Intrinsic pseudovolume forms and K-correspondences

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0 Introduction

In recent years, the notion of K-equivalence has appeared in several contexts, like motivic integration [9], McKay correspondence [2] and derived category of coherent sheaves on varieties [12], [23]. A K-equivalence between two algebraic varieties X and Y is a birational map $\phi : X \dashrightarrow Y$ whose graph $\Gamma_\phi \subset X \times Y$ admits a desingularization

$$\tau : Z \to \Gamma_\phi$$

such that, denoting $f = pr_1 \circ \tau$, $g = pr_2 \circ \tau$, f^*K_X and g^*K_Y are linearly equivalent. Equivalently, the two ramification divisors should satisfy :

$$R_f = R_g. \tag{0.1}$$

In this paper, we start the study of what we call K-(iso)correspondences between smooth varieties or complex manifolds X, Y of the same dimension, which are graphs of multivalued maps, or analytic subsets in the product $X \times Y$, generically finite over each factor, such that any desingularization $\widetilde{\Sigma}$ satisfies with the notations above the condition (0.1) or, in case of K-correspondences, the weakened condition

$$R_f \leq R_g.$$

Hence we simply forget the condition that the degree of the graph over X and Y should be 1. Such K-isocorrespondences appear naturally in the McKay situation (cf section 2).

Our main result proved in section 2, is the fact that many K-trivial projective varieties carry a lot of self-K-isocorrespondences $\Sigma \subset X \times Y$ satisfying the condition that $deg\, pr_{1|\Sigma} \neq deg\, pr_{2|\Sigma}$. With the notations above, one can see easily that this last condition is equivalent to the equality of volume forms on $\widetilde{\Sigma}$

$$f^*\Omega_X = \lambda g^*\Omega_X,$$

where $\lambda \neq 1$ is a real number, and Ω_X is the canonical volume form of X. Equivalently

$$f^*\omega_X = \mu g^*\omega_X$$

where ω_X is any generator of $H^0(X, K_X)$, and μ is a complex number of modulus $\neq 1$. Because of this dilatation property, these self-K-isocorrespondences look like the multiplication by an integer in an abelian variety.

Section 3 discusses potential applications of this result to the study of intrinsic pseudovolume forms on complex manifolds (see [13]). Kobayashi and Eisenman have introduced an intrinsic pseudovolume form Ψ_X on any complex manifold X, which is computed using all holomorphic maps from a polydisk D^n, $n = dim\, X$ to X. Here we introduce modified intrinsic pseudovolume forms

$$\Phi_{X,an} \leq \Phi_X \leq \Psi_X$$

which are defined essentially by replacing holomorphic maps with holomorphic K-correspondences in the Eisenman-Kobayashi definition. We show that on one hand, the following theorem, due to Griffiths [11] and Kobayashi-Ochiai [14], still holds for the pseudovolume form $\phi_{X,an}$:

Theorem 1 *If X is a projective variety which is of general type, $\Phi_{X,an} > 0$ on a dense Zariski open set of X.*

On the other hand we show that Φ_X is equal to 0 for many types of K-trivial varieties listed in section 3, and also for varieties which are fibered with fiber of these types. This gives us a weak version of the Kobayashi conjecture (i.e. the converse to the Griffiths-Kobayashi-Ochiai theorem) for the modified pseudovolume form Φ_X.

Section 4 is devoted to a few supplementary results, remarks and questions concerning K-isocorrespondences. In particular, we provide (cf corollary 1) new examples of K-trivial varieties satisfying Kobayashi's conjecture 1.

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1 K-correspondences

In this section, we introduce and discuss the notions of K-correspondences and K-isocorrespondences, which are straightforward generalizations of the so-called K-ordering and K-equivalence in birational geometry (cf [12], [23]). We assume that X and Y are smooth complex manifolds of dimension n.

Definition 1 *A K-correspondence from X to Y is a reduced n-dimensional closed analytic subset $\Sigma \subset X \times Y$, such that on each irreducible component of Σ, the projections to X and Y are generically of maximal rank, and satisfying the following two conditions :*

1. *The restriction $pr_{1|\Sigma}$ is proper.*
2. *Let $\widetilde{\Sigma} \xrightarrow{\tau} \Sigma$ be a desingularization, and let*

$$f := pr_1 \circ \tau : \widetilde{\Sigma} \to X,\, g = pr_2 \circ \tau : \widetilde{\Sigma} \to Y.$$

Then we have the inequality of ramification divisors on $\widetilde{\Sigma}$:

$$R_f \leq R_g.$$

Note that property 2 has to be checked on one desingularization, and then will be satisfied by all desingularizations, as a standard argument shows. Another way to phrase it is to say that the generalized Jacobian map

$$J_{\widetilde{\Sigma}} := g_* \circ f_*^{-1} : f^*(\bigwedge^n T_X) \to g^*(\bigwedge^n T_Y) \qquad (1.2)$$

is holomorphic.

A holomorphic map ϕ from X to Y leads to a correspondence, obtained by taking the graph of ϕ. It turns out that K-correspondences behave with many respects as maps. Their main common feature with ordinary maps is the fact that for any desingularization $\tau : \widetilde{\Sigma} \to \Sigma$ as above, we get a natural inclusion

$$g^* K_Y \subset f^* K_X,$$

as subsheaves of $K_{\widetilde{\Sigma}}$. We also have the following important fact :

Proposition 1 *K-correspondences can be composed. More precisely, if $\Sigma \subset X \times Y$ and $\Sigma' \subset Y \times Z$ are K-correspondences, then define $\Sigma' \circ \Sigma$ to be the union of the components of $p_{13}(p_{12}^{-1}(\Sigma) \cap p_{23}^{-1}(\Sigma'))$ on which the projections to X and Z are generically of maximal rank. Then $\Sigma' \circ \Sigma$ is a K-correspondence.*

Proof. Note first that the properness of the first projections on Σ and Σ' implies that $p_{13}(p_{12}^{-1}(\Sigma) \cap p_{23}^{-1}(\Sigma'))$ is a closed analytic subset of $X \times Z$. It also shows that the projection to X is proper on this analytic subset. Finally it is easy to see that a component of this set which is generically of maximal rank over both X and Z must be of dimension n.

Next let $\widetilde{\Sigma}$, $\widetilde{\Sigma}'$ be desingularizations of Σ, Σ'. Denote by f, g the maps from $\widetilde{\Sigma}$ to X and Y, and by f' and g' the maps from $\widetilde{\Sigma}'$ to Y and Z. Let Σ'' be a component of $\widetilde{\Sigma} \times_Y \widetilde{\Sigma}'$ on which the maps $F := f \circ \phi$ and $G := g' \circ \psi$ are generically of maximal rank. Here $\phi : \Sigma'' \to \widetilde{\Sigma}$ and $\psi : \Sigma'' \to \widetilde{\Sigma}'$ are the two natural maps. Choose a desingularization $\widetilde{\Sigma}''$ of Σ''. Let now $\sigma \in \widetilde{\Sigma}''$ and let

$$x = F(\sigma),\ z = G(\sigma),\ y = g \circ \phi(\sigma) = f' \circ \psi(\sigma).$$

Let ω_x be a holomorphic n-form which generates K_X near x, and similarly choose ω_y near y and ω_z near z. Then property 2 says that we have the following equality of n-forms on $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$ respectively:

$$g^* \omega_y = \chi \cdot f^* \omega_x,\ g'^* \omega_z = \chi' \cdot f'^* \omega_y,$$

where χ is a holomorphic function on $\widetilde{\Sigma}$ and χ' is a holomorphic function on $\widetilde{\Sigma}'$, defined respectively on the inverse image in $\widetilde{\Sigma}$ of a neighbourhood of (x, y) in $X \times Y$ and on the inverse image in $\widetilde{\Sigma}'$ of a neighbourhood of (y, z) in $Y \times Z$.

Pulling-back, via ϕ, ψ respectively, these equalities to $\widetilde{\Sigma}''$ now gives :

$$\chi \circ \phi \cdot F^* \omega_x = \phi^*(g^* \omega_y),$$

$$G^* \omega_z = \chi' \circ \psi \cdot \psi^*(f'^* \omega_y).$$

Then, using $g \circ \phi = f' \circ \psi$, we conclude that $\phi^*(g^* \omega_y) = \psi^*(f'^* \omega_y)$ and hence

$$G^* \omega_z = \chi' \circ \psi \cdot \chi \circ \phi \cdot F^* \omega_x$$

as n-forms on $\widetilde{\Sigma}''$, where $\chi' \circ \psi \cdot \chi \circ \phi$ is a holomorphic function on $\widetilde{\Sigma}''$. ∎

Our next definition is the following :

Definition 2 *A K-isocorrespondence between X and Y is a K-correspondence Σ from X to Y such that ${}^t\Sigma$ is a K-correspondence from Y to X, where ^{t} means the image under the natural isomorphism $X \times Y \cong Y \times X$.*

In other words, Σ has to satisfy the properties that the two projections pr_i on X and Y are proper on Σ and that if $\tau : \widetilde{\Sigma} \to \Sigma$ is a desingularization, with $f = pr_1 \circ \tau$, $g = pr_2 \circ \tau$, we have now the equality

$$R_f = R_g.$$

K-isocorrespondences look like isomorphisms with certain respects. The most important point for us will be the fact that with the notations above, a K-isocorrespondence induces a canonical isomorphism

$$f^* K_X \cong g^* K_Y.$$

Indeed they are equal as subsheaves of $K_{\widetilde{\Sigma}}$.

