On the universal CH_0 group of cubic hypersurfaces

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Abstract

We study the existence of a Chow-theoretic decomposition of the diagonal of a smooth cubic hypersurface, or equivalently, the universal triviality of its CH_0-group. We prove that for odd dimensional cubic hypersurfaces or for cubic fourfolds, this is equivalent to the existence of a cohomological decomposition of the diagonal, and we translate geometrically this last condition. For cubic threefolds X, this turns out to be equivalent to the algebraicity of the minimal class $\theta^4/4!$ of the intermediate Jacobian $J(X)$. In dimension 4, we show that a special cubic fourfold with discriminant not divisible by 4 has universally trivial CH_0 group.

1 Introduction

Let X be a smooth rationally connected projective variety over $\mathbb{C}$. Then $\mathrm{CH}_0(X) = \mathbb{Z}$, as all points of X are rationally equivalent. However, if L is a field containing $\mathbb{C}$, e.g. a function field, the group $\mathrm{CH}_0(X_L)$ can be different from $\mathbb{Z}$. As explained in [4], the group $\mathrm{CH}_0(X_L)$ is equal to $\mathbb{Z}$ for any field L containing $\mathbb{C}$ if and only if, for $L = \mathbb{C}(X)$, the diagonal (or generic) point δ_L is rationally equivalent over L to a constant point x_L for some (in fact any) point $x \in X(\mathbb{C})$. Following [4], we will then say that X has universally trivial CH_0 group. Observe that, on the other hand, the equality

$$\delta_L = x_L \text{ in } \mathrm{CH}_0(X_L) = \mathrm{CH}^n(X_L),\ n = \dim X$$

is, by the localization exact sequence applied to Zariski open sets of $X \times X$ of the form $U \times X$, equivalent to the vanishing in $\mathrm{CH}^n(U \times X)$ of the restriction of $\Delta_X - X \times x$, where U is a sufficiently small dense Zariski open set of X and $\Delta_X \subset X \times X$ is the diagonal of X. This provides a Bloch-Srinivas decomposition of the diagonal

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X), \tag{1}$$

where Z is supported on $D \times X$ for some proper closed subset D of X. As in [29], we will call an equality (1) a *Chow-theoretic decomposition of the diagonal.* So, having universally trivial CH_0 group is equivalent to admitting a Chow-theoretic decomposition of the diagonal, but the second viewpoint is much more geometric, and leads to the study of weakened properties, like the existence of a *cohomological decomposition of the diagonal*, which is the cohomological counterpart of (1), studied for threefolds in [29] :

$$[\Delta_X] = [X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}), \tag{2}$$

where Z is supported on $D \times X$ for some proper closed algebraic subset D of X. In these notions, integral coefficients are essential in order to make the property restrictive, as the existence of decompositions as above with rational coefficients already follows from the assumption that $\mathrm{CH}_0(X) = \mathbb{Z}$ (see [7]).

Projective space has universally trivial CH_0 group. It follows that rational or stably rational varieties admit a Chow-theoretic (and a fortiori cohomological) decomposition of the

diagonal. More generally, if X is a unirational variety admitting a unirational parametrization $\mathbb{P}^n \dashrightarrow X$ of degree N, then there is a decomposition

$$N\Delta_X = N(X \times x) + Z \text{ in } \mathrm{CH}^n(X \times X),$$

with Z supported on $D \times X$ for some $D \subsetneqq X$.

On the other hand, the existence of a decomposition of the diagonal is certainly not a sufficient condition for stable rationality, as there are surfaces of general type (hence very far from being rational or stably rational) which admit a Chow-theoretic decomposition of the diagonal (see Corollary 2.2). It could be also the case that a smooth projective variety X admits unirational parametrizations of coprime degrees N_i without being stably rational (although we do not know such examples). Nevertheless, the existence of a decomposition of the diagonal is a rather strong condition, and there are now a number of unirational examples where the non-existence provides an obstruction to rationality or stable rationality:

1) Examples of rationally connected varieties with no cohomological decomposition of the diagonal include varieties with non-trivial Artin-Mumford invariant (this is the torsion in $H^3(X, \mathbb{Z})$ or the second unramified cohomology group with torsion coefficients), or non-trivial third unramified cohomology group $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$ (see [11] for examples), as both of these groups have to be 0 when X has a cohomological decomposition of the diagonal, see [29]).

2) Examples of rationally connected varieties with no Chow-theoretic decomposition of the diagonal include varieties with non-trivial unramified cohomology groups $H^i_{nr}(X, \mathbb{Q}/\mathbb{Z})$ with $i \geq 4$, as these groups have to be 0 when X has a Chow-theoretic decomposition of the diagonal, see [12]. We refer to [22] for such examples and to [33] for the cycle-theoretic interpretation of this group in degree 4.

3) Furthermore, we proved in [28] that the non-existence of a decomposition of the diagonal is a criterion for (stable) irrationality which is actually stronger than those given by the nontriviality of unramified cohomology: For example, we show in *loc. cit.* that very general smooth quartic double solids do not admit a Chow-theoretic or even a cohomological decomposition of the diagonal while their unramified cohomology vanishes in all positive degrees. We prove similar results for very general nodal quartic double solids with $k \leq 7$ nodes and in the case of very general double solids with exactly 7 nodes, we prove in [28] that the non-existence of a cohomological decomposition of the diagonal is equivalent to the non-existence of a universal codimension 2 cycle $Z \in \mathrm{CH}^2(J(X) \times X)$, where $J(X)$ is the intermediate Jacobian of X. ($J(X)$ is also known to be isomorphic to the group $\mathrm{CH}^2(X)_{hom}$ of codimension 2 cycles homologous to 0 on X.) Thus, in this case, the study of the decomposition of the diagonal led to the discovery of new stable birational invariants which are nontrivial for some unirational varieties.

The purpose of this paper is to investigate the existence of a decomposition of the diagonal for cubic hypersurfaces. Motivations for this study are the following problems:

1) It is well-known that 3-dimensional smooth cubics are irrational (see [10]), but they are not known not to be stably rational.

2) In dimension 4, some cubics are known to be rational and it is a famous open problem to prove that there exist irrational cubic fourfolds. Some precise conjectures concerning the rationality of cubic fourfolds have been formulated and compared (see [15], [18], [2], [1]). All these conjectures develop the idea that if a cubic fourfold is rational, it is related in some way (Hodge-theoretic, categorical) to a $K3$ surface. An interesting computation has been made recently by Galkin and Shinder [13], who prove that a rational cubic fourfold has to satisfy the property that its variety of lines is birational to $\mathrm{Hilb}^2(S)$ for some $K3$ surface S, unless a certain explicitly constructed nonzero element in the Grothendieck ring of complex varieties is annihilated by the class of $\mathbb{A}^1$, that is, provides a counterexample to the cancellation conjecture for the Grothendieck ring. (Note that such counterexamples are now known to exist by the work of Borisov [9].)

In general, the existence of a cohomological decomposition is much weaker than the existence of a Chow-theoretic one. Our first result, which is unconditional for cubic fourfolds

and for odd-dimensional cubics, is the following :

Theorem 1.1. *Let X be a smooth cubic hypersurface. Assume that $H^*(X,\mathbb{Z})/H^*(X,\mathbb{Z})_{alg}$ has no 2-torsion (this holds for example if $\dim X$ is odd or $\dim X \leq 4$, or X is very general of any dimension). Then X admits a Chow-theoretic decomposition of the diagonal (equivalently, $\mathrm{CH}_0(X)$ is universally trivial) if and only if it admits a cohomological decomposition of the diagonal.*

Here $H^*(X,\mathbb{Z})_{alg} \subset H^*(X,\mathbb{Z})$ is the subgroup of classes of algebraic cycles. For any odd degree and odd dimension smooth hypersurface in projective space, the quotient group $H^*(X,\mathbb{Z})/H^*(X,\mathbb{Z})_{alg}$ has no 2-torsion. For cubic hypersurfaces, the first example where we do not know if the assumption is satisfied is 6-dimensional cubics and degree 6 integral cohomology classes on them.

The proof of Theorem 1.1 uses the fact that $\mathrm{Hilb}^2(X)$ is birationally a projective bundle over X, a property which is also crucially used in the recent paper [13].

One consequence of this result, established in Section 5, concerns the following notion:

Definition 1.2. *(i) If $Y \subset X$ is a closed algebraic subset of a variety defined over a field K, we say that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective if $\mathrm{CH}_0(Y_L) \to \mathrm{CH}_0(X_L)$ is surjective for any field L containing K.*

(ii) The essential CH_0-dimension of a variety X is the minimal integer k such that there exists a closed algebraic subset $Y \subset X$ of dimension k, such that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective.

Theorem 1.3. *The essential CH_0-dimension of a very general n-dimensional cubic hypersurface over $\mathbb{C}$ is either n or 0.*

More precisely, Theorem 5.2 proves the result above for a smooth cubic hypersurface of dimension n with no 2-torsion in $H^n(X,\mathbb{Z})/H^n(X,\mathbb{Z})_{alg}$ and such that $\mathrm{End}_{HS}H^n(X,\mathbb{Q})_{prim} = \mathbb{Q}$. In dimension 4, we get further precise consequences, for example we prove that a cubic fourfold which is special in the sense of Hassett [15], with discriminant not divisible by 4, has universally trivial CH_0 group.

The rest of the paper focuses on the cohomological decomposition of the diagonal. We first investigate the existence of a cohomological decomposition of the diagonal for varieties whose non-algebraic cohomology is supported in middle degree, like complete intersections. We prove the following result:

Theorem 1.4. *Let X be a smooth projective variety such that $H^*(X,\mathbb{Z})$ has no torsion. Assume that $H^{2i}(X,\mathbb{Z})$ is generated over $\mathbb{Z}$ by algebraic cycles for $2i \neq n = \dim X$. Then X admits a cohomological decomposition of the diagonal if and only if there exist varieties Z_i of dimension $n-2$, correspondences $\Gamma_i \in \mathrm{CH}^{n-1}(Z_i \times X)$, and integers n_i with the property that for α, $\beta \in H^n(X,\mathbb{Z})$,*

$$\sum_i n_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Z_i} = \langle \alpha, \beta \rangle_X. \tag{3}$$

Note that the condition (3) presents obvious similarities with the one considered in [23] by Shen, who studied the case of cubic fourfolds. It is however weaker in several respects: the integers n_i do not need to be positive, and the correspondences Γ_i do not need to factor through the variety of lines. Finally, the condition formulated by Shen is only conjecturally a necessary condition for rationality, while our condition is actually a necessary condition for the triviality of the universal CH_0 group, hence a fortiori for (stable) rationality.

Remark 1.5. Concerning the second assumption in Theorem 1.4, it is satisfied by uniruled threefolds by [31], but it is not clear that it is satisfied by Fano complete intersections in any dimension. The group $H^{2i}(X,\mathbb{Z})$, $2i \neq n$, is equal to $\mathbb{Z}$ by Lefschetz hyperplane restriction theorem, but Kollár [17] exhibits examples of hypersurfaces where this group is not generated by an algebraic class for $2i > n$. It is not known if such Fano examples can be constructed.

The case of rationally connected threefolds X is also particularly interesting. In this case, we complete the results of [29] by proving the following result. Let $J(X)$ be the intermediate Jacobian of X. It is isomorphic as a group to $\mathrm{CH}^2(X)_{hom}$ via the Abel-Jacobi map. It is canonically a principally polarized abelian variety, the polarization being determined by the intersection pairing on $H^3(X,\mathbb{Z})/\text{torsion} \cong H^1(J(X),\mathbb{Z})$. Let $\theta \in H^2(J(X),\mathbb{Z})$ be the class of the Theta divisor of $J(X)$.

Theorem 1.6. *(See also Theorem 4.1) Let X be a rationally connected threefold. Then X admits a cohomological decomposition of the diagonal if and only if the following three conditions are satisfied:*

1. *$H^3(X,\mathbb{Z})$ has no torsion.*
2. *There exists a universal codimension 2 cycle in $X \times J(X)$.*
3. *The minimal class $\theta^{g-1}/(g-1)!$ on $J(X)$, $\dim J(X) = g$, is algebraic, that is, the class of a 1-cycle in $J(X)$.*

The main new result in this theorem is the fact that condition 3 above is implied by the existence of a cohomological decomposition of the diagonal. In particular, it is a necessary condition for stable rationality. This can be seen as a variant of Clemens-Griffiths criterion which can be stated as saying that a necessary criterion for rationality of a threefold is the fact that the minimal class $\theta^{g-1}/(g-1)!$ on $J(X)$ is the class of an *effective* curve in $J(X)$.

Note that examples of unirational threefolds not satisfying 1 were constructed by Artin and Mumford [3], and examples of unirational threefolds not satisfying 2 were constructed in [28]. It is not known if examples not satisfying 3 exist. More generally, it is not known if there exists any principally polarized abelian variety (A,Θ) such that the minimal class $\theta^{g-1}/(g-1)!$ is not algebraic on A, where $g = \dim A$. Notice that for many Fano threefolds, the intermediate Jacobian $J(X)$ is a Prym variety, so the class $2\theta^{g-1}(g-1)!$ is known to be algebraic. In the case of cubic threefolds, the algebraicity of $\theta^4/4!$ is a classical completely open problem. Combining the theorems above, we get in this case:

Theorem 1.7. *Let X be a smooth cubic threefold. Then X has universally trivial CH_0 group if and only if the class $\theta^4/4!$ on $J(X)$ is algebraic. This happens (at least) on a countable union of closed subvarieties of codimension ≤ 3 of the moduli space of X.*

The paper is organized as follows: Theorem 1.1 is proved in Section 2. Theorem 1.4 is proved in Section 3 and Theorem 1.6 is proved in Section 4. In Section 5, we come back to the case of cubic hypersurfaces, where we prove Theorem 5.2 and establish further results, particularly in dimension 4.

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2 Chow-theoretic and cohomological decomposition of the diagonal

We prove in this section Theorem 1.1. In the case of cubic hypersurfaces of dimension ≤ 4, a shorter proof will be given, which uses the following result of independent interest:

Proposition 2.1. *Let X be a smooth projective variety. If X admits a decomposition of the diagonal modulo algebraic equivalence, that is*

$$\Delta_X - X \times x = Z \text{ in } \mathrm{CH}(X \times X)/\mathrm{alg}, \tag{4}$$

with Z supported on $D \times X$ for some closed algebraic subset $D \subsetneq X$, then X admits a Chow-theoretic decomposition of the diagonal.

