Contemporary Mathematics

Hyper-Kähler compactification of the intermediate Jacobian fibration of a cubic fourfold : the twisted case

Claire Voisin

Pour Lawrence, avec beaucoup d'estime et d'amitié

Abstract. The starting point of this note is our recent paper with Laza and Saccà constructing deformations of O'Grady's 10-dimensional hyper-Kähler manifolds as compactifications of intermediate Jacobian fibrations associated to cubic fourfolds. The note provides a complement to that paper consisting in the analogous construction in the twisted case, leading to isogenous but presumably not isomorphic or birational hyper-Kähler manifolds.

1. Introduction

Hyper-Kähler geometry is a geometry of a very restricted type which is part of the more general setting of K-trivial compact Kähler geometry. The existence of hyper-Kähler manifolds rests on Yau's theorem [**28**]. Hyper-Kähler manifolds are complex manifolds of even complex dimension $2n$ with a Ricci-flat Kähler metric and parallel everywhere nondegenerate holomorphic 2-form. Forgetting about the metric, the complex manifolds one obtains can always be deformed to projective complex manifolds. Hodge theory plays a major role in the deformation theory of these complex manifolds. In fact they are not only locally but also globally determined determined by their period point (see [**3**], [**25**]), namely the de Rham cohomology class of the closed holomorphic 2-form. It is also remarkable that studying the period map for these manifolds led Beauville and Bogomolov to the discovery of the so-called Beauville-Bogomolov quadratic form, whose existence is their most striking topological property. The situation concerning the construction and classification of deformation types of hyper-Kähler manifolds is very strange: Two infinite series are known (see [**3**]), each one having one type for each even dimension, and furthermore two sporadic (families of) examples in dimension 6 and 10 were constructed by O'Grady ([**22**], [**23**]). Another strange feature of the theory is the following: the simplest kyper-Kähler manifolds are $K3$ surfaces, with particular examples constructed as Kummer surfaces, hence associated with abelian

1991 *Mathematics Subject Classification.* Primary 14J99, 14D06; Secondary 14K99.

Key words and phrases. hyper-Kähler manifolds, cubic fourfolds, Lagrangian fibrations.

surfaces or 2-dimensional complex tori. There are many different ways of associating to $K3$ surfaces or abelian surfaces higher dimensional hyper-Kähler manifolds with $b_2 = 23$ (and also 24 for the O'Grady examples) built as (desingularizations of) moduli spaces of simple sheaves on them. Unfortunately algebraic $K3$ surfaces have only 19 parameters, while these algebraic hyper-Kähler manifolds have their deformation spaces (as polarized manifolds) of dimension $b_2 - 3 > 19$, so the general one does not come from an (algebraic) $K3$ surface. Very curiously, cubic hypersurfaces in $\mathbb{P}^5$, which have 20 parameters, have also been very much used for the construction of various 20 parameters families of algebraic hyper-Kähler manifolds. This is well-understood and even expected Hodge-theoretically, but rather unexpected geometrically. In fact the variation of Hodge structure on the cohomology of degree 4 of a cubic fourfold exactly looks like the variation of Hodge structure on the degree 2 cohomology of a polarized hyper-Kähler manifold with $b_2 = 23$. We will describe several instances of these constructions in Section 2.2. The most recent such construction has been provided in [**16**] and we will achieve in Sections 3 and 4 a twisted variant of that construction. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. Let $U \subset B := (\mathbb{P}^5)^*$ be the open set parametrizing smooth hyperplane sections $Y \subset X$. The family of intermediate Jacobians $J(Y_t)_{t\in U}$ is a smooth projective fibration $\pi_U : \mathcal{J}_U \to U$ which according to [**9**] has a nondegenerate closed holomorphic 2-form making the fibration Lagrangian. The following is the main result of [**16**]:

Theorem 1.1. *There exists a flat projective fibration $\pi : \overline{\mathcal{J}} \to B$ extending π_U, such that the total space $\overline{\mathcal{J}}$ is smooth and hyper-Kähler. Furthermore, $\overline{\mathcal{J}}$ is a deformation of a* 10*-dimensional O'Grady hyper-Kähler manifold.*

We can be slightly more precise, introducing the open set $U_1 \subset B$ parametrizing at worst 1-nodal hyperplane sections of X. The motivation for introducing U_1 is the fact that $\operatorname{codim}(B \setminus U_1 \subset B) \geq 2$, and this is a key point in the strategy of [**16**]. The family of intermediate Jacobians has a standard extension $\mathcal{J}_{U_1} \to U_1$ over U_1, first as a family of quasiabelian schemes $\mathcal{J}^\circ_{U_1} \to U_1$, and then by applying the Mumford compactification: the fibers of $\pi_{U_1} : \mathcal{J}^\circ_{U_1} \to U_1$ are $\mathbb{C}^*$-bundles over the four-dimensional intermediate Jacobians $J(\widetilde{Y}_t)$ for $t \in U_1 \setminus U$, and the Mumford compactification is obtained by compactifying the $\mathbb{C}^*$-bundle to a $\mathbb{P}^1$-bundle with the 0- and ∞-sections glued via a translation. The compactified hyper-Kähler manifold $\overline{\mathcal{J}}$ is in fact a compactification of $\mathcal{J}_{U_1}$.

The intermediate Jacobian fibration $\mathcal{J}_U$ has a twisted version $\mathcal{J}^T_U$ (which appears in [**27**] and plays an important role there, although it is not defined very carefully). There are several ways of understanding it (see Section 3). The set of points in the fiber of $\mathcal{J}^T_U$ over a point $t \in U$ identifies with the set of 1-cycles of degree 1 in the fiber Y_t modulo rational equivalence (see Section 3). We will construct in Section 3 $\mathcal{J}^T_U$ as an algebraic variety (a torsor over $\mathcal{J}_U$) and a natural extension $\mathcal{J}^T_{U_1}$ of $\mathcal{J}^T_U$ over U_1 which is étale (or analytically) locally isomorphic to $\mathcal{J}_{U_1}$ over U_1, thus getting a twisted version of $\mathcal{J}_{U_1}$. Note that $\mathcal{J}^T_{U_1}$ carries a nondegenerate closed holomorphic 2-form, for exactly the same reasons $\mathcal{J}_{U_1}$ does. The goal of this note is to prove the following twisted analogue of Theorem 1.1:

Theorem 1.2. *Let X be general cubic fourfold. There exists a flat projective fibration $\pi^T : \overline{\mathcal{J}}^T \to B$ extending $\pi^T_{U_1} : \mathcal{J}^T_{U_1} \to U_1$, such that the total space $\overline{\mathcal{J}}^T$ is smooth and hyper-Kähler.*

The conclusion above holds for general X, that is, all cubic fourfolds parametrized by a certain Zariski open set of the space of all smooth cubic fourfolds. When the cubic carries a degree 4 integral Hodge class which restricts to the generator of $H^4(Y_t,\mathbb{Z})$ on its hyperplane sections, but is still general in the sense that it belongs to this Zariski open set, which happens along a countable dense union of hypersurfaces in the moduli space, the two fibrations $\mathcal{J}_U$ and $\mathcal{J}_U^T$ are isomorphic over U and the two varieties $\overline{\mathcal{J}}^T$ and $\overline{\mathcal{J}}$ are thus birational. It is not clear that they are then isomorphic, but Huybrechts [**13**] shows that they are deformation equivalent. In particular the varieties $\overline{\mathcal{J}}^T$ are deformation equivalent to OG10 manifolds. One may wonder if the varieties $\overline{\mathcal{J}}^T$ and $\overline{\mathcal{J}}$ are birational for very general, nonspecial X. It is clear that they are not birational as Lagrangian fibrations, since one has a rational section, while the other does not have a rational section. It is likely that the varieties we are considering have a unique Lagrangian fibration so in fact are not isomorphic or even birational, but we have not pursued this.