With the same arguments as before, one shows that K-correspondences can be composed.

Example 1 *If $f : X \to Y$ is a proper étale map, the graph of f and its transpose are K-isocorrespondences.*

Example 2 *If G is a finite group acting on X, in such a way that the stabilizer G_x acts via $SL(n)$ on the tangent bundle $T_{X,x}$ at each point x of X, the quotient X/G has Gorenstein singularities. If $x \in X$, we can choose a G_x-invariant n-form ω_x near x. The canonical bundle $K_{X/G}$ admits then as a local generator the form $\omega_{X/G}$ such that $q^*\omega_{X/G} = \omega_x$, where q is the quotient map. Next assume that a crepant resolution $\pi : Y \to X/G$ exists. This means exactly that in a neighbourhood of $\pi^{-1}(y)$, $y := q(x)$, the n-form $\pi^*\omega_{X/G}$, defined on the open set of Y where π is a local isomorphism, extends to a holomorphic n-form ω_Y which generates the canonical bundle of Y. We now claim that the graph Γ of the meromorphic map*

$$q' : X \dashrightarrow Y$$

is a K-isocorrespondence. Indeed, choose as before x, ω_x. Then the equality

$$q^*\omega_{X/G} = \omega_x,$$

and the fact that $\pi^*\omega_{X/G} = \omega_Y$ on the smooth locus of Y show that on the smooth part of Γ, we have

$$pr_1^*\omega_x = pr_2^*\omega_Y.$$

Since ω_Y generates K_Y, this shows immediately that Γ is a K-isocorrespondence.

The simplest example of such a situation is the case of an involution ι acting with isolated fixed points on a surface X. Then the involution acts on the blow-up $\tilde{X}$ of X at the fixed points, and the lifted involution $\tilde{\iota}$ fixes the exceptional curves pointwise. Then the quotient map

$$\tilde{X} \to \tilde{X}/\tilde{\iota}$$

ramifies simply along the exceptional curves. Furthermore the ramification divisor of the blowing-down map $\tau : \tilde{X} \to X$ is also the union of the exceptional curves with multiplicity 1.

2 Calabi-Yau varieties and K-correspondences

We consider projective n-dimensional complex manifolds with trivial canonical bundle (Calabi-Yau manifolds). We shall denote by ω_X a generator for $H^0(X, K_X)$. It can be normalized up to a complex coefficient of modulus 1 in such a way that

$$\Omega_X = (-1)^{\frac{n(n-1)}{2}} i^n \omega_X \wedge \overline{\omega}_X$$

has integral 1 on X. This Ω_X is a canonically defined volume form on X. We want to show the existence of self-K-isocorrespondences for a large set of Calabi-Yau manifolds, which have furthermore the dilating property, like isogenies of abelian varieties, of multiplying the canonical volume form by a real number > 1. The Calabi-Yau varieties for which we are able to prove this fall into three classes. Consider the following properties :

1. X is swept out by abelian varieties.
2. There exists a rationally connected variety Y, such that some embedding $j : X \hookrightarrow Y$ realizes X as a member of the linear system $\mid -K_Y \mid$ on Y.
3. X is the Fano variety (assumed to be smooth of the right dimension) of linear subspaces $\mathbb{P}^r \subset M$ of a complete intersection $M \subset \mathbb{P}^N$ of type $(d_1, \ldots, d_k)$, with the exception of the case $(d_1, \ldots, d_k) = (2, \ldots, 2)$. (Here the numbers r, N, d_i are chosen in such a way that K_X is trivial. For fixed r, d_i's, this happens in exactly one dimension N (see below).)

Note that the class of Calabi-Yau varieties satisfying property 2 is very large. It contains all complete intersections in Fano varieties with Picard number 1. It can be shown however that the varieties in class 3 do not in general satisfy property stated in 2.

Our result is the following :

Theorem 2 *Assume X satisfies 1, 2 or is generic satisfying 3. Then there exists a self-K-isocorrespondence*

$$\Sigma \subset X \times X$$

which satisfies the property that

$$f^*\Omega_X = \lambda g^*\Omega_X,\ \lambda > 1. \tag{2.3}$$

Here as always, f and g denote the two projections to X, on a desingularisation of Σ.

We can rephrase formula (2.3) as follows : since Σ is a self-K-isocorrespondence, which we may assume to be irreducible, there is a non zero coefficient μ such that

$$f^*\omega_X = \mu g^*\omega_X. \tag{2.4}$$

Indeed, because ω_X nowhere vanishes, these two n-forms have the same zero divisor on $\widetilde{\Sigma}$, which is equal to $R_f = R_g$. So the statement concerning the volume form is simply the statement that we can find such a self-K-correspondence Σ whose corresponding $\lambda := \mid \mu \mid^2$ satisfies $\lambda \neq 1$. Notice that (still assuming Σ to be irreducible), this is also equivalent to the fact that the degrees of f and g are not equal. Indeed, (2.4) gives the formula

$$f^*\Omega_X = \lambda g^*\Omega_X$$

and λ is then computed by integrating both sides over $\widetilde{\Sigma}$, which gives

$$deg\, f = \lambda deg\, g. \tag{2.5}$$

In case 1, the construction of Σ is straightforward. Namely, let

$$\begin{array}{ccc} P & \stackrel{\varphi}{\to} & X \\ h\downarrow & & \\ B & & \end{array}$$

be a covering of X by abelian varieties. So ϕ is dominating and the fibers of h are abelian varieties. We may assume that there is a rational section of $h: P \to B$. Hence the smooth fibers of h have a zero, which allows to define multiplication by any integer $m \in \mathbb{Z}$. Now choose two integers m and m' and define

$$\Sigma = \{(\phi(mx), \phi(m'x)),\, x \in P\}.$$

(To be more rigorous, take the closure of the set above defined for $x \in P^0$, the open set of P where h is of maximal rank.) It is easy to see that Σ has dimension n. The fact that it is a self-K-isocorrespondence follows, using the fact that K_X is trivial, from the following formula (2.6), where a is the dimension of the abelian varieties P_b :

$$\frac{1}{m^a} pr_1^*\omega_{X_{|\Sigma}} = \frac{1}{m'^a} pr_2^*\omega_{X_{|\Sigma}}, \tag{2.6}$$

as n-forms on the smooth locus of Σ. The formula (2.6) also shows that the coefficient λ introduced above is equal to $(\frac{m}{m'})^{2a}$, hence can be made different from 1.

To prove formula (2.6), we note that Σ is the image under (ϕ, ϕ) of $\Sigma' \subset P \times_B P$,

$$\Sigma' = \{(mx, m'x),\, x \in P\}.$$

Next the restriction Σ'_b of Σ' to $P_b \times P_b$ is the graph

$$\Sigma'_b = \{(mx, m'x),\, x \in P_b\}.$$

It is obvious that it satisfies

$$\frac{1}{m^a} pr_1^*\omega_{P_b}{}_{|\Sigma'_b} = \frac{1}{m'^a} pr_2^*\omega_{P_b}{}_{|\Sigma'_b},$$

where ω_{P_b} is a holomorphic a-form on P_b. Now, since $\Sigma' \subset P \times_B P$, the two projections pr_1, pr_2 from Σ' to P induce

$$pr_1^* : R^0h_*K_{P/B} \to R^0(h \circ pr_1)_*K_{\Sigma'/B},\ pr_2^* : R^0h_*K_{P/B} \to R^0(h \circ pr_2)_*K_{\Sigma'/B}, \tag{2.7}$$

and the maps

$$pr_1^* : R^0h_*K_P \to R^0(h \circ pr_1)_*K_{\Sigma'},\ pr_2^* : R^0h_*K_P \to R^0(h \circ pr_2)_*K_{\Sigma'} \tag{2.8}$$

are simply the above tensorized with the identity of K_B. Since we just noticed that the maps pr_i^* in (2.7) satisfy the relation $\frac{1}{m^a} pr_1^* = \frac{1}{m'^a} pr_2^*$, it follows that the same relation holds for the maps pr_i^* of (2.8). Taking global sections, it follows that

$$\frac{1}{m^a} pr_1^*\omega_P = \frac{1}{m'^a} pr_2^*\omega_P$$

for any holomorphic n-form ω_P on P and in particular for $\phi^*\omega_X$. ■

Proof of Theorem 2 in case 2. The construction is the following : recall that we have an embedding

$$j : X \hookrightarrow Y.$$

Now choose a rational curve $C \subset Y$ with sufficiently ample normal bundle, so that deformations of C induce arbitrary deformations of the M-th order jet of C at two points of intersection x, y of C with X. Here M is a fixed integer, and such C exists since Y is rationally connected ([16]). We may assume furthermore that the intersection of C with X, as a divisor on C, is of the form

$$mx + m'y + z,$$

where $x \neq y$ and z is a reduced zero-cycle on C disjoint from x and y. Here m and m' are two distinct fixed integers $\leq M$. Now choose a hypersurface $W \subset X$ supporting z. We will then define, for an adequate choice of W, the correspondence Σ as the closure of the image in $X \times X$ via the map (F, G) defined below, of the following set

$$\Sigma' = \{(x', y', C'),\, C' \cdot X = mx' + m'y' + z',\, z' \subset W\},$$

where in this definition, C' has to be a deformation of C. The map

$$(F, G) : \Sigma' \to X \times X$$

is defined by

$$(F, G)((x', y', C')) = (x', y').$$

More precisely, we will consider below the (unique) component of Σ' passing through (x, y, C). We first show that $dim\, \Sigma' = n$ for a generic choice of W. This is an easy dimension count : the Hilbert scheme of C is smooth at C and has dimension

$$h^0(C, N_{C/Y}) = -K_Y \cdot C + n - 2.$$

Next we impose the conditions that the intersection of C' with X is finite (this is open) and of the form $mx' + m'y' + z'$, with $z' \subset W$. This imposes at most $(m-1) + (m'-1) + deg\, z'$ conditions to the deformations of C'. Furthermore, one sees easily that for an adequate choice of W, these conditions are infinitesimally independent at our initial point (x, y, C). Hence it follows that Σ' is smooth at (x, y, C), of dimension

$$dim\, \Sigma' = -K_Y \cdot C + n - 2 - ((m-1) + (m'-1) + deg\, z')$$

$$= -K_Y \cdot C + n - 2 - (X \cdot C - 2)$$

and this is equal to n because $X \in \mid -K_Y \mid$.