Proof. We use the fact proved in [26], [27] that cycles algebraically equivalent to 0 are nilpotent for the composition of self-correspondences. We write (4) as

$$\Delta_X - X \times x - Z = 0 \text{ in } \mathrm{CH}(X \times X)/\mathrm{alg} \tag{5}$$

and apply the nilpotence result mentioned above. This provides

$$(\Delta_X - X \times x - Z)^{\circ N} = 0 \text{ in } \mathrm{CH}(X \times X) \tag{6}$$

for some large N. As $\Delta_X - X \times x$ is a projector and $Z \circ (X \times x) = 0$, this gives

$$\Delta_X - X \times x - W \circ Z = 0 \text{ in } \mathrm{CH}(X \times X), \tag{7}$$

for some cycle W on $X \times X$. As $W \circ Z$ is supported on $D \times X$, for some $D \subsetneq X$, this concludes the proof. □

Corollary 2.2. *Let S be a surface of general type with $\mathrm{CH}_0(S) = \mathbb{Z}$ and $\mathrm{Tors}(H^*(S, \mathbb{Z})) = 0$ (for example the Barlow surface ([5]). Then S has universally trivial CH_0 group.*

Proof. (See [4] for a different proof.) As $p_g(S) = q(S) = 0$ by Mumford's theorem [21], the cohomology $H^*(S, \mathbb{Z})$ is generated by classes of algebraic cycles. As $H^*(S, \mathbb{Z})$ has no torsion, the cohomology of $S \times S$ admits a Künneth decomposition with integral coefficients, so that we can write the class of the diagonal of S as

$$[\Delta_S] = \sum_i [\alpha_i] \otimes [\beta_i] \text{ in } H^4(S \times S, \mathbb{Z}), \tag{8}$$

where α_i, β_i are algebraic cycles on S with $\dim \alpha_i + \dim \beta_i = 2$. Clearly, $[\alpha_i] \otimes [\beta_i] = [\alpha_i \times \beta_i]$ is supported over $D \times S$ with $D \subsetneq S$ when $\dim \alpha_i < 2$, so that (8) provides in fact a cohomological decomposition of Δ_S:

$$[\Delta_S] = [S \times s] + [Z] \text{ in } H^4(S \times S, \mathbb{Z}), \tag{9}$$

where Z is a cycle supported over $D \times S$, for some $D \subsetneq S$. Next, as $\mathrm{CH}_0(S \times S) = \mathbb{Z}$, codimension 2 cycles on $S \times S$ which are cohomologous to 0 are algebraically equivalent to 0 by [7]. Thus the cycle $\Gamma := \Delta_S - S \times s - Z$ is algebraically equivalent to 0 on $S \times S$. The surface S thus admits a decomposition of the diagonal modulo algebraic equivalence, and we then apply Proposition 2.1. □

For a smooth projective variety X, we denote by $X^{[2]}$ the second punctual Hilbert scheme of X. It is smooth, obtained as the quotient of the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal by its natural involution. Let $\mu : X \times X \dashrightarrow X^{[2]}$ be the natural rational map and $r : \widetilde{X \times X} \to X^{[2]}$ be the quotient morphism. We start with the following result:

Lemma 2.3. *Let X be a smooth projective variety of dimension n. Then there exists a codimension n cycle Z in $X^{[2]}$ such that $\mu^* Z = \Delta_X$ in $\mathrm{CH}^n(X \times X)$.*

Proof. Let E_Δ be the exceptional divisor over the diagonal of the blow-up map $\tau : \widetilde{X \times X} \to X \times X$. The key point is the fact that there is a (non effective) divisor δ on $X^{[2]}$ such that $r^*\delta = E_\Delta$. It follows that

$$r^*\delta^n = E_\Delta^n \text{ in } \mathrm{CH}^n(\widetilde{X \times X}).$$

Now we use the fact that

$$\tau_*((-1)^{n-1} E_\Delta^n) = \Delta_X \text{ in } \mathrm{CH}^n(X \times X),$$

which gives $\mu^*((-1)^{n-1}\delta^n) = \tau_*(r^*((-1)^{n-1}\delta^n)) = \Delta_X$ in $\mathrm{CH}^n(X \times X)$. □

Corollary 2.4. *Any symmetric codimension n cycle on $X \times X$ is rationally equivalent to $\mu^*\Gamma$ for a codimension n cycle Γ on $X^{[2]}$.*

Proof. Indeed, we can write $Z = Z_1 + Z_2$ where Z_1 is a combination of irreducible subvarieties of $X \times X$ invariant under the involution i of $X \times X$, and Z_2 is of the form $Z_2' + i(Z_2')$, where the diagonal does not appear in Z_2'. Write $Z_1 = n_1\Delta_X + Z_1'$, with $Z_1' = \sum_j n_j Z_{1,j}'$, the Z_j' being invariant under i but different from Δ_X. Then $Z_{1,j}'$ is the inverse image of a subvariety $Z_{1,j}''$ of $X^{(2)}$. Let $\widetilde{Z_{1,j}''}$ be the proper transform of $Z_{1,j}''$ under the Hilbert-Chow map $X^{[2]} \to X^{(2)}$. Then clearly $\mu^*(\widetilde{Z_{1,j}''}) = Z_{1,j}'$ in $\mathrm{CH}^n(X \times X)$ so

$$\mu^*(\sum_j n_j \widetilde{Z_{1,j}''}) = Z_1' \text{ in } \mathrm{CH}^n(X \times X).$$

Next, let $\overline{Z_2'}$ be the image of Z_2' in $X^{[2]}$ by μ. Then clearly

$$\mu^*(\overline{Z_2'}) = Z_2' + i(Z_2') = Z_2 \text{ in } \mathrm{CH}^n(X \times X).$$

Thus

$$Z = n_1\Delta_X + Z_1' + Z_2 = n_1\Delta_X + \mu^*(\sum_j n_j \widetilde{Z_{1,j}''}) + \mu^*(\overline{Z_2'}).$$

Finally, we use Lemma 2.3 to conclude. □

We have next:

Lemma 2.5. *Suppose X admits a cohomological decomposition of the diagonal*

$$[\Delta_X - x \times X] = [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}), \tag{10}$$

where Z is a cycle supported on $D \times X$ for some proper closed algebraic subset D of X. Then X admits a cohomological decomposition of the diagonal

$$[\Delta_X - x \times X - X \times x] = [W] \text{ in } H^{2n}(X \times X, \mathbb{Z}), \tag{11}$$

where W is a cycle supported on $D \times X$ for some $D \subsetneqq X$, and W is invariant under the involution i.

Proof. Let us denote by ${}^t\Gamma$ the image of a cycle Γ under the involution i of $X \times X$. Formula (10) gives as well

$$[\Delta_X - X \times x] = [{}^tZ] \text{ in } H^{2n}(X \times X, \mathbb{Z}),$$

and

$$[(\Delta_X - X \times x) \circ (\Delta_X - x \times X)] = [{}^tZ \circ Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}). \tag{12}$$

The cycle ${}^tZ \circ Z$ is invariant under the involution and supported on $D \times X$. On the other hand, the left hand side in (12) is equal to $[\Delta_X - X \times x - x \times X]$ (we assume here $n > 0$). □

The following proposition is a key point in our proof of Theorem 1.1.

Proposition 2.6. *Let X be a smooth odd degree complete intersection in projective space. If X admits a cohomological decomposition of the diagonal, and $H^{2*}(X, \mathbb{Z})/H^{2*}(X, \mathbb{Z})_{alg}$ has no 2-torsion, there exists a cycle $\Gamma \in \mathrm{CH}^n(X^{[2]})$ with the following properties:*

(i) $\mu^*\Gamma = \Delta_X - x \times X - X \times x - W$ in $\mathrm{CH}^n(X \times X)$, with W supported over $D \times X$, for some closed algebraic subset $D \subsetneqq X$.

(ii) $[\Gamma] = 0$ in $H^{2n}(X^{[2]}, \mathbb{Z})$.

The proof of this proposition will use a few lemmas concerning the cohomology of $X^{[2]}$, when X is the projective space or a smooth complete intersection in projective space. We give here an elementary and purely algebraic proof. These results can be obtained as well as an application of [8] or [20] (cf. [25]), but these papers are written in a topologist's language and it is not obvious how to translate them into the concrete statements below. With a better understanding of these papers, our arguments would presumably prove Proposition 2.6 for a smooth projective variety X such that $H^*(X,\mathbb{Z})$ is torsion free and $H^{2*}(X,\mathbb{Z})/H^{2*}(X,\mathbb{Z})_{alg}$ has no 2-torsion.

Let X be a smooth projective variety and let

$$j_{E,X} : E_{\Delta,X} \hookrightarrow X^{[2]},\ i_{E,X} : E_{\Delta,X} \to \widetilde{X \times X},\ \tau_{E,X} : E_{\Delta,X} \to X,$$

be respectively the inclusion of the exceptional divisor over the diagonal in $X^{[2]}$, its inclusion in $\widetilde{X \times X}$, and its natural morphism to X. Let $\delta \in \operatorname{Pic} X^{[2]}$ be the natural divisor such that $2\delta = E$. Then $\delta_E := \delta_{|E_{\Delta,X}}$ is the line bundle $\mathcal{O}_{E_{\Delta,X}}(-1)$ of the projective bundle $\tau_{E,X} : E_{\Delta,X} \cong \mathbb{P}(T_X) \to X$.

Lemma 2.7. *Assume $X = \mathbb{P}^n$ and let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. For any cohomology class $a \in H^*(E_{\Delta,\mathbb{P}^n},\mathbb{Z})$, one has $(j_{E,\mathbb{P}^n})_* a \in 2H^{*+2}((\mathbb{P}^n)^{[2]},\mathbb{Z})$ if and only if*

$$a = \sum_{i\geq 0, m\geq 0} \alpha_{i,m} \delta_E^i \cdot \tau_{E,\mathbb{P}^n}^* h^m \cdot (\tau_{E,\mathbb{P}^n}^* h - \delta_E)^m \text{ mod. } 2H^*(E_{\Delta,\mathbb{P}^n},\mathbb{Z}), \tag{13}$$

where the $\alpha_{i,m}$ are integers.

Proof. To see that the condition is sufficient (and also to get a nice interpretation of the condition), we observe that for any smooth subvariety $\Sigma \subset \mathbb{P}^n$ of codimension m, we have the inclusion $\Sigma^{[2]} \subset (\mathbb{P}^n)^{[2]}$ and the class $b := [\delta \cdot \Sigma^{[2]}] \in H^{4m+2}((\mathbb{P}^n)^{[2]},\mathbb{Z})$ satisfies $2b = [E_{\Delta,\mathbb{P}^n} \cdot \Sigma^{[2]}]$, so that

$$2b = (j_{E,\mathbb{P}^n})_*([\Sigma^{[2]}]_{|E_{\Delta,\mathbb{P}^n}}) \text{ in } H^{4m+2}((\mathbb{P}^n)^{[2]},\mathbb{Z}).$$

Next we observe that $\Sigma^{[2]} \cap E_{\Delta,\mathbb{P}^n}$ is equal to $\mathbb{P}(T_\Sigma) \subset \mathbb{P}(T_{\mathbb{P}^n})$. Take now for Σ a $\mathbb{P}^{n-m} \subset \mathbb{P}^n$. Then the class of $\mathbb{P}(T_\Sigma) \subset \mathbb{P}(T_{\mathbb{P}^n})$ is $\tau_{E,\mathbb{P}^n}^* h^m \cdot (\tau_{E,\mathbb{P}^n}^* h - \delta_E)^m$. We thus proved that $(j_{E,\mathbb{P}^n})_*(\tau_{E,\mathbb{P}^n}^* h^m \cdot (\tau_{E,\mathbb{P}^n}^* h - \delta_E)^m)$ is divisible by 2 in $H^*((\mathbb{P}^n)^{[2]},\mathbb{Z})$, and so is the class

$$(j_{E,\mathbb{P}^n})_*(\delta_E^i \cdot \tau_{E,\mathbb{P}^n}^* h^m \cdot (\tau_{E,\mathbb{P}^n}^* h - \delta_E)^m) = \delta^i \cdot (j_{E,\mathbb{P}^n})_*(\tau_{E,\mathbb{P}^n}^* h^m \cdot (\tau_{E,\mathbb{P}^n}^* h - \delta_E)^m)$$

for any $i \geq 0$.

In the other direction, it is better to see $(\mathbb{P}^n)^{[2]}$ as a $\mathbb{P}^2$-bundle over the Grassmannian $G(2,n+1)$, namely, if $\pi_1 : P \to G(2,n+1)$ is the universal $\mathbb{P}^1$-bundle, it is clear that $(\mathbb{P}^n)^{[2]}$ is isomorphic to the second symmetric product $\pi_2 : P_2 \to G(2,n+1)$ of P over $G(2,n+1)$. Furthermore, $E_{\Delta,\mathbb{P}^n} \subset (\mathbb{P}^n)^{[2]}$ identifies with the Veronese embedding $P \subset P_2$. Write $P = \mathbb{P}(\mathcal{E})$ with polarization $H = \tau_{E,\mathbb{P}^n}^* h$; then $P_2 = \mathbb{P}(S^2\mathcal{E})$ with polarization H_2, and clearly

$$j_{E,\mathbb{P}^n}^* H_2 = 2H \text{ in } H^2(P,\mathbb{Z}) \tag{14}$$

since $j_{E,\mathbb{P}^n} : P \to P_2$ is the Veronese embedding. The cohomology of P decomposes as

$$H^*(P,\mathbb{Z}) = \pi_1^* H^*(G(2,n+1),\mathbb{Z}) \oplus H \cdot \pi_1^* H^{*-2}(G(2,n+1),\mathbb{Z}).$$