Thanks. *I am happy to thank Radu Laza and Giulia Saccà with whom I had many interesting discussions at the origin of this note.*

2. Deforming and constructing hyper-Kähler manifolds

2.1. Deformation theory and the period map. The Bogomolov-Tian-Todorov theorem says that a compact Kähler manifold X with trivial canonical bundle has unobstructed deformations. This means that a first order deformation of the complex structure of X, which is given by an element of $H^1(X,T_X)$ called the Kodaira-Spencer class, see [**26**, 9.1.2], extends to a deformation of arbitrarily large order, and in fact there exists in this case a universal family $\mathcal{X}\to B$ where B is a ball in $H^1(X,T_X)\cong\mathbb{C}^N$, $\mathcal{X}$ is a complex manifold and ϕ is smooth proper holomorphic with central fiber $\mathcal{X}_0\cong X$ such that the Kodaira-Spencer map $T_{B,0}\to H^1(X,T_X)$ (the classifying map for first order deformations) is an isomorphism.

Assume now that X is hyper-Kähler. It is a general fact that the fibers $\mathcal{X}_t$ for t small are still hyper-Kähler: indeed, the close fibers are still Kähler as small deformations of a compact Kähler manifold, and the holomorphic 2-form still exists on $\mathcal{X}_t$ for small t because the Hodge numbers $h^{p,q}(X):=\dim H^{p,q}(X)$, $H^{p,q}(X)=H^q(X,\Omega_X^p)$, are constant under a deformation of compact Kähler manifolds. Ehresmann's fibration theorem tells us that the family $\phi:\mathcal{X}\to B$ is $\mathcal{C}^\infty$ trivial and in particular topologically trivial: $\mathcal{X}\overset{homeo}{\cong}\mathcal{X}_0\times B$. In particular, we have canonical identifications

$$H^2(\mathcal{X}_t,\mathbb{C})\cong H^2(\mathcal{X}_0,\mathbb{C})=H^2(X,\mathbb{C}). \tag{2.1}$$

The period map $\mathcal{P}$ associates to $t\in B$ the class $[\sigma_t]$ of the closed holomorphic 2-form σ_t on $\mathcal{X}_t$, seen as an element of $H^2(X,\mathbb{C})$ via the isomorphism (2.1). Note that σ_t is defined up to a multiplicative coefficient, hence $[\sigma_t]$ is well-defined only in $\mathbb{P}(H^2(X,\mathbb{C}))$. Hence the period map takes value in $\mathbb{P}(H^2(X,\mathbb{C}))$. It is a general result due to Griffiths that the period map is holomorphic. Furthermore the computation of its differential shows in our case that the period map is an immersion. Note that $\dim H^1(X,T_X)=\dim H^1(X,\Omega_X)=b_2(X)-2$, as Hodge theory provides the Hodge decomposition $H^2(X,\mathbb{C})=H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$ with $H^{1,1}(X)\cong H^1(X,\Omega_X)$, the two other spaces being 1-dimensional. It follows that

$\mathcal{P}(B) \subset \mathbb{P}(H^2(X,\mathbb{C}))$ is a germ $\mathcal{D}_{loc}$ of analytic hypersurface. One can use it (see [**3**], [**12**]) to construct the Beauville-Bogomolov quadratic form q on $H^2(X,\mathbb{Q})$.

THEOREM 2.1. *(see [**3**]) Let X be a hyper-Kähler manifold of dimension $2n$. There exists a nondegenerate quadratic form q of signature $(3, b_2(X)-3)$ on $H^2(X,\mathbb{Q})$ which is defined up to a multiplicative coefficient by the following condition: for some positive rational number λ, one has*

$$\int_X \alpha^{2n} = \lambda q(\alpha)^n \tag{2.2}$$

for any $\alpha \in H^2(X,\mathbb{Q})$.

This form is usually normalized in such a way that it takes integral values on $H^2(X,\mathbb{Z})$ and is not divisible as an intersection form on $H^2(X,\mathbb{Z})$; it is then uniquely determined. Although Beauville gives an explicit formula for q, the existence of q follows directly from the study of the period map. Indeed, we observe that classes $[\sigma_t] \in \mathcal{D}_{loc}$ satisfy $[\sigma_t]^{n+1} = 0$ in $H^2(X,\mathbb{C})$. Indeed, they are classes of holomorphic 2-forms σ_t on (some deformation X_t of) X, and clearly $\sigma_t^{n+1} = 0$ as a form on X. It follows that the hypersurface $\mathcal{H}_{2n}$ of degree $2n$ in $\mathbb{P}(H^2(X,\mathbb{C}))$ which is defined by the degree $2n$ homogeneous form f given by the formula $f(\alpha) = \int_X \alpha^{2n}$ contains $\mathcal{D}_{loc}$ and has multiplicity at least n along it, hence has a component of multiplicity $\geq n$, which is not a hyperplane. One then concludes that this component is a quadric hypersurface Q and that $\mathcal{D}_{loc}$ is open in Q. Thus the equation q of the quadric Q and f are related by $f = \lambda q^n$ for some coefficient λ. Finally, using the fact that f is rational, one sees that both λ and q can be taken to be rational. The statement concerning the signature of q follows from the Hodge index theorem and further identities derived from (2.2).

The Verbitsky Torelli theorem for marked hyper-Kähler manifolds involves the integral structure on cohomology. It says the following:

THEOREM 2.2. *([**25**], [**14**]) Let X, X' be two hyper-Kähler manifolds which are deformation equivalent. Assume there is an isomorphism $\phi : H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ which is obtained by transporting cohomology along a path of deformations from X to X' and such that $\phi([\sigma_X]) = [\sigma_{X'}]$. Then X and X' are birationally equivalent. If furthermore ϕ sends one Kähler class on X to a Kähler class on X', X and X' are isomorphic.*

2.2. Constructing hyper-Kähler manifolds. One particularity of hyper-Kähler geometry is the fact that although their deformation theory has an analytic and transcendental character, all known examples have been constructed by means of algebraic geometry. Note again that algebraic geometry provides constructions for the underlying complex manifolds, but of course not for the hyper-Kähler metrics. In dimension 2, hyper-Kähler manifolds are $K3$ surfaces (see [**29**]). These surfaces are all obtained by deforming smooth quartic surfaces in $\mathbb{P}^3$, defined by one degree 4 homogeneous equation $f(X_0,\ldots,X_3)$. The projective dimension of the space of such polynomials is 34, while $\mathrm{Aut}\,\mathbb{P}^3 = PGl(4)$ has dimension 15, so that the family of isomorphism classes of quartic $K3$ surfaces has dimension 19. This is a general fact of hyper-Kähler geometry. The number $h^{1,1}$ in this case is equal to 20 and this is the dimension of the space of all deformations of the $K3$ surface S. The deformations of S as a quartic surface are restricted since these $K3$ surfaces carry a holomorphic line bundle L such that $c_1(L)^2 = 4$. The class

$c_1(L)$ is a Hodge class, that is integral of type $(1,1)$, and equivalently it is integral and satisfies $q(c_1(L),\sigma_S)=0$. Note that the Beauville-Bogomolov form q is in this case the intersection pairing on middle cohomology. The general situation will be similar: given a hyper-Kähler manifold X equipped with an ample line bundle L, the polarized deformations of X, that is the deformations of the pair (X,L) consisting of a complex manifold and a holomorphic line bundle on it, have $h^{1,1}(X)-1$ parameters, and are parameterized by the period domain $\mathcal{D}_l\subset\mathcal{D}$ determined by the condition $q(\sigma,l)=0$ where $l=c_1(L)$.