An easy infinitesimal computation involving the assumption made on the normal bundle of C shows that Σ is also of dimension n, or more precisely has a component of dimension n.

It remains now to show that Σ gives a self-K-isocorrespondence satisfying furthermore the condition

$$m^2 f^* \Omega_X = m'^2 g^* \Omega_X. \tag{2.9}$$

(Choosing then $m \lneq m'$, will give a coefficient $\lambda = \frac{m'^2}{m^2} \gneq 1$. Formula (2.9) and the fact that Σ is a K-isocorrespondence will follow from the following fact :

Lemma 1 *We have*

$$mF^*\omega_X + m'G^*\omega_X = 0 \tag{2.10}$$

on Σ'*, for any holomorphic* n*-form* ω_X *on* X*.*

Indeed, since $(\Sigma, (pr_1, pr_2))$ is the Stein factorization of $(\Sigma', (F, G))$, the formula will be true as well for (Σ', F, G) replaced with (Σ, pr_1, pr_2) or better by a desingularization $(\widetilde{\Sigma}, f, g)$. Now, the canonical bundle of X being trivial, the divisor of $f^*\omega_X$ (resp. $g^*\omega_X$) is equal to R_f (resp. R_g), so that the formula

$$mf^*\omega_X + m'g^*\omega_X = 0 \tag{2.11}$$

implies that Σ is a K-isocorrespondence. ■

Proof of the lemma. We have three 0-correspondences between Σ' and X. The first one is $\Gamma_C \subset \Sigma' \times X$, which has for fiber over $\sigma = (x, y, C) \in \Sigma'$ the 0-dimensional subscheme $C \cap X$ of X. If

$$\mathcal{C} \subset \Sigma' \times Y$$

is the universal subscheme, corresponding to the map from Σ' to the Hilbert scheme of curves in Y, then

$$\Gamma_C = \mathcal{C} \cap (\Sigma' \times X).$$

The second one is $\Gamma_{x,y}$, whose fiber over $\sigma = (x, y, C) \in \Sigma'$ is the 0-cycle $mx + m'y$. This correspondence is nothing but the sum $m\Gamma_F + m'\Gamma_G$ of the graphs of F and G. The third one, which we denote by Γ_z has for fiber over $\sigma = (x, y, C) \in \Sigma'$ the residual cycle $z = C \cdot X - mx - m'y$. Hence we obviously have the relation

$$\Gamma_C = \Gamma_z + \Gamma_{x,y}$$

as n-cycles in $\Sigma' \times X$. It follows from this that for $\omega_X \in H^0(X, K_X)$, the Mumford pull-backs $\Gamma_C^*\omega_X$, $\Gamma_{x,y}^*\omega_X$ and $\Gamma_z^*\omega_X$, which are holomorphic n-forms on the smooth part of Σ', satisfy the relation

$$\Gamma_C^*\omega_X = \Gamma_z^*\omega_X + \Gamma_{x,y}^*\omega_X. \tag{2.12}$$

Since $\Gamma_{x,y} = m\Gamma_F + m'\Gamma_G$, we have

$$\Gamma_{x,y}^*\omega_X = mF^*\omega_X + m'G^*\omega_X,$$

and hence (2.12) gives

$$mF^*\omega_X + m'G^*\omega_X = \Gamma_C^*\omega_X - \Gamma_z^*\omega_X.$$

To prove (2.10), it suffices now to prove that $\Gamma_C^*\omega_X$ and $\Gamma_z^*\omega_X$ vanish.

For the second one, this is quite easy. Indeed, by definition of Σ', the cycle Γ_z is supported on $\Sigma' \times W$. On the other hand the n-form ω_X vanishes on W, because $dim\, W < n$. So $\Gamma_z^*\omega_X = 0$.

As for the second one, we already noticed the fact that

$$\Gamma_C = (Id, j)^*\mathcal{C}, \tag{2.13}$$

where we see $\mathcal{C}$ as a codimension n-cycle in $\Sigma' \times Y$. This last cycle induces a cohomological correspondence

$$[\mathcal{C}]^* : H^1(Y, \Omega_Y^{n+1}) \to H^0(\Sigma', K_{\Sigma'}).$$

Formula (2.13) then shows immediately that

$$\Gamma_C^* \omega_X = [\mathcal{C}]^*(j_* \omega_X).$$

So to conclude the proof that $\Gamma_C^* \omega_X = 0$, it suffices to see that $j_* \omega_X = 0$ in $H^1(Y, \Omega_Y^{n+1})$. But this last space is in fact 0, because it is Serre dual to $H^n(Y, \mathcal{O}_Y)$ and Y is rationally connected. ∎

Remark 1 *Another way to understand the proof above is to say that that we have for any $(x, y) \in \Sigma$ the relation*

$$mx + m'y \in j^* CH_1(Y) + k_* CH_0(W) \tag{2.14}$$

in the group $CH_0(X)$, where k is the inclusion of W in X. But since Y is rationally connected, Bloch-Srinivas argument (cf [4]) shows that $CH_1(Y)$ is a direct factor in $CH_0(W)$ for some $n-1$-dimensional variety W. Hence the relation (2.14) shows that the 0-cycles $mx + m'y$, $(x, y) \in \Sigma$ of X are up to rational equivalence parameterized by 0-cycles in a $n-1$-dimensional variety. Hence the higher dimensional version of Mumford's theorem [20] applies to give the relation $mF^*\omega_X + m'G^*\omega_X = 0$.

Proof of Theorem 2 in case 3. The construction in this case is as follows. We assume the complete intersection $M \subset \mathbb{P}^N$ is a generic complete intersection of multidegree $d_1 \leqslant \ldots \leqslant d_k$, so that its Fano variety X of r-planes is smooth of the right dimension. Let

$$G = Grass(r + 1, N + 1).$$

Then the canonical bundle K_G is equal to $-(N+1)L$, where $L = det\,\mathcal{E}$ is the Plücker line bundle, $\mathcal{E}$ is the dual of the tautological subbundle. Now $X \subset G$ is defined as the 0-set of the section $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k)$ of the vector bundle $S^{d_1}\mathcal{E} \oplus \ldots \oplus S^{d_k}\mathcal{E}$ corresponding to the section $(\sigma_1, \ldots, \sigma_k)$ of $\mathcal{O}_{\mathbb{P}^N}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^N}(d_k)$ defining M.

It follows from adjunction that the canonical bundle of X is given by the formula

$$K_X = -(N + 1)L_{|X} + \sum_i det\, S^{d_i}\mathcal{E}.$$

We use the following lemma :

Lemma 2 *Let E be a vector bundle of rank k. Then for any integer l, we have*

$$det\, S^l E \cong (det\, E)^{\otimes \alpha},$$

where $\alpha = h^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(l - 1))$.

We conclude from this and the fact that

$$rk\,\mathcal{E} = r + 1,\ det\,\mathcal{E} = L,$$

that the triviality of the canonical bundle of X is equivalent to the equality

$$N+1=\sum_i h^0(\mathbb{P}^{r+1},\mathcal{O}_{\mathbb{P}^{r+1}}(d_i-1)). \tag{2.15}$$

Since $d_k\geq 3$, we can choose two integers $m<m'$ such that $m+m'=d_k$. Let Z be the complete intersection defined by $(\sigma_1,\ldots,\sigma_{k-1})$. We consider now

$$\widetilde{\Sigma}=\{(P_1,P_2,P),\ P\cong\mathbb{P}^{r+1}\subset Z,$$

$$P_1,\ P_2\subset M,\ P\cap M\supseteqq mP_1+m'P_2\}.$$

(Note that $M\subset Z$ is defined by one equation of degree d_k so that we have then either $P\cap M=mP_1+m'P_2$ or $P\subset X$.)

