

On Chern classes of Lagrangian fibered hyper-Kähler manifolds

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Abstract: We study the rank stratification for the differential of a Lagrangian fibration over a smooth basis. We also introduce and study the notion of Lagrangian morphism of vector bundles. As a consequence, we prove some of the vanishing, in the Chow groups of a Lagrangian fibered hyper-Kähler variety X , of certain polynomials in the Chern classes of X and the Lagrangian divisor, predicted by the Beauville-Voisin conjecture. Under some natural assumptions on the dimensions of the rank strata, we also establish nonnegativity and positivity results for Chern classes.

Keywords: Lagrangian fibrations, Chern classes, Chow ring.

1. Introduction

In the paper [4], Beauville and myself proved that for any projective $K3$ surface S , there exists a canonical degree 1 zero-cycle $o_S \in \mathrm{CH}_0(S)$ satisfying the following properties:

1. For any elements $D, D' \in \mathrm{CH}^1(S)$, $D \cdot D' = \deg(D \cdot D')o_S$ in $\mathrm{CH}_0(S)$.
2. $c_2(S) = 24o_S$ in $\mathrm{CH}_0(S)$.

Although we can work with integral coefficients in the statement above, $\mathrm{CH}(X)$ will denote from now on the Chow groups of a variety X with $\mathbb{Q}$ -coefficients. Beauville subsequently proposed in [3] a generalization of property 1 to any projective hyper-Kähler manifold X , whose weak version is called “Beauville weak splitting conjecture”, stating that any cycle Z on X which is a polynomial in divisor classes and is cohomologous to 0 on X is

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rationally equivalent to 0 modulo torsion. I proposed in turn in [41] to generalize the combination of 1 and 2 to any projective hyper-Kähler manifold X in the following form:

Conjecture 1.1. *Let X be a hyper-Kähler manifold and let $Z \in \mathrm{CH}(X)$ be a cycle on X which belongs to the subalgebra generated by divisor classes and the Chern classes of X . Then if Z is cohomologous to 0, Z is rationally equivalent to 0 modulo torsion on X .*

This conjecture is known to hold for Hilbert schemes of $K3$ surfaces by [32] and for generalized Kummer varieties by [17]. It is also proved in [41] for the Fano variety of lines in a smooth cubic fourfold. Riess proved in [37] a very nice result concerning the weak splitting conjecture. We know the polynomial cohomological relations between divisor classes on a hyper-Kähler manifold X , as they were described by Verbitsky (see [8]). When there is at least one divisor class $h = c_1(H) \in \mathrm{CH}^1(X)$ such that $\int_X h^{2n} = 0$, or equivalently $q(h) = 0$, where q is the Beauville-Bogomolov quadratic form, the ideal of these relations is generated by

$$(1) \quad l^{n+1} = 0 \text{ in } H^{2n+2}(X, \mathbb{Q}) \text{ if } q(l) = 0.$$

(In the absence of a rational isotropic class, the relations do not admit such a concrete description and only have a representation-theoretic characterization, unless we pass to real coefficients.) Riess noticed that the relation (1) obviously holds in $\mathrm{CH}^{n+1}(X)$ when $l = c_1(L)$ and L is a Lagrangian line bundle, that is, there exist a Lagrangian fibration $f : X \rightarrow B$, an integer $d > 0$, and an ample line bundle H_B on B , such that $L^{\otimes d} = f^*H_B$. Indeed, one has $\dim B = n$ so $H_B^{n+1} = 0$ in $\mathrm{CH}^{n+1}(B)$ and

$$(2) \quad L^{n+1} = 0 \text{ in } \mathrm{CH}^{n+1}(X).$$

If an isotropic line bundle L is Lagrangian, L is in particular nef. The SYZ conjecture states conversely that a nef isotropic line bundle on a hyper-Kähler manifold is Lagrangian. When the Picard number of X is 2 and X has an isotropic class, X has two isotropic classes (up to a scalar), and there are many instances when only one class is nef, the other ray of the positive cone not even belonging to the birational Kähler cone (see for example [40], where the case of the O'Grady 10-dimensional manifold constructed in [28] is discussed; see also [14]). We thus a priori do not have the relation (2) for the other isotropic class. Riess, using work of Huybrechts [19] on the existence of self-correspondences inducing automorphisms of the ring $\mathrm{CH}(X)$, could nevertheless extend to the nonnef ray the relation (2) and finally prove the following:

Theorem 1.1. (*Riess [37]*) *If X is a projective hyper-Kähler manifold which has an isotropic class in $\mathrm{NS}(X)$ and whose deformations satisfy the SYZ conjecture, X satisfies Beauville's weak splitting conjecture.*

One of the purposes in this paper is to study Conjecture 1.1 for a certain type of polynomial cohomological relations involving both Chern classes of X and divisor classes. More precisely, as proved in [19], the following generalized Verbitsky relations hold in $H^*(X, \mathbb{Q})$ for X hyper-Kähler of dimension $2n$

$$(3) \quad l^{n+1-i} c_I = 0 \text{ in } H^{2n+2-2i+4j}(X, \mathbb{Q}) \text{ if } q(l) = 0 \text{ and } \deg c_I = 4j \geq 4i,$$

where c_I is any polynomial in the topological Chern classes $c_{2k}(X) \in H^{4k}(X, \mathbb{Q})$ of X and the degree is the weighted cohomological degree. The relations (3) hold in the cohomology algebra of X because they hold for the class σ_t of any $(2, 0)$ -form on a deformation X_t of X , and these classes σ_t fill-in an Euclidean open (hence Zariski dense) set in the quadric $q = 0$ in $H^2(X, \mathbb{C})$. Indeed c_I is of Hodge type $(2j, 2j)$ on X_t , while σ^{n+1-i} is of Hodge type $(2(n+1-i), 0)$ on X_t .

Remark 1.2. The odd Chern classes of a hyper-Kähler manifold X vanish in $\mathrm{CH}(X)$ since its cotangent bundle is isomorphic to its dual. This is why we discuss only even Chern classes.

Remark 1.3. The relations (3) do not exhaust the cohomological relations in the tautological ring generated by divisor classes and topological Chern classes. For example, in top degree $4n$, the cohomology ring has rank 1 and thus all polynomials of weighted degree $4n$ in the topological Chern classes $c_2(X), \dots, c_{2n}(X)$ generate only a 1-dimensional vector space. In degree $4n$, there are thus plenty of polynomial cohomological relations involving only the $c_{2i}(X)$, while the relations in (3) not involving l appear only for $i = n + 1$, that is, in degree $4n + 4$, (so they trivially hold in this case).

Conjecture 1.1 combined with (3) leads us to the following

Conjecture 1.4. *Let X be a projective hyper-Kähler manifold. Then for any line bundle L on X with $q(L) = 0$, for any integer $j \geq i$, and any Chern monomial $c_I \in \mathrm{CH}^{2j}(X)$*

$$(4) \quad L^{n-i+1} c_I = 0 \text{ in } \mathrm{CH}(X).$$

When L is a Lagrangian line bundle, the vanishing (4) is the vanishing (2) for $i = 0$. The next case where $i = 1$ is also quite easy. Indeed, we have

to prove that

$$(5) \quad L^n c_I = 0 \text{ in } \mathrm{CH}^{n+2i}(X)$$

for any Chern polynomial of degree $2i > 0$ on X . This follows from the fact that L^n is proportional to the class of a general fiber X_b of the Lagrangian fibration $f : X \rightarrow B$ and from the exact sequence

$$(6) \quad 0 \rightarrow f_b^* \Omega_{B,b} \rightarrow \Omega_{X|X_b} \rightarrow \Omega_{X_b} \rightarrow 0,$$

where the vector bundles on the right and on the left are trivial on X_b which is an abelian variety. This implies that $c_i(\Omega_X)|_{X_b} = 0$ in $\mathrm{CH}^i(X_b)$ for $i > 0$.

Note that we have more generally an exact sequence

$$0 \rightarrow f^* \Omega_{B^0} \rightarrow \Omega_{X^0} \rightarrow f^* T_{B^0} \rightarrow 0$$

on $X^0 := f^{-1}(B^0)$, where $B^0 \subset B$ is the Zariski open set where B is smooth and over which f has maximal rank. It follows that the Chern classes of positive degree of X are supported over a proper closed algebraic subset of B . A strengthened conjecture in the case of a Lagrangian fibration is the following.

Conjecture 1.5. *Assume X is projective hyper-Kähler and $f : X \rightarrow B$ is a Lagrangian fibration. Then for any i and any monomial $c_I \in \mathrm{CH}^{2i}(X)$ of weighted degree $2i$ in the Chern classes of X , there exists a codimension i closed algebraic subset $B^i \subset B$, such that c_I vanishes in $\mathrm{CH}(X \setminus f^{-1}(B^i))$.*

This conjecture obviously implies Conjecture 1.4 in the case where L is Lagrangian, since then L^{n-i+1} is supported on $f^{-1}(B_{i-1})$, for a closed algebraic subset B_{i-1} of B of dimension $i-1$ and in general position.

Our first results in this paper concern the study of the relations (4). We first prove

Theorem 1.2. *Let $f : X \rightarrow B$ be a Lagrangian fibration of a projective hyper-Kähler manifold, with Lagrangian line bundle L . Then for any pair of integers $i, j \leq n$, $j \geq i$, we have*

$$(7) \quad L^{n-i+1} c_{2j}(X) = 0 \text{ in } \mathrm{CH}(X).$$

This proves Conjecture 1.4 only for the Chern classes, not for monomials in the Chern classes. We will establish Theorem 1.2 in Section 2. We will

discuss the case of $c_2^2 L^{n-1}$ in Section 5.1. As we will see, unlike (7), the vanishing

$$(8) \quad L^{n-1} c_2^2(X) = 0 \text{ in } \text{CH}(X)$$

predicted by Conjecture 1.4 does not follow from formal arguments involving the Lagrangian fibration. We will establish this vanishing in Section 5.1 for a specific class of Lagrangian fibered hyper-Kähler manifolds, namely the deformations of the 10-dimensional O’Grady manifolds constructed in [28] as compactifications of the intermediate Jacobian fibration associated with a cubic fourfold (which we will call below a LSV-manifold).

Theorem 1.3. *Let $X \rightarrow \mathbb{P}^5$ be a LSV-manifold. Then $c_2^2 L^4 = 0$ in $\text{CH}^8(X)$.*

In Section 5.2, we will show how the arguments of Riess in [37] allow us to deduce from Theorem 1.2 the following result in direction of Conjecture 1.4.

Theorem 1.4. *Let X be a projective hyper-Kähler manifold of dimension $2n$. Assume the hyper-Kähler manifolds in the same deformation class as X satisfy the SYZ conjecture. Then for any isotropic class $l = c_1(L) \in \text{NS}(X) = \text{CH}^1(X)$, and for any pair of integers $i, j \leq n, j \geq i$, one has*

$$L^{n+1-i} c_{2j}(X) = 0 \text{ in } \text{CH}(X).$$

Section 3 of the paper is devoted to introducing and studying the notion of *Lagrangian morphism* between bundles F, E , of respective ranks n and $2n$ on a variety Y , where E is equipped with an everywhere nondegenerate skew-symmetric 2-form, with the following

Definition 1.6. *A morphism $\phi : F \rightarrow E$ of vector bundles as above over Y is Lagrangian if*

1. *The generic rank of ϕ is equal to n .*
2. *At any point $x \in Y$, the image $\text{Im } \phi_x \subset E_x$ is isotropic for $\langle, \rangle$ (hence it is Lagrangian at a point where the rank of ϕ is n).*

The morphism given by the differential of a Lagrangian fibration map (over the smooth locus of the base) is the motivating example, although it is not general since its rank loci are not of the expected codimension, except for the first (see Remark 3.4). Nevertheless, the following result applies to them when there are no nonreduced fibers in codimension 1, (and we will explain in Section 3.2 a variant if nonreduced fibers appear in codimension 1).

Theorem 1.5. *Let $\phi : F \rightarrow E$ be a Lagrangian morphism of vector bundles on Y . Assume that ϕ is injective in codimension 1. Then*

(i) *The locus Y_{n-1} where ϕ has rank $n - 1$ is of codimension 2 (or is empty).*

(ii) *Y_{n-1} is a local complete intersection.*

(iii) *There is an exact sequence*

$$(9) \quad 0 \rightarrow F \rightarrow E \rightarrow F^* \rightarrow \mathcal{G} \rightarrow 0,$$

where $\mathcal{G}$ is a torsion sheaf supported on the locus $Y_{\leq n-1}$ where ϕ has rank $\leq n - 1$, and $\mathcal{G}$ is a line bundle on Y_{n-1} .

The geometry of this sheaf $\mathcal{G}$ seems very interesting, as it has a locally free resolution of length 3.

In Section 4, we will study the loci X_k , resp. $X_{\leq k}$, where a Lagrangian fibration morphism has rank k , resp. $\leq k$. We will explain why, starting from corank 2, they do not satisfy the general codimension estimates for general Lagrangian morphisms of vector bundles established in Section 3. We will also establish the following

Theorem 1.6. *Let X be a complex manifold of dimension $2n$ equipped with a symplectic holomorphic structure and let $f : X \rightarrow B$ be a holomorphic Lagrangian fibration, with B smooth. Then for any irreducible component $X_{k,i}$ of X_k , with image $B_{k,i}$ in B , the following hold*

(1) *The dimension $\dim B_{k,i}$ is at most k .*

(2) *The relative dimension of $X_{k,i}$ over $B_{k,i}$ is at least k .*

(3) *If we have equality in (1) and (2), any connected component of the general fiber $X_{k,i,b}$, $b \in B_{k,i}$, is a complex torus T of dimension k .*

(4) *Under the same assumptions as in (3), the locally closed complex submanifold X_k is smooth along $X_{k,i,b}$ and the restricted normal bundle $N_{X_k/X|_{X_{k,i,b}}}$ is on each connected component T a homogeneous vector bundle on the complex torus T .*

Furthermore, we prove in Theorem 4.1 that the dimension of the locus X_k where f has rank k is expected to be at least $2k$ and the dimension of its image B_k in B is expected to be at least k , unless some unexpected cohomological vanishing (30), (which is stronger than (3)), holds. As a consequence of our arguments, we get a positivity result for the Chern classes which provides some evidence for the questions asked in [35], see also [36].

