only recently in [CM13] for a more general class of smooth projective varieties, namely projective hyper-Kähler manifolds of $K3^{[n]}$ deformation type (see [Bea83]).

The Lefschetz standard conjecture has very important consequences and thus deserves to be formulated independently of the Hodge conjecture. We start with

Proposition 3.12. *Assume the Lefschetz standard conjecture is true for X. Then the intersection pairing*

$$H^{2k}(X,\mathbb{Q})_{\mathrm{alg}} \times H^{2n-2k}(X,\mathbb{Q})_{\mathrm{alg}} \to \mathbb{Q}$$

obtained by restricting the Poincaré pairing of X to the $\mathbb{Q}$-vector subspaces

$$H^{2k}(X,\mathbb{Q})_{\mathrm{alg}} \subset H^{2k}(X,\mathbb{Q}),\ H^{2n-2k}(X,\mathbb{Q})_{\mathrm{alg}} \subset H^{2n-2k}(X,\mathbb{Q})$$

generated by cycle classes, is perfect.

Remark 3.13. Proposition 3.12 illustrates the importance of the Lefschetz standard conjecture for the theory of motives. Indeed, it says that, assuming the Lefschetz standard conjecture, numerical equivalence of cycles (where a cycle is said numerically equivalent to zero if it has trivial intersection number with cycles of the complementary dimension) is the same as cohomological equivalence of cycles. Numerical equivalence is a purely algebraic notion, while cohomological equivalence needs a cohomology theory with a cycle class in order to be defined.

Proof of Proposition 3.12. We can assume $2k \le n$. We choose an ample class $l = c_1(L)$ and consider the hard Lefschetz isomorphism $l^{n-2k} \smile : H^{2k}(X,\mathbb{Q}) \cong H^{2n-2k}(X,\mathbb{Q})$. Being an isomorphism of Hodge structures, it induces an isomorphism $l^{n-2k} \smile : \mathrm{Hdg}^{2k}(X,\mathbb{Q}) \cong \mathrm{Hdg}^{2n-2k}(X,\mathbb{Q})$. Furthermore it preserves the spaces of algebraic classes, since $l^{n-2k} \smile [Z] = [D_1 \cap \ldots D_{n-2k} \cap Z]$ for general hypersurfaces D_i of X with $[D_i] = c_1(L)$.

If we now assume the Lefschetz standard conjecture, we can even conclude that

$$l^{n-2k} \smile : H^{2k}(X,\mathbb{Q})_{\mathrm{alg}} \to H^{2n-2k}(X,\mathbb{Q})_{\mathrm{alg}} \tag{19}$$

is an isomorphism, with left inverse given by $[Z]_*$, where $Z \subset X \times X$ is a Lefschetz cycle for degree $2k$ cohomology.

Once one has this statement for all even degrees $\le 2k$, one concludes that the Lefschetz pairing

$$\langle \alpha, \beta \rangle_{\mathrm{lef}} = \int_X l^{n-2k} \smile \alpha \smile \beta$$

is a perfect pairing on $H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$, (which, thanks to the isomorphism (19), implies the desired statement that $H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$ and $H^{2n-2k}(X,\mathbb{Q})_{\mathrm{alg}}$ are dual,) by the following argument. Having the Lefschetz isomorphisms on the subalgebra $H^{2*}(X,\mathbb{Q})_{\mathrm{alg}} \subset \mathrm{Hdg}^{2*}(X)$, we conclude that each $H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$ is stable under the Lefschetz decomposition, hence can be decomposed as

$$H^{2k}(X,\mathbb{Q})_{\mathrm{alg}} = \bigoplus_{2r \le 2k} l^r H^{2k-2r}(X,\mathbb{Q})_{\mathrm{alg,prim}}. \tag{20}$$

Looking at the Hodge-Riemann relations (Theorem 2.7(ii)), we get that $\langle\,,\rangle_{\mathrm{lef}}$ is definite on each $l^r H^{k-r,k-r}(X,\mathbb{R})_{\mathrm{prim}}$, hence remains nondegenerate on each subspace $l^r H^{2k-2r}(X,\mathbb{R})_{\mathrm{alg,prim}}$. We then conclude using the fact that the decomposition (20) is orthogonal for $\langle\,,\rangle_{\mathrm{lef}}$. $\square$

Remark 3.14. As the same proof as above shows, it is always true (that is, without assuming the Lefschetz standard conjecture) that the intersection pairing

$$\mathrm{Hdg}^{2k}(X) \times \mathrm{Hdg}^{2n-2k}(X) \to \mathbb{Q}$$

obtained by restricting the Poincaré pairing of X to the $\mathbb{Q}$-vector subspaces

$$\mathrm{Hdg}^{2k}(X) \subset H^{2k}(X,\mathbb{Q}),\ \mathrm{Hdg}^{2n-2k}(X) \subset H^{2n-2k}(X,\mathbb{Q})$$

of Hodge classes, is perfect.

Let us give two formal but important corollaries.

Corollary 3.15. *Let $\phi : X \to Y$ be a morphism between smooth complex projective manifolds, and let $d := \dim X - \dim Y$. Assume X and Y satisfy the Lefschetz standard conjecture.*

(i) If $\alpha \in H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$ can be written as $\alpha = \phi^*\beta$, for some $\beta \in H^{2k}(Y,\mathbb{Q})$, then there exists $\beta' \in H^{2k}(Y,\mathbb{Q})_{\mathrm{alg}}$ such that $\alpha = \phi^*\beta'$.

(ii) If $\alpha \in H^{2k}(Y,\mathbb{Q})_{\mathrm{alg}}$ can be written as $\alpha = \phi_*\beta$, for some $\beta \in H^{2k+2d}(X,\mathbb{Q})$, then there exists $\beta' \in H^{2k+2d}(X,\mathbb{Q})_{\mathrm{alg}}$ such that $\alpha = \phi_*\beta'$.

Proof. We prove only (i), and in fact the more general statement concerns any correspondences between X and Y. Let $m := \dim Y$. The class β produces an element of $H^{2m-2k}(Y,\mathbb{Q})^*$, hence by restriction an element β^* of $H^{2m-2k}(Y,\mathbb{Q})^*_{\mathrm{alg}}$. By Proposition 3.12 applied to Y, there is an element $\beta' \in H^{2k}(Y,\mathbb{Q})_{\mathrm{alg}}$ such that

$$\beta^* = (\beta')^*.$$

We finally prove that $\alpha = \phi^*\beta'$ using Proposition 3.12 applied to X. □

The following corollary of Corollary 3.15(i) appears in [And06].

Corollary 3.16. *Assume the Lefschetz standard conjecture. Let $\mathcal{X}$ be a smooth projective variety and $\phi : \mathcal{X} \to B$ be a dominant morphism, where B is a connected projective variety, everything being defined over $\mathbb{C}$. Let $\alpha \in \mathrm{Hdg}^{2k}(\mathcal{X})$ be a Hodge class. Then, if there exists a point $b \in B$ such that the fiber $\mathcal{X}_b$ is smooth and the restriction $\alpha_{|\mathcal{X}_b}$ is algebraic, the restriction $\alpha_{|\mathcal{X}_{b'}}$ is algebraic for all smooth fibers $\mathcal{X}_{b'}$.*

Indeed, we apply Corollary 3.15(i) to $\mathcal{X} = Y$ and $\mathcal{X}_b = X$. Note that, in Corollary 3.16, we only need that $\mathcal{X}$ and $\mathcal{X}_b$ satisfy the Lefschetz standard conjecture.

3.3 Weil Hodge classes on Weil abelian varieties

One dissatisfactory point concerning the Hodge conjecture is the fact that, apart from the formal manipulations explained above, it is very hard to produce interesting Hodge classes which are not obviously algebraic. In this section, we will describe a rather explicit construction of Hodge classes on certain abelian varieties. We refer to [vG94] for more detail. An abelian variety over the complex numbers can be seen as a complex torus which is projective. A compact complex torus is a quotient $T = \mathbb{C}^n/\Gamma$, where

$\Gamma \subset \mathbb{C}^n$ is a discrete lattice of rank $2n$. The dual lattice Γ^* is canonically isomorphic to $H^1(T,\mathbb{Z})$. The inclusion $\Gamma \subset \mathbb{C}^n$ gives rise to a surjective morphism of complex vector spaces

$$\Gamma_{\mathbb{C}} \to \mathbb{C}^n$$

with kernel $K \subset \Gamma_{\mathbb{C}}$. Dually, we get an exact sequence

$$0 \to K^{\perp} \to \Gamma_{\mathbb{C}}^* \to K^* \to 0$$

and it is not hard to see that the n-dimensional subspace $K^{\perp} \subset \Gamma_{\mathbb{C}}^*$ identifies to $H^{1,0}(T) \subset H^1(T,\mathbb{C})$ (this corresponds to the space of holomorphic 1-forms on T). Furthermore $K^{\perp}$ determines T and all the Hodge structures on $H^*(T,\mathbb{Z})$ since $H^k(T,\mathbb{Z}) \cong \bigwedge^k H^1(T,\mathbb{Z})$ as Hodge structures.