In dimension 4, there are two known topological types of hyper-Kähler manifolds, namely $S^{[2]}$ and $K_2(A)$, where S is a $K3$ surface and A is a 2-dimensional abelian surface (or complex torus). The second punctual Hilbert scheme $X^{[2]}$ is defined for any algebraic variety or complex manifold X as the set of length 2 subschemes of X, which consist either of two distinct points of X or a point with a tangent vector. If X is smooth, this is the desingularization of the symmetric product $X^{(2)}$ obtained by blowing-up the diagonal. The generalized Kummer variety $K_2(A)$ is a fourfold obtained as follows: using the group structure of A, the smooth variety (or complex manifold) $A^{[3]}$ parametrizing length 3 subschemes of A admits the sum morphism $A^{[3]}\to A$, and $K_2(A)$ is any fiber of this morphism. These two examples appear in [**3**], as their next generalizations: It is a general theorem due to Fogarty [**10**] that the n-th Hilbert scheme $S^{[n]}$ is smooth for any n and any smooth surface S. For $n\leq 3$, the result is true without any assumptions on the dimension because length 3 subschemes are supported on smooth surfaces. Beauville shows that $S^{[n]}$ and $K_n(A)\subset A^{[n+1]}$ are hyper-Kähler manifolds of dimension $2n$ for S a $K3$ surface, A a 2-dimensional complex torus. The holomorphic 2-form on $S^{[n]}$ comes from the holomorphic 2-form σ_S on S by descending the $(2,0)$-form $\sum_i pr_i^*\sigma_S$ on S^n, which is invariant under the action of the symmetric group $\mathfrak{S}_n$, to the smooth part of the quotient $S^{(n)}=S^n/\mathfrak{S}_n$ and then showing that it extends to a $(2,0)$-form on the desingularization $S^{[n]}$.

Apart from these two series of examples, only two other deformation types are known, of respective dimensions 6 and 10, and they have been constructed by O'Grady [**22**], [**23**]. The 10-dimensional example is constructed as follows: Consider the moduli space of simple coherent sheaves $\mathcal{E}$ of rank 2 on a $K3$ surface S, satisfying $\det\mathcal{E}=\mathcal{O}_S$ (equivalently $c_1(\mathcal{E})=0$ in $H^2(S,\mathbb{Z})$), and $\deg c_2(\mathcal{E})=4$. By results of Mukai [**19**], this is a smooth algebraic variety with an everywhere nondegenerate closed holomorphic (in fact algebraic) 2-form. The variety is quasiprojective and admits a natural projective completion which is a moduli space of semistable sheaves (with respect to a given polarization). The later is singular along the locus of nonstable objects (for example $\mathcal{I}_z\oplus\mathcal{I}_{z'}$, where z and z' are two subschemes of length 2 in S). O'Grady constructs a hyper-Kähler desingularization of this singular moduli space. This variety has $b_2=24$, hence its algebraic deformations have 21 parameters. Those constructed starting from an algebraic $K3$ surface have 19 parameters.

Cubic fourfolds, that is smooth cubic hypersurfaces in $\mathbb{P}^5$, play an unexpected role in this study. Although they are Fano varieties, hence have a priori nothing to do with hyper-Kähler varieties, the Hodge decomposition on their degree 4 cohomology takes the form

$$H^4(X,\mathbb{C})=H^{3,1}(X)\oplus H^{2,2}(X)\oplus H^{1,3}(X),$$

with $\dim H^{3,1}(X) = \dim H^{1,3}(X) = 1$. Furthermore, the space $H^{2,2}$ has dimension 21, and it contains one Hodge (in fact algebraic) class, namely h^2, $h = c_1(\mathcal{O}_X(1))$. This Hodge structure is polarized by the intersection form $\langle\, ,\, \rangle_X$ on the middle degree cohomology $H^4(X,\mathbb{Z})$ (which will thus play the role of the Beauville-Bogomolov form). This notion of polarization of the Hodge structure says that the line $H^{3,1}(X) \subset H^4(X,\mathbb{C})$ is isotropic for $\langle\, ,\, \rangle_X$, and futhermore there is a (open) sign condition saying in our case that $\langle \alpha, \overline{\alpha}\rangle_X < 0$ for $0 \neq \alpha \in H^{3,1}(X)$. Thus up to shift of bigrading by $(1,1)$, (and change of sign for the quadratic form), what we get is a polarized Hodge structure of hyper-Kähler type. Furthermore, the cubic has 20 parameters (this is the dimension of the projective space of cubic homogeneous polynomials in 6 variables minus the dimension of $PGl(6)$) and in fact the period map $t \mapsto H^{3,1}(X_t) \subset H^4(X_t,\mathbb{C}) = H^4(X,\mathbb{C})$ is a local isomorphism to the polarized period domain $\mathcal{D}_{h^2}$ which as before is an open set in the quadric defined by the Poincaré intersection pairing on $H^4(X,\mathbb{Z})^{\perp h^2}$. The cubic is there only to give us a polarized variation of Hodge structure of hyper-Kähler type with 20 parameters. Surprisingly enough, there are many hyper-Kähler manifolds associated with a cubic fourfold, which have their variation of rational Hodge structure isomorphic to the one described above on the degree 4 cohomology of the cubic (with a shift of degree). The known examples are:

(1) The variety of lines of a cubic fourfold X [**4**]. This is a hyper-Kähler fourfold $F(X)$ and the incidence correspondence $P \subset F(X) \times X$ (which is the universal $\mathbb{P}^1$-bundle over $F(X)$) induces an isomorphism $P^* : H^4(X,\mathbb{Z}) \cong H^2(F(X),\mathbb{Z})$ of Hodge structures. Beauville and Donagi show that for some special cubic fourfolds X (more precisely "Pfaffian" cubic fourfolds), $F(X)$ becomes isomorphic to $S^{[2]}$ for some $K3$ surface S.

(2) The variety of rational curves of degree 3 in X [**17**]. This is a $\mathbb{P}^2$-bundle on a variety which is the blow-up of a hyper-Kähler 8-fold $Z(X)$ along a Lagrangian embedding of X in $Z(X)$. Again, the variations of Hodge structures on $H^4(X)$ and $H^2(Z(X))$ coincide, using the fact that up to a unininteresting summand, the H^2 of $Z(X)$ and $F_3(X)$ coincide, and then using the universal correspondence between $F_3(X)$ and X. This variety $F(X)$ has been shown in [**1**] to be of the same deformation type as $S^{[4]}$, where S is a $K3$ surface. Another proof of this last statement has been also given in [**15**].

(3) The intermediate Jacobian fibration of a cubic fourfold has been already mentioned in the introduction. This is a quasiprojective holomorphically symplectic 10-fold. It is proved in [**16**] that it admits a hyper-Kähler compactification $\overline{\mathcal{J}}$, and furthermore that $\overline{\mathcal{J}}$ is deformation equivalent to O'Grady's 10-dimensional varieties. One remark is that contrarily to the previous cases where we exhibited a complete family of projective hyper-Kähler manifolds, (hence a family with b_2-3 parameters), the expected dimension for the O'Grady examples should be 21 while our family has 20 parameters. This is due to the fact that the varieties that we construct are by definition Lagrangian fibered over $\mathbb{P}^5$. Hence their Picard number is at least 2, containing one class pulled-back from $\mathbb{P}^5$ and one ample class, and in fact it has to be generally equal to 2, since the family we construct has 20 parameters and the manifolds have $b_2 = 24$.

The rest of this paper is devoted to the description of the twisted version of (3), giving rise to a second 20 parameters family of deformations of O'Grady 10-dimensional varieties.

Remark 2.3. The cubic fourfold is not the only Fano variety X which has associated hyper-Kähler manifolds whose variation of Hodge structure on H^2 is isomorphic (up to a shift of degree) to the variation of Hodge structure on some cohomology group of X. Another example can be found in [**8**], where X is a Plücker hyperplane section of the Grassmannian $G(3, V_{10})$ of 3-dimensional vector subspaces of a given 10-dimensional vector space V_{10}, so that X is defined by a general element s of $\bigwedge^3 V_{10}^*$. It is proved in *loc. cit.* that the fourfold $K_s \subset G(6, V_{10})$ consisting of 6-dimensional vector subspaces of V_{10} on which s vanishes identically is a hyper-Kähler fourfold, with VHS on H^2_{prim} isomorphic to the VHS on $H^{20}(X)_{prim}$.

3. The twisted intermediate Jacobian fibration: smooth and 1-nodal case

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, $B = (\mathbb{P}^5)^\vee$ be the set of hyperplane sections of X, and let $\mathcal{Y} \subset B \times X$ be the universal hypersurface. Let $U \subset U_1 \subset B$ be the Zariski open sets of B parametrizing smooth, resp. 1-nodal, hyperplane sections of X. The reason for introducing U_1 is, as in [**16**] which we follow closely, the fact that $B \setminus U_1$ has codimension 2 in B, so that the flat fibration with smooth, hence normal, total space $\overline{\mathcal{J}}^T$ will be determined (see Section 4) by its restriction $\mathcal{J}^T_{U_1}$ over U_1.