We define the maps f and g from $\widetilde{\Sigma}$ to X by

$$f(P_1,P_2,P)=P_1\in X,\ g(P_1,P_2,P)=P_2\in X.$$

Lemma 3 *We have* $dim\,\widetilde{\Sigma}=n=dim\,X$ *and* f, g *are dominating.*

The variety W parametrizing the $\mathbb{P}^{r+1}$'s contained in Z is the 0-set of the natural section

$$(\tilde{\sigma}'_1,\ldots,\tilde{\sigma}'_{d-1})$$

of the bundle $S^{d_1}\mathcal{E}'\oplus\ldots\oplus S^{d_{k-1}}\mathcal{E}'$ on the Grassmannian $Grass(r+2,N+1)$. By genericity of Z, it is smooth of dimension

$$(r+2)(N-r-1)-\sum_{i\leq k-1} rk\,S^{d_i}\mathcal{E}', \tag{2.16}$$

where

$$rk\,S^{d_i}\mathcal{E}'=h^0(\mathbb{P}^{r+1},\mathcal{O}_{\mathbb{P}^{r+1}}(d_i)).$$

On W there is a $\mathbb{P}^{r+1}\times\mathbb{P}^{r+1}$-bundle, whose fiber over $P\in W$ parametrizes pairs of hyperplanes in P. Let us call it W'. Then we have

$$\widetilde{\Sigma}\subset W',$$

and $\widetilde{\Sigma}$ is defined by the condition that $(P_1,P_2,P)\in\widetilde{\Sigma}$ if and only if the restriction $\sigma_{k|P}$ is proportional to $\tau_1^m\tau_2^{m'}$, where τ_i are linear equations defining P_i in P. In other words, $\widetilde{\Sigma}\subset W'$ is the zero locus of the section of the vector bundle $\pi^*S^{d_k}\mathcal{E}'/H$, where π is the projection from W' to W and H is the line subbunble with fiber $<\tau_1^m\tau_2^{m'}>$ at (P_1,P_2,P). It follows that

$$dim\,\widetilde{\Sigma}\geqslant dim\,W+2(r+1)-rk\,S^{d_k}\mathcal{E}'+1, \tag{2.17}$$

and since our equations are generic, a standard argument shows that we have in fact equality and that $\widetilde{\Sigma}$ is smooth. Combining (2.16) and (2.17), we get

$$dim\,\widetilde{\Sigma}=(r+2)(N-r-1)-\sum_i rk\,S^{d_i}\mathcal{E}'+2(r+1)+1.$$

Next we note that, since X is the zero locus of a transverse section of the vector bundle $\oplus_i S^{d_i}\mathcal{E}$ on $Grass(r+1,N+1)$, we have

$$dim\,X=(r+1)(N-r)-\sum_i rk\,S^{d_i}\mathcal{E}.$$

Noting finally that if E is of rank $r+1$, then $rk\, S^k E = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k))$, and that $\mathcal{E}$, $\mathcal{E}'$ are of respective ranks $r+1$, $r+2$, we get

$$dim\,\widetilde{\Sigma} - dim\,X$$

$$= -(\sum_i h^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}(d_i)) - h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_i))) + (r+2)(N-r-1) - (r+1)(N-r) + 2r+3$$

$$= -\sum_i h^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}(d_i - 1) + (r+2)(N-r-1) - (r+1)(N-r) + 2r+3.$$

Using equality (2.15), this gives us $dim\,\widetilde{\Sigma} - dim\,X = 0$.

To conclude the proof of the lemma, we have to show that the maps f, g are dominating. We do it for f: the fiber of f over $P_1 \in X$ is the zero locus of a section s of a vector bundle over the variety W'_{P_1} parametrizing $\mathbb{P}^{r+1}$'s containing P_1, together with a hyperplane P_2 in them. Precisely, this vector bundle has for fiber

$$\oplus_{i<k} H^0(P, \mathcal{O}_P(d_i - 1)) \oplus H^0(P, \mathcal{O}_P(d_k - 1)) / < \tau_1^{m-1} \tau_2^{m'} >$$

at (P, P_2). The section s takes the value

$$((\sigma_{i|P}/\tau_1)_{i<k}, \sigma_{k|P}/\tau_1 \bmod \tau_1^{m-1}\tau_2^{m'}$$

at (P, P_2), where τ_i is a defining equation for $P_i \subset P$. We use here the fact that $P_1 \subset M$, so that $\sigma_{i|P}$ vanishes along P_1. The vector bundle has the same rank as the variety W'_{P_1}, as shows the previous computation. To show that f is dominating, it suffices to show that this vector bundle has a non zero top Chern class, which is not hard. ∎

To conclude the proof of the theorem in case 3, it remains to prove the following

Lemma 4 *The two ramification divisors R_f and R_g are equal.*

Indeed, the lemma shows that $\widetilde{\Sigma}$ provides a self-K-isocorrespondence of X. Next, we have explained after the statement of the theorem that for a self-K-isocorrespondence of a K-trivial variety, the fact that it multiplies the volume by a real coefficient different from 1 as in formula (2.3) is equivalent to the fact that the degrees of the maps f and g are different (cf (2.5)). Now the degree of f and g are the top Chern classes of the vector bundles described above. From this it is easy to show that for $m > m'$ we have $deg\, f < deg\, g$.

∎

Proof of lemma 4. We observe first that the set K of $(P_1, P_2) \in \Sigma$ such that the linear space generated by P_1 and P_2 is a $\mathbb{P}^{r+1}$ contained in M is of dimension $< n-1$. Suppose that we show that $R_f = R_g$ away from $(f,g)^{-1}(K)$: then $R_f - R_g$ is a divisor which is rationally equivalent to 0 (since both R_f and R_g are members of the linear system $K_{\widetilde{\Sigma}}$), and supported on $(f,g)^{-1}(K)$ which is contracted by (f,g). But it is well known that the components of a contractible divisor are rationally independent. Hence this suffices to imply that $R_f = R_g$. Next, we show by a dimension count (recall that our parameters are generic) and the description given above of the fibers of f and g that, away from $(f,g)^{-1}(K)$, the ramification of f and g is simple, i.e. the ramification divisor is reduced. In conclusion, it suffices to show that we have the set theoretic equality $R_f = R_g$ away from $(f,g)^{-1}(K)$. Next we note that the set of $(P_1, P_2, P) \in \widetilde{\Sigma}$ such that $P_1 = P_2$ is of codimension greater than 1 in $\widetilde{\Sigma}$. Hence it suffices to show the set theoretic equality $R_f = R_g$ away from $(f, g)^{-1}(K)$ and at points (P_1, P_2, P) where $P_1 \neq P_2$.

We do it by an explicit computation : let $P_1 \in X$, and let (P_1, P_2, P), $P_1 \neq P_2$, $P \not\subset M$ be a point of $\widetilde{\Sigma}$ where f ramifies. So there is a first order deformation P_ϵ of P in Z, fixing P_1, and such that $\sigma_{k|P_\epsilon}$ remains to first order of the form $\tau_{1,\epsilon}^m \tau_{2,\epsilon}^{m'}$, where $\tau_{1,\epsilon}$ is a defining equation of P_1 in P_ϵ. Since the first order deformation P_ϵ fixes P_1, it is contained in a $\mathbb{P}^{r+2}$ that we shall denote by P'. Let us choose coordinates $X_0, \ldots, X_{r+2}$ on P' so that P is defined by $X_{r+2} = 0$, P_1 is defined by $X_{r+2} = X_{r+1} = 0$ and P_2 is defined by $X_{r+2} = X_r = 0$.

The deformation P_ϵ is then given by the equation

$$X_{r+2} = \varepsilon X_{r+1}.$$

We have by assumption :

$$\sigma_{k|P'} = X_{r+1}^m X_r^{m'} + X_{r+2} G.$$

It follows that in the coordinates $X_0, \ldots, X_{r+1}$ for P_ϵ, we have to first order in ϵ :

$$\sigma_{k|P_\epsilon} = X_{r+1}^m X_r^{m'} + \epsilon X_{r+1} G',$$

where G' is the restriction of G to P. Since $\tau_{1,\epsilon}$ is proportional to X_{r+1}, the condition that $\sigma_{k|P_\epsilon}$ remains to first order of the form $\tau_{1,\epsilon}^m \tau_{2,\epsilon}^{m'}$ is then clearly

$$G' = X_{r+1}^{m-1} X_r^{m'-1} A$$

for some linear form A on P. Hence our condition is that

$$\sigma_{k|P'} = X_{r+1}^m X_r^{m'} + X_{r+1}^{m-1} X_r^{m'-1} X_{r+2} A + X_{r+2}^2 H \quad (2.18)$$

$$= X_{r+1}^{m-1} X_r^{m'-1} (X_{r+1} X_r + X_{r+2} A) \,\mathrm{mod} X_{r+2}^2. \quad (2.19)$$

Furthermore, we note that the fact that P has a deformation in P' which remains contained in Z can be written as the fact that $\sigma_{i|P'}$, $i < k$ vanish at order 2 along P, hence it does not depend on P_1. Now the equation (2.18) is symmetric in P_1 and P_2. This shows that g ramifies as well at (P_1, P_2, P), and concludes the proof of lemma 4.

■

3 Intrinsic pseudo-volume forms and a problem of Kobayashi

The Kobayashi-Eisenman pseudo-volume form Ψ_X on a complex manifold X is defined as follows : for $x \in X$, $u \in \bigwedge^n T_{X,x}$, put

$$\Psi_X(u) = \frac{1}{\lambda}, \quad (3.20)$$

where

$$\lambda = Max_{\phi: D^n \mapsto X, \phi(0)=x} \{ \mid \mu \mid, \, \phi_*(\frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_n}) = \mu u \}. \quad (3.21)$$

Here D is the unit disk in $\mathbb{C}$.

Remark 2 *A similar definition can be made using the ball instead of the polydisk , cf [7]. The resulting pseudo-volume forms so obtained are equivalent, and all the results that follow will be true as well with this definition of* Ψ_X.