We assume now that T admits an endomorphism ϕ that satisfies $\phi^2 = -d\mathrm{Id}_T$ for some positive integer d. The endomorphism ϕ induces a morphism $\phi^* : H^1(T,\mathbb{Z}) \to H^1(T,\mathbb{Z})$ which satisfies the same quadratic equation and is a morphism of Hodge structures, giving

$$\phi^*(H^{1,0}(T)) \subset H^{1,0}(T),\ \phi^*(H^{0,1}(T)) \subset H^{0,1}(T).$$

Conversely, such a morphism of Hodge structures induces an endomorphism of T.

A Weil complex torus (see [Wei]) is a complex torus of even dimension $n = 2m$, equipped with a quadratic endomorphism ϕ as above, with the property that ϕ^* acting on $H^{1,0}(T)$ has m eigenvalues equal to $i\sqrt{d}$ and thus m eigenvalues equal to $-i\sqrt{d}$. For example, if we start from a complex torus T_+ of dimension m, admitting an endomorphism ϕ_+ satisfying $\phi_+^2 = -d\mathrm{Id}_{T_+}$ and acting by $i\sqrt{d}$ on $H^{1,0}(T_+)$, then

$$T = T_+ \times T_+,\ \phi = (\phi_+, -\phi_+)$$

provides a Weil complex torus. Being non-simple, it is of course non-generic.

Given an endomorphism ϕ^* acting on $H^1(T,\mathbb{Z})$, with eigenspaces $W^+, W^- \subset H^1(T,\mathbb{C})$ associated respectively with the eigenvalues $i\sqrt{d}$, $-i\sqrt{d}$, a Weil complex torus with these given topological data is determined by the data of the two m-dimensional vector spaces

$$H^{1,0^+} \subset W^+,\ H^{1,0^-} \subset W^-,$$

such that

$$H^{1,0}(T) = H^{1,0^+} \bigoplus H^{1,0^-},\ H^{0,1}(T) = \overline{H^{1,0^+}} \bigoplus \overline{H^{1,0^-}},$$

and thus

$$W^+ = H^{1,0^+} \bigoplus \overline{H^{1,0^-}}. \tag{21}$$

Consider now the 1-dimensional vector subspace

$$\bigwedge^{2m} W^+ \subset \bigwedge^{2m} H^1(T,\mathbb{C}) = H^{2m}(T,\mathbb{C}).$$

Using (21), we get $\bigwedge^{2m} W^+ \subset H^{m,m}(T)$, hence also $\bigwedge^{2m} \overline{W^+} \subset H^{m,m}(T)$. Finally we have

Lemma 3.17. *There exists a $\mathbb{Q}$-vector subspace $M \subset H^{2m}(T,\mathbb{Q})$ such that*

$$M_{\mathbb{C}} = \bigwedge^{2m} W^+ \bigoplus \bigwedge^{2m} \overline{W^+} \subset H^{2m}(T,\mathbb{C}).$$

Proof. Indeed, let $L := \mathbb{Q}(\sqrt{-d})$. Then $H^1(T,\mathbb{Q})$ is a n-dimensional L-vector space and we construct M as the trace of the 1-dimensional L-vector space $\bigwedge_L^n H^1(T,\mathbb{Q})$. □

Lemma 3.17 shows that T has a 2-dimensional space of Hodge classes, called Weil classes. It is proved in [Voi02a] that for a very general Weil complex torus T as above, with $m \geq 2$, any coherent sheaf $\mathcal{F}$ on T has trivial Chern classes, so the Weil classes are not Chern classes of coherent sheaves on T. We turn now to the projective case, where the Hodge conjecture predicts that the Weil classes are algebraic.

In order to make a Weil complex torus T algebraic, it suffices by the Kodaira embedding theorem to have an integral Hodge class of degree 2 on T whose $(1,1)$-representative by a constant 2-form on the universal cover of T is a positive $(1,1)$-form (this is called a polarization on T). As ϕ_T^* acts on $H^2(T,\mathbb{Z})$ preserving integral Hodge classes and positive $(1,1)$-classes, and furthermore satisfies $(\phi_T^*)^2 = d^2 \mathrm{Id}_{H^2(T,\mathbb{Z})}$, ϕ_T^* has the eigenvalues d and $-d$ on $H^2(T,\mathbb{Z})$ and we can always find, when T is algebraic, a polarization λ such that $\phi_T^*\lambda = d\lambda$. The very general projective Weil torus has in fact Picard number 1, and the polarization is unique, assuming it is nondivisible. We thus get another numerical invariant, which is related to the degree of the polarization and is called the discriminant of the polarized Weil abelian variety.

The known results on the Hodge conjecture for abelian varieties are as follows. First of all, Moonen and Zharhin [MZ95] proved that, in the case of abelian fourfolds, the Hodge conjecture reduces to the Hodge conjecture on Weil abelian fourfolds. More precisely, their algebra of Hodge classes is generated by degree 2 Hodge classes and Weil type Hodge classes. Next, after a first work by Schoen [Sch88] (see also [Sch07]), Markman proved in [Mar23] the Hodge conjecture for Weil abelian 4-folds of discriminant 1, using a long detour through hyper-Kähler manifolds of generalized Kummer type and some work of O'Grady [O'G21]. Finally, Markman [Mar25] proved recently by a completely different method the Hodge conjecture for Weil abelian 6-folds of discriminant 1. By a specialization argument, this implies the Hodge conjecture for Weil abelian 4-folds of any discriminant, and thus the Hodge conjecture for all abelian fourfolds by [MZ95], which is a remarkable achievement.

3.4 Hodge loci and absolute Hodge classes

We discuss in this section the variational theory of Hodge classes, and more precisely the structure of *Hodge loci*. We first discuss the Grothendieck-Serre isomorphism, which is a crucial bridge between algebraic geometry over $\mathbb{C}$ and topology. Let X be a smooth algebraic variety defined over a field K. Then one can define the locally free sheaves $\Omega_{X/K}$ of Kähler differentials, their exterior powers $\Omega_{X/K}^l := \bigwedge^l \Omega_{X/K}$, and the exterior differential $d : \Omega_{X/K}^l \to \Omega_{X/K}^{l+1}$, which satisfies as usual $d \circ d = 0$. If $K = \mathbb{C}$, and $X \subset \mathbb{P}_{\mathbb{C}}^N$ is projective, X is covered by affine open sets $X_i = U_i \cap X$, where $U_i \subset \mathbb{P}_{\mathbb{C}}^N$ is the complement of a hyperplane, so $U_i \cong \mathbb{A}^N$. The algebraic differential forms on X, restricted to the Zariski open sets X_i, can then be described as restrictions to X_i of algebraic differential forms $\sum_I \alpha_I dz_I$ on the ambient space $U_i \cong \mathbb{A}^N$ with linear coordinates z_i, $i = 1, \ldots, N$, where the α_I are polynomials on $\mathbb{A}^N$.