Theorem 1.7. (Cf. Theorem 4.1) *Let $f : X \rightarrow B$ be a Lagrangian fibration of a hyper-Kähler $2n$ -fold with Lagrangian line bundle L . Assume the base B*

is smooth and the rank loci X_k satisfy $\dim X_k = 2k$, $\dim f(X_k) = k$ for all k . Then the classes $L^{n-i}(X)c_{2i}(X)$ are $\mathbb{Q}$ -effective. In particular, for $i = n$, one has $\chi_{\text{top}}(X) \geq 0$, and furthermore $\chi_{\text{top}}(X) > 0$ if the locus X_0 where the rank of f is 0 is not empty.

The last result is easy and follows from Beauville's argument in [5] if we know that the group scheme $\mathcal{G}$, which is constructed under some assumptions in [2], [26] and [27], acts on X .

Remark 1.7. The question of the existence of the relative group scheme $\mathcal{G}$ over the base B , acting on X over B (extending the relative Albanese variety defined over the regular locus of f and acting by translation on the smooth fibers) seems to be open when some fibers are reducible or nonreduced. One of our motivations in this paper is to analyze the geometry of Lagrangian fibrations without assuming it.

Remark 1.8. The numerical positivity of the class $c_2(X)$ is well-known and follows from the Lübke (or Bogomolov-Miyaoka-Yau) inequality and existence of Kähler-Einstein metrics.

In Section 4.1 we will establish Conjecture 1.5 assuming that the base of the Lagrangian fibration is smooth and under a dimension assumption on the rank strata X_k .

Theorem 1.8. *Let $f : X \rightarrow B$ be a Lagrangian fibration and i be a positive integer. Assume that B is smooth in codimension $i - 1$ and that for any $k \geq n - i + 1$, and any irreducible component Z of X_k , we have either $\dim Z \leq 2k$ or $\dim f(Z) \geq k$. Then for $j \geq i$, $c_{2j}(X)$ vanishes in $\text{CH}(X \setminus f^{-1}(B^i))$ for some codimension i closed algebraic subset B^i of B .*

2. Proof of Theorem 1.2

Proof of Theorem 1.2(i). Let $f : X \rightarrow B$ be a Lagrangian fibration, where X is projective hyper-Kähler of dimension $2n$. Let H be a very ample line bundle on B , whose pull-back to X is thus a multiple μL . Let $i \leq n$ be a positive integer. We will prove the vanishing

$$c_{2j}(\Omega_X)L^{n-i+1} = 0 \text{ in } \text{CH}(X)$$

for $j \geq i$ by induction on i . By Bertini, a general set of $n - i + 1$ sections of f^*H on X defines a smooth complete intersection $\Sigma \subset X$. Along Σ , we thus have an injective morphism of vector bundles

$$\phi : N_{\Sigma/X}^* \rightarrow \Omega_{X|\Sigma},$$

where $N_{\Sigma/X}^* \cong \mathcal{O}_{\Sigma}(-H)^{n-i+1}$.

As X is hyper-Kähler, the vector bundle Ω_X carries an everywhere non-degenerate skew-symmetric pairing $\langle, \rangle$. We observe that, as f is Lagrangian, the image of ϕ is totally isotropic for $\langle, \rangle$. Indeed, it suffices to prove the result at a generic point x of Σ , whose image $b = f(x)$ does not belong to $\text{Sing } B$ and where f is of maximal rank. Then the image of ϕ is contained in $f^*\Omega_{B,b}$, which is Lagrangian in $\Omega_{X,x}$ because f is a Lagrangian fibration.

The subbundle $(\text{Im } \phi)^\perp$ of $\Omega_{X|\Sigma}$ is thus of rank $n+i-1$ and contains $\text{Im } \phi$. Furthermore, the quotient $\Omega_{X|\Sigma}/(\text{Im } \phi)^\perp$ is isomorphic to $N_{\Sigma/X} = N_{\Sigma/X}^{**}$ by duality using $\langle, \rangle$. Let

$$(10) \quad \mathcal{E} := (\text{Im } \phi)^\perp / \text{Im } \phi.$$

This is a vector bundle of rank $2i-2$ on Σ , hence we have

$$(11) \quad c_{2i}(\mathcal{E}) = 0 \text{ in } \text{CH}(\Sigma).$$

We now use the exact sequences

$$\begin{aligned} 0 &\rightarrow N_{\Sigma/X}^* \xrightarrow{\phi} (\text{Im } \phi)^\perp \rightarrow \mathcal{E} \rightarrow 0, \\ 0 &\rightarrow (\text{Im } \phi)^\perp \rightarrow \Omega_{X|\Sigma} \rightarrow N_{\Sigma/X} \rightarrow 0 \end{aligned}$$

explained above and the Whitney formula, which gives equalities in $\text{CH}(\Sigma)$

$$(12) \quad \begin{aligned} c(\Omega_{X|\Sigma}) &= c(N_{\Sigma/X})c((\text{Im } \phi)^\perp) \\ &= c(N_{\Sigma/X})c(N_{\Sigma/X}^*)c(\mathcal{E}). \end{aligned}$$

We also get by inverting the total Chern classes in (12)

$$(13) \quad c(\mathcal{E}) = c(\Omega_{X|\Sigma})c(N_{\Sigma/X})^{-1}c(N_{\Sigma/X}^*)^{-1} \text{ in } \text{CH}(\Sigma).$$

If we expand the equality $c(\Omega_{X|\Sigma}) = c(N_{\Sigma/X})c(N_{\Sigma/X}^*)c(\mathcal{E})$ of (12), and take into account the fact that $c_{2i}(\mathcal{E}) = 0$ and $c(N_{\Sigma/X}) = (1 + \mu L_{|\Sigma})^{n-i+1}$, we find that

$$(14) \quad c_{2i}(\Omega_{X|\Sigma}) = \sum_{l>0} \alpha_l L_{|\Sigma}^{2l} c_{2i-2l}(\mathcal{E})$$

for some integers α_l . Using (13), we can replace in (14) the terms $c_{2i-2l}(\mathcal{E})$ by polynomials in L and $c_k(\Omega_{X|\Sigma})$, which provides us with an equation

$$(15) \quad c_{2i}(\Omega_{X|\Sigma}) = \sum_{l>0} \beta_l L_{|\Sigma}^{2l} c_{2i-2l}(\Omega_{X|\Sigma})$$

for some integers β_l . Using the fact that the class of Σ in X is a nonzero multiple of L^{n-i+1} in $\mathrm{CH}(X)$, equation (15) gives a relation

$$(16) \quad c_{2i}(\Omega_X)L^{n-i+1} = \sum_{l>0} \gamma_l L^{2l+n-i+1} c_{2i-2l}(\Omega_X)$$

for some rational numbers γ_l .

The proof of the vanishing $c_{2i}(\Omega_X)L^{n-i+1}$ in $\mathrm{CH}(X)$ thus follows by induction on i . The proof of the vanishing of $c_{2j}(\Omega_X)L^{n-i+1}$ in $\mathrm{CH}(X)$ for $j \geq i$ works in the same way. $\square$

3. Lagrangian morphisms of vector bundles

We study in this section general properties of Lagrangian morphisms of vector bundles introduced in Definition 1.6. We do not know if the notion has been classically studied. Our motivation for introducing this notion and its relevance for the subject of this paper come from the following

Example 3.1. *Let X be hyper-Kähler and let $f : X \rightarrow B$ be a Lagrangian fibration. Then denoting by $B' \subset B$ the smooth locus of B and $X' \subset X$ its inverse image in X , the morphism*

$$(17) \quad \phi := f^* : f^*\Omega_{B'} \rightarrow \Omega_{X'}$$

is Lagrangian on X' .

Note that it is conjectured that, if B is normal (which we will always assume) B is smooth (and if it is smooth, it is known to be isomorphic to $\mathbb{P}^n$ by [22]). In any case, as B is normal, the codimension of $B \setminus B'$ in B is at least 2, while the discriminant locus, over which ϕ is not everywhere of maximal rank, has codimension 1 in B , so that the notion is already interesting over B' .

To study the rank loci for a general Lagrangian morphism, we consider the universal situation, where the basis is the set M_{lag} of matrices M of size $(2n, n)$ which are Lagrangian, in the sense that the morphism $M : \mathbb{C}^n \rightarrow \mathbb{C}^{2n}$ is Lagrangian.

Lemma 3.2. (i) *The subvariety $M_{\mathrm{lag}} \subset M_{2n,n}$ is an irreducible, local complete intersection subvariety of codimension $n(n-1)/2$, hence of dimension $2n^2 - \frac{n(n-1)}{2}$.*

(ii) *M_{lag} is smooth at a matrix M which is of rank $\geq n-1$, but it is singular at a matrix which is of rank $\leq n-2$. Along the set $M_{\mathrm{lag}, n-2}$ of matrices of rank $n-2$, M_{lag} has an ordinary quadratic singularity of codimension 7.*

(iii) The locus $M_{\text{lag},n-i}$ of isotropic matrices of rank $n-i$ is smooth and has codimension $\frac{3i^2+i}{2}$ in M_{lag} . In particular, the locus $M_{\text{lag},n-1}$ is smooth of codimension 2 in M_{lag} and is contained in its smooth locus, and $M_{\text{lag},n-2}$ is smooth of codimension 7 in M_{lag} and is contained in its singular locus.

Proof. (i) Denoting m_k the k -th column vector of a matrix m , the equations defining M_{lag} in $M_{2n,n}$ are

$$(18) \quad \langle m_k, m_l \rangle = 0$$

for $n \geq l > k \geq 1$. This locus is thus defined by $n(n-1)/2$ equations. In order to prove that M_{lag} is a local complete intersection, it suffices to show that its dimension is $2n^2 - \frac{n(n-1)}{2}$, which is done by proving (iii) since M_{lag} is stratified by the rank $n-i$ of the matrix. The locus of rank $n-i$ isotropic matrices is homogeneous under the symplectic group $\text{Sp}(2n)$ and of dimension

$$(n-i)(n+i) - (n-i)(n-i-1)/2 + n(n-i) = 2n^2 - \frac{n(n-1)}{2} - \frac{3i^2+i}{2},$$

since a rank $n-i$ isotropic matrix of size $n \times 2n$ determines its image W_{n-i} which is an isotropic vector subspace of $\mathbb{C}^{2n}$ of dimension $n-i$, and a matrix $\mathbb{C}^n \rightarrow W_{n-i}$ of size $n \times n-i$ and rank $n-i$. This dimension count proves the codimension statement in (iii). The irreducibility follows from the above dimension count and the fact that the rank n stratum is homogeneous, hence irreducible.

(ii) and (iii) Denote by m_{ij} the entries of a matrix $m \in M_{2n,n}$, where $i \in \{1, \dots, 2n\}$ and $j \in \{1, \dots, n\}$, and by m_i its i -th column vector. Let $M \in M_{\text{lag}}$ be any Lagrangian matrix. As the image of M is isotropic, we can assume by choosing an adequate basis of $\mathbb{C}^{2n}$ in which the intersection form $\langle, \rangle$ is the standard one, that for $k \leq \text{rk } M$, the vector M_k is the basis vector f_k of $\mathbb{C}^{2n}$, and that it is 0 for $k > \text{rk } M$. For a matrix $m \in M_{2n,n}$, we then write $m_k = f_k + h_k$ for $k \leq \text{rk } M$, so that the column vector $h_k(m)$ vanishes at the point M . For $k > \text{rk } M$, we write $m_k =: h_k$ and the vector $h_k(m)$ vanishes at the point M . In the coordinates h_{kl} on $M_{2n,n}$ centered at M , the equations (18) are

$$(19) \quad h_{k+n,l} - h_{l+n,k} + \langle h_k, h_l \rangle = 0 \text{ for } k, l \leq \text{rk } M$$

$$(20) \quad h_{l,k+n} + \langle h_k, h_l \rangle = 0 \text{ for } k \leq \text{rk } M, l > \text{rk } M$$

$$(21) \quad \langle h_k, h_l \rangle = 0 \text{ for } k > \text{rk } M, l > \text{rk } M.$$

where the terms $\langle h_k, h_l \rangle$ are quadratic in the coordinates h_{kl} .

When the rank of M is n , only the equations in the first line (19) appear. When the rank of M is $n - 1$, only the equations in the first and second lines (19), (20) appear. In both cases, the linear parts of these equations are obviously independent, which proves the result in that case. When the rank of M is $n - 2$, there is one quadratic equation in the third line (21), namely $\langle h_{n-1}, h_n \rangle = 0$. (ii) and the last statement in (iii) follows by studying the rank of this quadratic equation. $\square$

Remark 3.3. The variety M_{lag} carries a universal Lagrangian morphism, but, as we just proved that it is singular, we cannot say that it is universal for the study of Lagrangian morphisms of vector bundles on *smooth* bases Y . However, as $M_{\text{lag}, \geq n-1}$ is smooth, it is universal for the study of Lagrangian morphisms of rank $\geq n - 1$ of vector bundles of ranks $n, 2n$ on smooth bases Y .

Remark 3.4. Consider the case of Example 3.1. Assume for simplicity that the singular fibers have normal crossing singularities. Then the locus Z where $\phi = f^*$ has corank 1 is the union of the singular loci of the fibers which is expected to have codimension 2, by Lemma 3.2, so that singular fibers are singular in codimension 1. If furthermore the normalization of the locus Z is smooth of dimension $2n - 2$, the locus where $\phi = f^*$ has corank i is the locus of i -branches singularities in the fibers, which is expected to appear in codimension $2i$ (being the locus of intersection of i branches of Z). So the geometric dimension count does not fit with the abstract dimension count of Lemma 3.2. We will discuss from a different viewpoint this lack of transversality in Section 4.

In the rest of this section, we will focus on the first rank stratum parameterizing matrices of rank $n - 1$, which is the only one to be studied in order to establish the vanishing (8) because, by Sard's theorem, the locus where a Lagrangian map $f : X \rightarrow B$ has rank $\leq n - 2$ maps to a locus of dimension $\leq n - 2$ in the base B . Our first result is the following (cf. Theorem 1.5).

Proposition 3.5. *Let $\phi : F \rightarrow E$ be a Lagrangian morphism on a smooth variety Y , with $\text{rank } E = 2n$. Assume that*

() the codimension of the locus $Y_{\leq n-1} \subset Y$ where ϕ has rank $\leq n - 1$ is at least 2.*

Then

(i) Y_{n-1} is a local codimension 2 complete intersection in Y .

(ii) There is an exact sequence on Y

$$(22) \quad 0 \rightarrow F \xrightarrow{\phi} E \xrightarrow{\phi^*} F^* \rightarrow \mathcal{G} \rightarrow 0,$$

where $\mathcal{G}$ is supported on $Y_{\leq n-1}$ and is a line bundle on $Y_{n-1} = Y_{\leq n-1} \setminus Y_{\leq n-2}$.