Denoting κ_n the hyperbolic form on the polydisk :

$$\kappa_n = i^n \Pi_1^n \frac{1}{(1- \mid z_j \mid^2)^2} dz_j \wedge \overline{dz}_j, \tag{3.22}$$

we see immediately, using the fact that κ_n coincides with the standard volume form at 0, is invariant under the automorphisms of D_n, and that the later act transitively on D^n, that we can express Ψ_X as follows :

$$\Psi_{X,x} = inf_{\phi: D^n \to X,\, \phi(b)=x} \{(\phi_b^{-1})^* \kappa_n\}. \tag{3.23}$$

Here we consider only the holomorphic maps $\phi : D^n \to X$ which are unramified at b, $\phi(b) = x$, and ϕ_b is then defined as the local inverse of ϕ near b. It is obvious from either definition that Ψ_X satisfies the decreasing volume property with respect to holomorphic maps :

For any holomorphic map $\phi : X \to Y$ *between* n*-dimensional complex manifolds, we have*

$$\phi^* \Psi_Y \leq \Psi_X.$$

Also, the following theorem is a consequence of Ahlfors-Schwarz lemma (cf [7]) :

Theorem 3 *If* X *is isomorphic to* D^n *(resp. to the quotient of* D^n *by a group acting freely and properly discontinuously, eg* X *is a product of curves), then* $\Psi_X = \kappa_n$*, (resp. to the hyperbolic volume form on the quotient induced by* κ_n*).*

There is also a meromorphic version $\widetilde{\Psi}_X$ introduced by Yau [24], which has the advantage of being invariant under birational maps : namely put

$$\widetilde{\Psi}_{X,x} = inf_{\phi: D^n \dashrightarrow X,\, \phi(b)=x} \{(\phi_b^{-1})^* \kappa_n\}. \tag{3.24}$$

Here we consider the meromorphic maps $\phi : D^n \dashrightarrow X$ which are defined at b and unramified at b, $\phi(b) = x$, and ϕ_b is then defined as the local inverse of ϕ near b.

The following result is proved in [11], [15], [24] :

Theorem 4 *If* X *is a projective complex manifold which is of general type, then* $\widetilde{\Psi}_X$ *is non degenerate outside a proper closed algebraic subset of* X.

(The result is proved in [11] for Ψ_X and for the varieties with ample canonical bundle, and in [15] for Ψ_X.) Kobayashi [13] conjectures the converse to this statement :

Conjecture 1 *If* X *is a projective complex manifold which is not of general type, then* $\widetilde{\Psi}_X = 0$ *on a dense Zariski open set of* X.

Remark 3 *A priori,* $\widetilde{\Psi}_X$ *is only uppersemicontinuous, hence the equality* $\widetilde{\Psi}_X = 0$ *on a dense Zariski open set of* X *does not imply that* $\widetilde{\Psi}_X = 0$ *everywhere.*

This conjecture is known in dimension ≤ 2, [10]. In dimension 2, it uses the classification of surfaces, and the fact that $K3$-surfaces are swept out by elliptic curves. The proof shows more generally that $\Psi_X = 0$ on a dense Zariski open set, for a variety which is swept out by abelian varieties, and $\widetilde{\Psi}_X = 0$ on a dense Zariski open set, for a variety which is rationally swept out by abelian varieties.

We start this section with the definition of modified versions Φ_X, $\Phi_{X,an}$ of Ψ_X, together with their meromorphic counterparts $\widetilde{\Phi}_X$, $\widetilde{\phi}_{X,an}$.

Definition 3 *We put*

$$\Phi_{X,x} = inf_{\Sigma\subset X\times X, K-iso,\, \sigma\in\widetilde{\Sigma},\, g(\sigma)=x}(f^*\Psi_X)_\sigma.$$

Here Σ runs through the self-K-isocorrespondences of X, and we denote as usual

$$\begin{array}{ccc} \widetilde{\Sigma} & \stackrel{g}{\rightarrow} & X \\ f\downarrow & & \\ X & & \end{array}$$

a desingularization. We use then the fact that $\widetilde{\Sigma}$ induces a canonical isomorphism

$$f^*K_X \cong g^*K_X$$

to see that $(f^*\Psi_X)_\sigma$ gives a pseudo-volume element for X at x, $g(\sigma) = x$.

Another equivalent way to define Φ_X is by the following formula, closer to (3.20), (3.21) : for $x \in X$, $\zeta \in \bigwedge^n T_{X,x}$, $\Phi_{X,x}(\zeta) = \frac{1}{\lambda}$, where

$$\lambda = \sup_{\Sigma\subset X\times X, \phi: D^n\rightarrow X, \sigma\in\widetilde{\Sigma}', f(\sigma)=0, g(\sigma)=x}\{\mid\mu\mid, J_{\widetilde{\Sigma}',\sigma}(\frac{\partial}{\partial z_1}\wedge\ldots\wedge\frac{\partial}{\partial z_n}) = \mu\zeta\}.$$

Here ϕ is any holomorphic map from D^n to X which is generically of maximal rank, Σ is any self-K-isocorrespondence of X, $\widetilde{\Sigma}'$ is a desingularization of the K-correspondence $\Sigma\circ graph(\phi)$ between D^n and X, and $J_{\widetilde{\Sigma}',\sigma}$ is the Jacobian morphism defined in (1.2).

The definition of $\Phi_{X,an}$ is similar : instead of considering only K-correspondences from D^n to X which are of the form $\Sigma\circ graph(\phi)$, we consider all K-correspondences from D^n to X :

Definition 4 *We put*

$$\Phi_{X,an,x} = inf_{\Sigma\subset Y\times X, K-corresp,\, \sigma\in\widetilde{\Sigma},\, g(\sigma)=x}(f^*\Psi_Y)_\sigma.$$

Here Σ runs through the set of all K-correspondences from Y to X, and the condition on the point σ is that Σ is unramified at σ, namely that near $\sigma \in \widetilde{\Sigma}$, we have the equality $R_f = R_g$. Then exactly as above, $(f^*\Psi_Y)_\sigma$ gives a pseudo-volume element for X at x, $g(\sigma) = x$ so that our definition makes sense.

Equivalently, $\Phi_{X,an,x}(\zeta) = \frac{1}{\lambda}$, where

$$\lambda = \sup_{\Sigma\subset D^n\times X, \sigma\in\widetilde{\Sigma}, f(\sigma)=0, g(\sigma)=x}\{\mid\mu\mid, J_{\widetilde{\Sigma},\sigma}(\frac{\partial}{\partial z_1}\wedge\ldots\wedge\frac{\partial}{\partial z_n}) = \mu\zeta\}.$$

Note that since κ_n is the Euclidean volume form at 0, $\Phi_{X,x,an}$ can also be computed as

$$\Phi_{X,an,x} = inf_{\Sigma\subset D^n\times X, K-corresp,\, \sigma\in\widetilde{\Sigma},\, g(\sigma)=x}(f^*\kappa_n)_\sigma. \quad (3.25)$$

Remark 4 *There are other intermediate possible definitions for a modified version of Ψ_X using K-correspondences. For example, we could restrict in the definition of $\Phi_{X,an}$ to the proper K-correspondences, i.e. those for which g is also proper. In the definition of Φ_X, we could consider all K-isocorrespondences from Y to X, instead of the self-K-isocorrespondences from X to X. We restricted to the two extremal cases, which seem to be the most interesting, because on one side Φ_X is of course the closest to Ψ_X, while on the other side $\Phi_{X,an}$ satisfies the following version of the decreasing volume property, as follows immediately from its definition.*

Lemma 5 *If Y is a complex manifold of dimension n, and $\Sigma \subset Y \times X$ is a K-correspondence, then with the notations used before for the desingularization :*

$$g^*\Phi_{X,an} \leq f^*\Phi_{Y,an}.$$

■

Note also that from the definition of Φ_X we get the following :

Lemma 6 *If $\Sigma \subset X \times X$ is a self-K-isocorrespondence, we have with the same notations :*

$$f^*\Phi_X = g^*\Phi_X.$$

■

Finally, we define the meromorphic versions $\widetilde{\Phi}_X$, $\widetilde{\Phi}_{X,an}$ by the formula :

$$\widetilde{\Phi}_{X,x} = inf_{\phi:X \dashrightarrow Y,\, \phi(x)=y}\,\{\phi^*\Phi_{Y,y}\},$$

$$\widetilde{\Phi}_{X,an,x} = inf_{\phi:X \dashrightarrow Y,\, \phi(x)=y}\,\{\phi^*\Phi_{Y,an,y}\}.$$

In both formulas, we consider only the birational maps $\phi : X \dashrightarrow Y$ which are defined at x and such that ϕ^{-1} is defined at $y = \phi(x)$.

Of course we have, for any birational map $\phi : X \dashrightarrow Y$, the equalities

$$\phi^*\widetilde{\Phi}_Y = \Phi_X,$$

$$\phi^*\widetilde{\Phi}_{Y,an} = \Phi_{X,an},$$

which are satisfied on the open set U of X where ϕ is defined and is a local isomorphism. In particular, if $U \hookrightarrow X$ is the inclusion of a Zariski open set, we have

$$\widetilde{\Phi}_{X|U} = \Phi_U,\ \widetilde{\Phi}_{X,an|U} = \Phi_{U,an}. \tag{3.26}$$

Our main result towards the comparison of Φ_X, $\Phi_{X,an}$ and Ψ_X is the following :

Theorem 5 *If X is the polydisk D^n, or any quotient of the polydisk by a free properly discontinuous action of a group on D^n, eg X is a product of curves, then*

$$\Phi_{X,an} = \Psi_X.$$

Since

$$\Phi_{X,an} \leq \Phi_X \leq \Psi_X,$$

it follows that $\Phi_X = \Psi_X$ too.