We can thus define algebraic de Rham cohomology (see [Gro66], [Har75]) of X as

$$H^l_{\mathrm{dR}}(X/K) := \mathbb{H}^l(X, \Omega^\bullet_{X/K}). \tag{22}$$

The right hand side is hypercohomology of the complex $\Omega^\bullet_{X/K}$ on the algebraic variety X (see [Voi03, 8.1]). This is a K-vector space, and if we extend the definition field $K \subset K'$, we get

$$H^l_{\mathrm{dR}}(X_{K'}/K') = H^l_{\mathrm{dR}}(X/K) \otimes_K K'.$$

Assume now that $K = \mathbb{C}$. Then we observe that the analytization $\Omega^l_{X/\mathbb{C},\mathrm{an}}$, that is, the sheaf of holomorphic sections of the algebraic vector bundle $\Omega^l_{X/\mathbb{C}}$, is naturally isomorphic to the analytic coherent sheaf $\Omega^l_{X_{\mathrm{an}}}$ of holomorphic l-forms on the associated complex manifold X_{an}. When X is projective, GAGA comparison theorem [Ser56] thus gives an isomorphism

$$H^l_{\mathrm{dR}}(X/\mathbb{C}) \cong \mathbb{H}^l(X_{\mathrm{an}}, \Omega^\bullet_{X_{\mathrm{an}}}). \tag{23}$$

Finally, by the holomorphic Poincaré lemma, the complex $\Omega^\bullet_{X_{\mathrm{an}}}$ of holomorphic differential forms on X_{an} is locally exact in degree > 0 in the Euclidean topology, hence is a resolution of the constant sheaf $\mathbb{C}$ on X_{an}. We thus conclude that

$$\mathbb{H}^l(X_{\mathrm{an}}, \Omega^\bullet_{X_{\mathrm{an}}}) \cong H^l(X_{\mathrm{an}}, \mathbb{C})$$

which, combined with (23), gives the Grothendieck-Serre isomorphism

$$H^l_{\mathrm{dR}}(X/\mathbb{C}) \cong H^l(X_{\mathrm{an}}, \mathbb{C}). \tag{24}$$

It is a remarkable fact that this also holds true when X is only quasiprojective (see [Gro66]). The isomorphism (24) is obviously compatible with the Hodge filtrations

$$F^p H^l_{\mathrm{dR}}(X/\mathbb{C}) := \mathrm{Im}\,(\mathbb{H}^l(X, \Omega^{\bullet\geq p}_{X/K}) \to \mathbb{H}^l(X, \Omega^\bullet_{X/K})), \tag{25}$$

$$F^p H^l(X_{\mathrm{an}}, \mathbb{C}) := \mathrm{Im}\,(\mathbb{H}^l(X_{\mathrm{an}}, \Omega^{\bullet\geq p}_{X_{\mathrm{an}}})) \to \mathbb{H}^l(X_{\mathrm{an}}, \Omega^\bullet_{X_{\mathrm{an}}})). \tag{26}$$

Finally, thanks to the Hodge decomposition theorem, the filtration (26) is nothing but the Hodge filtration that we defined in Remark 2.3.

For a smooth algebraic variety X defined over K and an algebraic subvariety $Z \subset X$ of codimension k also defined over K, there is an algebraic cycle class

$$[Z]_{\mathrm{dR}} \in F^k H^{2k}_{\mathrm{dR}}(X/K)$$

which is compatible with field extensions and, when $K = \mathbb{C}$, the following comparison holds between the algebraic and topological cycle classes:

Proposition 3.18. *Let X be a smooth quasiprojective variety over $\mathbb{C}$ and $Z \subset X$ be a subvariety of codimension k. Then, under the (filtered) isomorphism (24), one has*

$$[Z]_{\mathrm{dR}} = (2i\pi)^k [Z_{\mathrm{an}}]. \tag{27}$$

A similar comparison holds for Chern classes of vector bundles, and it is even easier to see in this context the reason for the coefficient $(2i\pi)^k$. Indeed the Chern classes are determined by the first Chern class using the standard axiomatic formalism (see [Gro58]) so it suffices to check that for an algebraic line bundle L on X, we have the comparison

$$c_1(L)_{\mathrm{dR}} = (2i\pi)c_1(L_{\mathrm{an}}) \tag{28}$$

between its algebraic de Rham Chern class and the first Chern class of the holomorphic line bundle L_{an} on X_{an}. The algebraic line bundle L is an element of the algebraic Picard group $\mathrm{Pic}\,X$, hence corresponds to a cocycle $\alpha_L \in H^1(X, \mathcal{O}_X^*)$, with analytic counterpart $\alpha_{L,\mathrm{an}} \in H^1(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}^*)$ defining the holomorphic line bundle L_{an}. On the left hand side of (28), $c_1(L)_{\mathrm{dR}}$ is by construction the image of α_L in

$$H^1(X, \Omega_{X/\mathbb{C}}^{\mathrm{closed}}) \subset \mathbb{H}^1(X, \Omega_{X/\mathbb{C}}^{\geq 1})$$

via the map $d\log : \mathcal{O}_X^* \to \Omega_{X/\mathbb{C}}^{\mathrm{closed}}$ which sends f to $\frac{df}{f}$. On the right hand side of (28), looking at the construction of $c_1(L_{\mathrm{an}})$ via the exponential exact sequence, one checks that $c_1(L_{\mathrm{an}})$ is the image of $\alpha_{L,\mathrm{an}}$ in

$$H^1(X_{\mathrm{an}}, \Omega_{X_{\mathrm{an}}}^{\mathrm{closed}}) \subset \mathbb{H}^1(X_{\mathrm{an}}, \Omega_{X_{\mathrm{an}}}^{\geq 1})$$

via the $\frac{d\log}{2i\pi}$ map which sends f to $\frac{1}{2i\pi}\frac{df}{f}$ (see [Voi02b, 7.1.3]). This proves (28).

Let us explain the consequences of these constructions on the structure of "Hodge loci" predicted by the Hodge conjecture. Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism of complex algebraic varieties, with B smooth. We can even assume that $\mathcal{X}, B$ and π are defined over a number field K, since by the theory of the Hilbert scheme (or just by spreading the coefficients of the defining equations), every smooth complex projective variety is a fiber of a family of smooth projective varieties defined over a number field. The associated morphism

$$\pi_{\mathrm{an}} : \mathcal{X}_{\mathrm{an}} \to B_{\mathrm{an}}$$

of complex manifolds is smooth and proper, hence is a topological fibration by Ehresmann's theorem. There is thus for each integer k a local system $\mathbb{H}^{2k} := R^{2k}\pi_{\mathrm{an},*}\mathbb{Q}$ of $\mathbb{Q}$-vector spaces, and using the relative version of the various comparison theorems we have been discussing above, we can compute the holomorphic vector bundle $\mathcal{H}^{2k} := \mathbb{H}^{2k} \otimes \mathcal{O}_{B_{\mathrm{an}}}$ as

$$\mathcal{H}^{2k} = R^{2k}\pi_{\mathrm{an},*}(\Omega^{\bullet}_{\mathcal{X}_{\mathrm{an}}/B_{\mathrm{an}}}),$$

with Hodge subbundle

$$F^k\mathcal{H}^{2k} = R^{2k}\pi_{\mathrm{an},*}(\Omega^{\bullet \geq k}_{\mathcal{X}_{\mathrm{an}}/B_{\mathrm{an}}}).$$

The fibers of these bundles over $b \in B$ are respectively $H^{2k}(\mathcal{X}_t, \mathbb{C})$, $F^kH^{2k}(\mathcal{X}_t, \mathbb{C})$. By a slight abuse of notations, we use the same notation for the total space of the holomorphic vector bundles above, and, following [CDK95], we make the following

Definition 3.19. *The locus of Hodge classes for the family $\pi : \mathcal{X} \to B$ is the subset of $F^k\mathcal{H}^{2k}$ consisting of classes $\alpha_t \in F^kH^{2k}(\mathcal{X}_t) \cap H^{2k}(\mathcal{X}_t, \mathbb{Q})$, $t \in B$.*

As $\mathrm{Hdg}^{2k}(\mathcal{X}_t) = F^k\mathcal{H}^{2k} \cap H^{2k}(\mathcal{X}_t, \mathbb{Q})$ by Remark 2.6, the locus of Hodge classes is the set of all Hodge classes of degree $2k$ in fibers of π. The locus of Hodge classes can be locally written as a countable union of closed analytic subsets of $F^k\mathcal{H}^{2k}$, determined locally by the choice of a section α of the local system $R^{2k}\pi_*\mathbb{Q}$, determining a closed analytic subset of $\mathcal{H}^{2k}$ made of points $t \in B$ where α_t belongs to $F^k\mathcal{H}^{2k}_t$. The Hodge conjecture predicts a more algebraic structure, as we now explain. Indeed, it predicts that the locus of Hodge classes is also the locus of cycle classes $[Z_t] \in F^kH^{2k}(\mathcal{X}_t)$ for all codimension k algebraic cycles in some fiber of π. Next, the holomorphic vector bundle $F^k\mathcal{H}^{2k}$ has the structure of an algebraic vector bundle on B, defined over K. Indeed, there is the relative algebraic de Rham complex $\Omega^\bullet_{\mathcal{X}/B}$ whose analytisation is the holomorphic relative de Rham complex, which provides algebraic vector bundles $\mathcal{H}^{2k}_{\mathrm{alg}}$, $F^k\mathcal{H}^{2k}_{\mathrm{alg}}$ on B, defined over K, and given by the formulas

$$\mathcal{H}^{2k}_{\mathrm{alg}} = R^{2k}\pi_*(\Omega^\bullet_{\mathcal{X}/B}),\ F^k\mathcal{H}^{2k}_{\mathrm{alg}} = R^{2k}\pi_*(\Omega^{\bullet\geq k}_{\mathcal{X}/B}).$$

The relative version of the comparison theorem (24) says that $\mathcal{H}^{2k}$ is the analytization of $\mathcal{H}^{2k}_{\mathrm{alg}}$ and $F^k\mathcal{H}^{2k}$ is the analytization of $F^k\mathcal{H}^{2k}_{\mathrm{alg}}$.