We claim that modulo 2, the set of classes a as in (13) is exactly $\pi_1^* H^*(G(2,n+1),\mathbb{Z}/2\mathbb{Z})$. Let $l = c_1(\mathcal{E})$, $c = c_2(\mathcal{E})$ be the two generators of $H^*(G(2,n+1),\mathbb{Z})$. It is easy to check that $\delta = H_2 - \pi_2^* l$ in $H^2(P_2,\mathbb{Z})$. Restricting this equality to P, we get by (14)

$$\delta_E = \pi_1^* l \text{ mod. } 2H^2(P,\mathbb{Z}). \tag{15}$$

Finally we have $\pi_1^* c = H \cdot (\pi_1^* l - H)$ in $H^*(P, \mathbb{Z})$, hence we get by (15)

$$\pi_1^* c = \tau_{E,\mathbb{P}^n}^* h \cdot (\tau_{E,\mathbb{P}^n}^* h - \delta_E) \text{ mod. } 2H^*(P, \mathbb{Z}), \tag{16}$$

which together with (15) proves the claim. Having this, the fact that condition (13) is sufficient tells us that

$$(j_{E,\mathbb{P}^n})_* \circ \pi_1^*(H^*(G(2, n+1), \mathbb{Z})) \subset 2H^*(P_2, \mathbb{Z})$$

and in order to prove that it is necessary, we need to prove that the set of classes $z \in H^*(P, \mathbb{Z})$ such that $(j_{E,\mathbb{P}^n})_* z \in 2H^*(P_2, \mathbb{Z})$ is equal to $\pi_1^*(H^*(G(2, n+1), \mathbb{Z}))$ modulo $2H^*(P, \mathbb{Z})$. Equivalently, we have to prove that if $z \in H^*(G(2, n+1), \mathbb{Z})$ satisfies $(j_{E,\mathbb{P}^n})_*(H \cdot \pi_1^* z) \in 2H^*(P_2, \mathbb{Z})$, then $z \in 2H^*(G(2, n+1), \mathbb{Z})$. But for any $\alpha \in H^*(G(2, n+1), \mathbb{Z})$, we have

$$\langle \alpha, z \rangle_{G(2,n+1)} = \langle \pi_1^* \alpha, H \cdot \pi_1^* z \rangle_P = \langle \pi_2^* \alpha, (j_{E,\mathbb{P}^n})_*(H \cdot \pi_1^* z) \rangle_{P_2},$$

which implies that $\langle \alpha, z \rangle_{G(2,n+1)}$ is even since $(j_{E,\mathbb{P}^n})_*(H \cdot \pi_1^* z)$ is divisible by 2. Hence z is divisible by 2 by Poincaré duality on $G(2, n+1)$. □

Lemma 2.8. *Let $X \subset \mathbb{P}^N$ be a smooth odd degree complete intersection of dimension n.*

(i) For any integer $m \geq 0$, the class $(j_{E,X})_*(\tau_{E,X}^* h^m \cdot (\tau_{E,X}^* h - \delta_E)^m) \in H^{4m+2}(X^{[2]}, \mathbb{Z})$ is equal to $2\delta \cdot [\Sigma_l^{[2]}]$, where $\Sigma_m \subset X$ is the smooth proper intersection of X with a linear space of codimension m in $\mathbb{P}^N$.

(ii) For any integral cohomology class $a \in H^{2n-2}(E_{\Delta,X}, \mathbb{Z})$, one has $(j_{E,X})_* a \in 2H^{2n}(X^{[2]}, \mathbb{Z})$ if and only if

$$a = \sum_{i \geq 0, m \geq 0, i+2m=n-1} \alpha_{i,m} \delta_E^i \cdot \tau_{E,X}^* h^m \cdot (\delta_E - \tau_{E,X}^* h)^m \text{ modulo } 2H^*(E_{\Delta,X}, \mathbb{Z}), \tag{17}$$

where the $\alpha_{i,m}$'s are integers.

Proof. (i) This has been already proved in the case of $\mathbb{P}^N$ and follows from the fact that $\tau_{E,X}^* h^m \cdot (\tau_{E,X}^* h - \delta_E)^m \in H^{4m}(E_{\Delta,X}, \mathbb{Z})$ is the class of $\mathbb{P}(T_{\Sigma_m}) = E_{\Delta,\Sigma_m}$ in $\mathbb{P}(T_X) = E_{\Delta,X}$.

(ii) In the case where $X \overset{j_X}{\hookrightarrow} \mathbb{P}^N$ has odd dimension, observe that the map $j_{X*} : H^*(X, \mathbb{Z}/2) \to H^{*+2k}(\mathbb{P}^N, \mathbb{Z}/2)$, $k := N - n$, is injective since X has odd degree. It follows as well that denoting by $\tilde{j}_X : \mathbb{P}(T_X) \to \mathbb{P}(T_{\mathbb{P}^N})$ the natural map, $\tilde{j}_{X*} : H^*(\mathbb{P}(T_X), \mathbb{Z}/2) \to H^{*+4k}(\mathbb{P}(T_{\mathbb{P}^N}), \mathbb{Z}/2)$ is also injective. Now, let

$$a \in H^{2n-2}(E_{\Delta,X}, \mathbb{Z}) = H^{2n-2}(\mathbb{P}(T_X), \mathbb{Z})$$

such that $(j_{E,X})_* a \in 2H^{2n}(X^{[2]}, \mathbb{Z})$. Then $\tilde{j}_{X*} a \in H^*(E_{\Delta,\mathbb{P}^N}, \mathbb{Z})$ satisfies $(j_{E,\mathbb{P}^N})_*(\tilde{j}_{X*} a) \in 2H^*((\mathbb{P}^N)^{[2]}, \mathbb{Z})$. Thus we conclude by Lemma 2.7 that the class $\tilde{j}_{X*} a$ mod. 2 belongs to the subgroup of $H^*(E_{\Delta,\mathbb{P}^N}, \mathbb{Z}/2)$ generated by the $\delta_E^i \cdot \tau_{E,\mathbb{P}^n}^* h^m \cdot (\delta_E - \tau_{E,\mathbb{P}^n}^* h)^m$ with $i \geq 0$, $m \geq 0$. It easily follows that a mod. 2 belongs to the subgroup of $H^*(E_{\Delta,X}, \mathbb{Z}/2)$ generated by the $\delta_E^i \cdot \tau_{E,X}^* h^m \cdot (\delta_E - \tau_{E,X}^* h)^m$ with $i \geq 0$, $l \geq 0$, since the class of $E_{\Delta,X}$ in $E_{\Delta,\mathbb{P}^N}$ is equal modulo 2 to $\tau_{E,\mathbb{P}^n}^* h^{N-n} \cdot (\delta_E - \tau_{E,\mathbb{P}^n}^* h)^{N-n}$.

In the case where X has even dimension, the maps $j_{X*} : H^n(X, \mathbb{Z}/2) \to H^{n+2k}(\mathbb{P}^N, \mathbb{Z}/2)$ and $\tilde{j}_{X*} : H^{2n-2}(\mathbb{P}(T_X), \mathbb{Z}/2) \to H^{2n-2+4k}(\mathbb{P}(T_{\mathbb{P}^n}), \mathbb{Z}/2)$ are no longer injective, but their kernels are equal respectively to $H^n(X, \mathbb{Z}/2)_{prim}$ and $\delta_E^{n/2-1} \cdot \tau_{E,X}^* H^n(X, \mathbb{Z}/2)_{prim}$. The proof above shows that if $a \in H^{2n-2}(E_{\Delta,X}, \mathbb{Z})$ satisfies $(j_{E,X})_* a \in 2H^{2n}(X^{[2]}, \mathbb{Z})$, then $a \in \delta_E^{n/2-1} \cdot \tau_{E,X}^* H^n(X, \mathbb{Z}/2)_{prim}$ modulo the subgroup of $H^*(E_{\Delta,X}, \mathbb{Z}/2)$ generated by the $\delta_E^i \cdot \tau_{E,X}^* h^m \cdot (\delta_E - \tau_{E,X}^* h)^m$ with $i \geq 0$, $m \geq 0$. By (i), it thus suffices to prove that a class $a \in \delta_E^{n/2-1} \cdot \tau_{E,X}^* H^n(X, \mathbb{Z}/2)_{prim}$ satisfying $(j_{E,X})_* a = 0$ in $H^{2n}(X^{[2]}, \mathbb{Z}/2)$ must be 0. This is proved as follows: By a monodromy argument, either any class $a \in$

$\delta_E^{n/2-1} \cdot \tau_{E,X}^* H^n(X, \mathbb{Z}/2)_{prim}$ satisfies this property, or no nonzero class satisfies it. Assume that the first possibility occurs. Then we get a contradiction as follows: Let

$$a = \delta_E^{n/2-1} \cdot \tau_{E,X}^* \alpha,\ b = \delta_E^{n/2-1} \cdot \tau_{E,X}^* \beta \in H^{2n-2}(E_{\Delta,X}, \mathbb{Z}),$$

with α, $\beta \in H^n(X, \mathbb{Z})_{prim}$. Then

$$\langle j_{\Delta,X*}a, j_{\Delta,X*}b\rangle_{X^{[2]}} = -2\langle \alpha, \beta\rangle_X. \tag{18}$$

If both $(j_{\Delta,X})_*\alpha$ and $(j_{\Delta,X})_*\beta$ are divisible by 2 in $H^{2n}(X^{[2]}, \mathbb{Z})$, then $\langle (j_{\Delta,X})_*a, (j_{\Delta,X})_*b\rangle_{X^{[2]}}$ is divisible by 4, hence $\langle \alpha, \beta\rangle_X$ is divisible by 2 by (18). Thus, under our assumption, the intersection pairing modulo 2 would be identically 0 on $H^n(X, \mathbb{Z}/2)_{prim}$. As X has odd degree, the intersection pairing is nondegenerate on $H^n(X, \mathbb{Z}/2)_{prim}$ and we get a contradiction. □

Proof of Proposition 2.6. Let X be an odd degree complete intersection in $\mathbb{P}^N$ which admits a cohomological decomposition of the diagonal. We know by Lemma 2.5 that there is a symmetric cycle W supported on $D \times X$, $D \subsetneq X$, such that $[\Delta_X - x \times X - X \times x] = [W]$ in $H^{2n}(X \times X)$. Corollary 2.4 provides a cycle $\Gamma_0 \in \mathrm{CH}^n(X^{[2]})$ such that $\mu^*\Gamma_0 = \Delta_X - x \times X - X \times x - W$ in $\mathrm{CH}^n(X \times X)$, which is property (i). It remains to see that we can modify Γ_0 keeping property (i) and imposing condition (ii), namely

$$[\Gamma_0] = 0 \text{ in } H^{2n}(X^{[2]}, \mathbb{Z}).$$

We know that $\mu^*[\Gamma_0] = 0$ in $H^{2n}(X \times X, \mathbb{Z})$, which implies that $r^*[\Gamma_0]$ vanishes in $H^{2n}(\widetilde{X \times X} \setminus E_\Delta, \mathbb{Z})$. Thus there is a cohomology class $\beta \in H^{2n-2}(E_\Delta, \mathbb{Z})$ such that

$$i_{E*}\beta = r^*[\Gamma_0] \text{ in } H^{2n}(\widetilde{X \times X}, \mathbb{Z}),$$

where we come back to the notation $i_E : E_\Delta \to \widetilde{X \times X}$, $j_E : E_\Delta \to X^{[2]}$ for the natural inclusions of the exceptional divisor over the diagonal of X. This implies that $j_{E*}\beta$ is divisible by 2 in $H^{2n}(X^{[2]}, \mathbb{Z})$. Indeed, we have

$$j_{E*}\beta = r_*(i_{E*}\beta) = r_*(r^*([\Gamma_0])) = 2[\Gamma_0] \text{ in } H^{2n}(X^{[2]}, \mathbb{Z}). \tag{19}$$

According to Lemma 2.8,(ii), one has then

$$\beta = \sum_{i \geq 0, i+2m=n-1} \alpha_i \delta_E^i (h - \delta_E)^m h^m + 2\gamma \text{ in } H^{2n-2}(E_\Delta, \mathbb{Z}), \tag{20}$$

which by Lemma 2.8,(i) gives

$$j_{E*}\beta = 2\Big(\sum_{i \geq 0, i+2m=n-1} \alpha_i \delta^{i+1}[\Sigma_m^{[2]}]\Big) \text{ in } H^{2n}(X^{[2]}, \mathbb{Z}) \text{ modulo } j_{E*}(2H^*(E_\Delta, \mathbb{Z})) \tag{21}$$

for some integers α_i. By (19), we thus have

$$2\Big(\sum_{i \geq 0, i+2m=n-1} \alpha_i \delta^{i+1}[\Sigma_m^{[2]}] + j_{E*}(\gamma)\Big) = 2[\Gamma_0] \text{ in } H^{2n}(X^{[2]}, \mathbb{Z}) \tag{22}$$

for some integral homology class $\gamma \in H^{2n-2}(E, \mathbb{Z})$. It is proved in [25, Theorem 2.2] that the cohomology of $X^{[2]}$ as no 2-torsion when X is an odd degree complete intersection in projective space, so (22) gives

$$\sum_{i \geq 0, i+2m=n-1} \alpha_i \delta^{i+1}[\Sigma_m^{[2]}] + j_{E*}(\gamma) = [\Gamma_0] \text{ in } H^{2n}(X^{[2]}, \mathbb{Z}). \tag{23}$$

We now observe that β is algebraic in $H^{2n-2}(E_\Delta, \mathbb{Z})$, which easily follows from the fact that $i_{E*}\beta$ is algebraic in $H^{2n}(\widetilde{X \times X}, \mathbb{Z})$ by recalling that $\widetilde{X \times X}, \mathbb{Z}$ is simply the blow-up

of $X \times X$ along its diagonal. We thus conclude from (20) that 2γ is algebraic, and by our assumption that $H^{2*}(X,\mathbb{Z})/H^{2*}(X,\mathbb{Z})_{alg}$ has no 2-torsion, γ is algebraic, that is, $\gamma = [\Gamma']$ for some $\Gamma' \in \mathrm{CH}^{n-1}(E_\Delta)$. Thus we have

$$\sum_{i\geq 0, i+2m=n-1} \alpha_i \delta^{i+1}[\Sigma_m^{[2]}] + j_{E*}[\Gamma'] = [\Gamma_0] \text{ in } H^{2n}(X^{[2]},\mathbb{Z}). \tag{24}$$

We have $\mu^*\delta^n = \pm[\Delta_X]$ and, for any $z \in \mathrm{CH}^{n-1}(E_\Delta)$, $\mu^*(j_{E*}z) = N\Delta_X$ for some $N \in \mathbb{Z}$. As $\mu^*[\Gamma_0] = 0$, we can thus assume, up to modifying Γ' but without changing $\mu^*\Gamma_0$, that in formula (24), $\alpha_{n-1} = 0$ and Γ' satisfies $\tau_{E*}\Gamma' = 0$, hence $\mu^*(j_{E*}(\Gamma')) = 0$ in $\mathrm{CH}^n(X \times X)$. Next, for $m > 0$, $\mu^*\delta^{i+1}[\Sigma_m^{[2]}]$ is supported over $D \times X$ for some proper closed algebraic subset $D \subset X$. It follows that the cycle

$$\Gamma := \Gamma_0 - \sum_{i\geq 0, i+2m=n-1} \alpha_i \delta^{i+1}[\Sigma_m^{[2]}] - j_{E*}(\Gamma')$$

is cohomologous to 0 on $X^{[2]}$, and satisfies $\mu^*\Gamma = \mu^*\Gamma_0 + Z'$ where $Z' \in \mathrm{CH}^n(X \times X)$ is supported over $D \times X$. □

We now consider the case where X is a smooth cubic hypersurface in $\mathbb{P}^{n+1}$. We then have the following description of $X^{[2]}$, which is also used in [13]. We denote below by $F(X)$ the variety of lines of X. Let

$$P = \{([l], x),\, x \in l,\, l \subset X\}$$

be the universal $\mathbb{P}^1$-bundle, with first projection $p : P \to F(X)$, and let $P_2 \to F(X)$ be the $\mathbb{P}^2$-bundle defined as the symmetric product of P over $F(X)$. There is a natural embedding $P_2 \subset X^{[2]}$ which maps each fiber of $P \to F(X)$, that is the second symmetric product of a line in X, isomorphically onto the set of subschemes of length 2 of X contained in this line. Let $p_X : P_X \to X$ be the projective bundle with fiber over $x \in X$ the set of lines in $\mathbb{P}^{n+1}$ passing through x. Note that P is naturally contained in P_X, as one sees by considering the second projection $q : P \to X$.