Proof. We do it for D^n, the general case follows exactly in the same way, using the fact that Ψ_X in this case is the hyperbolic volume form, which satisfies the Kähler-Einstein equation (3.29). The proof is very similar to the proof that $\Psi_{D^n} = \kappa_n$ (theorem 3), namely it uses the Ahlfors-Schwarz lemma. We want however to explain carefully why it works as well in the context of K-correspondences.

By formula 3.25, what we have to prove is the following :

If

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{g} & D^n \\ f \downarrow & & \\ D^n & & \end{array}$$

is the desingularization of a K-correspondence from D^n to itself, then

$$g^*\kappa_n \leq f^*\kappa_n. \tag{3.27}$$

Let

$$\begin{array}{ccc} \widetilde{\Sigma}_\epsilon & \xrightarrow{g_\epsilon} & D^n \\ f_\epsilon \downarrow & & \\ D^n & & \end{array}$$

be the restriction of the K-correspondence Σ to the polydisk of radius $1-\epsilon$. In other words, we intersect Σ with $D^n_{1-\epsilon} \times D^n$ and we identify $D^n_{1-\epsilon}$ with D^n via the dilatation of coefficient $\frac{1}{1-\epsilon}$. It suffices to show that

$$g_\epsilon^*\kappa_n \leq f_\epsilon^*\kappa_n. \tag{3.28}$$

Next because Σ is a K-correspondence, the ratio

$$\psi_\epsilon := \frac{g_\epsilon^*\kappa_n}{f_\epsilon^*\kappa_n}$$

is a non negative C^∞-function (which is even real analytic) on $\widetilde{\Sigma}_\epsilon$. Furthermore, as we have restricted to $D^n_{1-\epsilon}$, the numerator stays bounded near the boundary while the denominator tends to ∞ generically on the boundary of D^n, so we have

$$\lim_{f(x)\to\partial D^n} \psi_\epsilon(x) = 0.$$

It follows then from the properness of the map f_ϵ that ψ_ϵ has a maximum on $\widetilde{\Sigma}_\epsilon$. Let $\psi_\epsilon(x)$ be maximum. Formula (3.28) is equivalent to

$$\psi_\epsilon(x) \leq 1.$$

Assume the contrary and let $c := \psi_\epsilon(x) > 1$. Choose α generic, $1 < \alpha < c$. Let

$$\widetilde{\Sigma}_{\epsilon,\alpha} := \{y \in \widetilde{\Sigma}_\epsilon, \, \psi_\epsilon(x) \geq \alpha\}.$$

Then since α is generic, and ψ_ϵ tends to 0 near $\partial\widetilde{\Sigma}_\epsilon$, $\widetilde{\Sigma}_{\epsilon,\alpha}$ is compact and has a smooth boundary. Now let $\chi = i^n \Pi_{j=1}^{j=n} \frac{1}{(1-|z_j|^2)^2}$. By definition of κ_n (cf (3.22)), we have

$$\psi_\epsilon = \frac{g_\epsilon^*\chi}{f_\epsilon^*\chi} \mid G \mid^2,$$

where G is holomorphic. Furthermore, we have the Kähler-Einstein equation

$$(\frac{i}{2}\partial\overline{\partial}log\,\chi)^n = n!\kappa_n. \tag{3.29}$$

Denoting by $\omega = \frac{i}{2}\partial\overline{\partial}log\,\chi$, we have

$$\frac{i}{2}\partial\overline{\partial}log\,\psi_\epsilon = g_\epsilon^*\omega - f_\epsilon^*\omega,\ \omega^n = n!\kappa_n. \tag{3.30}$$

Now, in $\widetilde{\Sigma}_{\epsilon,\alpha}$, we have $\psi_\epsilon > 1$, which implies that

$$f_\epsilon^*\kappa_n \leq g_\epsilon^*\kappa_n, \tag{3.31}$$

with strict inequality away from the ramification divisor R_f. Let

$$\theta := g_\epsilon^*\omega^{n-1} + g_\epsilon^*\omega^{n-2}f_\epsilon^*\omega + \ldots + f_\epsilon^*\omega^{n-1}.$$

This is a semipositive $(n-1, n-1)$-form, which is positive away from R_f. Furthermore formulae (3.30) and (3.31) say that

$$(\frac{i}{2}\partial\overline{\partial}log\,\psi_\epsilon)\theta \geq 0 \tag{3.32}$$

in $\widetilde{\Sigma}_{\epsilon,\alpha}$ with strict inequality away from the ramification divisor R_f. Of course, if we knew that $x \notin R_f$ then we would conclude that the hypothesis that $log\,\psi_\epsilon$ has a maximum at x is absurd, because its Hessian should then be seminegative at x, contradicting the strict inequality in (3.32). In general, one can apply the following (standard) argument : choose a number α', such that $\alpha < \alpha' < log\,c$. Put

$$\mu^+ = Sup\,(0, log\,\psi_\epsilon - \alpha').$$

Then μ^+ is non negative, vanishes identically near the boundary of $\widetilde{\Sigma}_{\epsilon,\alpha}$, and is positive at x. Now consider

$$\int_{\widetilde{\Sigma}_{\epsilon,\alpha}} \mu^+(\frac{i}{2}\partial\overline{\partial}log\,\psi_\epsilon)\theta.$$

This is strictly positive. On the other hand, integration by parts, using the fact that the derivatives of μ^+ are integrable, gives :

$$\int_{\widetilde{\Sigma}_{\epsilon,\alpha}} \mu^+(\frac{i}{2}\partial\overline{\partial}log\,\psi_\epsilon)\theta = -\int_{\widetilde{\Sigma}_{\epsilon,\alpha}} \frac{i}{2}(\partial\mu^+ \wedge \overline{\partial}log\,\psi_\epsilon)\theta.$$

But since $\mu^+ = log\,\psi_\epsilon - \alpha'$ when it is non zero, the integral on the right is equal to

$$-\int_{\widetilde{\Sigma}_{\epsilon,\alpha'}} \frac{i}{2}(\partial log\,\psi_\epsilon \wedge \overline{\partial}log\,\psi_\epsilon)\theta,$$

where

$$\widetilde{\Sigma}_{\epsilon,\alpha'} = \{y \in \widetilde{\Sigma}_{\epsilon,\alpha},\ log\,\psi_\epsilon(y) \geq \alpha'\}.$$

But this last integral is obviously negative, which is a contradiction.

■

Next we have the following strenghtening of Theorem 4.

Theorem 6 *If X is a projective complex manifold which is of general type, we have $\Phi_{X,an} > 0$ (and in particular $\Phi_X > 0$) away from a proper closed algebraic subset of X.*

Proof. We just sketch the argument, since it is a combination of the construction in [11], [14] and of the arguments given above in the specific case of K-correspondences.

Since X is of general type, there exists an inclusion of sheaves

$$L \subset K_X^{\otimes \alpha}$$

for sufficiently large α, where L is an ample line bundle on X. Then, if h_L is a hermitian metric on L such that the associated Chern form

$$\omega_{L,h} = \frac{1}{2i\pi}\partial\overline{\partial}h_L$$

is a Kähler form, we can see $\mu := \frac{1}{h_L^{\frac{1}{\alpha}}}$ as a pseudovolume form on X, vanishing along a divisor, which satisfies, in local coordinates where $\mu = i^n \chi dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n$, the equation

$$i\partial\overline{\partial}log\,\chi = \frac{1}{\alpha}\omega_{L,h}. \tag{3.33}$$

Now after a rescaling, we may assume that

$$(\frac{1}{2\alpha}\omega_{L,h})^n \geq n!\mu. \tag{3.34}$$

So the theorem is a consequence of the following proposition, which is proved exactly as theorem 5 :

Proposition 2 *Assume X is equipped with a pseudo-volume form μ satisfying equations (3.33) and (3.34). Then for any K-correspondence $\Sigma \subset D^n \times X$, we have*

$$g^*\mu_n \leq f^*\kappa_n.$$

∎

Remark 5 *One can show similarly that the same result holds for $\widetilde{\Phi}_{X,an}$.*

The two theorems above obviously lead to the following

Conjecture 2 *Assume that X is projective. Then $\Phi_{X,an}$ is equivalent to Ψ_X. This means that there exists a non zero constant α depending on X such that*

$$\alpha\Psi_X \leq \Phi_{X,an} \leq \Psi_X.$$

We conclude this section with the proof of the following theorems, which prove a number of special cases of Kobayashi's conjecture 1 for our pseudovolume forms Φ_X, $\widetilde{\Phi}_X$.