Finally there are the so-called relative Chow varieties parameterizing all pairs (t, Z_t) consisting of a point $t \in B$ and a codimension k cycle $Z_t \subset \mathcal{X}_t$. There are countably many such varieties $f_i : M_i \to B$, where M_i is algebraic, f_i is an algebraic morphism, (M_i, f_i) is defined over a finite extension of K', and there exists a codimension k cycle $\mathcal{Z}_i \subset M_i \times_B \mathcal{X}$, with the property that any pair (t, Z_t) as above is the fiber $\mathcal{Z}_{i,s} \subset \mathcal{X}_t$, for some point $s \in M_i$ such that $t = f_i(s)$. By resolution of singularities, we can assume that M_i is smooth, and the cycle $\mathcal{Z}_i$ then has an algebraic cycle class $[\mathcal{Z}_i]_{\mathrm{dR}} \in H^{2k}(M_i \times_B \mathcal{X})$, whose restriction to $\mathcal{X}_s$ is $[\mathcal{Z}_{i,s}]_{\mathrm{dR}} = (2i\pi)^k[\mathcal{Z}_{i,s}]$, and in particular is locally constant along the fibers of f_i. The image of the morphism

$$M_i \to F^k\mathcal{H}^{2k}_{\mathrm{alg}},\ s \mapsto [\mathcal{Z}_{i,s}]_{\mathrm{dR}}$$

is thus an algebraic subvariety of $F^k\mathcal{H}^{2k}_{\mathrm{alg}}$ which is defined over a finite extension K' of K. To summarize this discussion, the vector bundle $F^k\mathcal{H}^{2k}$ over B contains the locus HL of Hodge classes of degree $2k$ in the fibers, and the locus CL of codimension k cycle classes in the fibers. The locus HL is locally a countable union of closed analytic subsets. The locus CL has more structure, namely $(2i\pi)^kCL$ is the image of the algebraic de Rham cycle class, hence is a countable union of closed algebraic subvarieties of $F^k\mathcal{H}^{2k}_{\mathrm{alg}}$ defined over a finite extension of K, namely those constructed above. Taking into account the comparison (27), the Hodge conjecture thus predicts that $(2i\pi)^k$ times the locus of Hodge classes is a countable union of closed algebraic subvarieties of $F^k\mathcal{H}^{2k}_{\mathrm{alg}}$ defined over finite extensions of K. Part of this prediction is a theorem, which is the best known evidence for the Hodge conjecture.

Theorem 3.20. *[CDK95] The locus of Hodge classes is a countable union of closed algebraic subvarieties of $F^k\mathcal{H}^{2k}_{\mathrm{alg}}$.*

What is missing is the statement concerning the definition field of these loci, despite some results (see eg [KOU23]). For example, a completely open problem is whether, given a family $\pi : \mathcal{X} \to B$ as above, that is, everything is defined over a number field, the image in B of the locus of Hodge classes could have isolated points not defined over a number field. This would disprove the Hodge conjecture...

The question of the field of definition of Hodge loci is almost equivalent to the following

Conjecture 3.21. *Any Hodge class on an algebraic variety is absolute Hodge.*

Here the notion of "absolute Hodge class" has been introduced by Deligne in [Del82], who proved that Hodge classes on abelian varieties are absolute Hodge. let X be a smooth projective variety defined over a field K of characteristic 0. For any field embedding $\tau : K \hookrightarrow \mathbb{C}$, we get a complex manifold X^{τ}_{an}, and for a de Rham cohomology class $\alpha \in F^k H^{2k}_{\mathrm{dR}}(X/K)$ we get using (23) a Betti cohomology class $\alpha_\tau \in F^k H^{2k}(X^{\tau}_{\mathrm{an}}, \mathbb{C})$. If $Z \subset X$ is a codimension k cycle on X, and $\alpha = [Z]_{\mathrm{dR}}$, then (27) shows that $\frac{1}{(2i\pi)^k}\alpha_\tau \in H^{2k}(X^{\tau}_{\mathrm{an}}, \mathbb{Q})$. The class $\frac{1}{(2i\pi)^k}\alpha_\tau \in H^{2k}(X^{\tau}_{\mathrm{an}}, \mathbb{Q})$ is thus a Hodge class, but it also satisfies the property that for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, the class $\frac{1}{(2i\pi)^k}\alpha_\sigma$ belongs to $H^{2k}(X^{\sigma}_{\mathrm{an}}, \mathbb{Q})$, hence is again a Hodge class. This property (independence of the field embedding) characterizes the absolute Hodge classes.

Cycle classes are absolute Hodge, which motivates Conjecture 3.21. The Hodge classes that appear in the standard conjectures are absolute Hodge. If we have a family $\pi : \mathcal{X} \to B$ with B irreducible and a rational cohomology class $\alpha \in H^{2k}(\mathcal{X}_{\mathrm{an}}, \mathbb{Q})$ which has the property that $\alpha_{|\mathcal{X}_b}$ is a Hodge class for every $b \in B$, then $\alpha_{|\mathcal{X}_b}$ is an absolute Hodge class for every $b \in B$ if and only if $\alpha_{|\mathcal{X}_b}$ is an absolute Hodge class for some $b \in B$ (hence in particular if $\alpha_{|\mathcal{X}_b}$ is algebraic for some $b \in B$).

4 The generalized Hodge conjecture

We discuss in this section a conjecture stated in [Gro69], and called the "generalized Hodge conjecture". Technically, it is a mild generalization of the Hodge conjecture (see Proposition 4.8).

4.1 Coniveau

Recall the definition of the (Hodge) coniveau of a Hodge structure (Definition 2.4). Given a smooth projective variety X, a natural geometric way to construct Hodge substructures $L \subset H^k(X, \mathbb{Q})$ of Hodge coniveau c is as follows. Let Y be a smooth complex projective variety, with $\dim Y = \dim X - c$, and let

$$\phi : Y \to X \tag{29}$$

be a morphism (i.e. a holomorphic map). Then

$$\phi_* : H^{k-2c}(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is a morphism of Hodge structures, that maps $H^{p,q}(Y)$ to $H^{p+c,q+c}(X)$. It follows that $L := \mathrm{Im}\,\phi_* \subset H^k(X, \mathbb{Q})$ is a Hodge substructure with $L^{p',q'} = 0$ if $p' < c$ or $q' < c$. Hence L has Hodge coniveau $\geq c$.

Definition 4.1. *A cohomology class $\alpha \in H^k(X, \mathbb{Q})$ has geometric coniveau $\geq c$, if there is a closed algebraic (equivalently, analytic) subset $Z \subset X$, of codimension c, such that*

$$\alpha_{|X \setminus Z} = 0 \text{ in } H^k(X \setminus Z, \mathbb{Q}). \tag{30}$$

By considering the cohomology of the pair $(X, X \setminus Z)$, a class α satisfying (30) has to come from a class in $H^k(X, X \setminus Z, \mathbb{Q})$, and by Poincaré duality, this also says that α comes from $H_{2n-k}(Z, \mathbb{Q})$ via the composite map

$$H_{2n-k}(Z, \mathbb{Q}) \to H_{2n-k}(X, \mathbb{Q}) \overset{PD_X}{\cong} H^k(X, \mathbb{Q}).$$

Example 4.2. The class $[Z] \in H^{2c}(X,\mathbb{Q})$ of a cycle $Z = \sum_i n_i Z_i$ of codimension c vanishes away from the support $\mathrm{Supp}\, Z := \cup_i Z_i$ of Z, hence has geometric coniveau c. More generally, if $[Z]$ is a cycle class as above, and $\beta \in H^*(X,\mathbb{Q})$ is any cohomology class, then $\beta \smile [Z]$ vanishes in $H^{*+2c}(X \setminus \mathrm{Supp}\, Z,\mathbb{Q})$, hence has geometric coniveau $\geq c$.