As we mentioned in the introduction, the twisted intermediate Jacobian fibration $\mathcal{J}^T_U$ over U can be interpreted as parametrizing 1-cycles of degree 1 in the fibers of the universal (smooth) hypersurface $u : \mathcal{Y}_U \to U$ over U. If we want to describe it as an abstract torsor over $\mathcal{J}_U$ or even better as a complex manifold, it can be defined by considering Deligne-Beilinson cohomology along the fibers of $u : \mathcal{Y}_U \to U$. This gives an exact sequence of sheaves of groups on U

$$0 \to \mathcal{J}_U \to \mathcal{H}^4_{\mathcal{D}} \xrightarrow{c} R^4 u_* \mathbb{Z} \to 0,$$

where the sheaf $R^4 u_* \mathbb{Z}$ is canonically isomorphic to $\mathbb{Z}$. We can then define $\mathcal{J}^T_U$ as $c^{-1}(1)$ (note that we identify here the analytic group fibration and its sheaf of holomorphic sections). The definition above is not good to understand the extension $\mathcal{J}^T_{U_1}$ and does not describe $\mathcal{J}^T_U$ as an algebraic object. We will thus give here an alternative description of $\mathcal{J}^T_{U_1}$ actually as a twist of $\mathcal{J}_{U_1}$. Our goal in this section is to establish the following result:

Proposition 3.1. *There exists a quasiprojective variety $\mathcal{J}^T_{U_1}$ with a projective morphism $\pi^T_{U_1} : \mathcal{J}^T_{U_1} \to U_1$ which has the following properties.*

(1) The family $\mathcal{J}^T_{U_1}$ is étale locally isomorphic to $\mathcal{J}_{U_1}$ over U_1.

(2) Let $f : U' \to U$ be a base change, with U' smooth, and assume that there is a codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{Y}_{U'})$ that has degree 1 in the fibers of $\mathcal{Y}_{U'} \to U'$. Then there is a canonical section $U' \to \mathcal{J}^T_{U'}$ of the fibration $\mathcal{J}^T_{U'} := f^* \mathcal{J}^T_U \to U'$. Equivalently, there is a morphism $\Phi_{\mathcal{Z}} : U' \to \mathcal{J}^T_U$ over U.

More precisely, the morphism $\Phi_{\mathcal{Z}}$ is in fact compatible with the Abel-Jacobi map in the sense that for two points u', $u'' \in U'$ with $f(u') = f(u'') = u$, $\Phi_{\mathcal{Z}}(u') = \Phi_{\mathcal{Z}}(u'') + \Phi_{\mathcal{Y}_u}(\mathcal{Z}_{u'} - \mathcal{Z}_{u''})$, where $\Phi_{\mathcal{Y}_u}$ is the Abel-Jacobi map of $\mathcal{Y}_u$. This last fact is automatic, due to the universal property of the Abel-Jacobi map (see [**21**]). The proof of the proposition will be done through a few lemmas.

To start with, we observe that $\mathcal{J}_{U_1}$ contains a Zariski open set $\mathcal{J}^\circ_{U_1}$, which is a group scheme over U_1 and differs from $\mathcal{J}_{U_1}$ only over $U_1 \setminus U$. Over $U_1 \setminus U$, the fibers of $\mathcal{J}_{U_1}^\circ$ are quasiabelian varieties, and more precisely, the fiber $\mathcal{J}_{U_1,t}^\circ$ over $t \in U_1 \setminus U$ is a $\mathbb{C}^*$-bundle over the intermediate Jacobian $J(\widetilde{\mathcal{Y}_t})$, where $\widetilde{\mathcal{Y}_t}$ is the desingularization of $\mathcal{Y}_t$ obtained by blowing-up the node. Denoting by o_t the singular point of $\mathcal{Y}_t$, the fiber $\mathcal{J}_{U_1,t}^\circ$ can be understood in terms of algebraic cycles as the group of 1-cycles homologous to 0 in $\mathcal{Y}_t \setminus \{o_t\}$ modulo rational equivalence relative to o_t, that is modulo the cycles of the form $\operatorname{div}\phi$, where ϕ is a rational function on $W \subset \mathcal{Y}_t$ which is well defined and invertible near o_t if $o_t \in W$ (see [**18**], [**5**, Lecture 3]). The fibers $\mathcal{J}_{U_1,t}$ of the compactified Jacobian fibration $\mathcal{J}_{U_1}$ are obtained as the Mumford compactification [**20**], obtained by replacing the $\mathbb{C}^*$-bundle mentioned above by the corresponding $\mathbb{P}^1$-bundle with the sections 0 and ∞ glued via a translation of the base $J(\widetilde{\mathcal{Y}_t})$. The translation is given by an element $\eta \in J(\widetilde{\mathcal{Y}_t})$ and the $\mathbb{C}^*$-bundle is given by its class $\eta' \in \operatorname{Pic}^0 J(\widetilde{\mathcal{Y}_t})$; the two classes η and η' must be identified via the (principal) Theta divisor of $J(\widetilde{\mathcal{Y}_t})$. This is indeed the condition that this compactification admits an ample divisor which is the limit of the Theta divisors on the smooth fibers $J(\mathcal{Y}_s)$. (The reader will find in [**2**], [**7**] the geometric interpretation of this class in the case of the family of intermediate Jacobians associated to a Lefschetz degeneration of threefolds.) Notice that the (sheaf of holomorphic sections of) the analytic group scheme $\mathcal{J}_{U_1}^\circ$ is given by the formula

$$\mathcal{J}_{U_1}^\circ = \mathcal{H}^{1,2}/R^3u_{1*}\mathbb{Z}, \tag{3.1}$$

where $u_1 : \mathcal{Y}_{U_1} \to U_1$ is the universal hypersurface over u_1, (thus, by Picard-Lefschetz theory, $R^3u_{1*}\mathbb{Z} = j_{1*}H^3_{\mathbb{Z}}$ is the natural extension of the local system $H^3_{\mathbb{Z}} = R^3u_*\mathbb{Z}$ existing on U,) and $\mathcal{H}^{1,2}$ is the Deligne extension of the Hodge bundle

$$\mathcal{H}^{1,2}_U = R^1u_*\Omega^2_{\mathcal{Y}_U/U}$$

existing on U. Let us prove the following lemma:

Lemma 3.2. *The group scheme $\mathcal{J}_{U_1}^\circ$ acts (over U_1) on the compactification $\mathcal{J}_{U_1}$.*

Proof. One easy way to prove this is to observe that we have a family $\pi_U : \mathcal{J}_U \to U$ of principally polarized varieties with a canonical Theta divisor $\Theta_U \subset \mathcal{J}_U$ inducing a relative isomorphism

$$\mathcal{J}_U \cong \operatorname{Pic}^0(\mathcal{J}_U/U).$$

Then $\mathcal{J}_{U_1}^\circ$ identifies to $\operatorname{Pic}^0(\mathcal{J}_{U_1/U_1})$ and by uniqueness, the compactification $\mathcal{J}_{U_1}$ identifies with the Mumford compactification of $\operatorname{Pic}^0(\mathcal{J}_{U_1/U_1})$ parameterizing rank 1 torsion free sheaves on fibers of $\pi_{U_1} : \mathcal{J}_{U_1} \to U_1$. The action over U_1 of $\mathcal{J}_{U_1}^\circ$ on $\mathcal{J}_{U_1}$ is then the action of Pic^0 on sheaves by tensor product. $\square$