Theorem 7 *Assume X is a K-trivial projective variety which is as in the statement of theorem 2, that is satisfies 1, 2 or is generic satisfying 3. Then $\Phi_X = 0$.*

Theorem 8 *Assume X is birational to X', and there exists a projective morphism $\phi : X \to B$ such that $\dim B < \dim X$ and the generic fiber X'_b is a K-trivial variety as in the previous theorem. Then $\widetilde{\Phi}_X = 0$ on a dense Zariski open set of X.*

Proof of theorem 7. By theorem 2, there exists a self-K-isocorrespondence $\Sigma \subset X \times X$ such that, with the notation

$$\begin{array}{ccc} \widetilde{\Sigma} & \stackrel{g}{\to} & X \\ f \downarrow & & \\ X & & \end{array}$$

for a desingularization of Σ, we have

$$f^*\Omega_X = \lambda g^*\Omega_X,$$

for some $\lambda > 1$. By lemma 6, we know that

$$f^*\Phi_X = g^*\Phi_X.$$

Writing $\Phi_X = \chi\Omega_X$ and combining these two equalities gives

$$f^*\chi = \lambda g^*\chi.$$

But the function χ is uppersemicontinuous and bounded, hence it has a maximum. Let x be a point where $\chi(x)$ is maximum. Let $\sigma \in \widetilde{\Sigma}$ be such that $g(\sigma) = x$. Then for $y = f(x)$, we get $\chi(y) = \lambda\chi(x)$. Since $\chi(x)$ is maximum, we also have $\chi(y) \leq \chi(x)$, which implies $\chi(x) = 0$ because $\lambda > 1$. So $\chi = 0$.

■

Proof of theorem 8. By the birational invariance of $\widetilde{\Phi}_X$, it suffices to show that $\widetilde{\Phi}_{X'} = 0$ on a dense Zariski open set X'' of X', or equivalently that, for some dense open X'', one has $\widetilde{\Phi}_{X''} = 0$.

But the construction of self-K-isocorrespondence given in the proof of theorem 2 can be made in families at least over a Zariski open set B'' of B. Letting $X'' = \phi^{-1}(B'')$, we get a relative self-K-isocorrespondence

$$\Sigma \subset X'' \times_{B''} X''.$$

Denote by $\Omega_{X''/B}$ the relative volume form on X'' which restricts to the canonical volume form on each fiber of $X'' \to B$. Then Σ satisfies the property that as relative pseudovolume forms on $\widetilde{\Sigma}$ over B'', we have

$$f^*\Omega_{X''/B} = \lambda g^*\Omega_{X''/B},\ \lambda > 1. \tag{3.35}$$

(Indeed, note that the coefficient λ is constant in families, by the formula (2.5).) Now, since $\Sigma \subset X'' \times_{B''} X''$, we have $\phi \circ f = \phi \circ g =: \pi$ and for Ω_B a volume form on B, we have

$$f^*(\Omega_{X''/B} \otimes \phi^*\Omega_B) = (f^*\Omega_{X''/B}) \otimes \pi^*\Omega_B,$$

and similarly for g. Hence (3.35) gives

$$f^*(\Omega_{X''/B}\otimes\phi^*\Omega_B)=\lambda g^*(\Omega_{X''/B}\otimes\phi^*\Omega_B). \tag{3.36}$$

Denoting by Ω'' the volume form $\Omega_{X''/B}\otimes\phi^*\Omega_B$ on X'', we have a relation $\Phi_{X''}=\chi\Omega''$ for some function χ, and formula (3.36), together with the relation

$$f^*\Phi_{X''}=g^*\Phi_{X''}$$

show that

$$f^*\chi=\lambda g^*\chi.$$

One concludes then as in the previous proof that $\chi=0$, hence $\Phi_{X''}=0$, using the fact that $\Sigma\subset X''\times_B X''$ and the properness of $\phi:X''\to B''$.

■

4 Concluding remarks and questions

4.1 Fano varieties of r-planes in a hypersurface

Our first question concerns the Chow-theoretic interpretation of our construction of a self-K-correspondence in case 3, that is when X is the variety of r-planes in a hypersurface of degree d (or more generally a complete intersection). Unlike case 2, we did not deduce formula

$$f^*\omega_X=\mu g^*\omega_X \tag{4.37}$$

from a relation between 0-cycles, of the form

$$\forall\sigma\in\Sigma,\ \alpha f(\sigma)+\beta g(\sigma)\equiv z, \tag{4.38}$$

where z is supported on a proper algebraic subset of X, and α, β are fixed integers depending on the integers m, m'. Of course, by Mumford's theorem [20], (4.38) implies (4.37), with $\mu=\frac{-\beta}{\alpha}$. Bloch-Beilinson's conjectures predict also that conversely (4.37) implies relations like (4.38). So our first question is : how to prove a formula like (4.38), for Σ constructed as in the proof of theorem 2, case 3?

Let us do it in the case where M is the cubic fourfold, $m=2$, $m'=1$ and $r=1$. In this case X is 4-dimensional and is hyperKähler (cf [3]).

Recall that Σ parametrizes the pairs (L_1,L_2) of lines in M such that there exists a plane $P\subset\mathbb{P}^5$, with

$$P\cap M=2L_1+L_2.$$

For each line $L\subset M$, let us denote by l the corresponding point in X. For a generic $l\in X$ there is an incidence surface in X (cf [21])

$$S_l:=\{l'\in X,\ L\cap L'\neq\emptyset\}.$$

Note that if

$$\begin{array}{ccc} P & \xrightarrow{q} & M \\ p\downarrow & & \\ X & & \end{array}$$

is the incidence correspondence, we have

$$S_l = p_* q^* L \,\text{in}\, CH^2(X).$$

It follows that, denoting by h the class of a plane section of M, we have for any $(l_1, l_2) \in \Sigma$ the relation

$$2S_{l_1} + S_{l_2} = p_* q^* h \,\text{in}\, CH^2(X). \tag{4.39}$$

Now, it is not hard to prove the following

Lemma 7 *There exists an integer $\alpha \neq 0$ and a proper algebraic subset $Z \subset X$ such that for any $l \in X$ the following relation holds in $CH(X)$:*

$$S_l^2 = \alpha l + z, \tag{4.40}$$

where z is a 0-cycle supported on Z.

We now combine formulas (4.40) and (4.39) to get for any $(l_1, l_2) \in \Sigma$ the relations

$$4S_{l_1}^2 = S_{l_2}^2 + z',$$
$$4\alpha l_1 = \alpha l_2 + z' + z'',$$

where z' and z'' are supported on a fixed algebraic subset of X. This gives us the formula (4.38) in this case. This also shows that $\mu = \frac{1}{4}$ hence $\lambda = \frac{1}{16} = \frac{deg\, f}{deg\, g}$ in this case.

4.2 Some examples satisfying the Kobayashi conjecture

In a different direction we observe that our construction in case 3 provides for $d = 3$ a true rational map $\phi : X \dashrightarrow X$. Here we consider the Fano variety of r-planes in a hypersurface M of degree 3 in $\mathbb{P}^n$, with the relation

$$n + 1 = h^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}(2)) \tag{4.41}$$

which implies that K_X is trivial (cf (2.15)). Now let as in section 2

$$\Sigma = \{(P_1, P_2) \in X \times X, \exists P \subset \mathbb{P}^n, \, P \cap M = 2P_1 + P_2\}.$$

Here P has to be a $r + 1$-plane and we in fact have to consider the Zariski closure of the set above.

We have the following

Lemma 8 *The first projection $pr_1 : \Sigma \to X$ is of degree 1. Hence Σ is the graph of a rational map ϕ.*

Proof. Let P_1 be generic in X. Consider

$$P := \bigcap_{x \in P_1} T_{M,x}.$$

Here $T_{X,x}$ is the projective hyperplane tangent to M at x. Then P has dimension $n - h^0((\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$ because the Gauss map of M is given by polynomials of degree 2. But we have by (4.41)

$$n - h^0((\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) = -1 + h^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}(2)) - h^0((\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$$

$$= -1 + h^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}(1)) = r + 1.$$

Hence P is a $\mathbb{P}^{r+1}$ everywhere tangent to M along P_1. Since P_1 is generic, it is not contained in a $\mathbb{P}^{r+1}$ contained in M. Hence we have

$$P \cap M = 2P_1 + P_2,$$

for some P_2 which must be the only point in the fiber of Σ over P_1.

■

Corollary 1 *For such X, we have*

$$\widetilde{\Psi}_X = 0$$

on a Zariski open set of X. In other words, Kobayashi's conjecture 1 is true for X.

Proof. The decreasing volume property for $\widetilde{\Psi}_X$ will say that

$$\phi^* \widetilde{\Psi}_X \leq \widetilde{\Psi}_X$$

on the open set where ϕ is defined. On the other hand, we have seen that

$$\phi^* \Omega_X = \lambda \Omega_X,\ \lambda = deg\, \phi,$$

where Ω_X is the canonical volume form of X. Now we conclude as in the proof of Theorem 7, using the fact that $deg\, \phi > 1$, (for example $deg\, \phi = 16$ in the case of the cubic fourfold). ■

Remark 6 *The existence of the self-map ϕ of degree > 1, hence multiplying the volume form by a coefficient > 1, suggests that not only the Kobayashi pseudovolume form of X vanishes but also the Kobayashi pseudodistance of X vanishes, as conjectured in [5], [13]. This would follow, as the following argument shows, from a dynamical study of the map ϕ but we have not been able to do it. In fact, what is easily seen is the fact that the Kobayashi pseudodistance d_K of X as above is* 0 *if, for general $y \in X$, the orbit $\{\phi^k(y),\, k \in \mathbb{Z}\}$ is dense in X. Indeed, one sees easily that ϕ has one fixed point x. Next consider the function $\chi(y) = d_K(x, y)$ on X. By the decreasing distance property, we have*

$$d_K(x, \phi(y)) \leq d_K(x, y).$$

So it follows that we have the inequality of pseudo-volume forms :

$$\phi^*(\chi \cdot \Omega_X) \leq \chi \cdot \phi^* \Omega_X.$$

Now we have

$$\phi^* \Omega_X = deg\, \phi \cdot \Omega_X.$$

So

$$\phi^*(\chi \cdot \Omega_X) \leq deg\, \phi \cdot \chi \cdot \Omega_X.$$

But the integrals of both sides over X are equal. Hence we conclude that

$$f^* \chi = \chi$$

almost everywhere on X. So we have $d_K(x, \phi(y)) = d_K(x, y)$ for almost all y. So if the $\phi^k(y)$ are dense in X for almost every y, (for k negative or positive), hence arbitrary close to x, we find that $d_K(x, y) = 0$ for almost every y.