Example 4.3. Here is an example that does not fit in the above category. Let $X \subset \mathbb{P}^n$ be a generic hypersurface of degree $d \leq n$. The Fano variety of lines $F_1(X)$ of X is smooth of dimension $2n-d-3$. Choose a smooth complete intersection of ample hypersurfaces $W \subset F_1(X)$ of dimension $n-3$, and consider the restriction $P_W \to W$ to W of the universal $\mathbb{P}^1$-bundle over the Grassmannian $G(2,n+1)$ of lines in $\mathbb{P}^n$. There is an incidence diagram

$$p : P_W \to W,\, q : P_W \to X$$

and it is known (see for example [Shi90]) that

$$q_* \circ p^* : H^{n-3}(W,\mathbb{Q}) \to H^{n-1}(X,\mathbb{Q})$$

is surjective. Thus the cohomology $H^{n-1}(X,\mathbb{Q})$ has geometric coniveau ≥ 1, as it vanishes away from $X \setminus q(P_W)$, with $\dim q(P_W) = n-2$.

Obviously, if $\alpha = \phi_* \beta$ for some morphism $\phi : Y \to X$ as in (29), then α has geometric coniveau $\geq c$, because α vanishes away from the closed algebraic subset $\phi(Y) \subset X$, which has codimension $\geq c$. It turns out that there is a converse to this statement, which follows from the following

Theorem 4.4. *(Deligne [Del71]) Let $j : Z \hookrightarrow X$ be the inclusion of a closed algebraic subset of codimension c, and let $\tau : Z' \to Z$ be a desingularization of Z, $j' := \tau \circ j : Z' \to X$. Then for any integer $l \geq 0$*

$$\mathrm{Im}\,(j_* : H_l(Z,\mathbb{Q}) \to H_l(X,\mathbb{Q})) = \mathrm{Im}\,(j'_* : H_l(Z',\mathbb{Q}) \to H_l(X,\mathbb{Q})). \tag{31}$$

This highly nontrivial equality follows from the theory of mixed Hodge structures, and the fact that a morphism j_* as above is a morphism of mixed Hodge structures (see [Del71], [Voi03, 4.3.2]).

Corollary 4.5. *Let X be a smooth projective complex variety. Then*

(i) Classes of geometric coniveau $\geq c$ on X are the classes of the form $\phi_* \beta$, for some morphism $\phi : Y \to X$, with Y smooth projective and $\dim Y = \dim X - c$.

(ii) The set of degree k cohomology classes of geometric coniveau $\geq c$ is a Hodge substructure of $H^k(X,\mathbb{Q})$ of Hodge coniveau $\geq c$.

An example of a smooth complex projective variety X, together with a nonzero rational cohomology class $\alpha \in H^3(X,\mathbb{Q})$ which is of Hodge coniveau 1, that is, $\alpha_{\mathbb{C}} = \alpha^{2,1} + \alpha^{1,2}$, while $H^3(X,\mathbb{Q})$ does not contain any Hodge substructure of coniveau 1 is described in [Voi03, Exercise 1, p 184]. Such a class α is of Hodge coniveau 1 but not of geometric coniveau 1.

4.2 Grothendieck's generalized Hodge conjecture

The generalized Hodge conjecture is the converse of Corollary 4.5(ii).

Conjecture 4.6. *[Gro69] Let X be a smooth complex projective variety, and let $L \subset H^k(X,\mathbb{Q})$ be a Hodge substructure of coniveau c. Then L is of geometric coniveau $\geq c$, that is, there exists a closed algebraic subset $Z \subset X$ of codimension $\geq c$ such that*

$$L \subset \operatorname{Ker}(H^k(X,\mathbb{Q}) \to H^k(X \setminus Z,\mathbb{Q})).$$

Equivalently (by Corollary 4.5(ii)), there exist a smooth projective variety Y of dimension $n-c$ and a morphism $j : Y \to X$, such that

$$L \subset \operatorname{Im}(j_* : H^{k-2c}(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})).$$

The Hodge conjecture (Conjecture 3.1) is a particular case of the generalized Hodge conjecture because the data of a Hodge class $\alpha \in H^{2k}(X,\mathbb{Q})$ is the same as the data of a Hodge substructure $\mathbb{Q}\alpha \subset H^{2k}(X,\mathbb{Q})$ of Hodge coniveau k (and of rank 1). The generalized Hodge conjecture then predicts that α vanishes on $X \setminus Z$ for some codimension k closed algebraic subset. As explained in the previous section, this says that α is supported on Z and must be a rational combination of homology classes of irreducible components of Z.

The Hodge conjecture itself does not imply the generalized Hodge conjecture, but we observe that the generalized Hodge conjecture, together with Corollary 4.5 and the semisimplicity property (Proposition 2.11), implies the following

Conjecture 4.7. *Let X be a smooth complex projective variety and let $L \subset H^k(X,\mathbb{Q})$ be a Hodge substructure of Hodge coniveau c. Then there exist a smooth projective variety Y and a Hodge substructure $L' \subset H^{k-2c}(Y,\mathbb{Q})$ such that there exists an isomorphism of Hodge structures $L' \cong L$ (of bidegree (c,c)).*

We have the following implication

Proposition 4.8. *Conjecture 4.7 and the Hodge conjecture together imply the generalized Hodge conjecture.*

Proof. Let X be a smooth complex projective variety and let $L \subset H^k(X,\mathbb{Q})$ be a Hodge substructure of Hodge coniveau c. Assuming Conjecture 4.7, there exist a smooth projective variety Y and a Hodge substructure $L' \subset H^{k-2c}(Y,\mathbb{Q})$ such that there exists an isomorphism of Hodge structures $L' \cong L$ (of bidegree (c,c)). By semi-simplicity, the Hodge substructure $L' \subset H^{k-2c}(Y,\mathbb{Q})$ is a direct summand, as a Hodge structure, of $H^{k-2c}(Y,\mathbb{Q})$. By Lemma 2.9, the isomorphism of Hodge structures $\eta_* : L' \cong L$ is induced by a Hodge class η of degree $2n$ on $Y \times X$, where $n = \dim X$. The Hodge conjecture then predicts that there exists a codimension n cycle $Z = \sum_i n_i Z_i$ in $Y \times X$ with rational coefficients, such that $[Z] = \eta$. Hence we have $[Z]_* = \eta_*$ and for any class $\alpha \in H^{k-2c}(Y,\mathbb{Q})$ we have

$$\eta_*\alpha = [Z]_*\alpha = \sum_i n_i \mathrm{pr}_{X*}(\mathrm{pr}_Y^*\alpha \smile [Z_i]),$$

where pr_X, pr_Y are the respective projections from $Y \times X$ to X, Y. It follows that $\eta_*\alpha$ vanishes away from $\cup_i \mathrm{pr}_X(Z_i)$, which is a closed algebraic subset of the codimension $\geq c$ of X, so $L = \eta_*(L')$ has geometric coniveau $\geq c$. ∎

4.3 The generalized Hodge conjecture as a construction problem

Let X be a smooth projective complex variety. The Hodge conjecture for Hodge classes of degree $2k$ on X rises a construction problem for codimension k subvarieties of X. We want to show in this paragraph that the generalized Hodge conjecture (Conjecture 4.6) is also a construction problem. Let $L \subset H^k(X,\mathbb{Q})$ and c be its coniveau. In order to solve the generalized Hodge conjecture for L, we can assume that $k \leq n = \dim X$, by the hard Lefschetz isomorphism. Indeed, if $k = n+r$, with $r > 0$, then we have the Lefschetz isomorphism

$$l^r : H^{n-r}(X,\mathbb{Q}) \to H^{n+r}(X,\mathbb{Q})$$

determined by an ample class $l = c_1(L)$. This isomorphism of Hodge structures provides a Hodge substructure

$$L' \subset H^{n-r}(X,\mathbb{Q})$$

which is isomorphic to L, and has coniveau $c-r$. If we solve the generalized Hodge conjecture for L', then L' is supported on a subvariety of codimension $\geq c-r$ so $L = l^r \smile L'$ is supported on a subvariety of codimension $\geq c$.