Proposition 2.9. *(i) The rational map $\Phi : X^{[2]} \dashrightarrow P_X$ which to a unordered pair of points $x, y \in X$ not contained in a common line of X associates the pair $([l_{x,y}], z)$, where $l_{x,y}$ is the line in $\mathbb{P}^{n+1}$ generated by x and y and $z \in X$ is the residual point of the intersection $l_{x,y} \cap X$, is desingularized by the blow-up of P_2 in $X^{[2]}$.*

(ii) The induced morphism $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ identifies $\widetilde{X^{[2]}}$ with the blow-up $\widetilde{P_X}$ of P in P_X.

(iii) The exceptional divisors of the two maps $\widetilde{X^{[2]}} \to X^{[2]}$ and $\widetilde{P_X} \to P_X$ are identified by the isomorphism $\widetilde{\Phi}' : \widetilde{X^{[2]}} \cong \widetilde{P_X}$ of (ii).

Proof. (i) There is a morphism from $X^{[2]}$ to the Grassmannian $G(1, n+1)$ of lines in $\mathbb{P}^{n+1}$, which to $x + y$ associates the line $\langle x, y\rangle$. Let $\pi : Q \to X^{[2]}$ be the pull-back of the natural $\mathbb{P}^1$-bundle on $G(1, n+1)$. Let $\alpha : Q \to \mathbb{P}^{n+1}$ be the natural map. Then $\alpha^{-1}(X)$ is a reducible divisor in Q, which is generically of degree 3 over $X^{[2]}$, with one component D_1 which is finite of degree 2 over $X^{[2]}$, parameterizing the pairs $(x, x + y)$ and a second component $\mathcal{D}$ which is of degree 1 over $X^{[2]}$ and parameterizes generically the pairs $(z, x + y)$. The divisor $\mathcal{D}$ is isomorphic to $X^{[2]}$ away from P_2. Over P_2, the restricted $\mathbb{P}^1$-bundle Q_{P_2} is contained in $\alpha^{-1}(X)$ but not in D_1, so it is contained in $\mathcal{D}$. We claim that $\mathcal{D}$ is smooth and identifies with the blow-up of $X^{[2]}$ along P_2. Indeed, this simply follows from the fact that the divisor $\mathcal{D}$ is the zero set of a section s of $\mathcal{O}_Q(3)(-D_1)$ on Q. The line bundle $\mathcal{O}_Q(3)(-D_1)$ on Q has degree 1 along the fibers of $Q \to X^{[2]}$, so $R^0\pi_*\mathcal{O}_Q(3)(-D_1)$ is a rank 2 vector bundle $\mathcal{E}$ on $X^{[2]}$ such that $Q = \mathbb{P}(\mathcal{E})$. The section s provides a section s' of $\mathcal{E}$ and one easily checks that $P_2 \subset X^{[2]}$ is scheme-theoretically defined as the zero-locus of s'. This implies that $\mathcal{D}$ is the blow-up of $X^{[2]}$ along P_2 and the smoothness of P_2 implies the smoothness of $\mathcal{D}$. The claim

is thus proved. On the other hand, the rational map Φ clearly pulls back to a morphism on $\mathcal{D}$, so (i) is proved.

(ii) and (iii) If the length 2 subscheme $Z = x + y$ (or $Z = (2x, v)$ with v tangent to X at x) does not belong to P_2, then the line $l_{x,y}$ (or $l_{x,v}$) is not contained in X, the morphism Φ is well defined at Z and its image is a pair $([l], u)$ where l is a line passing through u and is not contained in X. At such point $([l], u)$ of P_X, Φ^{-1} is well-defined, and associates to $([l'], u')$ in a neighborhood of $([l], u)$ in P_X, the residual scheme of u' in $l' \cap X$. This proves (iii). It remains to understand what happens along the exceptional divisor Q_{P_2} of $\mathcal{D}$. Now we have $Q_{P_2} = \{(u, x + y, [l]), l \subset X, x + y \in l^{(2)}, u \in l\}$. By definition, $\widetilde{\Phi}$ maps such a triple to the pair $(u, [l])$, which by definition belongs to P_X. Furthermore, the fiber of $\widetilde{\Phi}$ over $(u, [l])$ when $l \subset X$, that is when $(u, [l]) \in P$, identifies with the plane $l^{(2)} \cong \mathbb{P}^2$. Thus $\widetilde{\Phi}^{-1}(P)$ is equal to the smooth irreducible hypersurface Q_{P_2} in the smooth variety $\mathcal{D}$ and this implies that $\widetilde{\Phi}$ factors through a morphism $f : \mathcal{D} \to \widetilde{P_X}$ which has to be an isomorphism, since it cannot contract any curve ; indeed, otherwise a contracted curve would be a curve in a fiber $\mathbb{P}^2$ as described above, so the whole corresponding $\mathbb{P}^2$ would be contracted by f, hence also all deformations of this $\mathbb{P}^2$ in $\mathcal{D} = \widetilde{X^{[2]}}$. But then the divisor Q_{P_2} would be contracted by f, while its image has to be the exceptional divisor of $\widetilde{P_X} \to P_X$. □

We now first give the proof of Theorem 1.1 in the case of cubics of dimension ≤ 4, because the argument is shorter in this case.

Proof of Theorem 1.1 in the case $n \leq 4$. Let X be a smooth cubic hypersurface of dimension ≤ 4 and assume X admits a cohomological decomposition of the diagonal. The assumptions of Proposition 2.6 are satisfied by X, since the integral cohomology of a smooth cubic hypersurface has no torsion and the integral Hodge conjecture is proved in [32] for cubic fourfolds. Using the notation introduced previously, there exists by Proposition 2.6 a cycle $\Gamma \in \mathrm{CH}^n(X^{[2]})$ such that

$$\mu^*\Gamma = \Delta_X - x \times X - X \times x - W \text{ in } \mathrm{CH}^n(X \times X), \tag{25}$$

with W supported over $D \times X$, $D \subsetneq X$, and $[\Gamma] = 0$ in $H^{2n}(X^{[2]}, \mathbb{Z})$.

By Proposition 2.9, the blow-up $\sigma : \widetilde{X^{[2]}} \to X^{[2]}$ of $X^{[2]}$ along P_2 identifies via $\widetilde{\Phi}$ with a blow-up of the projective bundle P_X over X. Furthermore, the exceptional divisor of $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ is also the exceptional divisor of $\sigma : \widetilde{X^{[2]}} \to X^{[2]}$, hence maps via σ to $P_2 \subset X^{[2]}$. It follows that the pull-back $\sigma^*(\Gamma)$ of the cycle Γ to $\widetilde{X^{[2]}}$ can be written as

$$\sigma^*(\Gamma) = \Gamma_1 + \Gamma_2, \tag{26}$$

where Γ_1 and Γ_2 are cohomologous to 0, Γ_1 is a cycle cohomologous to 0 on the exceptional divisor of $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$, and Γ_2 is the pull-back of a cycle Γ_2' cohomologous to 0 on P_X. As the exceptional divisor of $\widetilde{\Phi}$ equals the exceptional divisor of σ, it follows from (26), by applying σ_*, that

$$\Gamma = i_{P_2 *}(\Gamma_1') + \Phi^*(\Gamma_2') \text{ in } \mathrm{CH}(X^{[2]}), \tag{27}$$

where Γ_1' is a cycle cohomologous to 0 on P_2. Here i_{P_2} denotes the inclusion map of P_2 in $X^{[2]}$. It is known that for a smooth cubic hypersurface of dimension ≤ 4, cycles homologous to 0 are algebraically equivalent to 0. For codimension 2 cycles, this is proved by Bloch and Srinivas [7] and is true more generally for any rationally connected variety; for 1-cycles on cubic fourfolds, this is proved in [24], and this is true more generally for 1-cycles on Fano complete intersections of index ≥ 2. The result then also holds for cycles on a projective bundle over a cubic of dimension ≤ 4. Thus Γ_2' is algebraically equivalent to 0 and thus we conclude that

$$\Gamma = i_{P_2 *}(\Gamma_1') \text{ in } \mathrm{CH}(X^{[2]})/\mathrm{alg}, \tag{28}$$

for some n-cycle Γ_1' homologous to 0 on P_2. We apply now the following result:

Lemma 2.10. *Let X be a n-dimensional smooth cubic hypersurface and Z be a n-cycle homologous to 0 on $P_2 \overset{i_{P_2}}{\hookrightarrow} X^{[2]}$. Then $\mu^*(i_{P_2*}Z) \in \mathrm{CH}^n(X \times X)$ is supported on $D \times X$ for some proper closed algebraic subset D of X.*

Proof. Recall that P_2 is the union of the symmetric products $L^{(2)}$ over all lines $L \subset X$. As before, denote by

$$q : P \to X,\ p : P \to F$$

the natural maps, and by q_2 the natural map $P \times_F P \to X \times X$ induced by q. We will also denote by $\pi : P \times_F P \to F$ the map induced by p. Via π, $P \times_F P$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over F. Let $H := c_1(\mathcal{O}_X(1)) \in \mathrm{CH}^1(X)$ and let $h = q^*H \in \mathrm{CH}^1(P)$ be its pull-back to P. For a cycle Z supported on P_2, we have

$$\mu^*(i_{P_2*}Z) = q_{2*}(r'^*Z) \text{ in } \mathrm{CH}(X \times X)$$

where r' is the quotient map $P \times_F P \to P_2$. Let $T := r'^*Z \in \mathrm{CH}(P \times_F P)$. This is a cycle homologous to 0 on $P \times_F P$, and thus it can be written as

$$T = h_1h_2\pi^*\alpha + h_1\pi^*\beta + h_2\pi^*\gamma + \pi^*\zeta \text{ in } \mathrm{CH}(P \times_F P), \tag{29}$$

for some cycles α, β, γ, ζ homologous to 0 on F, with $h_i = pr_i^*h$, for $i = 1, 2$, $pr_i : P \times_F P \to P$ being the i-th projection.

We now push-forward these cycles to $X \times X$ via q_2 and observe that the three cycles

$$q_{2*}(\pi^*\zeta),\ q_{2*}(h_1h_2\pi^*\alpha),\ q_{2*}(h_1\pi^*\beta)$$

are cycles supported on $D \times X$ for some $D \subsetneq X$. Indeed, for the two last ones, this is due to the projection formula (and the equality $h_1 = q_2^*(H_1)$, where $H_1 := pr_1^*H \in \mathrm{CH}^1(X \times X)$), and for the first one, this is because $q_{2*}(\pi^*\zeta)$ is supported on $q(p^{-1}(\mathrm{Supp}\,\zeta)) \times X$. Now ζ is a $(n-2)$-cycle, so $q(p^{-1}(\mathrm{Supp}\,\zeta))$ is a proper closed algebraic subset of X. It remains to examine the cycle $q_{2*}(h_2\pi^*\gamma)$. We observe now that the diagonal $\Delta_P \subset P \times_F P \subset P \times P$ is a divisor d in $P \times_F P$ whose class is of the form $d = h_1 + h_2 + \pi^*\lambda$ for some divisor class $\lambda \in \mathrm{CH}^1(F)$. Furthermore, we obviously have $q_2(\Delta_P) \subset \Delta_X$. Thus we can write

$$h_2\pi^*\gamma = (d - h_1 - \pi^*\lambda)\pi^*\gamma$$

in $\mathrm{CH}(P \times_F P)$. We thus have

$$q_{2*}(h_2\pi^*\gamma) = q_{2*}(d\pi^*\gamma) - q_{2*}(h_1\pi^*\gamma) - q_{2*}(\pi^*(\lambda\gamma)) \text{ in } \mathrm{CH}(X \times X).$$

As already explained, the cycles $q_{2*}(h_1\pi^*\gamma)$, $q_{2*}(\pi^*(\lambda\gamma))$ are supported over $D \times X$ for some $D \subsetneq X$. Finally, the last cycle $q_{2*}(d\pi^*\gamma)$ has to be 0 in $\mathrm{CH}(X \times X)$. Indeed, this is a n-cycle of $X \times X$ which is supported on the diagonal, hence proportional to it, and also cohomologous to 0. □

Combining (25), (28) and Lemma 2.10, we conclude that

$$\Delta_X = x \times X + X \times x - W' \text{ in } \mathrm{CH}(X \times X)/\mathrm{alg}, \tag{30}$$

where W' is supported on $D' \times X$ for some $D' \subsetneq X$. In conclusion, X admits a decomposition of the diagonal modulo algebraic equivalence, and we can now apply Proposition 2.1 to conclude that X admits a Chow-theoretic decomposition of the diagonal. □

We now turn to the general case.