4.3 K-correspondences and the Kodaira dimension

The following two propositions relate the Kodaira dimension and K-correspondences.

Proposition 3 *Let $\Sigma \subset Y \times X$ be a K-correspondence, where X and Y are smooth and projective. Then*

$$\kappa(Y) \geq \kappa(X).$$

Proof. Let

$$\begin{array}{ccc} \widetilde{\Sigma} & \stackrel{g}{\rightarrow} & X \\ f \downarrow & & \\ Y & & \end{array}$$

be a desingularization of Σ. If $\kappa(X) = -\infty$ there is nothing to prove. If $\kappa(X) \geq 0$ there is a non zero section of $K_X^{\otimes m}$ for some $m \geq 1$. Since $g^* K_X \subset f^* K_Y$, there is a non zero section of $f^* K_Y^{\otimes m}$, and it follows that there is a non zero section of $K_Y^{\otimes Nm}$, where N is the degree of f. So $\kappa(Y) \geq 0$. So we can consider the Iitaka fibration

$$Y \dashrightarrow B$$

whose generic fiber Y_b satisfies

$$\kappa(K_{Y|Y_b}) = 0.$$

Let $\widetilde{Y_b} := f^{-1}(Y_b)$. Then

$$\kappa(f^* K_{Y|\widetilde{Y_b}}) = 0.$$

Since $g^* K_X \subset f^* K_Y$, it follows that

$$\kappa(g^* K_{X|\widetilde{Y_b}}) = 0.$$

Hence the components of $g(\widetilde{Y_b})$ are contained in a fiber of the Iitaka fibration $X \dashrightarrow B'$ of X. It follows that $dim\, B' \leq dim\, B$. ∎

Proposition 4 *If X is a projective variety which is of general type, any self-K-isocorrespondence*

$$\begin{array}{ccc} \widetilde{\Sigma} & \stackrel{g}{\rightarrow} & X \\ f \downarrow & & \\ X & & \end{array}$$

satisfies

$$deg\, f = deg\, g.$$

Proof. For a line bundle L on a projective variety, whose Iitaka dimension is equal to $n = dim\, X$, define

$$d^+(L) = Sup_m \{ \frac{deg\, \phi_m(X)}{m^n} \}$$

where ϕ_m is the rational map to projective space given by the sections of $L^{\otimes m}$ assuming there are any. This is a finite positive number. Also, it is immediate to see that if

$$\phi : X' \rightarrow X$$

is a generically finite cover, we have

$$d^+(\phi^*L) = deg\,\phi\, d^+(L).$$

We can apply this to K_X and to $f : \widetilde{\Sigma} \to X$ and $g : \widetilde{\Sigma} \to X$, since the Iitaka dimension of K_X is equal to n, and using the fact $f^*K_X \cong g^*K_X$, we find that

$$deg\, f\, d^+(K_X) = deg\, g\, d^+(K_X).$$

Hence $deg\, f = deg\, g$. ■

Note that in the above propositions, we used only the fact that $f^*K_X \cong g^*K_X$, which is weaker than the equality of the ramification divisors. Note also that the hypothesis in proposition 4 is necessary. Indeed we know the existence of self-K-isocorrespondences Σ of arbitrary large degree $deg\, g/deg\, f$ for K-trivial varieties X (eg take for X an elliptic curve). Considering a product $Y \times X$, and the self-K-isocorrespondences $\Delta_Y \times \Sigma$ of $Y \times X$, we find examples of self-K-isocorrespondences of degree $\neq 1$ on varieties with any possible Kodaira dimension, except for the maximal one.

Let us conclude now with the case where X is a curve. We have proved above that any self-K-isocorrespondence has to be of the same degree over each factor. In [6], Clozel and Ullmo provide examples of curves C having infinitely many self-K-isocorrespondences satisfying furthermore the very restrictive property that they are unramified over each factor (while the K-isocorrespondence property just asks that the correspondence has the same ramification over each factor). They call them modular correspondences. They show that possessing such non trivial modular correspondence is a very restrictive condition on the curve.

We have the following results :

Proposition 5 *Assume X is a smooth curve of genus > 1, then any self-K-isocorrespondence of X is rigid.*

Proof. Let $\widetilde{\Sigma} \stackrel{(f,g)}{\to} X \times X$ be the desingularization of Σ. By rigid, we mean here that there is no deformation of the triple $(\widetilde{\Sigma}, f, g)$, keeping the property that $R_f = R_g$. But, since both f and g ramify exactly along R_f, the torsion free part of the normal bundle

$$(f,g)^*(T_{X\times X})/(f,g)_*T_{\widetilde{\Sigma}}$$

is isomorphic to f^*T_X, which has negative degree on any component of $\widetilde{\Sigma}$. Hence it has no non zero section.

■

Proposition 6 *Let X be a generic smooth complex curve of genus $g \geq 3$. Then X does not carry any non trivial self-K-isocorrespondence.*

Proof. It suffices to show that if X is any smooth curve of genus $g \geq 3$ and Σ is a self-K-isocorrespondence, with desingularization

$$\begin{array}{ccc} \widetilde{\Sigma} & \stackrel{f}{\to} & X \\ g \downarrow & & \\ X & & \end{array}$$

then the map $(f,g) : \widetilde{\Sigma} \to X \times X$ does not deform with X to first order in every direction.

If $u \in H^1(X, T_X)$ is a first order deformation of X, inducing the corresponding deformation $pr_1^* u + pr_2^* u \in H^1(X \times X, T_{X\times X})$ of $X \times X$, the obstruction to deform (f,g) with $X \times X$ lies in the image of $pr_1^* u + pr_2^* u$ in $H^1(\widetilde{\Sigma}, N^f_{\widetilde{\Sigma}})$, where $N^f_{\widetilde{\Sigma}}$ is the torsion free part of the sheaf

$$(f,g)^* T_{X\times X}/(f,g)_* T_{\widetilde{\Sigma}}.$$

We observed in the previous proof that $N^f_{\widetilde{\Sigma}}$ is isomorphic to $f^*T_X \cong g^*T_X$, (where the isomorphism is canonical, given by $g_* \circ f_*^{-1}$). It is easy to see that up to a sign, the composite map

$$H^1(X, T_X) \stackrel{pr_1^*}{\to} H^1(X \times X, T_{X\times X}) \stackrel{(f,g)^*}{\to} H^1(\widetilde{\Sigma}, (f,g)^* T_{X\times X}) \to H^1(\widetilde{\Sigma}, N^f_{\widetilde{\Sigma}})$$

is equal to the natural map $H^1(X, T_X) \stackrel{f^*}{\to} H^1(\widetilde{\Sigma}, f^*T_X)$.

So it suffices to prove is that via the isomorphism $f^*T_X \cong g^*T_X$, the two maps f^* and g^* from $H^1(X, T_X)$ to $H^1(\widetilde{\Sigma}, f^*T_X)$ cannot be proportional. Indeed that will show that the map

$$H^1(X, T_X) \stackrel{pr_1^*+pr_2^*}{\to} H^1(X \times X, T_{X\times X}) \stackrel{(f,g)^*}{\to} H^1(\widetilde{\Sigma}, (f,g)^* T_{X\times X}) \to H^1(\widetilde{\Sigma}, N^f_{\widetilde{\Sigma}}),$$

which is the obstruction map, is non zero.

Let us dualize the maps f^* and g^* above. We get as duals the trace maps

$$f_*,\ g_* : H^0(\widetilde{\Sigma}, K_{\widetilde{\Sigma}} \otimes f^*K_X) \to H^0(X, K_X^{\otimes 2}).$$

Let now $x \in X$ be a generic point. We may assume that no component of Σ is the diagonal of $X \times X$, and then it follows that $f^{-1}(x)$ and $g^{-1}(x)$ are disjoint divisors D_1, D_2 of $\widetilde{\Sigma}$. Since the genus of X is at least three, the divisor $f^*K_X(-D_1 - D_2)$ has positive degree, and it follows that the restriction map

$$H^0(\widetilde{\Sigma}, K_{\widetilde{\Sigma}} \otimes f^*K_X) \to H^0(D_1 \cup D_2, (K_{\widetilde{\Sigma}} \otimes f^*K_X)_{|D_1\cup D_2})$$

is surjective. It follows that we can find a section σ of $K_{\widetilde{\Sigma}} \otimes f^*K_X$ which vanishes on D_1 and at every point of D_2 except for one. Then $f_*\sigma$ vanishes at x, while $g_*\sigma$ does not vanish at x. So f_* and g_* are not proportional.

■

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