(iii) The normal bundle of Y_{n-1} in Y is isomorphic to $\mathcal{H}om(\mathcal{E}_2, \mathcal{G})$, where $\mathcal{E}_2$ is a rank 2 vector bundle with trivial determinant on Y_{n-1} .

In (22), ϕ^* denotes the transpose of ϕ , which is defined using the self-duality of E given by $\langle, \rangle$.

Proof. Statement (i) follows, by local trivialization of the vector bundles F and E (equipped with its symplectic structure), from Lemma 3.2, (ii) and (iii), which say that $M_{\text{lag}, n-1}$ is a smooth codimension 2 locally closed subvariety of M_{lag} and is contained in its smooth locus, hence is a local complete intersection of codimension 2.

(ii) As the image of ϕ is isotropic, we have $\phi^* \circ \phi = 0$. Furthermore, at any point where ϕ has rank n , we have $\text{Ker } \phi^* = (\text{Im } \phi)^\perp = \text{Im } \phi$ hence the sequence

$$(23) \quad 0 \rightarrow F \xrightarrow{\phi} E \xrightarrow{\phi^*} F^*$$

is exact on the right and at the middle on the Zariski open set $Y \setminus Y_{\leq n-1}$ of Y . Under assumption (*), this implies that the sequence remains exact at the middle everywhere on Y . Indeed, let α be a local section of $\text{Ker } \phi^*$ defined on a Zariski open set U of Y . Then, on $U \setminus (Y_{\leq n-1} \cap U)$, $\alpha = \phi(\beta)$ by the exactness of the sequence (23) on $U \setminus (Y_{\leq n-1} \cap U)$. As U is smooth and $Y_{\leq n-1} \cap U$ has codimension ≥ 2 in U , β extends to a section $\tilde{\beta}$ of F on U and $\alpha = \phi(\tilde{\beta})$. This proves the exactness of (22) in the second term. Finally, the cokernel of ϕ^* is a line bundle supported on Y_{n-1} , because it is isomorphic to the cokernel of the rank $n-1$ morphism

$$E|_{Y_{n-1}} \xrightarrow{\phi^*|_{Y_{n-1}}} F^*|_{Y_{n-1}}$$

of vector bundles on Y_{n-1} .

(iii) The vector bundle $\mathcal{E}_2$ on Y_{n-1} is defined as follows: the morphism $\phi|_{Y_{n-1}}$ has rank $n-1$, hence its image in $\text{Im}(\phi|_{Y_{n-1}}) \subset E|_{Y_{n-1}}$ is a vector subbundle of rank $n-1$, which is totally isotropic for $\langle, \rangle$. We define $\mathcal{E}_2$ as $(\text{Im } \phi)^\perp / \text{Im } \phi = \text{Ker } \phi^* / \text{Im } \phi$. The fact that $\mathcal{E}_2$ has trivial determinant follows from the fact $\langle, \rangle$ induces a nondegenerate skew-symmetric pairing on $\mathcal{E}_2$, which trivializes its determinant. It remains to prove that the normal bundle of Y_{n-1} in Y is isomorphic to $\mathcal{H}om(\mathcal{E}_2, \mathcal{G})$. We first do the case where Y_{n-1} is smooth of codimension 2. At any point $y \in Y_{n-1}$, we have a well-defined morphism

$$d\phi_y : T_{Y,y} \rightarrow \text{Hom}(\text{Ker } \phi_y^*, \text{Coker } \phi_y^*)$$

whose kernel is the tangent space to Y_{n-1} at y . It remains to see that the image is contained in

$$\mathrm{Hom}(\mathrm{Ker} \phi_y^* / \mathrm{Im} \phi_y, \mathrm{Coker} \phi_y^*) = \mathrm{Hom}(\mathcal{E}_{2,y}, \mathcal{G}_y).$$

This follows immediately from the fact that $\phi^* \circ \phi = 0$ on Y . As $\mathrm{Hom}(\mathcal{E}_{2,y}, \mathcal{G}_y)$ has dimension 2, we get from this construction a canonical isomorphism $N_{Y_{n-1}/Y} \cong \mathcal{H}om(\mathcal{E}_2, \mathcal{G})$ in the case where $Y_{n-1} \subset Y$ is smooth of codimension 2. The general case follows by local trivializations using Lemma 3.2(ii), (iii). $\square$

Corollary 3.6. *Under the same assumptions as in Proposition 3.5, consider the following combination C of total Chern classes*

$$C = c(E)c(F)^{-1}c(F^*)^{-1}.$$

Then, if the line bundle $\mathcal{G}$ is trivial on Y_{n-1} , C_i vanishes in $\mathrm{CH}(Y \setminus Y_{\leq n-2})$ for $i \geq 3$.

Proof. Indeed, by the exact sequence (22), one has $C = s(\mathcal{G})$, where s denotes the total Segre class. As $\mathcal{G}$ is trivial on Y_{n-1} by assumption, and Y_{n-1} is a local complete intersection of codimension 2 by Proposition 3.5(i), the normal bundle $N_{Y_{n-1}/Y}$ has trivial determinant by Proposition 3.5(i). Using the fact that $\mathcal{G} \cong \mathcal{O}_{Y_{n-1}}$, the corollary follows from Lemma 3.7 below applied to the closed subvariety $Y_{n-1} \subset Y \setminus Y_{\leq n-2}$. $\square$

Lemma 3.7. *Let Y be a smooth variety and $Z \subset Y$ be a local complete intersection of codimension 2. Assume that the determinant of $N_{Z/Y}$ is trivial. Then the total Segre class $s(\mathcal{O}_Z) \in \mathrm{CH}(Y)$ satisfies $s_i = 0$ for $i \geq 3$.*

Proof. The obvious case is when $Z \subset Y$ is the zero locus of a section of a rank 2 vector bundle M on Y with trivial determinant. Indeed, we have in this case the two exact sequences

$$(24) \quad 0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0,$$

$$(25) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow M \rightarrow \mathcal{I}_Z \rightarrow 0.$$

From (24), we deduce that $s(\mathcal{O}_Z) = c(\mathcal{I}_Z)$ and from (25), we deduce that $c(\mathcal{I}_Z) = c(M)$. Finally $c_i(M) = 0$ for $i \geq 3$ since M has rank 2.

The general case follows from this case. There exists a rank 2 vector bundle M on Y with the desired property if the natural morphism

$$(26) \quad \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Y) \rightarrow H^0(Z, \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}_Y)) = H^0(Z, \mathcal{O}_Z)$$

is surjective. Here the isomorphism $\mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}_Y) \cong \mathcal{O}_Z$ is given by the triviality of the determinant of $N_{Z/Y}$. The surjectivity of the local to global map (26) is satisfied if Y is affine, hence the lemma is proved in the affine case. If Y is not affine, we can use the Jouanolou trick [25, Lemme 1.5] and replace Y by an affine bundle A_Y over Y , which has isomorphic Chow groups since it is fibered into affine spaces over Y and whose total space is affine. The result then applies to $A_Z \subset A_Y$. $\square$

3.1. The case of Lagrangian fibrations

We study in this section the Lagrangian morphism of vector bundles given in Example 3.1, namely $\phi = f^*$ for some Lagrangian fibration $f : X \rightarrow B$, where B is normal. More precisely, we will consider the restriction $f : X' \rightarrow B'$, where B' is the smooth locus of B and $X' := f^{-1}(B')$. As before, we denote by X'_i the locally closed algebraic subset of X' where f^* has rank i . We will give a second proof of Theorem 1.2 for $i = 2$, based on the following

Proposition 3.8. *There exists a Zariski open set $B'' \subset B'$ such that $B \setminus B''$ has codimension ≥ 2 in B' (hence in B) and contains $f(X'_{n-2})$, with the following property. Let $X'' := f^{-1}(B'')$, so ϕ has rank $\geq n - 1$ everywhere on X'' since $X'' \cap X'_{n-2} = \emptyset$. Then the line bundle $\mathcal{G}$ on X''_{n-1} constructed in Proposition 3.5(ii) is trivial along the fibers of $f|_{X''_{n-1}} : X''_{n-1} \rightarrow B''$.*

Proof. Let $\tau : \tilde{X}'_{n-1, \text{red}} \rightarrow X'_{n-1}$ be a desingularization of the locus X'_{n-1} equipped with its reduced structure and let

$$\tilde{f}_{n-1} := f \circ \tau : \tilde{X}'_{n-1, \text{red}} \rightarrow B'.$$

Using Sard's theorem, the locus $Z \subset \tilde{X}'_{n-1, \text{red}}$ where $\tilde{f}_{n-1}$ has rank $< n - 1$ satisfies $\dim \tilde{f}_{n-1}(Z) \leq n - 2$. Similarly, the closed algebraic subset $f(X'_{\leq n-2})$ of B has dimension $\leq n - 2$. Let $B'_1 := B' \setminus (f(X'_{\leq n-2}) \cup \tilde{f}_{n-1}(Z))$. By construction, $\tilde{f}$ has rank exactly $n - 1$ on $\tilde{X}_{n-1, \text{red}} \cap \tilde{f}^{-1}(B'_1)$. Its image is thus locally a finite union of smooth hypersurfaces in B'_1 and, up to removing a codimension 2 closed algebraic subset of B'_1 where this hypersurface has at least two branches, we get the desired Zariski open set B'' . By construction, the hypersurface $\Delta = \tilde{f}(\tilde{X}_{n-1, \text{red}} \cap \tilde{f}^{-1}(B''))$ is a smooth hypersurface in B'' and $\tilde{f} : \tilde{X}_{n-1, \text{red}} \cap \tilde{f}^{-1}(B'') \rightarrow \Delta$ is everywhere of maximal rank $n - 1$. Having constructed B'' , it remains to prove the statement about $\mathcal{G}$. We now observe that, for any point y in the fiber F_b of $\tilde{f}$ over $b \in \Delta$, the image of $f_* : T_{X, \tau(y)} \rightarrow T_{B, b}$ has dimension $n - 1$ and contains the image of $\tilde{f}_* :$

$T_{\widetilde{X}_{n-1,\text{red}},y}$, namely $T_{\Delta,b}$. Hence the two spaces are equal and we find that the image of $f_* : T_{X,\tau(y)} \rightarrow T_{B,b}$ is constant, equal to $T_{\Delta,b}$ along F_b . This shows that $\mathcal{G}$ is trivial along the fiber F_b . This is a priori not enough to prove the desired statement since we worked only with a desingularization of the reduced structure of X_{n-1} and not with X_{n-1} itself. To complete the proof, we use the following

Lemma 3.9. *Let Z be projective scheme and L be a line bundle on Z . Assume that*

- (i) *L is generated by global sections.*
- (ii) *The pull-back of L to a desingularization $\widetilde{Z}_{\text{red}}$ of the underlying reduced subscheme Z_{red} is trivial.*

Then L is trivial.

Proof. As L is generated by global sections, it induces a morphism $g : Z \rightarrow \mathbb{P}^N$ such that L is isomorphic to $g^*\mathcal{O}_{\mathbb{P}^N}(1)$. As the pull-back of L to $\widetilde{Z}_{\text{red}}$ is trivial, this morphism has image supported on a finite number of points. Any line bundle on a 0-dimensional scheme is trivial and this finishes the proof. $\square$

Lemma 3.9 applies in our situation since the line bundle $\mathcal{G}$ on X'_{n-1} is by definition a quotient of the vector bundle $f^*\Omega_{B'}^*$, hence is generated by global sections along the fibers of $f : X'_{n-1} \rightarrow B'$. $\square$

Combining the results of the previous sections, we get another proof of Theorem 1.2(i) for $i = 2$.

Corollary 3.10. *Let $f : X \rightarrow B$ be a Lagrangian fibration of a hyper-Kähler manifold, with B normal. Let L be the Lagrangian line bundle. Assume that there is no divisor in X made of non-reduced fibers. Then for $i \geq 2$, the Chern classes $c_{2i}(X) \in \text{CH}^{2i}(X)$ are supported over a closed algebraic subset of codimension ≥ 2 in B . In particular*

$$(27) \quad L^{n-1}c_i(X) = 0 \text{ in } \text{CH}(X) \text{ for } i \geq 4.$$

Proof. Let B'' and X'' be as in Proposition 3.8. By our assumption on singular fibers, the locus $X''_{n-1} = X''_{\leq n-1} \subset X''$ where $f : X'' \rightarrow B''$ has rank $n - 1$ is a closed algebraic subset of codimension 2 in X'' . Proposition 3.8 tells that the line bundle $\mathcal{G}$ on X''_{n-1} is trivial on the fibers of $f_{n-1} : X''_{n-1} \rightarrow B''_{n-1}$, so it is trivial on a Zariski open set $f_{n-1}^{-1}(U)$ for some dense Zariski open set $U := B''_{n-1} \setminus W$ of B''_{n-1} . Let $B''' := B'' \setminus W$ and $X''' := f^{-1}(B''')$. Then $B''' \subset B$ is the complement of a closed algebraic subset of codimension ≥ 2 . Furthermore, by construction, the line bundle $\mathcal{G}$ is trivial on X'''_{n-1} . We can

thus apply Corollary 3.6 to conclude that $c_{2i}(\Omega_X)$ vanishes in $\mathrm{CH}(X''')$ for $i \geq 2$. $\square$

Remark 3.11. We will give a different proof of Corollary 3.10 in Section 4.1. More precisely, Corollary 4.10 reproves Corollary 3.10 without the assumption on non-reduced fibers.