Formula (3.1) shows that $R^3u_{1*}(\mathbb{Z}/3\mathbb{Z}) \subset \mathcal{J}_{U_1}^\circ$ identifies with the sheaf ${}_3\mathcal{J}_{U_1}^\circ$ of 3-torsion points and Lemma 3.2 shows that $\mathcal{J}_{U_1}^\circ \subset \mathcal{A}ut\,(\mathcal{J}_{U_1}/U_1)$. Let us now exhibit the twisting class in $H^1_{et}(U_1, \mathcal{A}ut\,(\mathcal{J}_{U_1}/U_1))$ needed to construct $\mathcal{J}_{U_1}^T$. The twisting class will be of 3-torsion and more precisely will come from a class in $H^1_{et}(U_1,\ {}_3\mathcal{J}_{U_1}^\circ) = H^1(U_1,\ {}_3\mathcal{J}_{U_1}^\circ)$, where in the right hand side we consider cohomology with respect to the usual topology. Consider the morphism $u_1 : \mathcal{Y}_{u_1} \to U_1$. As they admit at worst one node, the fibers of u_1 satisfy $H^4(\mathcal{Y}_t, \mathbb{Z}) = \mathbb{Z}$ with generator at the point t the class of a line (not passing through the node of $\mathcal{Y}_t$ if $\mathcal{Y}_t$ is singular). Let $h = c_1(\mathcal{O}_X(1))$ and $h_{\mathcal{Y}} \in H^2(\mathcal{Y}, \mathbb{Z})$ be its pull-back to $\mathcal{Y}$. The image

$\widetilde{h^2_{\mathcal{Y}}}$ of the class $h^2_{\mathcal{Y}}$ in $H^0(U_1, R^4u_{1*}\mathbb{Z})$ is equal to 3 times the above generator of $R^4u_{1*}\mathbb{Z} = \mathbb{Z}$. The twisting class comes from the lack of degeneracy at E_2 of the Leray spectral sequence of u_1 with integral coefficients.

Lemma 3.3. *Let $\sigma \in H^0(U_1, R^4u_{1*}\mathbb{Z})$ be the natural generator. Then the image $d_2\sigma \in H^2(U_1, R^3u_{1*}\mathbb{Z})$ is of 3-torsion, and comes from a canonically defined element*

$$t \in H^1(U_1, R^3u_{1*}(\mathbb{Z}/3\mathbb{Z})).$$

Proof. The class $d_2\sigma \in H^2(U_1, R^3u_{1*}\mathbb{Z})$ is of 3-torsion because $3\sigma = \widetilde{h^2_{\mathcal{Y}}}$ is the image of $h^2_{\mathcal{Y}} \in H^4(\mathcal{Y}_{U_1}, \mathbb{Z})$ in $H^0(U_1, R^4u_{1*}\mathbb{Z})$. The exact sequence

$$0 \to \mathbb{Z} \stackrel{3}{\to} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

of constant sheaves on $\mathcal{Y}_{U_1}$ induces the exact sequence

$$0 \to R^3u_{1*}\mathbb{Z} \stackrel{3}{\to} R^3u_{1*}\mathbb{Z} \to R^3u_{1*}(\mathbb{Z}/3\mathbb{Z}) \to 0 \tag{3.2}$$

because $R^3u_{1*}\mathbb{Z}$ and $R^4u_{1*}\mathbb{Z}$ have no torsion. The long exact sequence associated to (3.2) thus shows that $d_2\sigma$ lifts to an element of $H^1(U_1, R^3u_{1*}(\mathbb{Z}/3\mathbb{Z})) = H^1(U_1, {}_3\mathcal{J}^\circ_{U_1})$. This does not prove however that this lift is canonical since in the considered long exact sequence of Betti cohomology, there is a nontrivial cohomology group $H^1(U_1, R^3u_{1*}\mathbb{Z})$ (this is due to the non-triviality of $H^4(X, \mathbb{Z})$). We can however go around this difficulty by considering the universal situation where instead of considering the universal family $\mathcal{Y}_{U_1}$ of smooth or 1-nodal hyperplane sections of the given cubic fourfold X, we consider the universal family $v_1 : \mathcal{Y}^{univ}_{W_1} \to W_1$ of smooth or 1-nodal cubic threefolds contained in some hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. Here the whole parameter space B_{univ} is the projective bundle over $(\mathbb{P}^5)^\vee$ with fiber over $[H]$ the projective space $\mathbb{P}(H^0(\mathcal{O}_H(3)))$ and $W_1 \subset B_{univ}$ is a Zariski open set in it. In this case, we have

Sublemma 3.4. *One has $H^1(W_1, R^3v_{1*}\mathbb{Z}) = 0$.*

Proof. One proves the result first of all with $\mathbb{Q}$-coefficients, using the fact that the Leray spectral sequence degenerates in E_2 and that the degree 4 cohomology of $\mathcal{Y}^{univ}_{W_1}$ is algebraic and very simple. Then one concludes using the exact sequence on W_1

$$0 \to R^3v_{1*}\mathbb{Z} \to R^3v_{1*}\mathbb{Q} \to R^3v_{1*}(\mathbb{Q}/\mathbb{Z}) \to 0 \tag{3.3}$$

and the fact that $H^0(W_1, R^3v_{1*}(\mathbb{Q}/\mathbb{Z})) = 0$. □

This fact gives us a canonical lift of $d_2\sigma^{univ}$ to an element

$$t^{univ} \in H^1(W_1, R^3v_{1*}(\mathbb{Z}/3\mathbb{Z})),$$

hence by pull-back to U_1 (using the natural map $g : U_1 \to W_1$), the desired canonical lift t of $d_2\sigma$ to an element of $H^1(U_1, R^3u_{1*}(\mathbb{Z}/3\mathbb{Z}))$. □

Lemma 3.3 gives us a class in $H^1(U_1, {}_3\mathcal{J}_{U_1}) = H^1_{et}(U_1, {}_3\mathcal{J}_{U_1})$ which by Lemma 3.2 allows us to construct a twisted family $\pi_{U_1} : \mathcal{J}^T_{U_1} \to U_1$, which is above U a torsor over the group scheme $\mathcal{J}_U$. Note that the proof of Lemma 3.3 also shows that $\mathcal{J}^T_{U_1}$ is the pull-back under $g_1 : U_1 \to W_1$ of the corresponding object $\mathcal{J}^{T,univ}_{W_1}$ over W_1. We now prove that the torsor so constructed is actually the object we want, namely, the target of the Abel-Jacobi map for 1-cycles of degree 1 along the smooth fibers of u. This will follow from the following result: Let as before

$W_1 \subset B_{univ}$ be the Zariski open set parameterizing cubic threefolds in $\mathbb{P}^5$ with at most one ordinary double point. Let $o_1 : \mathcal{F}_1^{univ} \to W_1$ be the family of lines in the fibers of $v_1 : \mathcal{Y}_{W_1}^{univ} \to W_1$ and $\mathcal{F}_1^{0,univ} \subset \mathcal{F}_1^{univ}$ be the Zariski open set consisting of lines not passing through the node when the corresponding cubic threefold is singular. Let $o_{1,0}$ be the restriction of o_1 to $\mathcal{F}_1^0$.

Lemma 3.5. *There exists an isomorphism*

$$o_{1,0}^*\mathcal{J}_{W_1}^{univ} \cong o_{1,0}^*\mathcal{J}_{W_1}^{T,univ} \tag{3.4}$$

of quasi-projective varieties over $\mathcal{F}_1^{0,univ}$.