Next, the generalized Hodge conjecture predicts the existence of a smooth projective variety Y of dimension $n-c$ and a morphism $j : Y \to X$ such that

$$L \subset \operatorname{Im}(j_* : H^{k-2c}(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})). \tag{32}$$

The Lefschetz standard conjecture applied to Y implies the existence of a cycle $\mathcal{Z} \subset Y \times Y$ of codimension $k-2c$ with rational coefficients, such that

$$[\mathcal{Z}]_* : H_{k-2c}(Y,\mathbb{Q}) \to H^{k-2c}(Y,\mathbb{Q}) \tag{33}$$

is surjective. Combining (32) and (33), we conclude that the cycle $\mathcal{Z}' := (\mathrm{Id}, j)_*\mathcal{Z} \subset Y \times X$ has the property that

$$L \subset \operatorname{Im}([\mathcal{Z}']_* : H_{k-2c}(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})). \tag{34}$$

We now observe that, as $\dim Y = n-c$, and $\dim \mathcal{Z} = 2\dim Y - k + 2c = 2n-k$, $\mathcal{Z}$ has relative dimension $n-k+c$ over Y, hence the cycle $\mathcal{Z}'$ can be seen as a family of cycles of dimension $n-k+c$ on X, parameterized by Y. The generalized Hodge conjecture for degree k cohomology and coniveau c on X of dimension n thus predicts the existence of "interesting" (families of) subvarieties of X of dimension $n-k+c$. Conversely, if there exist a smooth projective variety Y and a cycle W in $Y \times X$ of relative dimension $n-k+c$ over Y, such that (34) holds, we observe that, by the Lefschetz theorem on hyperplane sections, we may assume that $\dim Y \leq k-2c$, so $\dim W \leq n-c$. Then $\operatorname{Im}[W]_*$ has geometric coniveau $\geq c$, as it vanishes away from $\cup_i \mathrm{pr}_X(W_i)$, where $W = \sum_i n_i W_i$. So L has geometric coniveau $\geq c$.

5 Chow groups and coniveau

We discuss in this section an algebrogeometric approach to the coniveau of a variety via its Chow groups.

5.1 Chow groups and Mumford's theorem

let X be an algebraic variety over a field K. We define the group $\mathcal{Z}_d(X)$ of d-cycles as the free abelian group generated by closed irreducible subsets $Z \subset X$ of dimension d defined over K.

Definition 5.1. *The Chow group* $\mathrm{CH}_d(X)$ *is the quotient of* $\mathcal{Z}_d(X)$ *by the subgroup of cycles rationally equivalent to* 0*, namely the subgroup generated by d-cycles of the form $j_*\mathrm{div}\,\phi$, for any projective morphism $j : W \to X$, where W is a normal variety and $\phi \in K(W)$ is a nonzero rational function on W, everything being defined over K.*

Here we use the push-forward j_* on d-cycles under proper maps j: if Z is closed irreducible in W, $j_*Z \in \mathcal{Z}_d(X)$ is 0 if $\dim j(Z) < \dim Z$, and otherwise it is $\deg(Z/j(Z))\,j(Z)$, where the degree $\deg(Z/j(Z))$ is the degree of the field extension $K(j(Z)) \subset K(Z)$. We denote also $\mathrm{CH}_d(X) = \mathrm{CH}^{n-d}(X)$ when X is irreducible of dimension n. The Chow groups have excellent functoriality properties. The push-forward $\phi_* : \mathrm{CH}_d(X) \to \mathrm{CH}_d(Y)$ under a projective morphism is induced by the push-forward on cycles as defined above. In a much more subtle way, if $\phi : X \to Y$ is a morphism and Y is smooth, then there is a pull-back morphism

$$\phi^* : \mathrm{CH}^c(Y) \to \mathrm{CH}^c(X),$$

that needs intersection theory in order to be rigorously defined (see [Ful84], which defines more generally the intersection product of cycles modulo rational equivalence on a smooth variety). When $K = \mathbb{C}$ and X is smooth, the cycle class introduced in Section 3.1 induces a group morphism

$$[\,] : \mathrm{CH}^k(X) \to H^{2k}(X_{\mathrm{an}}, \mathbb{Z}),$$

which is compatible with the pull-back and push-forward morphisms when defined, and is also compatible with the intersection product.

That there is a strong relationship between the Chow groups of a smooth projective complex algebraic variety and the coniveau of its cohomology was first observed by Mumford [Mum68] (see also [Roj72]). A generalized formulation is

Theorem 5.2. *Let X be a smooth projective variety over $\mathbb{C}$ such that $\mathrm{CH}_0(X) = \mathbb{Z}$ (or equivalently by [Roj80], $\mathrm{CH}_0(X)_{\mathbb{Q}} = \mathbb{Q}$). Then $H^{i,0}(X) = 0$ for any $i > 0$. It follows that $H^i(X, \mathbb{Q})$ has Hodge coniveau ≥ 1 for $i > 0$.*

Let us comment on the assumption. It means that any two points of X are rationally equivalent in X. This is clearly satisfied if any two points are contained in a rational curve in X, namely a projective curve that is dominated by $\mathbb{P}^1$. (Indeed, if x, y are two points on $\mathbb{P}^1$, the difference $x - y$ is the divisor of a rational function on $\mathbb{P}^1$.) This property, that is now called "rational connectedness", directly implies that $H^{i,0}(X) = H^0(X, \Omega_X^i) = 0$ for any $i > 0$ (see [KMM92]). However, there are examples of smooth projective varieties over $\mathbb{C}$ such that $\mathrm{CH}_0(X) = \mathbb{Z}$ and that are far from being rationally connected. For example, smooth quintic Godeaux surfaces satisfy $\mathrm{CH}_0(X) = \mathbb{Z}$ (see [Voi92]), while the general one can be proved to contain no rational curve.

5.2 Decomposition of the diagonal and the generalized Bloch conjecture

We discuss in this section a notion and method proposed by Bloch and Srinivas [BS83], who give an elegant proof of Mumford's Theorem 5.2 and more importantly prove a stronger statement, namely that, under the same assumptions, the geometric coniveau of $H^i(X,\mathbb{Q})$ is ≥ 1 for $i>0$. This method has been since generalized by many authors (see eg [Lat98], [Par94]) to relate Chow groups of small dimension and geometric coniveau.

Definition 5.3. *A smooth projective variety of dimension n is said to have a cohomological decomposition of the diagonal (in codimension* 1*), if there exist a divisor $D\subset X$, and a cycle $\Gamma\in\mathcal{Z}^n(X\times X)_{\mathbb{Q}}$ supported on $D\times X$, such that*

$$[\Delta_X]=[X\times x]+[\Gamma]\text{ in }H^{2n}(X\times X,\mathbb{Q}) \tag{35}$$

In the definition above, x is any point of X. The right generalization of this notion to higher codimension is

Definition 5.4. *A smooth projective variety of dimension n is said to have a cohomological decomposition of the diagonal in codimension c, if there exist a closed algebraic subset $D_c\subset X$ of codimension c, and a cycle $\Gamma\in\mathcal{Z}^n(X\times X)_{\mathbb{Q}}$ supported on $D_c\times X$, such that*

$$[\Delta_X]=[W]+[\Gamma]\text{ in }H^{2n}(X\times X,\mathbb{Q}), \tag{36}$$

where the cycle W is decomposable, namely $W=\sum_i n_iW_i\times W'_{n-i}$ for some closed algebraic subsets W_i, W'_{n-i} of X such that $\dim W_i+\dim W'_{n-i}=n$.

The relevance of this notion for the study of the generalized Hodge conjecture is illustrated by the following statement

Proposition 5.5. *If X has a cohomological decomposition of the diagonal in codimension c, then*

$$H^*(X,\mathbb{Q})=H^*(X,\mathbb{Q})_{\rm alg}+N^cH^*(X,\mathbb{Q}), \tag{37}$$

where $N^cH^*(X,\mathbb{Q})$ denotes cohomology of geometric coniveau $\geq c$.

If $c=1$, $H^{*>0}(X,\mathbb{Q})$ has geometric coniveau ≥ 1.