3.2. Non-reduced fibers in codimension 1

We show in this section how to get rid of the assumption that there are no non-reduced fibers in codimension 1. Our goal is to show how, in the case of a Lagrangian fibration of a hyper-Kähler manifold of Picard number 2, we can modify the morphism (17) over the complement of a closed algebraic subset of codimension ≥ 2 of the base so as to get a Lagrangian morphism of vector bundles having the property that its locus X_{n-1} of jumping rank has codimension at least 2, and to which the arguments of the previous section apply. Note that the existence of a divisor in the base parameterizing non-reduced fibers can occur, as in the following

Example 3.12. Consider an elliptic K3 surface $f : S \rightarrow \mathbb{P}^1$ with elliptic general fiber E_t , $t \in \mathbb{P}^1$. Then for any $n \geq 2$, we have a composed morphism

$$(28) \quad f_n : S^{[n]} \rightarrow S^{(n)} \rightarrow (\mathbb{P}^1)^{(n)} = \mathbb{P}^n,$$

which gives a Lagrangian fibration with non-reduced fibers over the big diagonal of $(\mathbb{P}^1)^{(n)}$. More precisely, for $n = 2$, the fiber of f_2 over $t + t' \in (\mathbb{P}^1)^{(2)}$, $t \neq t'$, is isomorphic to $E_t \times E_{t'}$, and for $t = t'$, the fiber is as a set the union of $E_t^{(2)}$ and another component which is also a $\mathbb{P}^1$ -bundle over E_t , namely the set of length 2 subschemes of S supported at one point of E_t . The first component is not reduced since near a general point (x, y) , $x \neq y$ of $E_t^{(2)}$, the morphism f_2 locally factors through the morphism $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow (\mathbb{P}^1)^{(2)}$ which ramifies over the diagonal. The second component is reduced because the morphism $f_2 : S^{[2]} \rightarrow (\mathbb{P}^1)^{(2)}$ is of rank 2 at a general point of the divisor parameterizing nonreduced schemes of length 2. Indeed, considering a general curve $C \subset S$, the morphism $f_{2|C^{(2)}} : C^{(2)} \rightarrow (\mathbb{P}^1)^{(2)}$ is a local analytic isomorphism at a point $2c \in C^{(2)}$, once $f_{|C} : C \rightarrow \mathbb{P}^1$ is a local analytic isomorphism at c .

In this example, there are non-reduced fibers in codimension 1 which are not multiple fibers. This however cannot occur when X has Picard number 2 by the following lemma.

Lemma 3.13. *Let $f : X \rightarrow B$ be a Lagrangian fibration with X projective of Picard number 2. Then the general non-reduced fibers appearing in codimension 1 are multiple fibers.*

Proof. As B is normal, hence smooth in codimension 1, the general non-reduced fibers appearing in codimension 1 appear over the smooth locus B' , over which the Lagrangian fibration is flat by Matsushita's theorem [31]. Assume that over a generic point $t \in B^1 \subset B'$ of a divisor in B' , the fiber X_t is not reduced but not a multiple fiber. This implies that it has several irreducible components $X_{t,l}$ appearing with different multiplicities l . The inverse image $D := f^{-1}(B^1)$ thus has several irreducible components D_k , where D_k is defined as the union of components of the general fiber of f over B^1 with multiplicity k . As the fiber X_t is a local complete intersection, its irreducible components meet in codimension 1 and as it is connected, it follows that D_k is effective and non-trivial when restricted to a component $X_{t,l}$ for some $l \neq k$. As the divisor D_k is trivial on the general fiber $X_{t'}$, $t' \in B$, we conclude that $\rho(X) \geq 3$, since $\text{Pic}(X)$ already contains the Lagrangian line bundle, which is trivial on $X_{t,l}$ and $X_{t'}$, and an ample divisor, which is nontrivial on both $X_{t,l}$ and $X_{t'}$. This contradiction concludes the proof. $\square$

Recall that, by Matsushita [30], the Lagrangian fibrations deform over a codimension 1 locally closed analytic subspace of the Kuranishi family of X , so that, for the general projective deformation (X_t, f_t) of (X, f) , X_t has Picard number 2 and Lemma 3.13 applies. We now consider the case of a Lagrangian fibration $f : X \rightarrow B$, such that the non-reduced fibers appearing in codimension 1, that is, over a divisor B^1 of B , are multiple fibers. We work again over the Zariski open set B'' of B' where the various components of the divisor B^1 do not intersect and over which f has rank $\geq n - 1$. (This is the complement of a codimension 2 subset in B .) We now make the following construction: As $f : X'' \rightarrow B''$ is of rank $n - 1$ over $B''^1 := B^1 \cap B''$, the morphism $\phi : f^*\Omega_{B''} \rightarrow \Omega_{X''}$ can be modified into a morphism

$$(29) \quad \phi_{\text{sat}} : (f^*\Omega_{B''})_{\text{sat}} \rightarrow \Omega_{X''},$$

which is now generically of rank n along D , where the vector bundle $(f^*\Omega_{B''})_{\text{sat}}$ can be defined over B'' as the saturation of $f^*\Omega_{B''}$ in Ω_X and has the following explicit description: let $D := f^{-1}(B^1)$ (equipped with its reduced structure). By our assumption on B'' , at any point of $X'' \cap D$, the kernel of $\phi : f^*\Omega_{B''} \rightarrow \Omega_{X''}$ over $t \in B''^1$ equals dg_t where g_t is a defining equation of B^1 near t . Indeed, if the fibers over the component of B^1 passing through t have multiplicity k , then we have locally in a neighborhood of $f^{-1}(t)$, $f^*g_t = h_t^k$ for

a function h_t on X defining D , hence $f^*dg_t = kh_t^{k-1}dh_t$. The vector bundle $(f^*\Omega_{B''})_{\text{sat}}$ is then locally generated by $f^*\Omega_{B''}$ and dh_t , that is, by $f^*\Omega_{B''}$ and $\frac{f^*dg_t}{h_t^{k-1}}$.

Using this construction and the fact that the Chern classes of $(f^*\Omega_{B''})_{\text{sat}}$ are pulled-back from B'' with rational coefficients (as is the class of the divisor D), all the vanishing results for Chern classes obtained in the previous section and relying on Proposition 3.5 on Lagrangian morphisms of vector bundles with degeneracy locus of codimension 2, in particular Corollary 3.10, can be proved without assuming the non-existence of non-reduced fibers in codimension 1. When the Picard number of X is > 2 , we first have to deform X to a general X_t with Picard number 2, prove the result for X_t using the construction above, and specialize it to X (see [37]). We do not give the full argument here as we will discuss another approach in the next section.

4. Higher rank loci for Lagrangian fibrations

We study in this section the stratification by the rank of a Lagrangian fibration with smooth base. Let $f : X \rightarrow B$ be a Lagrangian fibration of a hyper-Kähler manifold X with $\dim X = 2n$. We assume that B is smooth (or restrict to its smooth locus B'). For any integer k , we consider the Zariski locally closed subset $X_k \subset X$ of points where f has rank k .

Lemma 4.1. *The relative dimension of X_k over B is at least k .*

Proof. Let $x \in X_k$ and let $b = f(x) \in B$. Let $f_1, \dots, f_k$ be k algebraic or holomorphic functions on B , defined near b , such that the differentials f^*df_i are independent near x . Then the Hamiltonian vector fields χ_i defined on X near x by the formula

$$\chi_i \lrcorner \sigma_X = f^*df_i$$

are independent near x and commute, since the fibration f is Lagrangian. They thus generate a holomorphic foliation of an Euclidean neighborhood V of x in X and a free action $U \times V \rightarrow V'$ on $V \subset V'$ of a germ $U \subset \mathbb{C}^k$ of commutative group of automorphisms. As f is a Lagrangian fibration, the considered automorphisms $\psi \in U$ preserve f , namely $f \circ \psi = f$ on V for any $\psi \in U$. It follows that the locus $X_k \cap V$ is preserved by the action of U (that is, mapped to $X_k \cap V'$) and for any $x' \in X_k \cap V$, the orbit $U \cdot x'$ is contained in $X_k \cap V'$. As these orbits are contained in the fibers of $f|_V$, the lemma is proved. $\square$

Remark 4.2. This vertical group action is also used in [23]. Thanks to the fact that the action is vertical and the map f is proper, the germ of commutative group appearing above globalizes to a holomorphic action of a commutative group isomorphic to $\mathbb{C}^k$ on a neighborhood of the fiber X_t in X passing through x . This is obtained by observing that, by properness, the flow generated by Hamiltonian vector fields is defined for all time. This action is free assuming that the differentials f^*df_i remain independent everywhere along the fiber. This will be the case in Section 5.1, where we will consider the case $k = n - 1$.

We also note the following basic

Lemma 4.3. *Let $f : Y \rightarrow S$ be an algebraic or analytic morphism, where both Y and S are smooth. With the same notation Y_k as above, for any irreducible component Z of Y_k , one has $\dim f(Z) \leq k$.*

Proof. Indeed, the generic rank of $f|_Z$ is not greater than k , hence $f(Z)$ has dimension $\leq k$ by Sard's theorem. $\square$

Proof of Theorem 1.6. Statement (1) is Lemma 4.3 and statement (2) is Lemma 4.1. In order to prove (3) and (4), we need to globalize the argument used in the proof of Lemma 4.1. Let Z be an irreducible component of X_k which has dimension $2k$ and whose image $f(Z)$ in B has dimension k . We observe that, by applying again Lemma 4.3 to $X_{\leq k-1}$, for a general point $b \in f(Z)$, the fiber $Z_b := (f|_Z)^{-1}(b)$ is contained in X_k , that is, the rank of f along Z_b is everywhere k . There is thus a morphism

$$\begin{aligned} \tilde{f}_Z : Z &\rightarrow \text{Grass}(k, f^*T_B), \\ z &\mapsto \text{Im } f_{*,z} \subset T_{B,f(z)}, \end{aligned}$$

which is well-defined in a neighborhood of Z_b . This morphism is generically constant along Z_b , since at any point of $Z \cap X_k$ where the rank of $f|_Z$ is also k , we have

$$\text{Im } f|_{Z,*} = \text{Im } f_*$$

and the space $\text{Im } f|_{Z,*}$ has to be equal to $T_{f(Z),b}$ at a general point of $f(Z)$. It follows that the morphism $\tilde{f}_Z$ is actually constant along Z_b . We now argue as in the proof of Lemma 4.1, in a global setting. Let $b = f(z)$ and let $f_1, \dots, f_k$ be holomorphic functions defined on B near b whose restrictions to $f(Z)$ have independent differentials at b . By the above argument, the pulled-back functions $g_i = f_i \circ f$ on X have independent differentials along Z_b , hence in a neighborhood of Z_b in X . Furthermore, their Poisson brackets vanish

since f is Lagrangian. It follows that the corresponding Hamiltonian vector fields χ_i generate an integrable foliation, whose underlying vector bundle is trivial along Z_b , and which is vertical, in the sense that it is annihilated by f_* , or equivalently that the diffeomorphisms $\psi_{i,t}$, $t \in \mathbb{C}$, generated by the Hamiltonian vector fields χ_i satisfy $f \circ \psi_{i,t} = f$. Recall from Remark 4.2 that these diffeomorphisms $\psi_{i,t}$ are in fact defined in a neighborhood of X_b in X . As we have $f \circ \psi_{i,t} = f$, $\psi_{i,t}$ preserves X_k , hence it preserves its irreducible components and thus acts on Z and on the fiber Z_b . As the group $U \cong \mathbb{C}^k$ generated by the $\psi_{t,i}$'s is of dimension k and acts freely on X , and by assumption $\dim Z_b = k$, we conclude that the orbits of this germ of groups are open in Z_b . In particular Z_b is smooth and its tangent bundle is trivial, isomorphic to the restriction of $\mathcal{F}$ to Z_b , and generated by commuting vector fields. Thus Z_b is a disjoint union of compact complex tori T_i , and the group action above of U on Z_b factors through the action of T_i on itself by translations (thus U identifies to the universal cover of T_i). As we already observed, U not only acts on T_i by translations, but also on a neighborhood of T_i in X_k and X . We thus conclude that U acts on the restricted normal bundle $N_{X_k/X|T_i}$, making it homogeneous. $\square$

Coming back to Lemma 4.3, note that we can easily find examples where $f(Z)$ has dimension $< k$. For example, consider the case where the morphism f is the blow-up $Y := \text{Bl}_M(S) \rightarrow S$ of a smooth subvariety $M \subset S$ of codimension 2. Then the exceptional divisor E is the locus where f is not of maximal rank. The rank of f along E is $m - 1$, where $m = \dim S = \dim Y$, and the rank of $f|_E$ is $m - 2$. Unlike the Lagrangian fibration maps when the basis is smooth, this example is not flat, but there are also flat examples of this phenomenon: consider a 3-dimensional singular affine quadric $Q \subset \mathbb{A}^4$ of equation $x^2 + y^2 + z^2 + t^2 = 0$ and let $\tau : Y \rightarrow Q$ be a small resolution of the singular point $0 \in Q$. Let $f = l \circ \tau : Y \rightarrow \mathbb{A}^2$, where $l : Q \rightarrow \mathbb{A}^2$ is the linear projection $(x, y, z, t) \mapsto (x, y)$. Then the fibers of f over $(x, y) \neq 0$ are conics $z^2 + t^2 = a$ and the fiber of f over 0 is the union of a conic $z^2 + t^2 = 0$ and the exceptional curve E . They are thus of dimension 1, so f is flat. The locus where f has rank 1 is the union of two components: the first component is the set of singular points of the conics $z^2 + t^2 = -x^2 - y^2$ over the set of points (x, y) such that $x^2 + y^2 = 0$, and the restriction of f to this component has generic rank 1. The second component is the exceptional curve E and the restriction of f to E has generic rank 0 since it is contracted. Finally, the rank of f is generically 1 along E , because if instead of Y we consider the blow-up of Y along E , that is the blow-up $\tau' : \tilde{Q} \rightarrow Q$ of Q at its singular point, with exceptional divisor F , then the rank of $l \circ \tau'$ is generically 1 along F and the

same follows easily for Y . In the case of a Lagrangian fibration $f : X \rightarrow B$ with smooth B , we do not know if an example as above is possible:

Question 4.4. *Let $f : X \rightarrow B$ be a Lagrangian fibration with X hyper-Kähler and B smooth (or the restriction of a Lagrangian fibration over the smooth locus B' of its base). For any integer $k \geq 0$ and any component Z of the set $X_k \subset X$ of points where f has rank k , is it true that $f(Z)$ has dimension k (or equivalently, that the generic rank of $f|_Z$ is k)?*

Note that an affirmative answer to Question 4.4 would imply, using Lemma 4.1, that the dimension of X_k (assuming it is non-empty) is greater than or equal to $2k$, which is different from the generic codimension estimate of Lemma 3.2.

The stratification of X by the rank, that is, by the X_k , induces a collection of subsets $f(X_k) =: B_k \subset B$. It would be interesting to compare these subsets to another natural stratification on B related to the topological degeneration of the fibers of f . Assume that the fibers of f are reduced and irreducible. We will say that the fiber X_b has abelian rank k if the Albanese variety of any desingularization $\tilde{X}_b$ of X_b has dimension k . The following comparison statement can be found in a slightly different form in [2, Proposition 5.17].