Proof of Theorem 1.1 for general n. Let X be a smooth cubic hypersurface such that the group $H^{2*}(X, \mathbb{Z})/H^{2*}(X, \mathbb{Z})_{alg}$ has no 2-torsion. Then Propositions 2.6 and 2.9 apply. Next, if we examine the proof given in the case of dimension ≤ 4, we see that the only place where we used the fact that $\dim X \leq 4$ is in the analysis of the term $\Phi^*(\Gamma_2')$, where Γ_2' is cohomologous to 0 on P_X. In the above proof, we directly used the fact that this term is algebraically equivalent to 0, which we do not know in higher dimension. The following provides an alternative argument which works also in higher dimension:

Lemma 2.11. *Let X be a smooth cubic hypersurface of dimension $n \geq 2$. Let Z be a n-cycle cohomologous to 0 on P_X. Then $3\mu^*(\Phi^*(Z)) \in \mathrm{CH}^n(X \times X)$ is rationally equivalent to a cycle supported on $D \times X$, for some $D \subsetneq X$.*

Proof. Recall that $p_X : P_X \to X$ is the $\mathbb{P}^n$-bundle over X with fiber over $x \in X$ the set of lines in $\mathbb{P}^{n+1}$ through x. Let us denote by $l \in \mathrm{CH}^1(P_X)$ the class of $\mathcal{O}_{P_X}(1)$ (we choose for $\mathcal{O}_{P_X}(1)$ the pull-back of the Plücker line bundle on the Grassmannian of lines $G(1, \mathbb{P}^{n+1})$). The cycle Z can be written as

$$Z = p_X^* Z_0 + l p_X^* Z_1 + \ldots + l^n p_X^* Z_n, \tag{31}$$

where Z_i are cycles of codimension $n - i$ on X. Note that the cycles Z_i are all homologous to 0. As $\dim X \geq 2$, $\mathrm{CH}_0(X)_{hom} = 0$ and thus $Z_0 = 0$ in $\mathrm{CH}_0(X)$. Hence (31) shows that $Z = l \cdot Z'$ for some $Z' \in \mathrm{CH}(P_X)$. Let $\Psi := \Phi \circ \mu : X \times X \dashrightarrow P_X$ and let $\widetilde{\Psi} : \widetilde{X \times X} \to P_X$ be the desingularization of Ψ obtained by blowing-up first the diagonal of X, and then the inverse image of $P_2 \subset X^{[2]}$ (see Proposition 2.9). Let $\tilde{\tau} : \widetilde{X \times X} \to X \times X$ be the composition of the two blow-ups. There are two exceptional divisors of $\tilde{\tau}$, namely E_Δ and E_{P_2}. We thus have (using the fact that $\widetilde{\Psi}$ factors through $X^{[2]}$)

$$\widetilde{\Psi}^*(l) = \alpha \tilde{\tau}^*(H_1 + H_2) + \beta E_\Delta + \gamma E_{P_2},$$

where the explicit computation of the coefficients α, β, γ can be done but is not useful here.

It follows that

$$\begin{aligned} \Psi^*(Z) = \Psi^*(l \cdot Z') &= \tilde{\tau}_*(\widetilde{\Psi}^*(l \cdot Z')) = \tilde{\tau}_*(\widetilde{\Psi}^*(l) \cdot \widetilde{\Psi}^*(Z')) \\ &= \tilde{\tau}_*((\alpha \tilde{\tau}^*(H_1 + H_2) + \beta E_\Delta + \gamma E_{P_2}) \cdot \widetilde{\Psi}^*(Z')) \text{ in } \mathrm{CH}(X \times X). \end{aligned} \tag{32}$$

Now, we develop the last expression and observe again that $\tilde{\tau}_*(\beta E_\Delta \cdot \widetilde{\Psi}^*(Z'))$ is supported on the diagonal of X, it must be proportional to Δ_X, hence in fact identically 0 as it is cohomologous to 0. Next, the cycle $\tilde{\tau}_*(\gamma E_{P_2} \cdot \widetilde{\Psi}^*(Z'))$ comes from a n-cycle Z'' homologous to 0 on $P \times_F P$, i.e.

$$\tilde{\tau}_*(\gamma E_{P_2} \cdot \widetilde{\Psi}^*(Z')) = q_{2*}(Z''), \tag{33}$$

where $q_2 : P \times_F P \to X \times X$ is introduced above. We can then apply Lemma 2.10 to conclude that the cycle $\tilde{\tau}_*(\gamma E_{P_2} \cdot \widetilde{\Psi}^*(Z'))$ is supported on $D \times X$ for some $D \subsetneq X$. Thus we conclude from (32), the projection formula, and the analysis above that

$$\Psi^*(Z) = H_1 \cdot W_1 + H_2 \cdot W_2 + W \text{ in } \mathrm{CH}(X \times X), \tag{34}$$

where W is supported on $D \times X$ for some $D \subsetneq X$, and W_1, W_2 are cycles homologous to 0 on $X \times X$. It thus suffices to show that cycles Γ on $X \times X$ of the form $H_1 \cdot W_1$ and $H_2 \cdot W_2$ with W_i homologous to 0 on $X \times X$ have the property that 3Γ is rationally equivalent to a cycle supported on $D \times X$ for some proper closed algebraic subset of X. For $H_1 \cdot W_1$ this is obvious, and the coefficient 3 is not needed. For $H_2 \cdot W_2$, we observe that if $i_2 : X \times X \to X \times \mathbb{P}^{n+1}$ denotes the natural inclusion, we have

$$3H_2 \cdot W_2 = i_2^* \circ i_{2*}(W_2) \text{ in } \mathrm{CH}(X \times X).$$

Now the cycle $i_{2*}(W_2) \in \mathrm{CH}_{n+1}(X \times \mathbb{P}^{n+1})$ is homologous to 0. Using its decomposition

$$i_{2*}(W_2) = \sum_i pr_1^* \gamma_i \cdot pr_2^* L^{n-i},$$

with $\gamma_i \in \mathrm{CH}^i(X)_{hom}$ and $L = c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \in \mathrm{CH}^1(\mathbb{P}^{n+1})$, we thus conclude that $\gamma_0 = 0$, hence that $i_{2*}(W_2) = \sum_{i>0} pr_1^* \gamma_i \cdot pr_2^* L^{n-i}$ in $\mathrm{CH}^n(X \times \mathbb{P}^{n+1})$. Hence $i_{2*}(W_2)$ is rationally equivalent to a cycle supported on $D \times \mathbb{P}^{n+1}$ for some proper closed algebraic subset D of X, and thus $3H_2 \cdot W_2 = i_2^* \circ i_{2*}(W_2)$ is rationally equivalent to a cycle supported on $D \times X$ for some proper closed algebraic subset D of X. □

The rest of the proof goes as before, using again Lemma 2.10, and this allows to conclude that, if X admits a cohomological decomposition of the diagonal, then $3(\Delta_X - X \times x)$ is rationally equivalent to a cycle supported on $D \times X$, for some $D \subsetneq X$. On the other hand, as X admits a unirational parametrization of degree 2 (see [10]), we also know that $2(\Delta_X - X \times x)$ is rationally equivalent to a cycle supported on $D' \times X$, for some $D' \subsetneq X$. It follows that X admits a Chow-theoretic decomposition of the diagonal. □

3 Criteria for the cohomological decomposition of the diagonal

This section is devoted to the existence of cohomological decomposition of the diagonal. Our main result here is the following criterion for such a decomposition to exist:

Theorem 3.1. *(cf. Theorem 1.4) Let X be smooth projective of dimension n. If X admits a cohomological decomposition of the diagonal, then the following condition (*) is satisfied:*

(*) There exist smooth projective varieties Z_i of dimension $n-2$, correspondences $\Gamma_i \in \mathrm{CH}^{n-1}(Z_i \times X)$, and integers n_i, such that for any α, $\beta \in H^n(X, \mathbb{Z})$,

$$\langle \alpha, \beta \rangle_X = \sum_i n_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Z_i}. \tag{35}$$

Conversely, assume condition (*) and furthermore

(i) $H^{2i}(X, \mathbb{Z})$ is algebraic for $2i \neq n$ and $H^{2i+1}(X, \mathbb{Z}) = 0$ for $2i+1 \neq n$.

(ii) $H^*(X, \mathbb{Z})$ has no torsion.

Then X admits a cohomological decomposition of the diagonal.

Here $\Gamma_i^* : H^n(X, \mathbb{Z}) \to H^{n-2}(Z_i, \mathbb{Z})$ is as usual defined by

$$\Gamma_i^*(\alpha) := pr_{Z_i *}(pr_X^* \alpha \smile [\Gamma_i]),$$

where pr_{Z_i}, pr_X are the projections from $Z_i \times X$ to its factors. Let us comment on assumptions (i) and (ii). If X is a complete intersection of dimension n, the integral cohomology of X has no torsion and the groups $H^{2i}(X, \mathbb{Z})$ are cyclic generated by h^i, $h = c_1(\mathcal{O}_X(1))$ for $i < n/2$. For $i > n/2$, they are also cyclic but the generator is now $\frac{1}{d} h^i$, where $d = \deg X$ and it is not true in general that they are generated by a cycle class, except when X is Fano and $n = 4$ (resp. $n = 3$), in which case $H^6(X, \mathbb{Z})$ (resp. $H^4(X, \mathbb{Z})$) is generated by the class of a line in X, and some sporadic cases. Note that in any case the class h^i is algebraic for any i, and in some cases this can be used as a substitute assumption in Theorem 3.1, like smooth cubic hypersurfaces (see Corollary 3.2). Another interesting class of varieties which satisfy these two properties, needed in order to apply Theorem 3.1 below, is the class of rationally connected threefolds with trivial Artin-Mumford invariant, for which it is proved in [31] that $H^4(X, \mathbb{Z})$ is algebraic. In this case, one gets Theorem 4.1 which improves [29, Corollary 4.5 and Theorem 4.9].

Proof of Theorem 3.1. Let us first prove that assuming (i) and (ii), condition (*) implies that X admits a cohomological decomposition of the diagonal. So let Z_i, Γ_i be as above and satisfy (35). As $\dim Z_i = n-2$, and $\operatorname{codim} \Gamma_i = n-1$, the Γ_i's are $(n-1)$-cycles in $Z_i \times X$. We denote by $(\Gamma_i, \Gamma_i) \in \mathrm{CH}^{2n-2}(Z_i \times Z_i \times X \times X)$ the correspondence $p_{13}^* \Gamma_i \cdot p_{24}^* \Gamma_i$ between $Z_i \times Z_i$ and $X \times X$, where the p_{rs} are the projectors from $Z_i \times Z_i \times X \times X$ to the product of two of its factors. Observe that $(\Gamma_i, \Gamma_i)_* \Delta_{Z_i}$ is supported on $D_i \times D_i$, where $D_i \subsetneq X$ is defined as the image of $\operatorname{Supp} \Gamma_i$ in X by the second projection. Let

$$\Gamma := \sum_i n_i (\Gamma_i, \Gamma_i)_* \Delta_{Z_i} \in \mathrm{CH}^n(X \times X).$$

Equation (35) can be written as

$$\langle \alpha, \beta \rangle_X = \int_{X \times X} pr_1^* \alpha \smile pr_2^* \beta \smile [\Gamma] \tag{36}$$

for any degree n classes α, β on X. It follows that the class

$$[\Delta_X] - [\Gamma] \in H^{2n}(X \times X, \mathbb{Z}) \cong \mathrm{End}_0\left(H^*(X, \mathbb{Z})\right) \tag{37}$$

annihilates $H^n(X, \mathbb{Z})$. In (37), End_0 denotes the group of degree preserving endomorphisms. The isomorphism $H^{2n}(X \times X, \mathbb{Z}) \cong \mathrm{End}_0\left(H^*(X, \mathbb{Z})\right)$ is a consequence, by Künneth decomposition and Poincaré duality, of the fact that $H^*(X, \mathbb{Z})$ has no torsion.

It follows that we have (again by Künneth decomposition)

$$[\Delta_X] - [\Gamma] \in \oplus_{i \neq n} H^i(X, \mathbb{Z}) \otimes H^{2n-i}(X, \mathbb{Z}) \subset H^{2n}(X \times X, \mathbb{Z}). \tag{38}$$

On the other hand, condition (i) tells us that $H^{*\neq n}(X, \mathbb{Z})$ consists of classes of algebraic cycles, so that (38) becomes

$$[\Delta_X] - [\Gamma] = \sum_i pr_1^*[W_i] \smile pr_2^*[W_i'] \tag{39}$$

for some cycles W_i, W_i' of X with $\dim W_i + \dim W_i' = n$. The right-hand side of (39) is of the form

$$[X \times x] + \sum_{i,\, \dim W_i' > 0} [pr_1^* W_i \cdot pr_2^* W_i']$$

and clearly $\sum_{i,\, \dim W_i' > 0} pr_1^* W_i \cdot pr_2^* W_i'$ is supported on $D' \times X$ for some proper closed algebraic subset $D' \subsetneqq X$. Hence we get

$$[\Delta_X] - [X \times x] = [\Gamma] + \sum_{i,\, \dim W_i' > 0} [pr_1^* W_i \cdot pr_2^* W_i'] = [Z],$$

where the cycle $Z = \Gamma + \sum_{i,\, \dim W_i' > 0} pr_1^* W_i \cdot pr_2^* W_i'$ is supported on $(\cup D_i \cup D') \times X$.

We now prove conversely that condition (*) is implied by the existence of a cohomological decomposition of the diagonal of X. Let $D \subset X$ be a divisor and $Z \subset D \times X$ be an n-cycle such that

$$[Z] = [\Delta_X] - [X \times x] \text{ in } H^{2n}(X \times X, \mathbb{Z}).$$

We first claim that we can assume that D is a normal crossing divisor. In order to achieve this, let $\tau : X' \to X$ be a blow-up of X such that a global normal crossing divisor $D' \subset X'$ dominates D. Enlarging D if necessary, we can assume that Z lifts to a n-cycle $Z' \in \mathrm{CH}^n(D'' \times X')$, where $D'' = \sqcup_i D_i$ is the normalization of D'. (Indeed, it suffices to choose D in such a way that for each irreducible component Z_i of the support of Z, D has at least one component which is generically smooth along Z_i.) It follows easily that the cycle class

$$[\Delta_{X'}] - [X' \times x] \ \in \ H^{2n}(X' \times X')$$

is the class of a cycle Z_1 supported on $(D' \cup E) \times X'$, where E is the exceptional divisor of τ. The divisor $D_1' = D' \cup E$ can be also assumed to have global normal crossings and thus the cycle Z_1 lifts to a n-cycle $Z_1' \in \mathrm{CH}^n(D_1'' \times X')$, where $D_1'' = \sqcup_i D_{1,i}$ is the normalization of D_1'. On the other hand, as we have $\langle \alpha, \beta \rangle_X = \langle \tau^* \alpha, \tau^* \beta \rangle_{X'}$ for α, $\beta \in H^n(X, \mathbb{Z})$, it suffices to prove (*) for X'. This proves our claim.