Proof. The second statement follows from the first since cycle classes of codimension >0 are of coniveau ≥ 1. To prove (37), we use the action γ^* of a correspondence $\gamma\in H^{2n}(X\times X,\mathbb{Q})_{\rm alg}$ on cohomology, given by

$$\gamma^*(\alpha)=\mathrm{pr}_{1*}(\mathrm{pr}_2^*\alpha\smile\gamma).$$

From (36), we get for any $\alpha\in H^*(X,\mathbb{Q})$, by letting both sides acting on $H^*(X,\mathbb{Q})$

$$\alpha=[W]^*\alpha+[\Gamma]^*\alpha\text{ in }H^*(X,\mathbb{Q}). \tag{38}$$

As W is decomposable, we get

$$[W]^*\alpha=\sum_i n_i[W_i]\langle\alpha,[W'_{n-i}]\rangle,$$

hence $[W]^*\alpha\in H^*(X,\mathbb{Q})_{\rm alg}$. Finally the last term $[\Gamma]^*\alpha$ in (38) vanishes on $X\setminus D_c$, since Γ is supported on $D_c\times X$, hence $[\Gamma]^*\alpha$ has geometric coniveau $\geq c$ for any $\alpha\in H^*(X,\mathbb{Q})$. ☐

The cohomological version of Bloch-Srinivas decomposition of the diagonal is the following statement.

Theorem 5.6. *Let X be a smooth projective complex variety. Assume that $\mathrm{CH}_0(X)=\mathbb{Z}$. Then X admits a cohomological decomposition of the diagonal in codimension* 1.

Combining Theorem 5.6 with Proposition 5.5, one gets the following strengthening of Mumford's theorem 5.2:

Corollary 5.7. *The assumptions on X being as in Theorem 5.6, $H^{*>0}(X,\mathbb{Q})$ has geometric coniveau ≥ 1.*

For completeness, we note also the following application of the decomposition of the diagonal to the Hodge conjecture itself.

Theorem 5.8. *[BS83] The assumptions on X being as in Theorem 5.6, X satisfies the Hodge conjecture for Hodge classes of degree* 4*.*

Proof. We write the decomposition of the diagonal

$$N[\Delta_X - X\times x] = [\Gamma] \text{ in } H^{2n}(X\times X,\mathbb{Q}) \tag{39}$$

with Γ supported on $D\times X$, for some divisor D of X. Let $\tilde{j}:\widetilde{D}\to X$ be a desingularization of D. We can lift Γ to a cycle $\widetilde{\Gamma}$ supported on $\widetilde{D}\times X$, at least with rational coefficients. Then for a Hodge class α of degree 4 on X, we have, by letting both sides of (39) act on α

$$\alpha = \tilde{j}_*([\widetilde{\Gamma}]^*\alpha) \text{ in } \mathrm{Hdg}^4(X,\mathbb{Q}).$$

As $[\widetilde{\Gamma}]^*\alpha$ is a Hodge class of degree 2 on $\widetilde{D}$, it is algebraic on $\widetilde{D}$ by Theorem 3.3, hence α is algebraic on X. ☐

Remark 5.9. This theorem has a more general version (see [BS83]), where the assumption on X is that $\mathrm{CH}_0(X)$ is supported on a closed algebraic subset of dimension at most 3. In this form, it generalizes a result due to Conte and Murre, see [CM78].

Theorem 5.6 has been generalized with a very similar proof in [Lat98], [Par94] (see also [Voi03, 10.2] or [Voi14]).

Theorem 5.10. *Let X be a smooth projective complex variety of dimension n. Assume that the cycle class map $[\,]:\mathrm{CH}_i(X)_{\mathbb{Q}}\to H^{2n-2i}(X,\mathbb{Q})$ is injective for $i\leq c-1$. Then X admits a cohomological decomposition of the diagonal in codimension c.*

Combining this statement with Proposition 5.5, one gets

Corollary 5.11. *The assumptions on X being as in Theorem 5.10,*

$$H^*(X,\mathbb{Q}) = H^*(X,\mathbb{Q})_{\mathrm{alg}} + N^cH^*(X,\mathbb{Q}).$$

This suggests that the generalized Hodge conjecture could possibly be attacked in certain cases through the computation of Chow groups. In fact, Bloch [Blo80], and later Bloch and Beilinson, conjectured a converse to the statements above. One version of the generalized Bloch conjecture is

Conjecture 5.12. *Let X be a smooth complex projective variety. Assume that $H^{i,0}(X) = 0$ for $i > 0$. Then* $\mathrm{CH}_0(X) = \mathbb{Z}$.

More generally, assume

$$H^*(X,\mathbb{Q}) = \mathrm{Hdg}^{*/2}(X,\mathbb{Q}) + L_c^*, \tag{40}$$

where each $L_c^i \subset H^i(X,\mathbb{Q})$ is a Hodge substructure of coniveau $\geq c$. Then the cycle class map $[\,]$: $\mathrm{CH}_i(X)_{\mathbb{Q}} \to H^{2n-2i}(X,\mathbb{Q})$ is injective for $i \leq c-1$.

Remark 5.13. Even if we make the stronger assumption that L_c^* has geometric coniveau $\geq c$, Conjecture 5.12 is still open (despite promising work by Ayoub, see [Ayo17]). For example, it is open for surfaces X with $p_g = q = 0$, while this condition is equivalent, thanks to the Lefschetz $(1,1)$-theorem (cf. Theorem 3.3) to the fact that the whole cohomology $H^*(X,\mathbb{Q})$ is algebraic.

5.3 Generalized Hodge conjecture and cohomological decomposition of the diagonal

We will say that a smooth projective complex variety X has geometric coniveau $\geq c$ if (37) holds.

Remark 5.14. Assuming furthermore the Hodge conjecture on X, the Poincaré pairing is nondegenerate on $H^*(X,\mathbb{Q})_{\mathrm{alg}}$, so replacing $N^cH^*(X,\mathbb{Q})$ by

$$N^cH^*(X,\mathbb{Q})_{\mathrm{tr}} := N^cH^*(X,\mathbb{Q}) \cap H^{2*}(X,\mathbb{Q})_{\mathrm{alg}}^{\perp},$$

where $H^{2*}(X,\mathbb{Q})_{\mathrm{alg}}^{\perp}$ denotes the orthogonal complement of $H^{2*}(X,\mathbb{Q})_{\mathrm{alg}}$ with respect to the Poincaré pairing, we can replace the decomposition (37) by a direct sum decomposition

$$H^*(X,\mathbb{Q}) = H^{2*}(X,\mathbb{Q})_{\mathrm{alg}} \bigoplus N^cH^*(X,\mathbb{Q})_{\mathrm{tr}}. \tag{41}$$

For X as above, we have the following partial converse to Proposition 5.5.

Proposition 5.15. *Let X be a smooth projective of dimension n. Assume the Hodge conjecture holds for varieties of dimension $\leq 2n-2$. Then if X has geometric coniveau $\geq c$, it has a cohomological decomposition of the diagonal in codimension c.*

Proof. We use (41), where $N^cH^*(X,\mathbb{Q})_{\mathrm{tr}}$ has geometric coniveau $\geq c$, hence comes from the cohomology of Y via a morphism $j : Y \to X$, where Y is a smooth projective variety of dimension $n-c$ (see Corollary 4.5). Using (41) and the Künneth decomposition, the class $[\Delta_X] \in H^{2n}(X \times X,\mathbb{Q})$ of the diagonal writes as

$$[\Delta_X] = \delta_{\mathrm{alg}} + \delta_{\geq c} \tag{42}$$

where $\delta_{\mathrm{alg}} \in H^{2*}(X,\mathbb{Q})_{\mathrm{alg}} \otimes H^{2*}(X,\mathbb{Q})_{\mathrm{alg}}$ and $\delta_{\geq c} \in N^cH^*(X,\mathbb{Q})_{\mathrm{tr}} \otimes N^cH^*(X,\mathbb{Q})_{\mathrm{tr}}$. The fact that there is no term in $H^{2*}(X,\mathbb{Q})_{\mathrm{alg}} \otimes N^cH^*(X,\mathbb{Q})_{\mathrm{tr}}$ follows from the fact that there is no nonzero Hodge class in $N^cH^*(X,\mathbb{Q})_{\mathrm{tr}}$, hence no nonzero Hodge class in $H^{2*}(X,\mathbb{Q})_{\mathrm{alg}} \otimes N^cH^*(X,\mathbb{Q})_{\mathrm{tr}}$.