Proposition 4.5. *Assume that the base B is smooth, the fibers of f are reduced and irreducible so that, by [2], there exists a group scheme $\mathcal{G}$ over B with Lie algebra isomorphic to Ω_B which acts on X over B . If a fiber X_b , $b \in B$, has abelian rank k , then b does not belong to $f(X_l)$ for $l < k$. In other words, the rank of the differential $f_* : T_{X,x} \rightarrow T_{B,b}$ is $\geq k$ for any $x \in X_b$.*

Proof. (Cf. [2]) The Lie algebra of $\mathcal{G}$ is isomorphic to Ω_B and the infinitesimal action of $\mathcal{G}$ on X is given at any $x \in X$ by the pull-back map $f^* : f^*\Omega_B \rightarrow \Omega_X \cong T_X$ followed by evaluation at x . The set X_l of points $x \in X$ where the evaluation map has rank l is thus also the locus where the isotropy subgroup $\mathcal{I}_x$ of x has dimension $n - l$. By our assumption on the singularities of the fibers, any fiber X_t is birational to the corresponding fiber $\mathcal{G}_t$, which is a commutative algebraic group. By [9, Theorem 2], the group $\mathcal{G}_b$, $b = f(x)$, is an extension of an abelian variety A_b by an affine group N_b , so any smooth projective model of $\mathcal{G}_b$ is a rationally connected fibration over A_b , hence A_b is the Albanese variety of any desingularization of $\mathcal{G}_b$. We now observe that, as the fiber X_b is irreducible and reduced, there are points $y \in X_b$ such that the isotropy group of y is trivial. It follows first of all that the fiber X_b is birational to the corresponding fiber $\mathcal{G}_b$. Secondly, considering the action of $\mathcal{I}_x$ on a general point $y \in X_b$ close to x , we get that the isotropy group $\mathcal{I}_x$ is

affine. It is thus contained in N_b , hence we get, for $x \in X_l$,

$$\dim \mathcal{I}_x = n - l \leq \dim N_b = n - \dim A_b,$$

which proves the proposition. $\square$

We will prove in the next section the following result, which relates Question 4.4 to unexpected vanishing for Chern classes.

Theorem 4.1. *Assume that B is smooth and that for some integer $k \geq 0$, and for any $l > n - k$, either X_l is empty or, for any irreducible component W of X_l , one has $\dim W = 2l$ and $\dim f(W) = l$. Then*

(i) *If $\dim f(X_{n-k}) < n - k$, one has*

$$(30) \quad c_{2k}(X)\alpha^{n-k} = 0 \text{ in } H^*(X, \mathbb{C})$$

for any class $\alpha \in H^2(X, \mathbb{C})$ such that $q(\alpha) = 0$.

(ii) *If X_{n-k} is not empty and any component W of X_{n-k} satisfies*

$$\dim W = 2n - 2k, \quad \dim f(W) = n - k,$$

the class

$$(31) \quad c_{2k}(X)L^{n-k} \in \mathrm{CH}(X)$$

is $\mathbb{Q}$ -effective and nonzero.

(iii) *Under the same assumptions as in (ii) for $k = n$, $2n = \dim X$, one has $\chi_{\mathrm{top}}(X) \geq 0$ and $\chi_{\mathrm{top}}(X) = 0$ if and only if the locus X_0 where f has rank 0 is empty.*

Remark 4.6. The vanishing (30) is a topological property which is different from the cohomological vanishing relations (3) described in the introduction and is not expected to hold in general. In fact, using Theorem 4.1(ii), one sees that it does not hold for the known hyper-Kähler manifolds.

Remark 4.7. Assume there is a commutative group scheme $\mathcal{G}$ over B , with Lie algebra Ω_B , acting on X . Then according to [2, Proposition 5.17], the abelian part of $\mathcal{G}_b$ has for dimension the infimum of the ranks of f_{x*} for $x \in X_b$. In particular the abelian part is nontrivial away from $f(X_0)$ and is nontrivial everywhere if X_0 is empty. This reproves Theorem 4.1(iii) in this case by the classical argument (see [5]).

We next discuss a refinement of the condition appearing in Question 4.4. We denote by $\pi : \mathcal{G}_k \rightarrow B$ the relative Grassmannian $G(k, T_B)$ of k -dimensional subspaces of T_B . Consider the following condition for given k .

$(*_k)$ For any irreducible component W of the set $X_k \subset X$, the rational map

$$(32) \quad \begin{aligned} \psi_W : W &\dashrightarrow \mathcal{G}_k, \\ x &\mapsto \operatorname{Im} f_{*,x} \subset T_{B,f(x)}, \end{aligned}$$

has image of dimension $\leq k$.

Lemma 4.8. *Condition $(*_k)$ holds for a given component W of X_k if either $\dim W \leq 2k$, or $\dim f(W) = k$. In particular, it holds if Question 4.4 has an affirmative answer for W .*

Proof. Let $w \in W$. Recall from the proof of Lemma 4.1 that there is a free local action of an analytic germ of commutative group of dimension k on a neighborhood of w in X , which is vertical, that is, preserves f . This group action preserves W and the map ψ_W is constant along the orbits of this group acting on W , hence if $\dim W \leq 2k$, we have $\dim \psi_W(W) \leq k$.

In the other case, where the generic rank of $f|_W$ is k , the image of $f_* : T_{X,w} \rightarrow T_{B,f(w)}$, for a general point $w \in W$, is equal to the image of $(f|_W)_* : T_{W,w} \rightarrow T_{B,f(w)}$, that is, to the tangent space of $f(W)$ at the point $f(w)$. In this case, ψ_W is constant along the fibers of $f|_W$, hence $\dim \psi_W(W) = \dim f(W) = k$. $\square$

We conclude this section with a result that will be used in the proof of Theorem 1.8. We denote by

$$\begin{aligned} X_{\mathcal{G}_k} &= X \times_B \mathcal{G}_k, \\ f_{\mathcal{G}} : X_{\mathcal{G}_k} &\rightarrow \mathcal{G}_k, \quad \pi_X : X_{\mathcal{G}_k} \rightarrow X \end{aligned}$$

the fibered product of X and $\mathcal{G}_k$ and its two projections.

Lemma 4.9. *Let $i \leq n$ be an integer. Assume Condition $(*_k)$ of (32) holds for any $k \geq n - i + 1$. Then the locus $Z \subset X_{\mathcal{G}_{i-1}}$ of pairs (x, V) , $x \in X$, $V \subset T_{B,f(x)}$, $\dim V = i - 1$, such that $f^* : V^\perp \rightarrow \Omega_{X,x}$ is not injective, is mapped by $f_{\mathcal{G}}$ to a closed algebraic subset of codimension $\geq i$ in $\mathcal{G}_{i-1}$.*

Proof. This is proved by a dimension count. The locus Z introduced above is the union over all k and irreducible components W of X_k of the loci $Z_W := Z \cap \pi_X^{-1}(W)$. As $\dim f(W) \leq k$ for $W \subset X_k$, the loci $\pi_X^{-1}(W)$ are of codimension

$\geq i$ for $n - k \geq i$, so we can assume that $k \geq n - i + 1$. At a point $x \in \pi_X^{-1}(W)$, the map $f_x^* : \Omega_{B, f(x)} \rightarrow \Omega_{X, x}$ has rank exactly k , and for a $n - i + 1$ -dimensional subspace $V^\perp \subset \Omega_{B, f(x)}$, the condition that $f_x^* : V^\perp \rightarrow \Omega_{X, x}$ is not injective says that $V^\perp \cap \text{Ker } f_x^* \neq \{0\}$. This imposes $k - n + i$ Schubert conditions on V , that are determined by the point $\text{Im } f_{*, x} \in \mathcal{G}_k$. It follows that the codimension of the image of Z_W in $\mathcal{G}_{i-1}$ is at least

$$k - n + i + \text{codim } f(W) - \dim \psi_W(W_b),$$

where $b \in f(W)$ is a general point and $W_b := f^{-1}(b) \cap W$. As

$$\text{codim } f(W) = n - \dim f(W), \quad \dim \text{Im } \psi_W = \dim f(W) + \dim \psi_W(W_b),$$

we conclude that the codimension of the image of Z_W in $\mathcal{G}_{i-1}$ is at least $i + k - \dim(\text{Im } \psi_W)$. By assumption, $\dim(\text{Im } \psi_W) \leq k$ for all $k \geq n - i + 1$ and W , so we get that the codimension of the image of Z_W in $\mathcal{G}_{i-1}$ is at least i for all W . $\square$

4.1. Support for Chern classes of Lagrangian fibered varieties

We prove in this section the following result.

Theorem 4.2. *Let $f : X \rightarrow B$ be a Lagrangian fibration of a projective hyper-Kähler manifold and i be a positive integer. Assume that B is smooth in codimension $i - 1$ and that the condition $(*_k)$ of the previous section (see (32)) is satisfied for $k \geq n - i + 1$. Then for any $j \geq i$, $c_{2j}(X)$ vanishes in $\text{CH}(X \setminus f^{-1}(B^i))$ for some codimension i closed algebraic subset B^i of B .*

We first establish some consequences.

Corollary 4.10. *Let $f : X \rightarrow B$ be a Lagrangian fibration of a projective hyper-Kähler manifold, where we assume B normal. Then $c_2(X)$ vanishes in $\text{CH}(X \setminus f^{-1}(B^1))$ and $c_{2j}(X)$ vanishes in $\text{CH}(X \setminus f^{-1}(B^2))$ for any $j \geq 2$, where $B^i \subset B$, $i = 1, 2$, is a closed algebraic subset of codimension $\geq i$.*

Proof. As B is normal, its singular locus has codimension ≥ 2 , so in both cases, in order to apply Theorem 4.2, we only have to study the condition $(*_k)$ for $k \geq n - i + 1$ for $i = 1, 2$. The assumption $(*_k)$ obviously holds for $k = n$, implying the first statement. For the second statement, where we have $i = 2$, we only need to study the sets X_k for $k = n - 1$. Let W be an irreducible component of X_{n-1} . Then the image of W under f has dimension $\leq n - 1$, since $f|_{W_{\text{reg}}}$ has rank $\leq n - 1$ everywhere. If $\dim f(W) = n - 1$, then

$(*_n)$ is satisfied by W by Lemma 4.8. There might be other components W for which $\dim f(W) \leq n-2$, but we can restrict the Lagrangian fibration over the open set $B \setminus \cup_{W \subset X_{n-1}, \dim f(W) \leq n-2} f(W)$ and apply Theorem 4.2 over this open set. $\square$

Proof of Theorem 1.8. By Lemma 4.8, the assumptions of Theorem 1.8 imply that Condition $(*_k)$ holds for $k \geq n-i+1$. Thus Theorem 4.2 implies Theorem 1.8. $\square$

For the proof of Theorem 4.2, we will use the following general lemma about base change invariance of vanishing of Chern classes.

Lemma 4.11. *Let $f : Y \rightarrow M$ be a proper morphism with M smooth and let $\pi : N \rightarrow M$ be a smooth proper morphism. Denote by Y_N the fibered product $Y \times_M N$ and by*

$$f_N : Y_N \rightarrow N, \pi_Y : Y_N \rightarrow Y$$

the two projections. Let E be a vector bundle on Y and let l be a positive integer. Then the Chern class $c_l(E)$ vanishes in $\mathrm{CH}^l(Y \setminus f^{-1}(Z))$ for some codimension k closed algebraic subset Z of M if and only if the Chern class $c_l(\pi_Y^ E)$ vanishes in $\mathrm{CH}^l(Y_N \setminus f_N^{-1}(Z'))$ for some codimension k closed algebraic subset Z' of N .*

Proof. If $c_l(E)$ vanishes in $\mathrm{CH}^l(Y \setminus f^{-1}(Z))$ for some codimension k closed algebraic subset Z of M , $c_l(\pi_Y^* E)$ vanishes in $\mathrm{CH}^l(Y_N \setminus \pi_Y^{-1}(f^{-1}(Z)))$, and, as $f \circ \pi_Y = \pi \circ f_N$, we have $\pi_Y^{-1}(f^{-1}(Z)) = f_N^{-1}(Z')$, where $Z' = \pi^{-1}(Z)$ is of codimension k by smoothness of π .

Conversely, let H be a very ample line bundle on N and let d be the relative dimension of π . Assume that the Chern class $c_l(\pi_Y^* E)$ vanishes in $\mathrm{CH}^l(Y_N \setminus f_N^{-1}(Z'))$ for some codimension k closed algebraic subset of N . Then considering the complete intersection of d general members of $|H|$, we get a smooth closed algebraic subset $N_1 \subset N$, that maps in a generically finite way to M via $\pi_1 := \pi|_{N_1}$, and a codimension k closed algebraic subset $Z'_1 \subset N_1$, whose image $Z := \pi_1(Z'_1) \subset M$ has codimension k . As $c_l(\pi_Y^* E)$ vanishes in $\mathrm{CH}^l(Y_N \setminus f_N^{-1}(Z'))$, we get that $c_l(\pi_{1,Y}^* E)$ vanishes in $\mathrm{CH}^l(Y_{N_1} \setminus f_{N_1}^{-1}(Z'_1))$, where $Y_{N_1} := Y \times_M N_1$, with projections $\pi_{1,Y} : Y_{N_1} \rightarrow Y$, $f_{N_1} : Y_{N_1} \rightarrow N_1$. As $Z = \pi_1(Z'_1)$, we find that $\pi_{1,Y*}(c_l(\pi_{1,Y}^* E))$ vanishes on $Y \setminus f^{-1}(Z)$ and this concludes the proof since $\pi_{1,Y*}(c_l(\pi_{1,Y}^* E)) = (\deg \pi_1)c_l(E)$. $\square$

Let now $f : X \rightarrow B$ be a Lagrangian fibration, where X is projective hyper-Kähler of dimension $2n$. Let $B' := B \setminus B_{\mathrm{sing}}$ and $X' := f^{-1}(B')$. The variety $\mathcal{G}_{i-1}$ introduced in the previous section is smooth of dimension $n + (n-i+1)(i-1)$ and fibered into Grassmannians over B' via a smooth proper

morphism that we denote by π . It carries a tautological subbundle $\mathcal{S}$ of rank $n - i + 1$ of the bundle $\pi^*\Omega_{B'}$, with fiber $V^\perp \subset \Omega_{B',b}$ at a point

$$(b, [V]), V \subset T_{B',b}, \dim V = i - 1$$

of $\mathcal{G}_{i-1}$. We denote as before by $f_{\mathcal{G}} : X'_{\mathcal{G}} \rightarrow \mathcal{G}_{i-1}$, $\pi_X : X'_{\mathcal{G}} \rightarrow X'$ the two projections. We have on X' the morphism of vector bundles

$$\phi := f^* : f^*\Omega_{B'} \rightarrow \Omega_{X'}$$

and thus, by pull-back to $X'_{\mathcal{G}}$ and restriction to $\mathcal{S}$, we get a morphism

$$(33) \quad \phi_{\mathcal{S}} : \mathcal{S} \rightarrow \pi_X^*\Omega_{X'}.$$

We now conclude the proof of Theorem 4.2.