Proof. It suffices to show that the pull-back $o_{1,0}^*(t)$ vanishes in the cohomology group $H^1(\mathcal{F}_1^{0,univ}, o_{1,0}^*({}_3\mathcal{J}_{W_1}^{univ}))$. Note that the morphism $o_{1,0}$ is smooth, because the family of lines in the smooth locus of a cubic hypersurface is smooth. It follows that the total space of the universal family

$$\mathcal{Y}_{\mathcal{F}_1^{0,univ}}^{univ} := \mathcal{Y}_{W_1}^{univ} \times_{W_1} \mathcal{F}_1^{0,univ}$$

is smooth. Let $v_1' : \mathcal{Y}_{\mathcal{F}_1^{0,univ}}^{univ} \to \mathcal{F}_1^{0,univ}$ be the natural morphism. Let $H_{\mathbb{Z}}^4$, $H_{\mathbb{Z}}^3$, $H_{\mathbb{Z}/3\mathbb{Z}}^3$ be respectively the sheaves

$$R^4v'_{1*}\mathbb{Z} = o_{1,0}^*(R^4v_{1*}\mathbb{Z}),\ R^3v'_{1*}\mathbb{Z} = o_{1,0}^*(R^3v_{1*}\mathbb{Z}),\ R^3v'_{1*}(\mathbb{Z}/3\mathbb{Z}) = o_{1,0}^*(R^3v_{1*}(\mathbb{Z}/3\mathbb{Z}))$$

on $\mathcal{F}_1^{0,univ}$. The class

$$o_{1,0}^*\sigma \in H^0(\mathcal{F}_1^{0,univ}, o_{1,0}^*(R^4v_{1*}\mathbb{Z})) = H^0(\mathcal{F}_1^{0,univ}, H_{\mathbb{Z}}^4)$$

comes from a class in $H^4(\mathcal{Y}_{\mathcal{F}_1^{0,univ}}^{univ}, \mathbb{Z})$, namely the class $[\Delta^{univ}]$ of the universal line $\Delta^{univ} \subset \mathcal{Y}_{\mathcal{F}_1^{0,univ}}^{univ}$. We thus get that $d_2(o_{1,0}^*\sigma) = 0$ in $H^2(\mathcal{F}_1^{0,univ}, H_{\mathbb{Z}}^3)$. This means that the class

$$o_{1,0}^*(t) \in H^1(\mathcal{F}_1^{0,univ}, H_{\mathbb{Z}/3\mathbb{Z}}^3) = H^1(\mathcal{F}_1^{0,univ},\ {}_3\mathcal{J}_{\mathcal{F}_1^0}^{univ})$$

vanishes in $H^2(\mathcal{F}_1^{0,univ}, H_{\mathbb{Z}}^3)$ (using as before the long exact sequence associated to the short exact sequence

$$0 \to H_{\mathbb{Z}}^3 \to H_{\mathbb{Z}}^3 \to H_{\mathbb{Z}/3\mathbb{Z}}^3 \to 0.$$

In order to prove that $o_{1,0}^*(t)$ vanishes in $H^1(\mathcal{F}_1^{0,univ},\ {}_3\mathcal{J}_{\mathcal{F}_1^0}^{univ})$, it thus suffices to show that $H^1(\mathcal{F}_1^{0,univ}, o_{1,0}^*H_{\mathbb{Z}}^3) = 0$, which is done exactly as before. □

Remark 3.6. The isomorphism between the pullbacks to $\mathcal{F}_1^{0,univ}$ of the twisted and the untwisted families given in Lemma 3.5 is not canonical since over $\mathcal{F}_W^{univ}$ the intermediate Jacobian fibration has a nonzero section. Indeed, for each cubic smooth cubic threefold Y with a line $\Delta \subset Y$, the 1-cycle $3\Delta - h^2$ is cohomologous to zero on Y, hence has a nontrivial Abel-Jacobi invariant. This provides a nontrivial section which acts by translation on $o^*\mathcal{J}_W^{univ}$ over $\mathcal{F}_W$. One can show that the isomorphism is unique up to the action of the group generated by this translation.

Proof of Proposition 3.1. We already constructed $\mathcal{J}_{U_1}^T$ as the pull-back via $g_1 : U_1 \to W_1$ of the twisted family $\mathcal{J}_{W_1}^T$. That it is étale locally isomorphic to $\mathcal{J}_{U_1}$ over U_1 is by construction. The only thing to prove is point (2). However, Lemma 3.5 shows that for any base change morphism $f : U' \to U$, assuming there is a family of lines $\Delta_{U'} \subset \mathcal{Y}_{U'}$, there is a canonical morphism $\Phi_\Delta : U' \to \mathcal{J}_U^T$ over

U. Indeed, the data above give a morphism $c : U' \to \mathcal{F}_W^{univ}$ so we can use Φ_Δ^{univ} of the previous lemma, and define $\Phi_\Delta := \Phi_{\Delta^{univ}} \circ c$. To see that a family of 1-cycles of degree 1 in the fibers of $\mathcal{Y}_{U'} \to U'$ in fact suffices to trivialize the twisted Jacobian fibration, we can use the universal generation result of [**24**] which says that any 1-cycle on a smooth cubic hypersurface Y over a field K comes from a 0-cycle on the surface of lines $F(Y)$, also defined over K. In our case, K is the function field of U'. Using this result, we get a correspondence over U' between U' and $\mathcal{F}_{U'}$, which has to be of degree 1. Using this correspondence and the previous step, we construct the section $U' \to \mathcal{J}^T_{U'}$. □

4. Descent

Our goal in this section is to explain how to mimic the arguments from [**16**] in order to construct for a general cubic fourfold X a hyper-Kähler compactification of the twisted intermediate Jacobian fibration $\mathcal{J}^T_{U_1}$ constructed in the previous section. Recall the notion of very good line in a cubic threefold Y, introduced and used in [**16**] (see also [**6**] for a slightly weaker notion): A line $\Delta \subset Y$ is very good if Δ does not pass through the singular point of Y (equivalently the surface of lines $F(Y)$ is smooth at $[\Delta]$), the curve $\widetilde{C}_\Delta$ of lines in Y meeting Δ is irreducible and the natural involution acting on $\widetilde{C}_\Delta$ has no fixed points. One of the results proved in [**16**] (improving previous results of [**6**]) is:

Proposition 4.1. *If X is a general cubic 4-fold, and $Y \subset X$ is any hyperplane section, Y has a very good line.*

Let $o_{vg} : \mathcal{F}^{vg} \to B$ be the family of very good lines in the fibers of $u : \mathcal{Y} \to B$. Proposition 4.1 says that o_{vg} is surjective and by definition it is smooth. Restricting over U_1, we get a Zariski open set

$$\mathcal{F}_1^{vg} \subset \mathcal{F}_1^0,\ o_{vg,1} : \mathcal{F}_1^{vg} \to U_1$$

with its pulled-back intermediate Jacobian fibration $o^*_{vg,1}\mathcal{J}_{U_1} := \mathcal{F}_1^{vg} \times_{U_1} \mathcal{J}_{U_1}$, which is also isomorphic to $o^*_{vg,1}\mathcal{J}^T_{U_1}$ by Lemma 3.5. In [**16**], a smooth quasiprojective compactification $\pi_{\mathcal{F}^{vg}} : \overline{\mathcal{P}}_{\mathcal{F}^{vg}} \to \mathcal{F}^{vg}$ of $o^*_{vg,1}\mathcal{J}_{U_1} \to \mathcal{F}_1^{vg}$ is constructed, with a morphism $\pi_{\mathcal{F}^{vg}}$ which is flat and projective. The compactified intermediate Jacobian fibration $\overline{\mathcal{J}}$ is then obtained by descending the fibration $\pi_{\mathcal{F}^{vg}}$. The statement which makes it possible is:

Lemma 4.2. *There exists a line bundle $\mathcal{O}(\Theta_1)$ on $\mathcal{J}_{U_1}$ whose pull-back $o^*_{vg,1}\mathcal{O}(\Theta_1)$ to $o^*_{vg,1}\mathcal{J}_{U_1}$ extends uniquely to a relatively ample line bundle $\mathcal{O}(\Theta^{vg})$ on $\overline{\mathcal{P}}_{\mathcal{F}^{vg}}$.*

Note that the extension of the pull-back $o^*_{vg,1}\mathcal{O}(\Theta_1)$ to a line bundle on $\overline{\mathcal{P}}_{\mathcal{F}^{vg}}$ exists and is unique because $\overline{\mathcal{P}}_{\mathcal{F}^{vg}}$ is smooth and $o^*_{vg,1}\mathcal{J}_{U_1} \subset \overline{\mathcal{P}}_{\mathcal{F}^{vg}}$ is Zariski open with complement of codimension at least 2. This last point follows from the flatness of $\pi_{\mathcal{F}^{vg}}$ because $\mathcal{F}_1^{vg} \subset \mathcal{F}^{vg}$ is Zariski open with complement of codimension ≥ 2. The important point in Lemma 4.2 is thus relative ampleness. Once we have the line bundle $\mathcal{O}(\Theta_1)$ on $\mathcal{J}_{U_1}$ as in the lemma, we get the formula defining $\overline{\mathcal{J}}$ as a Proj over B, namely, letting $j_1 : U_1 \to B$ be the natural inclusion, we set

$$\overline{\mathcal{J}} = Proj\,(\oplus_k j_{1*}(R^0\pi_{U_1*}\mathcal{O}(k\Theta_1))). \tag{4.1}$$