From now on, we thus assume $X = X'$ and D is a global normal crossing divisor in X with normalization $\sqcup_i D_i$, so that Z lifts to a cycle $\widetilde{Z}$ in $(\sqcup_i D_i) \times X$. Let us denote by

$\Gamma_i \in \mathrm{CH}^n(D_i \times X)$ the restriction of $\widetilde{Z}$ to the connected component $D_i \times X$. Let $k_i : D_i \to X$ be the inclusion map. We have

$$\sum_i (k_i, Id_X)_*[\Gamma_i] = [Z] = [\Delta_X] - [X \times x] \text{ in } H^{2n}(X \times X). \tag{40}$$

We can of course assume $n > 0$, so that $[Z]^*\alpha = \alpha$ for any $\alpha \in H^n(X, \mathbb{Z})$. Then (40) gives the following equality, for any α, $\beta \in H^n(X, \mathbb{Z})$

$$\langle \alpha, \beta \rangle_X = \langle [Z]^*\alpha, [Z]^*\beta \rangle_X = \langle \sum_i ((k_i, Id_X)_*[\Gamma_i])^*\alpha, \sum_i ((k_i, Id_X)_*[\Gamma_i])^*\beta \rangle_X. \tag{41}$$

We now develop the last expression, which gives for all α, $\beta \in H^n(X, \mathbb{Z})$:

$$\langle \alpha, \beta \rangle_X = \sum_{i,j} \langle ((k_i, Id_X)_*[\Gamma_i])^*\alpha, ((k_j, Id_X)_*[\Gamma_j])^*\beta \rangle_X. \tag{42}$$

Note now that

$$((k_i, Id_X)_*[\Gamma_i])^*\alpha = k_{i*}([\Gamma_i]^*\alpha) \text{ in } H^n(X, \mathbb{Z})$$

and similarly for $(k_j, Id_X)_*[\Gamma_j])_*\beta$. Hence (42) becomes

$$\langle \alpha, \beta \rangle_X = \sum_{i,j} \langle k_{i*}([\Gamma_i]^*\alpha), k_{j*}([\Gamma_j]^*\beta) \rangle_X, \tag{43}$$

where $[\Gamma_i]^*\alpha \in H^{n-2}(D_i, \mathbb{Z})$ and similarly $[\Gamma_j]^*\beta \in H^{n-2}(D_j, \mathbb{Z})$. Let $\delta_i = k_i^*(D_i) \in \mathrm{CH}^1(D_i)$ and $[\delta_i] \in H^2(D_i, \mathbb{Z})$ its cohomology class. As k_i is an embedding, we have $k_i^* \circ k_{i*} = [\delta_i] \smile : H^{n-2}(D_i, \mathbb{Z}) \to H^n(D_i, \mathbb{Z})$, and thus

$$\langle k_{i*}([\Gamma_i]^*\alpha), k_{i*}([\Gamma_i]^*\beta) \rangle_X = \langle [\delta_i] \smile [\Gamma_i]^*\alpha, [\Gamma_i]^*\beta \rangle_{D_i}. \tag{44}$$

Write $\delta_i = \sum_l n_{il} Z_{il}$ where $n_{il} \in \mathbb{Z}$ and Z_{il} is a smooth $(n-2)$-dimensional subvariety of D_i. Then letting $\Gamma_{il} \in \mathrm{CH}^{n-1}(Z_{il} \times X)$ be the pull-back of Γ_i to $Z_{il} \times X$, (44) can be written as

$$\langle k_{i*}([\Gamma_i]^*\alpha), k_{i*}([\Gamma_i]^*\beta) \rangle_X = \sum_l n_{il} \langle [\Gamma_{il}]^*\alpha, [\Gamma_{il}]^*\beta \rangle_{Z_{il}}. \tag{45}$$

The right hand side of this equation is exactly of the form allowed in (35) and it remains to analyze in (43) the terms

$$\langle k_{i*}([\Gamma_i]^*\alpha), k_{j*}([\Gamma_j]^*\beta) \rangle_X + \langle k_{j*}([\Gamma_j]^*\alpha), k_{i*}([\Gamma_i]^*\beta) \rangle_X$$

for $i \neq j$. Denote by W_{ij} the intersection $D_i \cap D_j$. It admits two correspondences Γ_{ij}, $\Gamma_{ji} \in \mathrm{CH}^{n-1}(W_{ij} \times X)$, namely the restriction to $W_{ij} \times X$ of $\Gamma_i \in \mathrm{CH}^{n-1}(D_i \times X)$ and the restriction to $W_{ij} \times X$ of $\Gamma_j \in \mathrm{CH}^{n-1}(D_j \times X)$ respectively. With this notation, we have

$$\begin{aligned} \langle k_{i*}([\Gamma_i]^*\alpha), k_{j*}([\Gamma_j]^*\beta) \rangle_X &= \langle [\Gamma_{ij}]^*\alpha, [\Gamma_{ji}]^*\beta \rangle_{W_{ij}}, \\ \langle k_{j*}([\Gamma_j]^*\alpha), k_{i*}([\Gamma_i]^*\beta) \rangle_X &= \langle [\Gamma_{ji}]^*\alpha, [\Gamma_{ij}]^*\beta \rangle_{W_{ij}}, \end{aligned} \tag{46}$$

which provides

$$\begin{aligned} \langle k_{i*}([\Gamma_i]^*\alpha), k_{j*}([\Gamma_j]^*\beta) \rangle_X &+ \langle k_{j*}([\Gamma_j]^*\alpha), k_{i*}([\Gamma_i]^*\beta) \rangle_X \\ &= \langle [\Gamma_{ij}]^*\alpha, [\Gamma_{ji}]^*\beta \rangle_{W_{ij}} + \langle [\Gamma_{ji}]^*\alpha, [\Gamma_{ij}]^*\beta \rangle_{W_{ij}} \\ = \langle ([\Gamma_{ij}] + [\Gamma_{ji}])^*\alpha, ([\Gamma_{ij}] + [\Gamma_{ji}])^*\beta \rangle_{W_{ij}} &- \langle [\Gamma_{ij}]^*\alpha, [\Gamma_{ij}]^*\beta \rangle_{W_{ij}} \\ &- \langle [\Gamma_{ji}]^*\alpha, [\Gamma_{ji}]^*\beta \rangle_{W_{ij}}. \end{aligned} \tag{47}$$

Each of the terms appearing in the final expression of (47) is of the form allowed in (35), which concludes the proof. □

Corollary 3.2. *Let X be a smooth cubic hypersurface. Then X admits a cohomological decomposition of the diagonal, (or equivalently a Chow-theoretic one if $H^*(X,\mathbb{Z})/H^*(X,\mathbb{Z})_{alg}$ has no torsion,) if and only if X satisfies condition (*).*

Proof. The necessity of the condition (*) is proved above and the proof does not use the assumptions (i) and (ii) of Theorem 3.1, so it works in our case. For the converse, it is in fact not a direct corollary of the theorem, since the proof uses these assumptions and we do not know that cubic hypersurfaces satisfy assumption (ii), but we can make a small variant of the proof, using the following observation: As a smooth cubic hypersurface of dimension ≥ 2 admits a unirational parametrization of degree 2, twice its diagonal admits a decomposition

$$2\Delta_X = 2(X \times x) + Z \text{ in } \mathrm{CH}(X \times X),$$

with Z supported on $D \times X$, $D \subsetneqq X$. So X admits a cohomological (or Chow-theoretic) decomposition of the diagonal if there is such a decomposition for $3\Delta_X$. But we know that h^i is algebraic for any i, and thus each class $3pr_1^*\alpha_i \smile pr_2^*\alpha_{n-i}$ for $2i \neq n$, 0, is the class of a cycle supported on $D \times X$, $D \subsetneqq X$, where α_i is a generator of $H^{2i}(X,\mathbb{Z})$. The proof of the existence of a decomposition of $3[\Delta_X]$ assuming condition (*) then works as the proof of Theorem 3.1. □

Let us conclude this section with the following variant of (part of) Theorem 3.1.

Theorem 3.3. *Let X be a smooth projective variety of dimension n, and let N be an integer. Assume there is a decomposition*

$$N[\Delta_X] = N[X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}),$$

where Z is supported on $D \times X$, $D \subsetneqq X$. Then there exist smooth projective varieties Z_i of dimension $n-2$, correspondences $\Gamma_i \in \mathrm{CH}^{n-1}(Z_i \times X)$, and integers n_i, such that for any α, $\beta \in H^n(X,\mathbb{Z})$,

$$N^2\langle \alpha, \beta\rangle_X = \sum_i n_i \langle \Gamma_i^*\alpha, \Gamma_i^*\beta\rangle_{Z_i}. \tag{48}$$

Proof. We look at the proof that condition (*) is implied by the existence of a cohomological decomposition of the diagonal and we repeat it replacing everywhere $\langle \alpha, \beta\rangle_X$ by $N^2\langle \alpha, \beta\rangle_X$. The main point is the fact that with the same notation as in the proof of the theorem, letting $Z = \sum_i (k_i, Id_X)_*\Gamma_i$, we have by assumption

$$[Z]^*\alpha = N\alpha$$

for $\alpha \in H^n(X,\mathbb{Z})$ (with $n > 0$), and thus

$$N^2\langle \alpha, \beta\rangle_X = \langle [Z]^*\alpha, [Z]^*\beta\rangle_X.$$

□

4 Rationally connected threefolds

In the case of rationally connected threefolds, we have the following result, which was partially proved in [29]:

Theorem 4.1. *(Cf. Theorem 1.6) Let X be a rationally connected 3-fold and let $J(X)$ be its intermediate Jacobian, with principal polarization $\theta \in H^2(J(X),\mathbb{Z})$. Then X admits a cohomological decomposition of the diagonal if and only if*

(a) $H^3(X,\mathbb{Z})$ has no torsion.

(b) There exists a universal codimension 2 cycle Γ on $J(X) \times X$.

(c) The integral Hodge class $\theta^{g-1}/(g-1)! \in H^{2g-2}(J(X),\mathbb{Z})$, $g := \dim J(X)$, is algebraic on $J(X)$, that is, is the class of a 1-cycle $Z \in \mathrm{CH}_1(J(X))$.

Proof. Indeed, it is proved in [29, Corollary 4.5] that for a rationally connected 3-fold, conditions (a) and (b) are necessary for the existence of a cohomological decomposition of the diagonal, and in [29, Theorem 4.9] that (a), (b) and (c) are sufficient for the existence of a cohomological decomposition of the diagonal. So it suffices to show that (c) is also necessary. Assume X admits a cohomological decomposition of the diagonal, and let Z_i, Γ_i and n_i be as in Theorem 3.1. Then the Abel-Jacobi map Φ_X of X induces (after choosing a reference point in Z_i) a morphism

$$\gamma_i = \Phi_X \circ \Gamma_{i*} : Z_i \to \mathrm{CH}^2(X)_{hom} \to J(X)$$

with image $Z_i' := \gamma_{i*} Z_i \in \mathrm{CH}_1(J(X))$. We claim that (35) is equivalent to the equality

$$\sum_i n_i [Z_i'] = \theta^{g-1}/(g-1)!. \tag{49}$$

Indeed, as $\bigwedge^2 H^1(J(X),\mathbb{Z}) \cong H^2(J(X),\mathbb{Z}) = H^{2g-2}(J(X),\mathbb{Z})^*$, (49) is equivalent to the fact that for α, $\beta \in H^1(J(X),\mathbb{Z})$,

$$\sum_i n_i \langle \gamma_i^* \alpha, \gamma_i^* \beta \rangle_{Z_i} = \langle \frac{\theta^{g-1}}{(g-1)!} \alpha, \beta \rangle_{J(X)}. \tag{50}$$

Now, the right hand side is equal, by definition of the polarization θ, to $\langle \alpha', \beta' \rangle_X$, where we use the canonical isomorphism $H^1(J(X),\mathbb{Z}) \cong H^3(X,\mathbb{Z})$ to identify α, β to classes α', β' of degree 3 on X. Finally, using again this canonical isomorphism, we have :

$$\gamma_i^* \alpha = \Gamma_i^* \alpha', \ \gamma_i^* \beta = \Gamma_i^* \beta'$$

so that the left hand side in (50) is equal to $\sum_i n_i \langle \Gamma_i^* \alpha', \Gamma_i^* \beta' \rangle_{Z_i}$. Hence (50) is equivalent to the fact that for any α', $\beta' \in H^3(X,\mathbb{Z})$,

$$\sum_i n_i \langle \Gamma_i^* \alpha', \Gamma_i^* \beta' \rangle_{Z_i} = \langle \alpha', \beta' \rangle_X,$$

which is equality (35). □

The following variant is proved as above, using Theorem 3.3 instead of Theorem 3.1. It answers a question asked to us by A. Beauville.

Theorem 4.2. *Let X be a rationally connected threefold with no torsion in $H^3(X,\mathbb{Z})$. Assume that, for some integer N, $N\Delta_X$ admits a decomposition*

$$[N\Delta_X] = N[X \times x] + [Z], \tag{51}$$

with Z supported on $D \times X$, $D \subsetneq X$. Then

(i) There exists a codimension 2 cycle $\Gamma \in \mathrm{CH}^2(J(X) \times X)$ such that for any $t \in J(X)$,

$$\Phi_X(\Gamma_t) = N\, t \text{ in } J(X).$$

(ii) The integral cohomology class $N^2\theta^{g-1}/(g-1)! \in H^{2g-2}(J(X),\mathbb{Z})$ is algebraic on $J(X)$, where $g = \dim J(X)$.

Proof. (i) Let $\widetilde{D} \overset{j}{\hookrightarrow} X$ be a desingularization with a cycle $\widetilde{Z} \in \mathrm{CH}(\widetilde{D} \times X)$ such that $(j, Id_X)_*(\widetilde{Z}) = N\Delta_X$. It follows that for any $\alpha \in H^3(X,\mathbb{Z})$,

$$N\alpha = j_*(\widetilde{Z}^* \alpha),$$

and similarly, looking at the induced morphisms of complex tori,

$$N\, Id_{J(X)} = j_* \circ \widetilde{Z}^* : J(X) \to J(X),$$

where $\widetilde{Z}^*$ here gives a morphism

$$\psi : J(X) \to J^1(\widetilde{D}) \cong \mathrm{Pic}^0(\widetilde{D}).$$

Now we use the existence of a universal divisor $\mathcal{D}$ on $\mathrm{Pic}^0(\widetilde{D}) \times \widetilde{D}$. Let

$$\mathcal{Z} := (Id_{J(X)}, j)_*(\psi, Id_{\widetilde{D}})^*(\mathcal{D}) \in \mathrm{CH}^2(J(X) \times X).$$

Then for any $t \in J(X)$,

$$\Phi_X(\mathcal{Z}_t) = j_*(\Phi_{\widetilde{D}}(\mathcal{D}_{\psi(t)})) \text{ in } J(X).$$

The right hand side is equal to $j_*(\psi(t)) = j_*(\widetilde{Z}^*(t)) = N\,t$, proving (i).