Proof of Theorem 4.2. By Lemma 4.9, assumption $(*)_k$ for $k \geq n - i + 1$ implies that the morphism $\phi_{\mathcal{S}}$ is injective on $X_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i)$ for some closed algebraic subset Z_i of $\mathcal{G}_{i-1}$ of codimension $\geq i$. As X is hyper-Kähler, the vector bundle Ω_X carries an everywhere nondegenerate skew-symmetric pairing $\langle, \rangle$. As f is Lagrangian, the morphism $f^* : f^*\Omega_{B'} \rightarrow \Omega_{X'}$ is Lagrangian and it follows that the image of the morphism $\phi_{\mathcal{S}}$, which by definition of Z_i is a subbundle of rank $n - i + 1$ on $X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i)$, is totally isotropic at any point of $X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i)$. The subbundle $(\text{Im } \phi_{\mathcal{S}})^\perp$ of $\pi_X^*\Omega_{X'}$ is thus of rank $n + i - 1$ on $X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i)$ and contains $\text{Im } \phi_{\mathcal{S}}$. Furthermore, the quotient $\Omega_X/(\text{Im } \phi_{\mathcal{S}})^\perp$ is isomorphic to $f_{\mathcal{G}}^*\mathcal{S}^*$ via $\langle, \rangle$. Let

$$\mathcal{E} := (\text{Im } \phi_{\mathcal{S}})^\perp / \text{Im } \phi_{\mathcal{S}}.$$

This is a vector bundle of rank $2i - 2$ on $X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i)$, hence we have

$$(34) \quad c_{2i}(\mathcal{E}) = 0 \text{ in } \text{CH}(X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i)).$$

We use the exact sequences

$$\begin{aligned} 0 &\rightarrow f_{\mathcal{G}}^*\mathcal{S} \rightarrow (\text{Im } \phi_{\mathcal{S}})^\perp \rightarrow \mathcal{E} \rightarrow 0, \\ 0 &\rightarrow (\text{Im } \phi_{\mathcal{S}})^\perp \rightarrow \pi_X^*\Omega_{X'} \rightarrow f_{\mathcal{G}}^*\mathcal{S}^* \rightarrow 0 \end{aligned}$$

explained above and the Whitney formula, which gives equalities in $\text{CH}(X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i))$

$$(35) \quad \pi_X^*c(\Omega_{X'}) = f_{\mathcal{G}}^*c(\mathcal{S}^*)c((\text{Im } \phi_{\mathcal{S}})^\perp)$$

$$= f_{\mathcal{G}}^* c(\mathcal{S}^*) f_{\mathcal{G}}^* c(\mathcal{S}) c(\mathcal{E}).$$

We also get by inverting the total Chern classes in (35)

$$(36) \quad c(\mathcal{E}) = \pi_X^* c(\Omega_{X'}) f_{\mathcal{G}}^* c(\mathcal{S}^*)^{-1} f_{\mathcal{G}}^* c(\mathcal{S})^{-1} \text{ in } \mathrm{CH}(X_{\mathcal{G}}^0 \setminus f_{\mathcal{G}}^{-1}(Z)).$$

We now conclude the proof by induction on i . The induction assumption tells us that for $j < i$, the Chern class $c_{2j}(\Omega_X)$ vanishes in $\mathrm{CH}(X' \setminus f^{-1}(B^j))$ for some codimension j closed algebraic subset of B' . By pull-back, the Chern class $c_{2j}(\pi_X^* \Omega_{X'})$ vanishes in $\mathrm{CH}(X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_j))$ for some codimension j closed algebraic subset Z_j of G_{n-i+1} , for $j < i$. It follows from (36) that, for $j < i$, the Chern class $c_{2j}(\mathcal{E})$ vanishes in $\mathrm{CH}(X'_{\mathcal{G}} \setminus (f_{\mathcal{G}}^{-1}(Z_i \cup Z'_j)))$ for some codimension j closed algebraic subset Z'_j of $\mathcal{G}_{i-1}$ (note that $\mathcal{E}$, being self-dual, has trivial odd Chern classes). It follows that any class of the form $f_{\mathcal{G}}^* \alpha \cdot c_{2j}(\mathcal{E})$, where $\alpha \in \mathrm{CH}^{2i-2j}(\mathcal{G}_{i-1})$, vanishes in $\mathrm{CH}^{2i}(X'_{\mathcal{G}} \setminus (f_{\mathcal{G}}^{-1}(Z_i \cup Z'_i)))$ for some codimension i closed algebraic subset Z'_i of $\mathcal{G}_{i-1}$. By (34), this is also true for $c_{2i}(\mathcal{E})$. Using (12) and expanding the product in degree $2i$, we conclude that $c_{2i}(\pi_X^* \Omega_X)$ vanishes in $\mathrm{CH}(X'_{\mathcal{G}} \setminus f_{\mathcal{G}}^{-1}(Z_i \cup Z'_i))$, where Z_i and Z'_i have codimension $\geq i$ in $\mathcal{G}_{i-1}$. By Lemma 4.11, we conclude that $c_{2i}(\Omega_{X'})$ vanishes in $\mathrm{CH}(X' \setminus f^{-1}(B^i))$ for some codimension i closed algebraic subset of B' . As $B \setminus B'$ has codimension $\geq i$ in B by assumption, $c_{2i}(\Omega_X)$ also vanishes in $\mathrm{CH}(X \setminus f^{-1}(B^i))$ for some codimension i closed algebraic subset of B . $\square$

4.2. Proof of Theorem 4.1

Proof of Theorem 4.1. (i) First of all, we observe that the mapping class group acts on the cohomology algebra $H^*(X, \mathbb{Q})$ preserving the Chern classes $c_i(X) \in H^{2i}(X, \mathbb{Q})$. Indeed, the Chern classes of a hyper-Kähler manifold are in fact determined by its Pontryagin classes, that are topological invariants. Furthermore, it is known by results of Verbitsky that this action restricted to $H^2(X, \mathbb{Z})$ is that of a finite index subgroup of $O(q)$. The orbit under this action of any nonzero class l with $q(l) = 0$ is Zariski dense in the quadric $\{q = 0\} \subset H^2(X, \mathbb{C})$. It thus suffices to show that, under the assumptions of Theorem 4.1(i), we have

$$(37) \quad c_{2k}(X) l^{n-k} = 0 \text{ in } H^{2*}(X, \mathbb{Q}),$$

since it will imply the same result with l replaced by any $\alpha \in H^2(X, \mathbb{C})$ with $q(\alpha) = 0$. A fortiori, it suffices to show that

$$(38) \quad c_{2k}(X) L^{n-k} = 0 \text{ in } \mathrm{CH}^*(X).$$

We choose a general complete intersection $B^{n-k} \subset B$ of $n-k$ ample hypersurfaces, whose inverse images in X belong to $|dL|$ for some d , and denote by $X^{n-k} \subset X$ its inverse image in X , so that the class of X^{n-k} in $\mathrm{CH}(X)$ is a nonzero multiple of L^{n-k} . As X^{n-k} is a smooth complete intersection, we have the cotangent bundle exact sequence

$$(39) \quad 0 \rightarrow \mathcal{O}_{X^{n-k}}(-dL)^{n-k} \rightarrow \Omega_{X|X^{n-k}} \rightarrow \Omega_{X^{n-k}} \rightarrow 0.$$

As we already used several times, the image of the map $\mathcal{O}(-dL)^{n-k} \rightarrow \Omega_{X|X^{n-k}}$ is totally isotropic and there is thus a dual surjective morphism

$$(40) \quad \Omega_{X^{n-k}} \rightarrow \mathcal{O}_{X^{n-k}}(dL)^{n-k} \rightarrow 0$$

whose kernel is a rank $2k$ vector bundle on X^{n-k} , which will be denoted by $\mathcal{E}$. Using (39) and (40), we get formulas in $\mathrm{CH}(X^{n-k})$

$$(41) \quad \begin{aligned} c_{2k}(\Omega_{X|X^{n-k}}) &= c_{2k}(\mathcal{E}) + \sum_{l>0} \alpha_l L^{2l} c_{2k-2l}(\mathcal{E}), \\ c_{2i}(\mathcal{E}) &= c_{2i}(\Omega_{X|X^{n-k}}) + \sum_{l>0} \beta_l L^{2l} c_{2i-2l}(\Omega_{X|X^{n-k}}), \end{aligned}$$

As we know by Theorem 1.2 that $L^{n-k+2l} c_{2k-2l}(X) = 0$ for $l > 0$, we deduce from (41) that the desired vanishing (38) is a consequence of the vanishing

$$(42) \quad c_{2k}(\mathcal{E}) = 0 \text{ in } \mathrm{CH}(X^{n-k})$$

that we prove now, using the assumptions of Theorem 4.1(i). We denote by $f' : X^{n-k} \rightarrow B^{n-k}$ the restriction of f . Using the fact that f is a Lagrangian fibration, the differential

$$f'^* : f'^* \Omega_{B^{n-k}} \rightarrow \Omega_{X^{n-k}}$$

induces a morphism

$$(43) \quad \phi' : f'^* \Omega_{B^{n-k}} \rightarrow \mathcal{E}.$$

Note that $\mathcal{E}$ has an induced nondegenerate skew-symmetric form and that ϕ' is a Lagrangian morphism of vector bundles in the sense of Definition 1.6. Let

$$\pi : P := \mathbb{P}(\Omega_{B^{n-k}}) \rightarrow B^{n-k}, \quad X_P^{n-k} := P \times_{B^{n-k}} X^{n-k}$$

and denote by

$$f'_P : X_P^{n-k} \rightarrow P, \quad \pi_X : X_P^{n-k} \rightarrow X$$

the two projections. Let $\mathcal{S} \subset \pi^* \Omega_{B^{n-k}}$ be the rank 1 tautological subbundle. The morphism ϕ' induces a morphism $\phi'' : f'_P{}^* \mathcal{S} \rightarrow \pi_X^* \mathcal{E}$, or equivalently a section

$$(44) \quad \alpha \in H^0(X_P, f'_P{}^* \mathcal{S}^{-1} \otimes \pi_X^* \mathcal{E}).$$

The section α is not transverse and we are going to describe its vanishing locus $Z(\alpha)$ below. After applying an excess formula à la Fulton [18, Section 6.3], the corrected vanishing locus will have class

$$(45) \quad \begin{aligned} Z(\alpha)_{\text{vir}} &= c_{2k}(f'_P{}^* \mathcal{S}^{-1} \otimes \pi_X^* \mathcal{E}) \\ &= \sum_{0 \leq 2i \leq 2k} f'_P{}^*(c_1(\mathcal{S}^{-1})^{2i}) \pi_X^* c_{2k-2i}(\mathcal{E}). \end{aligned}$$

Recall that $\pi_X : X_P \rightarrow X$ is the projectivized bundle $\mathbb{P}(f'^* \Omega_{B^{n-k}})$, polarized by $f'_P{}^* \mathcal{S}^{-1}$. We note here for future use that, as we assumed that B is smooth, it is isomorphic to $\mathbb{P}^n$ by [22], hence we can assume that B^{n-k} is a $\mathbb{P}^k$, and $\mathcal{S}^{-1}$ is very ample on P , with space of global sections $H^0(\mathbb{P}^k, T_{\mathbb{P}^k})$. We get from (45)

$$(46) \quad \pi_{X*}(Z(\alpha)_{\text{vir}} c_1(f'_P{}^* \mathcal{S}^{-1})^{k-1}) = c_{2k}(\mathcal{E}) + \sum_{i>0} f'^* s_{2i}(\Omega_{B^{n-k}}) c_{2k-2i}(\mathcal{E})$$

where the s_{2i} 's denote the Segre classes. One proves as in Theorem 1.2 that

$$f'^* s_{2i}(\Omega_{B^{n-k}}) c_{2k-2i}(\mathcal{E}) = 0 \text{ in } \text{CH}^{2k}(X^{n-k})$$

for $i > 0$, so we get in fact the equality

$$(47) \quad \pi_{X*}(Z(\alpha)_{\text{vir}} f'_P{}^*(c_1(\mathcal{S}^{-1})^{k-1})) = c_{2k}(\mathcal{E}) \text{ in } \text{CH}^{2k}(X^{n-k}).$$

We now describe the locus $Z(\alpha)$. We observe that X^{n-k} is stratified by strata

$$X_l^{n-k} := X^{n-k} \cap X_{n-k+l}$$

where the rank of f_* is $n-k+l$, or equivalently the rank of f'_* is l . For $l = 0$, as we assumed that $\dim f(X_{n-k}) < k$, X_0^{n-k} is empty. We will now analyze the contributions of the strata X_l^{n-k} for $l > 0$ and show that they all vanish.