We now observe that $\delta_{\rm alg} \in H^{2*}(X,\mathbb{Q})_{\rm alg} \otimes H^{2*}(X,\mathbb{Q})_{\rm alg}$ is the class of a decomposable cycle $\sum_i n_i W_i \times W_i'$, while the Hodge class δ_c on $X \times X$ comes from a Hodge class on $Y \times Y$ via the morphism $(j,j) : Y \times Y \to X \times X$ by Corollary 2.12. The Hodge conjecture applied to $Y \times Y$ says that

$$\delta_c = (j,j)_*[Z']$$

for some cycle Z' with rational coefficients supported on $Y \times Y$. Putting things together, we get

$$[\Delta_X] = [\sum_i n_i W_i \times W_i'] + [(j,j)_* Z'] \text{ in } H^{2n}(X \times X, \mathbb{Q}),$$

which provides a cohomological decomposition of the diagonal in codimension c since the cycle $(j,j)_*Z'$ is supported on $j(Y) \times X$ and $j(Y) \subset X$ has codimension $\geq c$. $\square$

6 The case of complete intersections

6.1 Computing the Hodge coniveau

Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of r hypersurfaces of degree $d_1 \leq \ldots \leq d_r$. Thus X is smooth of dimension $n-r$ and of degree $d_1 \cdot \ldots \cdot d_r$. The Lefschetz theorem on hyperplane sections says that the restriction map

$$H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \tag{43}$$

is an isomorphism if $i < n-r$ and is injective for $i = n-r$. Thus for $i < n-r$, we have $H^i(X,\mathbb{Z}) = 0$ if i is odd and $H^i(X,\mathbb{Z}) = \mathbb{Z}$ if i is even. This implies by Poincaré duality on X that, for $2n-2r \geq i > n-r$, the groups $H^i(X,\mathbb{Z})$ are cyclic if i is even and vanish if i is odd. Here all the generators with $\mathbb{Q}$-coefficients are known, namely, they are powers h^j, $i = 2j$, where h is the class of a hyperplane section. In particular the Hodge structures on $H^i(X,\mathbb{Q})$ are uninteresting for $i \neq n-r$.

The Hodge structures on the middle cohomology $H^{n-r}(X,\mathbb{Q})$, and better, when $n-r$ is even, the subgroup $H^{n-r}(X,\mathbb{Q})_{\rm prim}$ of classes orthogonal for the Poincaré pairing to the image of the restriction map (43), are however very interesting. To start with, we know how to compute their Hodge coniveau, thanks to the work of Griffiths [Gri69] and later generalization in the case of complete intersections (see [Pet75]).

Theorem 6.1. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Then $H^{p,q}(X,\mathbb{Q})_{\rm prim}$ vanishes for any $q \leq c$ if and only if $n \geq cd$. Equivalently the Hodge coniveau of $H^{n-1}(X,\mathbb{Q})_{\rm prim}$ is greater than or equal to c if and only if $n \geq cd$.*

More generally, if $X \subset \mathbb{P}^n$ is a smooth complete intersection of hypersurfaces of degrees $d_1 \leq \ldots \leq d_r$, the Hodge coniveau of $H^{n-r}(X,\mathbb{Q})_{\rm prim}$ is greater than or equal to c if and only if

$$n \geq \sum_i d_i + (c-1)d_r. \tag{44}$$

As discussed in Section 4.3, the generalized Hodge conjecture predicts the following:

Conjecture 6.2. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degrees $d_1 \leq \ldots \leq d_r$. Then, if $n \geq \sum_i d_i + (c-1)d_r$ as in (44), X has a cohomological decomposition of the diagonal in codimension c.*

If the complete intersection X is very general, an equivalent statement is that there exist a smooth projective variety Y of dimension $n-r-2c$ and a family $\mathcal{Z} \subset Y \times X$ of cycles of dimension c of X such that

$$[\mathcal{Z}]^* : H^{n-r}(X,\mathbb{Q})_{\mathrm{prim}} \to H^{n-r-2c}(Y,\mathbb{Q})$$

is injective. So the geometric question is to construct "interesting" subvarieties of dimension c in these complete intersections.

By Theorem 5.10, one way to prove Conjecture 6.2 is to solve the following Bloch type conjecture

Conjecture 6.3. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degrees $d_1 \leq \ldots \leq d_r$. Then, if $n \geq \sum_i d_i + (c-1)d_r$, the cycle class map*

$$\mathrm{CH}_i(X)_{\mathbb{Q}} \to H_{2i}(X,\mathbb{Q})$$

is injective for $i \leq c-1$.

In the case $c = 1$, Theorem 6.1 is immediate, since we have

$$H^{n-r,0}(X) = H^0(X, K_X)$$

where the canonical bundle $K_X = \bigwedge^{n-r}\Omega_X$ is by the adjunction formula isomorphic to $\mathcal{O}_X(-n-1+\sum_i d_i)$. So for $c = 1$, the numerical condition (44) just says that K_X is negative, that is, X is Fano. This implies both conjectures 6.2 and 6.3 in this case. Indeed, if X is Fano, then it is rationally connected by [KMM92], that is, through any two points of X, there is a rational curve. Then obviously any two points are rationally equivalent which proves Conjecture 6.3 in this case. By Theorem 5.10, this in turn implies Conjecture 6.2. Historically, Conjecture 6.2 for $c = 1$ was proved directly using the family of lines in X (as in Example 4.3). Also a direct proof of the isomorphism $\mathrm{CH}_0(X) = \mathbb{Z}$ was given in [Roj72] for Fano hypersurfaces X.

6.2 Further examples where the generalized Hodge conjecture is known and further results

We discuss in this section what is known about Conjectures 6.2 and 6.3 for coniveau $c > 1$. We restrict for simplicity to the hypersurface case. First of all, thanks to work of Esnault-Levine-Viehweg [ELV97], [Par94], improved later on by Otwinovska [Otw99], Conjecture 6.3 is true for c given and $n >> d$ depending on c and by Theorem 5.10, this implies 6.2 for the same values of n, d. The best known general statement is the following:

Theorem 6.4. *Let n, d, c be integers such that*

$$(c+1)(n-c) - \binom{c+d}{d} \geq n-1-c. \tag{45}$$

Then for a smooth hypersurface of degree d in $\mathbb{P}^n$, the group $\mathrm{CH}_i(X)_{\mathbb{Q},\mathrm{hom}}$ of i-cycles cohomologous to 0 is zero for $i \leq c-1$.

The inequality (45) corresponds to the condition that the family of linear spaces $\mathbb{P}^c \subset \mathbb{P}^n$ contained in X sweeps-out a subvariety of X of codimension $\leq c$.

Unfortunately, except for very small values of d, inequality (45) is very different from inequality (44) which tells that the conclusion of Theorem 6.4 should hold once $n \geq dc$. For example, Theorem 6.4 implies Conjectures 6.3 and 6.2 for cubic hypersurfaces of dimension $n-1 \leq 16$. This also works for quartic hypersurfaces of dimension ≤ 10: they have trivial $\mathrm{CH}_1(X)_{\mathbb{Q},\mathrm{hom}}$ once $n \geq 8$.

We have been discussing above the case of a *general* hypersurface of degree d in $\mathbb{P}^n$. In [Voi96], it is proved that for any value of d and n, there exist smooth hypersurfaces of degree d in $\mathbb{P}^n$ satisfying Conjecture 6.3, hence also Conjecture 6.2. More precisely, hypersurfaces whose equation takes the form

$$f(X_0,\ldots,X_n) = f_1(X_0,\ldots,X_d) + f_2(X_{c+1},\ldots,X_{2c}) + \ldots + f_c(X_{d(c-1)+1},\ldots,X_{dc+r}),$$

with $n = dc + r$, $r < c$, satisfy the two conjectures.

We finish with a result showing that for general hypersurfaces, the two conjectures 6.3 and 6.2, where Conjecture 6.3 is a priori stronger by Theorem 5.10, are in fact equivalent (that is, equally difficult and possibly wrong!).

Theorem 6.5. *[Voi15] Assume that a general hypersurface X of degree d in $\mathbb{P}^n$ has a cohomological decomposition of the diagonal in codimension c. Then, for any smooth hypersurface X of degree d in $\mathbb{P}^n$, the cycle class map*

$$\mathrm{CH}_i(X)_{\mathbb{Q}} \to H_{2i}(X,\mathbb{Q})$$

is injective for $i \leq c-1$.

This theorem works as well for complete intersections of very ample hypersurfaces in homogeneous varieties and has a more general version for motives of complete intersections admitting a finite group action.

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