(ii) We use Theorem 3.3. We thus have curves C_i, correspondences $\Gamma_i \in \mathrm{CH}^2(C_i \times X)$ and integers n_i such that for any $\alpha, \beta \in H^3(X, \mathbb{Z})$,

$$N^2\langle \alpha, \beta\rangle_X = \sum_i n_i \langle \Gamma_i^*\alpha, \Gamma_i^*\beta\rangle_{C_i}.$$

As in the proof of Theorem 4.1, this equality exactly says that the 1-cycles $D_i := \gamma_{i*} \in \mathrm{CH}_1(J(X))$, where $\gamma_i = \Phi_X \circ \Gamma_{i*} : C_i \to J(X)$, satisfy

$$\sum_i n_i[D_i] = N^2 \frac{\theta^{g-1}}{(g-1)!} \text{ in } H^{2g-2}(J(X), \mathbb{Z}).$$

□

Remark 4.3. When X is a unirational threefold admitting a degree N unirational parametrization

$$\phi : \mathbb{P}^3 \dashrightarrow X,$$

$N\Delta_X$ admits a decomposition as in (51), simply because, denoting Y a blow-up of $\mathbb{P}^3$ on which ϕ is desingularized to a true morphism $\tilde{\phi}$, one has

$$(\tilde{\phi}, \tilde{\phi})_*(\Delta_Y) = N\Delta_X$$

and Y admits a decomposition of the diagonal. In this case, Theorem 4.2, (ii) has an immediate proof which provides the following stronger statement:

There is an effective cycle of class $N\theta^{g-1}/(g-1)!$ in $H^{2g-2}(J(X), \mathbb{Z})$, where $g = \dim J(X)$.

To see this, recall from [10] that $(J(Y), \theta_Y)$ is a direct sum of Jacobians of smooth curves. Thus there exists a (possibly reducible) curve $C \subset J(Y)$ with class $\theta_Y^{g'-1}/(g'-1)!$, where $g' := \dim J(Y)$. Let $\psi : J(Y) \to J(X)$ be the morphism induced by $\tilde{\phi} : Y \to X$. We claim that $\psi_*(C) \subset J(X)$ has class $N\theta^{g-1}/(g-1)!$ in $J(X)$. Indeed, by definition of the Theta divisor of $J(X)$, this is equivalent to saying that for any α, $\beta \in H^3(X, \mathbb{Z})$, denoting by α', β' the corresponding degree 1 cohomology classes on $J(X)$ via the isomorphism $H^3(X, \mathbb{Z}) \cong H^1(J(X), \mathbb{Z})$,

$$N\langle \alpha, \beta\rangle_X = \int_{\psi_*(C)} \alpha' \wedge \beta'. \tag{52}$$

However,

$$\int_{\psi_*(C)} \alpha' \wedge \beta' = \int_C \psi^*\alpha' \wedge \psi^*\alpha',$$

where $\psi^*\alpha'$ identifies with $\phi^*\alpha \in H^3(Y, \mathbb{Z})$ via the natural isomorphism $H^1(J(Y), \mathbb{Z}) \cong H^3(Y, \mathbb{Z})$, and similarly for β. Finally, we get by definition of the Theta divisor of $J(Y)$:

$$\int_C \psi^*\alpha' \wedge \psi^*\alpha' = \langle \tilde{\phi}^*\alpha, \tilde{\phi}^*\beta\rangle_Y = N\langle \alpha, \beta\rangle_X,$$

which proves (52).

In the case of a smooth cubic threefold, we get the following consequence of Theorem 4.1:

Corollary 4.4. *(cf. Theorem 1.7) A smooth cubic threefold admits a Chow-theoretic decomposition of the diagonal (that is, its* CH_0 *group is universally trivial) if and only if the class* $\theta^4/4!$ *is algebraic on* $J(X)$*.*

Proof. Indeed, this is a necessary condition by Theorem 4.1. The fact that this is a sufficient condition is proved as follows: the cubic 3-fold has no torsion in $H^3(X,\mathbb{Z})$. It is not known if it admits a universal codimension 2 cycle, but it is known by work of Markushevich-Tikhomirov [19] that it admits a parametrization of the intermediate Jacobian with rationally connected fibers, that is, there exists a smooth projective variety B, and a codimension 2 cycle $Z \in \mathrm{CH}^2(B \times X)$, such that the induced morphism

$$\Phi_Z : B \to J(X),$$

$$t \mapsto \Phi_X(Z_t)$$

is surjective with rationally connected general fiber. In [29, Theorem 4.1], it is proved that if such a parametrization exists for a given rationally connected 3-fold X, and if furthermore the minimal class $\theta^{g-1}/(g-1)!$ is algebraic on $J(X)$, then there exists a universal codimension 2 cycle on $J(X) \times X$. In the case of the cubic threefold, we conclude that if the minimal class $\theta^4/4!$ is algebraic, then there exists a universal codimension 2 cycle on $J(X) \times X$. Thus Theorem 4.1 implies that X admits a cohomological decomposition of the diagonal. By Theorem 1.1, X admits then a Chow-theoretic decomposition of the diagonal. □

We conclude with the following result:

Theorem 4.5. *There exists a non-empty countable union of proper subvarieties of codimension* ≤ 3 *in the moduli space of smooth cubic threefolds parametrizing threefolds* X *with universally trivial* CH_0 *group.*

Proof. We first claim that if $(J(X),\theta)$ is isogenous via an odd degree isogeny to $(J(C), m\theta_C)$ for some (possibly reducible) curve C, then $J(X)$ has a one-cycle whose class is an odd multiple of the minimal class $\theta^4/4!$. Indeed, we have the odd degree isogeny $\mu : J(C) \to J(X)$ with the property that $\mu^*\theta_X = m\theta_C$. As $\deg \mu$ is odd, m is odd. Furthermore, we have $\mu_*(\theta_C^4/4!) = m(\theta^4/4!)$ and $\theta_C^4/4!$ is algebraic on $J(C)$. As $2(\theta^4/4!)$ is an algebraic class because $(J(X),\theta)$ is a Prym variety (see [10]), we conclude that $\theta^4/4!$ is algebraic, which proves the claim.

An explicit example is as follows: Consider a smooth cubic threefold defined by a homogeneous polynomial $P(X_0,\dots,X_4)$, where P is invariant under the automorphism g of order 3 acting on coordinates by

$$g^*X_0 = X_0,\ g^*X_1 = jX_1,\ g^*X_2 = j^2X_2,\ g^*X_3 = X_3,\ g^*X_4 = X_4,$$

where $j = \exp\frac{2\iota\pi}{3}$. The invariant part $H^3(X,\mathbb{Q})^{inv}$ of $H^3(X,\mathbb{Q})$ under the action of g has rank 6. This can be seen by looking at the action of g^* on $H^{2,1}(X)$, the later space being computed via Griffiths residues (see [34, 6.1]): One gets a residue isomorphism

$$H^0(X,\mathcal{O}_X(1)) \cong H^{2,1}(X),\ \ A \mapsto \mathrm{Res}_X \frac{A\Omega}{P^2}, \tag{53}$$

where Ω is the canonical generator of $H^0(\mathbb{P}^4, K_{\mathbb{P}^4}(5))$. As $g^*\Omega = \Omega$, (53) induces an isomorphism

$$H^0(X,\mathcal{O}_X(1))^{inv} \cong H^{2,1}(X)^{inv}$$

so that $\dim H^{2,1}(X)^{inv} = 3$.

Let $\pi = Id + g^* + (g^2)^* \in \mathrm{End}\,(H^3(X,\mathbb{Z}))$. Then $\pi/3$ is the orthogonal projector of $H^3(X,\mathbb{Q})$ onto $H^3(X,\mathbb{Q})^{inv}$ with respect to the intersection pairing on $H^3(X,\mathbb{Q})$. Hence over $\mathbb{Q}$ we have an orthogonal decomposition

$$H^3(X,\mathbb{Q}) = H^3(X,\mathbb{Q})^{inv} \oplus H^3(X,\mathbb{Q})^\sharp,$$

where $H^3(X,\mathbb{Q})^\sharp = \mathrm{Im}\,(Id - \pi/3)$. Over $\mathbb{Z}$, we conclude that $H^3(X,\mathbb{Z})$ contains a sublattice $H^3(X,\mathbb{Z})^{inv} \oplus H^3(X,\mathbb{Z})^\sharp$, where

$$H^3(X,\mathbb{Z})^\sharp = H^3(X,\mathbb{Q})^\sharp \cap H^3(X,\mathbb{Z}) = (H^3(X,\mathbb{Z})^{inv})^\perp.$$

The index of this sublattice is a power of 3, since for $a \in H^3(X,\mathbb{Z})$, we can write

$$3a = (a + g^*a + (g^2)^*a) + (3a - (a + g^*a + (g^2)^*a)),$$

with

$$a + g^*a + (g^2)^*a \in H^3(X,\mathbb{Z})^{inv},\ 3a - (a + g^*a + (g^2)^*a) \in H^3(X,\mathbb{Z})^\sharp.$$

It follows that we can construct two finite index sublattices

$$H_1 \subset H^3(X,\mathbb{Z})^{inv},\ H_2 \subset H^3(X,\mathbb{Z})^\sharp$$

with the property that the restriction of the intersection form $\langle , \rangle_X$ to H_1 and H_2 is m times a unimodular intersection pairing, where m is a power of 3. These sublattices determine principally polarized abelian varieties A and B of respective dimensions 3 and 2, together with an isogeny

$$A \oplus B \to J(X)$$

such that the pull-back of θ_X is $m(\theta_A, \theta_B)$. The ppav's A and B are Jacobians of curves C_A, C_B, and $(A \oplus B, (\theta_A, \theta_B))$ is the Jacobian of the curve $C_A \cup_x C_B$.

We conclude with the following:

Lemma 4.6. *Each choice of sublattices H_1, H_2 as above provides us with a subvariety of codimension ≤ 3 in the moduli space of X along which the class $\theta^4/4!$ is algebraic.*

Proof. Let $C = C_A \cup_x C_B$ be a curve as above, with x general, and let

$$\mathcal{C} \to V,\ c \in V,\ \mathcal{C}_c \cong C$$

be a universal family of deformations of C. Similarly, denote by U the base of a universal family of deformations of X. Denote by $\widetilde{\mathcal{A}}_{5,X}$ the base of a universal family of deformations of the ppav $J(X)$ and $\widetilde{\mathcal{A}}_{5,C}$ the base of a universal family of deformations of the ppav $J(C)$. We have an isogeny $\alpha : J(C) \overset{isog}{\cong} J(X)$ and the (local) period maps

$$\mathcal{P}_C : V \to \widetilde{\mathcal{A}}_{5,C},\ \mathcal{P}_X : U \to \widetilde{\mathcal{A}}_{5,X}.$$

The isogeny α provides a local (for the Euclidean topology) isomorphism

$$\alpha : \widetilde{\mathcal{A}}_{5,C} \cong \widetilde{\mathcal{A}}_{5,X}.$$

As $\dim \widetilde{\mathcal{A}}_{5,X} = 15$, any component of the subvariety $G \subset V \times U$ defined by

$$G = \{(t,u) \in \overline{\mathcal{M}_5} \times U,\ \alpha(\mathcal{P}_C(t)) = \mathcal{P}_X(u)\}$$

has codimension ≤ 15 in $V \times U$, hence has dimension ≥ 9 since $\dim U = \dim V = \dim \overline{\mathcal{M}_5} = 12$. The image $U' \subset U$ of the second projection $p_2 : G \to U$ consists of cubics whose intermediate Jacobian is isogenous (via the given isogeny type) to a Jacobian of curve. As $\dim U = 12$, one has $\mathrm{codim}\, U' \leq 3$, unless p_2 is not generically finite on its image. In this case, the image of G by the first projection is contained in the locus of reducible curves,

as this is the only locus where the period map $\mathcal{P}_C$ has positive dimensional fibers. So we have to exclude this last possibility. Assume by contradiction this happens. Note that the intermediate Jacobian of a generic cubic threefold X with $\mathbb{Z}/3\mathbb{Z}$-action as above has only two simple factors, one of dimension 2, the other of dimension 3. Hence the fibers of the period map $\mathcal{P}_C$ over any isomorphism class of an abelian 5-fold isogenous to $J(X)$ has dimension at most 2. As $\dim G \geq 9$, it follows that $p_2(G) \subset U$ has dimension ≥ 7, hence codimension ≤ 5, and is contained in the locus of U parameterizing cubic threefolds with reducible intermediate Jacobian. This can be excluded by an infinitesimal computation at any point $x \in U$. In fact, it suffices to prove that the infinitesimal variation of Hodge structure at x

$$T_{U,x} \to \mathrm{Hom}\,(H^{2,1}(X), H^{1,2}(X))$$

maps $T_{U,x}$ surjectively onto the 6-dimensional space $\mathrm{Hom}\,(H^{2,1}(X)^{inv}, H^{1,2}(X)^{\sharp})$, where $H^{1,2}(X)^{\sharp}$ is defined as the orthogonal complement of $H^{2,1}(X)^{inv}$. Indeed, the space

$$\mathrm{Hom}\,(H^{2,1}(X)^{inv}, H^{1,2}(X)^{\sharp})$$

identifies with the normal bundle of the locus of reducible ppav's in $\widetilde{\mathcal{A}}_{5,X}$. Using Griffiths' theory (see [34, 6.1-2]), this computation is easily performed in the Jacobian ring of the Fermat equation $P = \sum_i X_i^3$. □

It is a standard fact that there are countably many choices of pairs (H_1, H_2) of lattices as above. Indeed, starting from the unimodular intersection pairing on H_1, we can write H_1 as the sum of two Lagrangian sublattices

$$H_1 = \Lambda \oplus \Lambda',$$

and then for each integer m, we can consider $H_1'(m, \Lambda, \Lambda') := m\Lambda \oplus \Lambda'$; we also have to make a similar construction for H_2.