For $l > 0$, our assumption is that X_{l+n-k} has dimension $2l + 2n - 2k$ and its image B_l^{n-k} in B has dimension $l + n - k$, so we have $\dim X_l^{n-k} = 2l + n - k$ and its image in B has dimension l . We now observe that, denoting $X_{P,l}^{n-k}$ the inverse image $\pi_X^{-1}(X_l^{n-k})$, the section α of (44) vanishes along the projective subbundle

$$\mathbb{P}(\mathcal{K}_l) \subset \mathbb{P}((f'^* \Omega_{B^{n-k}})|_{X_l^{n-k}}) = X_{P,l}^{n-k},$$

where the vector bundle $\mathcal{K}_l$ on X_l^{n-k} is the kernel of f'^* , hence has rank $k - l$. We thus conclude that the intersection $Z(\alpha)_l$ of the locus $Z(\alpha)$ with the stratum $X_{P,l}^{n-k}$ has dimension

$$(48) \quad \begin{aligned} \dim Z(\alpha)_l &= \dim X_l^{n-k} + \operatorname{rk} \mathcal{K}_l - 1 \\ &= 2l + n - k + k - l - 1 = n + l - 1. \end{aligned}$$

Note that the expected dimension of $Z(\alpha)$ is

$$\dim X_P^{n-k} - 2k = 2n - (n - k) + k - 1 - 2k = n - 1.$$

The key point is now the following: generically along X_l^{n-k} , the rank $k - l$ vector bundle $\mathcal{K}_l$ comes from the rank $k - l$ subbundle on B_l^{n-k} with fiber $\operatorname{Ker}(\Omega_{B^{n-k}|B_l^{n-k}} \rightarrow \Omega_{B_l^{n-k}})$. This follows indeed from our assumptions that

$$\dim X_{l+n-k} = 2(n + l - k), \quad \dim B_{l+n-k} = n + l - k,$$

which implies that the generic rank of $f|_{X_{n-k+l}}$ is $n - k + l$ (cf. Question 4.4), hence the generic rank of $f'|_{X_l^{n-k}}$ is l . It follows from this fact that the image $f'_P(Z(\alpha) \cap X_l^{n-k})$ in P has dimension at most

$$\dim B_l^{n-k} + k - l - 1 = l + k - l - 1 = k - 1.$$

When we intersect $Z(\alpha)$ with $k - 1$ hypersurfaces in $|f_P^* \mathcal{S}^{-1}|$ coming from P , we thus get a subvariety $Z'(\alpha)$ which is supported over finitely many points in P , and its image in X^{n-k} under π_X has finitely many irreducible components, all contained in fibers of f' . More precisely, the fibers of $Z'(\alpha)$ supported over B_l are supported over generic points of B_l^{n-k} , contained in X_l^{n-k} and of dimension $n - k + l$ by (48). Finally, recall that, by formula (47), we have to compute $\pi_{X*}(Z(\alpha)_{\operatorname{vir}} f_P'^*(c_1(\mathcal{S}^{-1})^{k-1}))$ and show that it is 0 in $\operatorname{CH}(X^{n-k})$. Here the cycle $Z(\alpha)_{\operatorname{vir}}$ is supported on $Z(\alpha)$ and computed by applied Fulton's refined intersection formula [18]. Now we observe that we can replace the term $Z(\alpha)_{\operatorname{vir}} f_P'^*(c_1(\mathcal{S}^{-1})^{k-1})$ appearing in this formula by the

virtual class of $Z'(\alpha)$. As we described $Z'(\alpha)$ as a disjoint union of components $Z'(\alpha)_{l,b_i}$ of dimension $n - k + l$ contained in fibers X_{P,l,b_i}^{n-k} , it suffices to show that the contribution of each $Z'(\alpha)_{l,b_i}$ to $Z'(\alpha)_{\text{vir}}$ is 0 for $l > 0$. This follows from the following

Claim 4.12. *The excess bundle for the section α is trivial of rank l along the fiber $X_{P,b}^{n-k} \cap Z(\alpha)$, for a general point $b \in B_l^{n-k}$.*

Proof. Let $b \in B_l^{n-k}$ be a general point. Choose l functions $g_1, \dots, g_l$ on B^{n-k} whose differentials at b are independent on B_l^{n-k} . Then their pull-backs to X^{n-k} have independent differentials along $X_{l,b}^{n-k}$, hence in a Zariski open neighborhood U_b^{n-k} of $X_{l,b}^{n-k}$ in X^{n-k} , since $b \in B_l^{n-k}$ is a general point so the morphism $X_l^{n-k} \rightarrow B_l^{n-k}$ has to be of rank l at any point over b . Let $\mathcal{F}$ be the trivial rank l vector bundle on U_b^{n-k} generated by these differentials. We have a natural morphism $\mathcal{F} \rightarrow \mathcal{E}$ induced by the inclusion $\mathcal{F} \rightarrow \Omega_{U_b^{n-k}}$, using the fact that f is Lagrangian. By duality, this induces a quotient map

$$q : \mathcal{E} \rightarrow \mathcal{F}^* \rightarrow 0$$

on U_b^{n-k} . The morphism

$$f'^* : f'^* \Omega_{B^{n-k}} \rightarrow \mathcal{E},$$

being Lagrangian, takes value in $\text{Ker } q$. This proves that the excess bundle identifies to $\mathcal{F}^*$ along $X_{l,b}^{n-k}$. This proves the result since $\mathcal{F}$ is trivial on U_b^{n-k} , hence on X_b^{n-k} . $\square$

The claim concludes the proof since, as we already mentioned, by our assumption that $\dim B_{n-k} < n - k$, B_0^{n-k} is empty so there is no contribution from the stratum where $l = 0$. $\square$

Proof of Theorem 4.1(ii) and (iii). Statement (iii) is a particular case of (ii). For the proof of (ii), we just follow the analysis made previously, except that now, the stratum X_0^{n-k} is not empty and of the right dimension by (48). In particular B_0^{n-k} is a non-empty set of points and X_0^{n-k} has dimension $n - k$, and there is no excess in the contribution of the stratum X_0^{n-k} to the cycle $p_{X*}(c_1(\mathcal{S}^{-1})^{k-1}Z(\alpha)_{\text{vir}})$, and this stratum contributes via an effective non-empty cycle of dimension $n - k$. The contributions of the other strata are zero as explained in the previous proof. This shows that the cycle $c_{2k}(X)L^{n-k}$ is effective and nontrivial. $\square$

5. Proof of Theorems 1.3 and 1.4

5.1. The cycle $c_2^2 L^{n-1}$

We study in this section the cycle $L^{n-1}c_2(X)^2$, where L is a Lagrangian line bundle on a projective hyper-Kähler $2n$ -fold X , whose vanishing is predicted by Conjecture 1.4. We use some of the constructions and notations introduced in the previous sections. We denote by $C = B^{n-1} \subset B$ the (smooth) complete intersection of $n-1$ ample general hypersurfaces in a very ample linear system $|H|$ on B with $f^*H = L^{\otimes d}$ and $X_C = X^{n-1} \subset X$ its inverse image in X with restricted morphism $f' : X_C \rightarrow C$. We have on X_C the rank 2 vector bundle $\mathcal{E}$ with trivial determinant, which is the kernel of the natural surjective morphism $\Omega_{X_C} \rightarrow \mathcal{O}_{X_C}(dL)^{n-1}$ of (40), and has an induced symplectic structure (see also (10)). We only study the case where there are no nonreduced fibers in codimension 1. The vanishing locus of the morphism $f'^* : f'^*\Omega_C \rightarrow \Omega_{X_C}$ thus has codimension ≥ 2 in X_C . Furthermore, as we saw already, this morphism takes values in $\mathcal{E}$ and its vanishing locus Z thus represents the class $c_2(\mathcal{E}(-f'^*K_C))$. Note that Z is supported on fibers of f' , on which f'^*K_C is trivial. Arguing as in the proof of Theorem 4.1, that is, comparing $j_*c_2(\mathcal{E})$, where j is the inclusion of X_C in X , and $L^{n-1}c_2(X)$, we get

Lemma 5.1. *The class $L^{n-1}c_2(X)^2 \in \mathrm{CH}^{n+3}(X)$ is proportional to the class $j_{Z*}(c_2(\mathcal{E})|_Z)$, where Z is as above the critical locus of f' and $j_Z : Z \rightarrow X$ is the inclusion map.*

The geometry of Z and $\mathcal{E}$ is very interesting and a particular case of a phenomenon studied in [11], [16]. Indeed, we can apply Theorem 1.6 with $k = n-1$ since for $k = n-1$, the equalities in (1) and (2) hold in codimension 1 on B once there are no nonreduced fibers in codimension 1. We thus conclude that Z is a finite union of abelian varieties, which are projective leaves of a foliation. The conormal bundle of a leaf F of a foliation $\mathcal{F} := \mathcal{E}^\perp$ on a manifold Y admits an integrable holomorphic connection, defined as the composition

$$\Omega_Y \supset \mathcal{E} \xrightarrow{d} \mathcal{E} \wedge \Omega_Y \rightarrow \mathcal{E} \otimes \Omega_F,$$

so it is a flat bundle. Unfortunately, we cannot say much more about it, since by the suspension construction described in [16, Example 9.1], any flat vector bundle E on an abelian variety A is isomorphic to the conormal bundle of a foliation on the total space of E , of which A is a leaf. Furthermore, the Chern classes of a flat holomorphic vector bundle with trivial determinant on

an abelian variety A can be nontrivial in $\mathrm{CH}(A)$. For example, for any line bundle $M \in \mathrm{Pic}^0(A)$, the vector bundle $M \oplus M^{-1}$ has trivial determinant but its second Chern class $-M^2 \in \mathrm{CH}^2(A)$ is in general nontorsion. In fact, once the dimension of A is at least 2, the subgroup of $\mathrm{CH}^2(A)$ generated by these classes is infinite dimensional in the Mumford sense, by Mumford's theorem [34]. In the present situation, we have

Lemma 5.2. *For each connected component $Z_i \subset X_c$ of Z , the vector bundle $\mathcal{E}|_{Z_i}$ is isomorphic to either*

- (i) *a direct sum $M_i \oplus M_i^{-1}$ for some line bundle $M_i \in \mathrm{Pic}^0(Z_i)$, or*
- (ii) *a tensor product $M_i \otimes U$ for some 2-torsion line bundle M_i on Z_i and rank 2 vector bundle U which is a an extension of the trivial line bundle by itself.*

Proof. The vector bundle $\mathcal{E}|_{Z_i}$ is homogeneous by Theorem 1.6. Homogeneous rank 2 vector bundles on abelian varieties are classified in [33] and are direct sums of homogeneous (that is topologically trivial) line bundles, which gives the first case, or of the form $M_i \otimes U$ as in (ii). In case (ii), M_i has to be a 2-torsion line bundle since $\det \mathcal{E}$ is trivial. $\square$

In case (ii), the class $c_2(\mathcal{E}|_{Z_i})$ is trivial, and thus the contribution of Z_i to the class $c_2(X)L^{n-1}$ is zero. Unfortunately, case (i) is the most naturally encountered. We discuss another more classical viewpoint on the line bundle M_i in the following Proposition 5.3, showing that there is no restriction on the topologically trivial line bundle M_i in (i). Let $c \in C$ be a critical value of f' , let X_c be the fiber $f^{-1}(c)$ and $Z_c \subset X_c$ be the singular locus of X_c . By [23], the normalization $\widehat{X}_c$ is smooth and it is a $\mathbb{P}^1$ -bundle over an étale cover of an abelian variety A_c , via its Albanese map

$$a : \widehat{X}_c \rightarrow A_c.$$

The inverse image of Z_c is a finite union of abelian varieties contained in $\widehat{X}_c$ which are multisections of a and étale over A_c . These facts follow again from the existence of the local vertical free group action of $\mathbb{C}^{n-1}$ on a neighborhood of X_c in X_C , which is transitive on the components of Z_c , and provides automorphisms of X_c which lift to $\widehat{X}_c$.

Proposition 5.3. *Assume we are in Case (i) of Lemma 5.2, and that M_i is not torsion, X_c is irreducible, X_c has at most two branches along Z_i , and the branches are smooth. Then the inverse image $\widehat{Z}_i := n^{-1}(Z_i) \subset \widehat{X}_c$ is a divisor which has degree 2 over Z_i and there is a canonical isomorphism*

$$(49) \quad \mathcal{E}|_{Z_i} \cong n_*(\mathcal{O}_{\widehat{Z}_i}(\widehat{Z}_i)).$$

Furthermore $\widehat{Z}_i$ is the union of two disjoint divisors $\widehat{Z}_{i,1}$, $\widehat{Z}_{i,2}$ isomorphic to A_c via $a : \widehat{X}_c \rightarrow A_c$ and to Z_i via n and, for any ample line bundle H on X , we have for some nonzero integer d

$$(50) \quad dM_i \cong H|_{Z_i} - t^*H|_{Z_i},$$

where t is the translation of $A_c \cong Z_i$ given by the composition $a|_{\widehat{Z}_{i,2}} \circ \iota \circ (a|_{\widehat{Z}_{i,1}})^{-1}$, ι being the natural isomorphism $\widehat{Z}_{i,1} \cong \widehat{Z}_{i,2}$ given by the isomorphisms

$$n|_{\widehat{Z}_{i,1}} : \widehat{Z}_{i,1} \cong Z_i, \quad n|_{\widehat{Z}_{i,2}} : \widehat{Z}_{i,2} \cong Z_i.$$

Note that, from (50), we conclude that the line bundle M_i is torsion if and only if the translation t has finite order. In most examples, the translation t does not have finite order (see [23] for examples). In fact, the translation t is well understood from the viewpoint of degenerations of abelian varieties. If the family of fibers is the Jacobian fibration of a family of curves $\mathcal{C} \rightarrow B$ degenerating generically along the discriminant hypersurface Δ to irreducible curves $\mathcal{C}_c$, $c \in \Delta$ with one node x , then the translation used to construct the generalized Jacobian of $\mathcal{C}_c$ starting from a $\mathbb{P}^1$ -bundle on $J(\widehat{\mathcal{C}}_c)$ by glueing two sections is translation by the point $\text{alb}(x_1 - x_2) \in J(\widehat{\mathcal{C}}_c)$, where $x_1, x_2 \in \widehat{\mathcal{C}}_c$ are the two preimages of x under the normalization map $\widehat{\mathcal{C}}_c \rightarrow \mathcal{C}_c$ (see [10]). Similarly, if the family of fibers is the Jacobian fibration of a family of threefolds $\mathcal{X} \rightarrow B$ with $h^{3,0} = 0$ degenerating to threefolds $\mathcal{X}_c$ with one ordinary double point x_0 and nonzero vanishing cycle, then the similarly defined translation is translation by the point $\Phi_{\widetilde{\mathcal{X}}_c}(R_1 - R_2) \in J^3(\widetilde{\mathcal{X}}_c)$, where $\widetilde{\mathcal{X}}_c \rightarrow \mathcal{X}_c$ is the blow-up of the singular point, $\Phi_{\widetilde{\mathcal{X}}_c}$ is its Abel-Jacobi map, and the curves R_i are lines in the two different rulings of the exceptional divisor, which is a 2-dimensional quadric (see [13]).

Proof of Proposition 5.3. Our assumption is that, locally analytically near Z_i , X_c is the union of two smooth divisors in X_c so that n has degree 2 over Z_i . For any point $x \in Z_i$, we denote by x_1, x_2 its preimages in $\widehat{X}_c$ and R_{x_1}, R_{x_2} the respective fibers of a through x_1, x_2 . Let g be a coordinate on C centered at c . Then f^*g vanishes along X_c , and its differential vanishes along the component Z_i of the singular locus of X_c . The Hessian of f along Z_i is a nonzero section of $S^2\mathcal{E}|_{Z_i}$ since our assumptions also imply that X_c has multiplicity 2 along Z_i . If M_i is not a torsion line bundle, the only nonzero section of $S^2\mathcal{E}$ is the section of the factor $M_i \otimes M_i^{-1} \subset S^2\mathcal{E}$, which vanishes along the two factors M_i, M_i^{-1} of $\mathcal{E}|_{Z_i}^*$. As we assumed that n is an immersion at the points x_1, x_2 above $x \in Z_i$ and we know that the inverse image $\widehat{Z}_i \subset \widehat{X}_c$

of Z_i is étale over A_c , we get up to a permutation identifications $T_{R_{x_j}, x_j} \cong M_j^{\epsilon_j}$ with $j = 1, 2$ and $\epsilon_j = (-1)^j$. This proves (49).