Using the fact that $B \setminus U_1$ has codimension 2 in B, flatness of $\pi_{\mathcal{F}^{vg}}$ and relative ampleness of (the extension of) the pull-back of Θ_1, we see that the sheaf of algebras $\oplus_k j_{1*}(R^0\pi_{U_1*}\mathcal{O}(k\Theta_1))$ is a sheaf of finitely generated algebras over $\mathcal{O}_B$ whose graded pieces are locally free and that the Proj is smooth, because all these properties become true after pull-back to $\mathcal{F}^{vg}$, thanks to the existence of the flat compactification $\pi_{\mathcal{F}^{vg}}$. Indeed, the sheaf of algebras $\oplus_k j_{1*}(R^0\pi_{U_1*}\mathcal{O}(k\Theta_1))$ pulls-back to $\oplus_k R^0\pi_{\mathcal{F}^{vg}*}\mathcal{O}(k\Theta^{vg})$ on $\mathcal{F}^{vg}$. (To be completely rigorous here, we should in fact replace the line bundle Θ_1 by a multiple.)

The way this descent construction works also makes clear what ingredient we need in order to treat the twisted case. Indeed, what we are going to do is to descend the same fibration $\pi_{\mathcal{F}^{vg}} : \overline{\mathcal{P}}_{\mathcal{F}^{vg}} \to \mathcal{F}^{vg}$ to a flat fibration $\pi^T : \overline{\mathcal{J}}^T \to B$ with a different descent data, given by a different relatively ample divisor. More precisely, consider $\pi_{\mathcal{F}^{vg}} : \overline{\mathcal{P}}_{\mathcal{F}^{vg}} \to \mathcal{F}^{vg}$ as a relative flat projective compactification of $o^*_{vg,1}\mathcal{J}^T_{U_1} \to \mathcal{F}^{vg}_1$ using Lemma 3.5. The only ingredient needed is the following:

PROPOSITION 4.3. *There exists a line bundle $\mathcal{O}(\Theta^{vg,T})$ on $\overline{\mathcal{P}}_{\mathcal{F}^{vg}}$, which is relatively ample over $\mathcal{F}^{vg}$, and whose restriction to $\mathcal{F}^{vg}_1$ is the pull-back (using the isomorphism (3.4)) of a line bundle $\mathcal{O}(\Theta^T_1)$ on $\mathcal{J}^T_{U_1}$.*

Indeed, once one has the line bundles $\mathcal{O}(\Theta^{vg,T})$ and $\mathcal{O}(\Theta^T_1)$ as above, one defines $\overline{\mathcal{J}}^T$ as

$$\overline{\mathcal{J}}^T = Proj\,(\oplus_k j_{1*}(R^0\pi_{U_1*}\mathcal{O}(k\Theta^T_1))) \tag{4.2}$$

and the same arguments as above show that this is a smooth projective variety, flat over B and extending $\mathcal{J}^T_{U_1}$.

PROOF OF PROPOSITION 4.3. It is proved in [**16**, Section 4] that, X being general, the fibers of $\pi_{\mathcal{F}_{vg}} : \overline{\mathcal{P}}_{\mathcal{F}^{vg}} \to \mathcal{F}^{vg}$ are irreducible. We now use the following lemma:

LEMMA 4.4. *Let M be a smooth irreducible quasiprojective variety, $f : M \to N$ be a flat projective morphism with irreducible fibers M_t, $\forall t \in N$, and let $\mathcal{L} \in \operatorname{Pic} M$. If $\mathcal{L}_{|M_t}$ is topologically trivial for the general point $t \in N$, $\mathcal{L}_{|M_t}$ is numerically trivial for all $t \in N$.*

PROOF. The statement is local on N. Let $t_0 \in N$ and let $C \subset M_{t_0}$ be a curve. Choosing a sufficiently relatively ample line bundle H on M and using the fact that M_{t_0} is irreducible, we can assume that C is contained in an irreducible surface $S_{t_0} \subset M_{t_0}$ which is a complete intersection of members of $|H_{|M_{t_0}}|$. Up to replacing H by a multiple and shrinking N if necessary, we can construct a flat family $f_S : S \to N$, $S \subset M$ of complete intersection surfaces with fiber S_{t_0} over the point t_0. As the line bundle $\mathcal{L}_{|M_t}$ is topologically trivial for general t, we conclude that

$$c_1(\mathcal{L}) \cdot c_1(H) \cdot S_t = 0,\ c_1(\mathcal{L})^2 \cdot S_t = 0$$

for general t, and by flatness, this is also true for S_{t_0}. Let $\tau : \widetilde{S_{t_0}} \to S_{t_0}$ be a desingularization. The line bundle $\mathcal{L}'_0 := \tau^*\mathcal{L}_{|S_{t_0}}$ on the smooth connected surface $\widetilde{S_{t_0}}$ satisfies

$$c_1(\mathcal{L}'_0)^2 = 0,\, c_1(\mathcal{L}'_0) \cdot c_1(\tau^*H) = 0.$$

As $c_1(\tau^*H)^2 > 0$, it follows from the Hodge index theorem that $\mathcal{L}'_0$ is topologically trivial modulo torsion on $\widetilde{S_{t_0}}$. In particular, if $C' \subset \widetilde{S_{t_0}}$ is a curve mapping onto $C \subset S_{t_0}$, we have $\deg \mathcal{L}'_{0|C'} = 0 = D\deg \mathcal{L}_{|C}$, where D is the degree of C' over C. Thus $\deg \mathcal{L}_{|C} = 0$. $\square$

Let now Θ_1^T be any relatively ample line bundle on $\mathcal{J}_{U_1}^T \to U_1$. Its pull-back to $\mathcal{J}_{\mathcal{F}_1^{vg}}^T \cong \mathcal{J}_{\mathcal{F}_1^{vg}}$ extends to a line bundle $\mathcal{O}(\Theta^{vg,T})$ on the compactified family of Prym varieties $\pi_{\mathcal{F}^{vg}} : \overline{\mathcal{P}}_{\mathcal{F}^{vg}} \to \mathcal{F}^{vg}$. We also have on $\mathcal{J}_{\mathcal{F}_1^{vg}}$ the pull-back of the relatively ample line bundle Θ_1 on $\mathcal{J}_{U_1} \to U_1$. The later extends (in fact uniquely) to a relatively ample line bundle $\mathcal{O}(\Theta^{vg})$ on $\overline{\mathcal{P}}_{\mathcal{F}^{vg}} \to \mathcal{F}^{vg}$. Next, an easy monodromy argument shows (see [**16**, Section 5]) that the Néron-Severi group of the intermediate Jacobian of a very general cubic 3-fold is isomorphic to $\mathbb{Z}$. It follows that for adequate positive integers a, b, the line bundle $\mathcal{O}(a\Theta^{vg,T} - b\Theta^{vg})$ is topologically trivial on the general fibers of $\pi_{\mathcal{F}^{vg}}$. We then conclude by Lemma 4.4 that $\mathcal{O}(a\Theta^{vg,T} - b\Theta^{vg})$ is numerically trivial on all the fibers of $\pi_{\mathcal{F}^{vg}}$. Thus $\mathcal{O}(a\Theta^{vg,T})$ is the sum of an ample line bundle and a numerically trivial line bundle on any fiber of $\pi_{\mathcal{F}^{vg}}$, hence it is relatively ample. ☐