As X_c is irreducible, $\widehat{X}_c$ is connected and it is a $\mathbb{P}^1$ -bundle over an abelian variety A_c by [23]. The inverse image $\widehat{Z}_i \subset \widehat{X}_c$ is étale over Z_i , hence is a disjoint union of at most two abelian varieties, all isogenous to A_c . If $\widehat{Z}_i$ is irreducible, we have $n_*\mathcal{O}_{\widehat{Z}_i} = \mathcal{O}_{Z_i} \oplus \mathcal{O}_{Z_i}(\eta)$ for some 2-torsion line bundle $\eta \in \text{Pic}^0(Z_i)$. By (49), using the fact that the line bundle $\mathcal{O}_{\widehat{Z}_i}(\widehat{Z}_i)$ on $\widehat{Z}_i$ is homogeneous, hence topologically trivial, we get in this case an isomorphism

$$\mathcal{E}|_{Z_i} \cong M'_i \oplus M'_i(\eta)$$

for some line bundle $M'_i \in \text{Pic}^0(Z_i)$. Comparing with the isomorphism

$$\mathcal{E}|_{Z_i} \cong M_i \oplus M_i^{-1}$$

of (i), we conclude that M_i is torsion, which is a contradiction. It follows that $\widehat{Z}_i$ has two connected components $\widehat{Z}_{i,1}$, $\widehat{Z}_{i,2}$, each dominating A_c since otherwise they would be ruled. As the canonical bundle of X_c is trivial, the divisor $\widehat{Z}_i$ has degree at most 2 over A_c so each component $\widehat{Z}_{i,1}$, $\widehat{Z}_{i,2}$ has degree 1 over A_c , hence is isomorphic to A_c .

It remains to prove (50). Let H be an ample line bundle on X_c . Assume for simplicity that the pull-back $\widehat{H} := n^*H$ has degree 1 on the fibers of $a : \widehat{X}_c \rightarrow A_c$. (This can be assumed in any case if we work with Picard groups with $\mathbb{Q}$ -coefficients.) Then we have equalities in $\text{Pic}(\widehat{X}_c)$

$$(51) \quad \mathcal{O}_{\widehat{X}_c}(\widehat{Z}_{i,1}) = \widehat{H} \otimes a^*M'_1, \quad \mathcal{O}_{\widehat{X}_c}(\widehat{Z}_{i,2}) = \widehat{H} \otimes a^*M'_2$$

for some line bundles M'_i on A_c . We now use the fact that $\widehat{H}$ is the pull-back of a line bundle under n , so $\widehat{H}|_{\widehat{Z}_{i,2}}$ and $\widehat{H}|_{\widehat{Z}_{i,1}}$ coincide via the isomorphism ι . Denoting $a_1 := a|_{\widehat{Z}_{i,1}}$, $a_2 := a|_{\widehat{Z}_{i,2}}$, it follows from the definition of t that

$$(52) \quad (a_2^{-1})^*\widehat{H} = t^*((a_1^{-1})^*\widehat{H}).$$

We now observe that $\widehat{Z}_{i,1}$ and $\widehat{Z}_{i,2}$ do not intersect, so we get

$$\mathcal{O}_{\widehat{X}_c}(\widehat{Z}_{i,1})|_{\widehat{Z}_{i,2}} \cong \mathcal{O}_{\widehat{Z}_{i,2}},$$

hence, using (51), (52)

$$(53) \quad (a_2^{-1})^*\widehat{H} \otimes M'_1 = \mathcal{O}_{A_c} = t^*((a_1^{-1})^*\widehat{H}) \otimes M'_1.$$

It follows that

$$(a_1^{-1})^*(n^*M_i) = (a_1^{-1})^*(\mathcal{O}_{\widehat{Z}_{i,1}}(\widehat{Z}_{i,1})) = (a_1^{-1})^*\widehat{H} \otimes M'_1 = (a_1^{-1})^*\widehat{H} \otimes t^*((a_1^{-1})^*\widehat{H})^{-1},$$

where the first equality follows from (49) and the third from (53). This proves (50) since $\widehat{H} = n^*H$. $\square$

Remark 5.4. The proof also shows that the singular locus Z is irreducible in this case. This will be used below.

We now give a criterion for the vanishing of the class $c_2(X)^2 L^{n-1}$ in $\mathrm{CH}(X)$.

Proposition 5.5. *Let $X \rightarrow B$ be a Lagrangian fibered hyper-Kähler manifold. Assume the following three conditions hold:*

1. *In codimension 1 on B , the singularities of the fibers are as in Proposition 5.3, that is, the fiber X_b is irreducible, it has at most two local branches at any point, and they are smooth.*
2. *For any irreducible component Δ_j of the discriminant hypersurface $\Delta \subset B$, any desingularization $\widetilde{\Delta}_j$ of Δ_j satisfies $\mathrm{CH}_0(\widetilde{\Delta}_j) = \mathbb{Z}$.*
3. *For any irreducible component Δ_j as above, the family of abelian varieties $A_b = \mathrm{Alb}(\widehat{X}_b)$ parameterized by a Zariski open set of Δ_j satisfies the following nondegeneracy property:*
() For a general point $b \in \Delta_j$, there exists $u \in T_{\Delta_j,b}$ such that the first order variation of Hodge structure*

$$\overline{\nabla}_u : H^{1,0}(A_b) \rightarrow H^{0,1}(A_b)$$

is an isomorphism.

Then $c_2(X)^2 L^{n-1} = 0$ in $\mathrm{CH}(X)$.

Proof. The proof follows closely the argument of [43]. The family $\mathcal{A}_j = (A_b)_{b \in \Delta_j}$, of abelian varieties is an integrable system generically over Δ_j because the symplectic holomorphic form has generic rank $2n - 2$ on the normalization of the restricted family X_{Δ_j} , which is ruled over $\mathcal{A}_j$, hence induces a 2-form on $\mathcal{A}_j$ which is everywhere nondegenerate over a Zariski open set of Δ_j . This 2-form makes $\mathcal{A}_j \rightarrow \Delta_j$ a Lagrangian fibration over a Zariski open set of Δ_j . By symmetry (see [15]) of the map giving the infinitesimal variation of Hodge structure

$$\overline{\nabla} : T_{\Delta_j,b} \rightarrow \mathrm{Hom}(H^{1,0}(A_{j,b}), H^{0,1}(A_{j,b})),$$

the condition (*) translates into the fact that for general $b \in \Delta_j$, and general $\lambda \in H^{1,0}(A_{j,b})$, the map

$$\overline{\nabla}(\lambda) : T_{\Delta_j,b} \rightarrow H^{0,1}(A_{j,b})$$

is surjective. By the main result of [1], this condition implies that, for any normal function ν , that is, algebraic section ν of $\text{Pic}^0(\mathcal{A}'_j/\Delta'_j) \rightarrow \Delta'_j$ defined over a generically finite cover Δ'_j of Δ_j , the set of points $b \in \Delta'_j$ such that $\nu(b)$ is torsion in $\text{Pic}^0(A_{j,b})$ is dense for the Euclidean topology of Δ_j . We apply this to the normal function defined as follows. Let $\mathcal{Z}_j$ be the singular locus of X_{Δ_j} . If we restrict to an adequate dense Zariski open set of Δ_j , $\mathcal{Z}_j$ is smooth of codimension 2 in X and its conormal bundle $\mathcal{E}_j$ has one of the forms (i), (ii) of Lemma 5.2. In case (ii), we already noticed that the contribution of the fibers $Z_{j,b}$ to $c_2(X)^2 L^{n-1}$ is trivial. In case (i), the decomposition

$$\mathcal{E}_{j|Z_{j,b}} = M_{j,b} \oplus M_{j,b}^{-1}$$

defines a degree 2 cover of Δ_j parameterizing the choice of one of the line bundles $M_{j,b}$, $M_{j,b}^{-1} \in \text{Pic}^0(\mathcal{Z}_{j,b}) = \text{Pic}^0(A_{j,b})$. It follows that there are points $b \in \Delta_j$ where the line bundles $M_{j,b}$ and $M_{j,b}^{-1}$ are torsion. By Lemmas 5.1 and 5.2, the cycle $c_2(\Omega_{X|Z_{j,b}})$ is trivial in $\text{CH}^2(Z_{j,b})$ for any of these points. By condition 2, the cycle $j_{Z_{j,b}*}(c_2(\Omega_{X|Z_{j,b}})) \in \text{CH}^{n+3}(X)$ does not depend on $b \in \Delta_j$, hence it is trivial for any $b \in \Delta_j$. The cycle $c_2(\Omega_X)^2 L^{n-1}$ is up to a coefficient the sum of these cycles over all j , and all points $b \in \Delta_j \cap C$, hence it is trivial. $\square$

Proof of Theorem 1.3. We just have to check that the assumptions of Proposition 5.5 are satisfied in the case of the LSV manifold and its Lagrangian fibration. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. The base of the associated LSV manifold is the dual 5-dimensional projective space $\mathbb{P}^{5*}$ and the discriminant hypersurface $\Delta \subset \mathbb{P}^{5*}$ is the dual of the cubic $X \subset \mathbb{P}^5$. It follows that the LSV fibration satisfies condition 2. The condition 1 is also well-known: the generic behavior along Δ of the compactified intermediate Jacobian fibration is the same as the generic behaviour, along the discriminant divisor, of the family of intermediate Jacobians of all cubic threefolds. This is studied in [13], [15]. It only remains to check condition 3. However, since we are considering a Lagrangian family of abelian fourfolds, it is observed in [43] that the infinitesimal condition 3, which says that for generic $b \in \Delta$, there exists a $u \in T_{\Delta,b}$ such that

$$\overline{\nabla}_u : H^{1,0}(A_b) \rightarrow H^{0,1}(A_b)$$

is an isomorphism, is equivalent to the fact that there does not exist any $u \in T_{\Delta,b}$, $u \neq 0$, such that $\overline{\nabla}_u : H^{1,0}(A_b) \rightarrow H^{0,1}(A_b)$ is identically 0, that is, the moduli map of the family of abelian varieties $(A_b)_{b \in \Delta}$ has nowhere maximal rank. This follows from a result due to Lossen [29] concerning the projective geometry of homogenous cubic polynomials in four variables: it says that if a projective cubic surface is not a cone, then it has a point where its Hessian (or second fundamental form) is a nondegenerate quadratic form. We apply the Lossen result to the cubic form

$$C(u, v, w) := \langle \overline{\nabla}_u(v), w \rangle,$$

with $v \in H^{1,0}(A_b)$, $u \in T_{\Delta,b}$, $w \in H^{1,0}(A_b)$. The symmetry of C in u, v, w , using the natural isomorphism $T_{\Delta,b} \cong H^{1,0}(A_b)$ given by the Lagrangian fibration structure, is observed in [15].

By this argument, if condition 3 were not satisfied, the family of abelian varieties $(A_b)_{b \in \Delta}$ would not have maximal modulus, and it is easy to show that this does not happen, at least for very general X . For example, one can note that, by [12], these abelian fourfolds are the Jacobians of complete intersection curves of type $(2, 3)$ in $\mathbb{P}^3$, so that their variation of Hodge structure can be described explicitly. $\square$

Remark 5.6. If instead of a LSV manifold we consider the punctual Hilbert scheme $S^{[g]}$ of a $K3$ surface S with Picard group generated by a line bundle of self-intersection $2g-2$, or rather its birational version admitting a Lagrangian fibration compactifying the Jacobian fibration of the universal family of curves over $|L|$ (see [5]), then the proof above does not apply since the condition 2 in Proposition 5.5 is not satisfied. However, in this case the discriminant hypersurface is birational to a projective bundle over S , hence the relations in the Chow group of a $K3$ surface established in [4] (see statements 1 and 2 in the introduction) can probably be used to conclude also in that case. As the result is proved in [32], we do not pursue this argument.

5.2. Riess' argument and proof of Theorem 1.4

Theorem 1.4 follows from Theorem 1.2 by Riess' arguments as in [37]. For completeness, we sketch the proof below. The main result from Riess' paper that we need is the following.

Theorem 5.1. *Let X be a projective hyper-Kähler manifold of dimension $2n$, and let $L \in \text{NS}(X)$ be an isotropic class. Then there exists a projective hyper-Kähler manifold X' , and a correspondence $\Gamma \in \text{CH}^{2n}(X \times X')$ such that*

1. Γ induces a graded ring isomorphism $\Gamma_* : \mathrm{CH}(X) \rightarrow \mathrm{CH}(X')$ which maps $c_i(X)$ to $c_i(X')$.
2. $\Gamma_*(L) = L'$, where for some nef line bundle L' on X' .

The variety X' is birational to X , hence deformation equivalent to X by [19]. The cycle Γ is effective and is the limit of graphs of isomorphisms $X_t \cong X'_t$ for some small deformations X_t of X , resp. X'_t of X' . Such isomorphisms satisfy property 1, hence also Γ , as was observed by Riess in [38]. Being the specialization of graphs of isomorphisms, the correspondence $[\Gamma]^*$ also induces a ring isomorphism on cohomology, so L' is also isotropic since this is equivalent to $\deg L'^{2n} = 0$ by the Beauville-Fujiki formula, and we have $\deg L'^{2n} = 0$.

Assume now the SYZ conjecture for the hyper-Kähler manifolds of the same deformation type as X . Let $l = c_1(L)$ be an algebraic isotropic class on X . By Theorem 5.1, there exist X' , $\Gamma \in \mathrm{CH}^{2n}(X \times X')$ satisfying properties 1 and 2. As $l' := \Gamma_* l$ is nef on X' , and X' satisfies the SYZ conjecture, one has by Theorem 1.2

$$L'^{n+1-i} c_{2j}(X') = 0 \text{ in } \mathrm{CH}(X') \text{ for } j \geq i.$$

As Γ is a ring isomorphism preserving Chern classes, it follows that

$$L^{n+1-i} c_{2j}(X) = 0 \text{ in } \mathrm{CH}(X),$$

for $j \geq i$, which proves Theorem 1.4.

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