This concludes the construction of the smooth projective variety $\overline{\mathcal{J}}^T$. The fact that this is a hyper-Kähler manifold follows easily, as in [**16**]: The existence of the holomorphic 2-form works as in [**16**]. Indeed, according to Proposition 3.1, $\mathcal{J}_U^T$ is naturally the set of 1-cycles of degree 1 modulo rational equivalence in the fibers of the universal family $\mathcal{Y}_U \to U$, in the sense that for any smooth morphism $\phi : \mathcal{M} \to U$, and codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{M} \times_U \times \mathcal{Y}_U)$ such that $\mathcal{Z}_m$ is of degree 1 on the fiber $\mathcal{Y}_t$, $t = \phi(m)$, there exists a morphism (twisted Abel-Jacobi map) $\mathcal{M} \to \mathcal{J}_U^T$. A universal cycle, for which $\mathcal{M}$ is $\mathcal{J}_U^T$ and the map is the identity, may not exist with integer coefficients, but it always exists with rational coefficients, so we have a cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{J}_U^T \times_U \times \mathcal{Y}_U)_{\mathbb{Q}}$ with rational coefficients such that $[\mathcal{Z}]^*$ is the natural isomorphism $R^3 u_* \mathbb{Q} \cong R^1 \pi_{U*}\mathbb{Q}$. Using the natural proper map (inclusion) $\Phi : \mathcal{J}_U^T \times_U \times \mathcal{Y}_U \to \mathcal{J}_U^T \times X$, we get a codimension 3 cycle $\mathcal{Z}' = \Phi_*(\mathcal{Z}) \in \mathrm{CH}^3(\mathcal{J}_U^T \times X)_{\mathbb{Q}}$. We construct the holomorphic 2-form on $\mathcal{J}_U^T$ as

$$\sigma_U = [\mathcal{Z}']^* \alpha_X \in H^0(\mathcal{J}_U^T, \Omega^2_{\mathcal{J}_U^T}),$$

where α_X is a generator of $H^1(X, \Omega_X^3)$. For any algebraic extension W of $\mathcal{J}_U^T$, we can extend the cycle $\mathcal{Z}'$ to $W \times X$, showing that the $(2,0)$-form σ_U extends to W. In particular, we get the desired holomorphic 2-forms σ_{U_1} on $\mathcal{J}_{U_1}^T$ and σ on $\overline{\mathcal{J}}^T$. The fact that σ is nondegenerate on $\overline{\mathcal{J}}^T$ is a consequence of the fact that σ_{U_1} is nondegenerate on $\mathcal{J}_{U_1}^T$, since $B \setminus U_1$ has codimension ≥ 2 in B and $\pi : \overline{\mathcal{J}}^T \to B$ is flat. The fact that σ_{U_1} is nondegenerate on $\mathcal{J}_{U_1}^T$ follows from the similar statement for the untwisted family since they are étale locally isomorphic. Finally, the fact that the variety we construct is actually hyper-Kähler (i.e. simply connected with only one holomorphic 2-form up to a coefficient) follows from the fact that the two varieties are birational (this will also imply *a posteriori* that they are deformation equivalent) when the cubic fourfold X acquires an integral Hodge class $\alpha \in H^4(X, \mathbb{Z})$ which has degree 1 on its hyperplane sections. Note that the above construction of $\overline{\mathcal{J}}^T$ and $\overline{\mathcal{J}}$ works a priori only for general X, but the set of special X's as above is Zariski dense (this is a countable union of hypersurfaces, dense for the usual topology in the moduli space of cubic fourfolds), hence there are points which correspond to special X's in Hassett's sense (see [**11**]), with a special class of degree 1 along hyperplane sections, and for which the constructions of $\overline{\mathcal{J}}^T$ and $\overline{\mathcal{J}}$ specialize well.

Remark 4.5. The two varieties $\overline{\mathcal{J}}^T$ and $\overline{\mathcal{J}}$ are isogenous in the sense that there is a rational map (of degree 3^{10}) $\overline{\mathcal{J}}^T \dashrightarrow \overline{\mathcal{J}}$. This is obvious from the fact that the open part $\mathcal{J}_U^T$ is constructed as a torsor over the group scheme $\mathcal{J}_U^T$, with a twisting class of order 3. We believe but did not prove that the two varieties are not birational. We only note that by construction they are not birational over B.

References

[1] Th. Addington, M. Lehn. On the symplectic eightfold associated to a Pfaffian cubic fourfold, arXiv:1404.5657, 2014, to appear in Crelle.

[2] F. Bardelli. Polarized mixed Hodge structures: on irrationality of threefolds via degeneration. Ann. Mat. Pura Appl. (4) 137 (1984), 287-369.

[3] A. Beauville. Variétes Kählériennes dont la première classe de Chern est nulle, J. Differential Geom. 18 (1983), no. 4, 755782 (1984).

[4] A. Beauville, R. Donagi. La variété des droites d'une hypersurface cubique de dimension 4, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), no. 14, 703-706.

[5] S. Bloch. *Lectures on algebraic cycles*, Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., (1980).

[6] S. Casalaina-Martin, R. Laza. The moduli space of cubic threefolds via degenerations of the intermediate Jacobian, J. Reine Angew. Math. 633 (2009), 29-65.

[7] A. Collino. A cheap proof of the irrationality of most cubic threefolds, Boll. Un. Mat. Ital. B (5) 16 (1979), no. 2, 451-465.

[8] O. Debarre, C. Voisin. Hyper-Kähler fourfolds and Grassmann geometry, J. reine angew. Math. 649 (2010), 63-87.

[9] R. Donagi, E. Markman. Cubics, integrable systems, and Calabi-Yau threefolds, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, 1996, pp. 199-221.

[10] J. Fogarty. Algebraic Families on an Algebraic Surface, Am. J. Math. 10 (1968), 511-521.

[11] B. Hassett. Special cubic fourfolds, Compositio Math. 120 (2000), no. 1, 1-23.

[12] D. Huybrechts. Compact hyperkähler manifolds. Habilitationsschrift Essen (1997).

[13] D. Huybrechts. Compact hyper-Kähler manifolds: basic results, Invent. Math. 135 (1999), no. 1, 63-113.

[14] D. Huybrechts. A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky], Astérisque (2012), no. 348, Exp. No. 1040, 375-403, Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027-1042.

[15] J. Kollár, R. Laza, G. Saccà, C. Voisin. Remarks on degenerations of hyper-Kähler manifolds, arXiv:1704.02731.

[16] R. Laza, G. Saccà, C. Voisin. A hyper-Kähler compactification of the Intermediate Jacobian fibration associated to a cubic fourfold,arXiv:1602.05534, to appear in Acta Mathematica.

[17] Ch. Lehn, M. Lehn, Ch. Sorger, and D. van Straten, Twisted cubics on cubic fourfolds, arXiv:1305.0178 (to appear in Crelle), 2013.

[18] M. Levine, C. Weibel. Zero cycles and complete intersections on singular varieties. J. Reine Angew. Math. 359 (1985), 106-120.

[19] S. Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. Math. 77 (1984), no. 1, 101-116.

[20] D. Mumford. An analytic construction of degenerating abelian varieties over complete rings. Compositio Math. 24 (1972), 239-272.

[21] J. Murre. Applications of algebraic K-theory to the theory of algebraic cycles, in *Algebraic geometry*, Sitges (Barcelona), 1983, 216-261, Lecture Notes in Math., 1124, Springer, Berlin, (1985).

[22] K. O'Grady. Desingularized moduli spaces of sheaves on a $K3$, J. Reine Angew. Math. 512 (1999), 49-117.

[23] K. O'Grady. A new six-dimensional irreducible symplectic variety, J. Algebraic Geom. 12 (2003), no. 3, 435-505.

[24] M. Shen. Rationality, universal generation and the integral Hodge conjecture, arXiv:1602.07331.

[25] M. Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. Duke Math. J. 162 (2013), no. 15, 2929-2986.
[26] C. Voisin. *Hodge theory and complex algebraic geometry* I, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2007.
[27] C. Voisin. Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal. J. Algebraic Geom. 22 (2013), no. 1, 141-174.
[28] S.-T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Amp'ere equation I. Comm. Pure and Appl. Math. 31 (1978), 339-411.
[29] *Géométrie des surfaces K3: modules et périodes*, Astérisque No. 126 (1985), Société Mathématique de France, Paris.

Collège de France, 3 rue d'Ulm, 75005 Paris FRANCE