We get this way countably many corresponding codimension ≤ 3 subvarieties and it is likely that they are Zariski dense in the moduli space of cubic threefolds, but we did not try to prove this. □

5 More results on cubic hypersurfaces

Let X be a smooth cubic hypersurface of dimension $n \geq 3$. The Hodge structure on $H^n(X, \mathbb{Q})_{prim}$ is a polarized nontrivial Hodge structure (that is, when $n = 2k+1$, it is nonzero, and when $n = 2k$, it is not purely of type (k,k)). Let $\mathrm{End}_{HS}(H^n(X, \mathbb{Q})_{prim})$ be the space of endomorphisms of this Hodge structure.

Lemma 5.1. *For the very general cubic hypersurface of dimension n, $\mathrm{End}_{HS}(H^n(X, \mathbb{Q})_{prim}) = \mathbb{Q}Id$.*

Proof. This follows from the fact that the Mumford-Tate group of the considered Hodge structures is the symplectic group if n is odd and the orthogonal group if n is even. This fact in turn follows from the fact that the Mumford-Tate group contains a finite index subgroup of the monodromy group (see [30]) and that by [6], the monodromy group of a smooth hypersurface is Zariski dense in the symplectic or orthogonal group of $H^n(X, \mathbb{Q})_{prim}$ except for even dimensional quadrics and cubic surfaces, for which it is finite. This immediately implies the lemma because by definition of the Mumford-Tate group $G := \mathrm{MT}(H^n(X, \mathbb{Q})_{prim})$, any endomorphism ϕ of the Hodge structure on $H^n(X, \mathbb{Q})_{prim}$ has to commute with G, that is $\phi \circ g = g \circ \phi$ for $g \in G$. □

We now have the following result, saying that for cubic hypersurfaces satisfying the conclusion of Lemma 5.1, the CH_0 group cannot be universally supported on a proper closed algebraic subset of X, unless it is trivial. Let $Y \subset X$ be a proper closed algebraic subset. We introduced in Definition 1.2 the notion of $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ being universally

surjective. When Y is a point, the fact that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective is equivalent to the fact that X has universally trivial CH_0 group.

Theorem 5.2. *Let X be a smooth cubic hypersurface such that $H^n(X,\mathbb{Z})/H^n(X,\mathbb{Z})_{alg}$ has no 2-torsion for $n = \dim X$, and $\mathrm{End}_{HS}(H^n(X,\mathbb{Q})_{prim}) = \mathbb{Q}Id$. Assume there is a proper closed algebraic subset $Y \subset X$ such that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective, Then $\mathrm{CH}_0(X)$ is universally trivial.*

The assumptions of the Theorem are satisfied by a very general cubic hypersurface, which proves Theorem 1.3 stated in the introduction.

Proof of Theorem 5.2. Let $L = \mathbb{C}(X)$. Then we have the diagonal point δ_L and the fact that it comes from a 0-cycle supported on Y_L says, by taking the Zariski closure in $X \times X$ and using the localization exact sequence, that there is a decomposition of the diagonal of X which takes the following form:

$$\Delta_X = Z_1 + Z_2 \text{ in } \mathrm{CH}^n(X \times X),$$

where Z_1 is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$, and Z_2 is supported on $X \times Y$. This decomposition gives in particular a cohomological decomposition:

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^{2n}(X \times X, \mathbb{Z}), \tag{54}$$

where Z_1 and Z_2 are as above. We now use Lemma 5.3 below which says that a decomposition as in (54) implies that X admits a cohomological decomposition of the diagonal because we assumed $\mathrm{End}_{HS}(H^n(X,\mathbb{Q})_{prim}) = \mathbb{Q}Id$, and Theorem 1.1 which says that X admits then a Chow-theoretic decomposition of the diagonal because we assumed that $H^n(X,\mathbb{Z})/H^n(X,\mathbb{Z})_{alg}$ has no 2-torsion; hence we proved that $\mathrm{CH}_0(X)$ is in fact universally trivial. □

Lemma 5.3. *Let X be a smooth cubic hypersurface. Assume that there is a decomposition*

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^{2n}(X \times X, \mathbb{Z}), \tag{55}$$

where Z_1 is supported on $D \times X$, and Z_2 is supported on $X \times Y$ for some proper closed algebraic subsets D, $Y \subset X$. Then if furthermore $\mathrm{End}_{HS}(H^n(X,\mathbb{Q})_{prim}) = \mathbb{Q}Id$, X admits a cohomological decomposition of the diagonal.

Proof. Indeed, recall that cubic hypersurfaces admit a unirational parametrization of degree 2. So $2[\Delta_X]$ can be decomposed as $2[X \times x] + [Z]$, where Z is supported on $D \times X$ for a proper closed algebraic subset $D \subset X$. Hence it suffices to show that for some odd integer m, we have a decomposition

$$m[\Delta_X] = m[X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}),$$

where $Z \in \mathrm{CH}^n(X)$ is supported on $D \times X$ for some closed algebraic subset $D \subsetneq X$. Consider the decomposition (55): each class appearing in this decomposition acts on $H^n(X,\mathbb{Z})_{prim}$ via a morphism of Hodge structures, the diagonal acting as identity. As $\mathrm{End}_{HS}(H^n(X,\mathbb{Z})_{prim}) = \mathbb{Z}Id$, we have

$$[Z_1]^* = m_1 Id,\ [Z_2]^* = m_2 Id,$$

where m_1, m_2 are two integers such that $m_1 + m_2 = 1$. We may assume that m_1 is odd, applying transposition to our cycles if necessary. It follows that m_2 is even, and the cycle class $[Z_2] - m_2[\Delta_X]$ acts trivially on $H^n(X,\mathbb{Q})_{prim}$. Over $\mathbb{Q}$, using the orthogonal decomposition

$$H^*(X,\mathbb{Q}) = H^n(X,\mathbb{Q})_{prim} \oplus H^*(\mathbb{P}^{n+1},\mathbb{Q})_{|X},$$

we conclude that for some rational numbers α_i

$$[Z_2] - m_2[\Delta_X] = \sum_i \alpha_i h_1^i \otimes h_2^{n-i} \text{ in } H^{2n}(X \times X, \mathbb{Q}), \tag{56}$$

where here $h \in H^2(X,\mathbb{Z})$ is $c_1(\mathcal{O}_X(1))$ and the right hand side makes sense in $H^{2n}(X\times X,\mathbb{Q})$ via Künneth decomposition. We observe now that, because $H^*(X,\mathbb{Z})$ has no torsion and the pairings $\langle h^i, h^{n-i}\rangle$ are equal to 3, the denominators in the coefficients α_i of (56) are equal to 3 (or 1), so that we get

$$3[Z_2] - 3m_2[\Delta_X] = \sum_i \beta_i pr_1^* h^i \smile pr_2^* h^{n-i} \text{ in } H^{2n}(X\times X,\mathbb{Z}), \tag{57}$$

where now the β_i are integers. Combining (57) with (55), we get now:

$$3[\Delta_X] = 3[Z_1] + 3m_2[\Delta_X] + \sum_i \beta_i pr_1^* h^i \smile pr_2^* h^{n-i} \text{ in } H^{2n}(X\times X,\mathbb{Z}), \tag{58}$$

where Z_1 is supported on $D\times X$ for some closed algebraic subset $D \subsetneqq X$. Hence we proved that

$$(3-3m_2)[\Delta_X] - 3\beta_0[X\times x] = 3[Z_1] + \sum_{i>0} \beta_i pr_1^* h^i \smile pr_2^* h^{n-i} \text{ in } H^{2n}(X\times X,\mathbb{Z}), \tag{59}$$

and clearly, the class $\sum_{i>0} \beta_i pr_1^* h^i \smile pr_2^* h^{n-i}$ is the class of a cycle supported on $D'\times X$, for some closed algebraic subset $D' \subsetneqq X$. As m_2 is even, $3-3m_2$ is odd, and thus (59) finishes the proof. □

In the case of cubic fourfolds, we can replace in Theorem 5.2 the assumption that $\mathrm{End}_{HS}(H^n(X,\mathbb{Q})_{prim}) = \mathbb{Q}Id$ by the following assumption which concerns the *simple* Hodge structure

$$H^4(X,\mathbb{Q})_{tr} := H^4(X,\mathbb{Q})^{\perp}_{alg},$$

namely $\mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{tr}) = \mathbb{Q}Id$. This is a weaker and more natural assumption because, when the Hodge structure on $H^4(X,\mathbb{Q})_{prim}$ is not simple, or equivalently, when $H^4(X,\mathbb{Q})_{prim}$ contains nonzero Hodge classes, the algebra $\mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{prim})$ contains projectors associated to sub-Hodge structures, and this happens along codimension 1 loci paramerizing special cubic fourfolds. On the contrary, the non-existence of nontrivial endomorphisms of the Hodge structure on $H^4(X,\mathbb{Q})_{tr}$ is satisfied in codimension 1, and in particular at the very general point of a Noether-Lefschetz locus by the following lemma:

Lemma 5.4. *In the moduli space of smooth cubic fourfolds, the set of points parameterizing cubics X such that $\mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{tr}) \neq \mathbb{Q}Id$ is of codimension ≥ 2.*

Proof. Equivalently, we have to show that this set does not contain the very general point of a Noether-Lefschetz locus $\mathcal{D}_\sigma$ defined by a class σ, or the very general point of the moduli space. By contradiction, let X be a very general point of $\mathcal{D}_\sigma$, and let $h \in \mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{tr})$ be a morphism of Hodge structures (which then has to remain a morphism of Hodge structures acting on $H^4(X_t,\mathbb{Q})_{tr}$ for any small deformation X_t of X parameterized by a point $t\in\mathcal{D}_\sigma$) but is not a homothety. Let $\lambda\in\mathbb{C}$ be the algebraic number such that $h^*\eta_X = \lambda\eta_X$, where η_X is a generator of the rank 1 vector space $H^{3,1}(X)$. As the Hodge structure on $H^4(X,\mathbb{Q})_{tr}$ is simple and h is not a homothety, λ is not a rational number. It follows that the eigenspace H_λ of h associated with the eigenvalue λ has complex dimension $\leq \frac{1}{2}\dim H^4(X,\mathbb{Q})_{tr}$. On the other hand, the period map restricted to $\mathcal{D}_\sigma$ has by assumption its image contained in $\mathbb{P}(H_\lambda)$. As the period map is injective, we conclude that

$$\dim \mathcal{D}_\sigma \leq \frac{1}{2}(\dim H^4(X,\mathbb{Q})_{tr} - 2)$$

which is absurd since the right-hand side is equal to 19/2 while the left-hand side is equal to 19. This contradicts our assumption that h is not a homothety. The same argument works if X is a general point of the moduli space. □

Our next result is the following:

Theorem 5.5. *Let X be a smooth cubic fourfold such that $\mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{tr}) = \mathbb{Q}Id$. Assume there is a proper closed algebraic subset $Y \subset X$ such that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective. Then $\mathrm{CH}_0(X)$ is universally trivial.*

Proof. The proof is very similar to the previous proof. The assumption is that

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^8(X \times X, \mathbb{Z}),$$

with Z_1 supported on $D_1 \times X$, Z_2 supported on $X \times D_2$ for some closed proper algebraic subsets D_1, $D_2 \subsetneqq X$. We consider the action of $[Z_i]^*$ on $H^4(X,\mathbb{Z})_{tr}$. As $\mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{tr}) = \mathbb{Q}Id$, each of them must act as a multiple of the identity and the sum $[Z_1]^* + [Z_2]^*$ equals $Id_{H^4(X,\mathbb{Z})_{tr}}$. So one of them, say $[Z_1]^*$, must act as an odd multiple of the identity and the other as an even multiple of the identity. Let $[Z_2]^* = 2m\, Id$ on $H^4(X,\mathbb{Z})_{tr}$. Then $(2m[\Delta_X] - [Z_2])^*$ acts as 0 on $H^4(X,\mathbb{Z})_{tr}$. Note also that $(2m[\Delta_X] - [Z_2])^*$ maps $Hdg^4(X,\mathbb{Z}) = H^*(X,\mathbb{Z})^{\perp}_{tr}$ to itself, and this implies that

$$2m[\Delta_X] - [Z_2] \in Hdg^4(X,\mathbb{Z}) \otimes Hdg^4(X,\mathbb{Z}) \oplus \sum_{i \neq 2} H^{2i}(X,\mathbb{Z}) \otimes H^{8-2i}(X,\mathbb{Z}) \subset H^8(X \times X, \mathbb{Z}).$$

All classes in $Hdg^4(X,\mathbb{Z})$ are algebraic by the Hodge conjecture for integral Hodge classes on cubic fourfolds proved in [32] and all classes in $H^{*\neq 4}(X,\mathbb{Z})$ are algebraic. Hence we conclude that

$$2m[\Delta_X] - [Z_2] = \sum_i pr_1^*[W_i] \smile pr_2^*[W_i']$$

for some integral cycles W_i, W_i' on X satisfying $\dim W_i + \dim W_i' = 4$. The rest of the proof works as before, allowing to conclude that X admits a cohomological decomposition of the diagonal, hence also a Chow-theoretic one by Theorem 1.1. □

We finally prove the following Theorem 5.6. Let X be a cubic fourfold. Assume X is special in the sense of Hassett, that is $H^4(X,\mathbb{Z})$ contains two independent Hodge classes (one being the class h^2, the other being denoted σ). Let $P \subset H^4(X,\mathbb{Z})$ be the sublattice generated by these two Hodge classes. The restriction of the intersection form $\langle\, ,\, \rangle_X$ to P has a discriminant $D(\sigma)$. This number, which is always even, has been very much studied in conjunction with rationality properties of cubic fourfolds (see [15], [16], [18], [1]). Hassett's work suggested that if a cubic fourfold is rational, then it is special and the discriminant $D(\sigma)$ satisfies severe restrictions.

Theorem 5.6. *If 4 does not divide $D(\sigma)$, the CH_0 group of X is universally trivial (that is, X admits a Chow-theoretic decomposition of the diagonal).*

Proof. The assumption on X being satisfied along a countable union of Noether-Lefschetz type divisors $\mathcal{D}_\sigma$ in the moduli space of cubic fourfolds (see [15]), it suffices to show that the conclusion holds for X very general in each $\mathcal{D}_\sigma$. Indeed, inside each divisor $\mathcal{D}_\sigma$, the existence of a cohomological (or Chow-theoretic) decomposition of the diagonal is satisfied along a countable union of closed algebraic subsets (see [28, Proof of Theorem 1.1]). So if it satisfied at the very general point of $\mathcal{D}_\sigma$, it is satisfied everywhere along $\mathcal{D}_\sigma$. Next, by Lemma 5.4, for a very general point X in a Noether-Lefschetz divisor, we have $\mathrm{End}_{HS}(H^4(X,\mathbb{Q})_{tr}) = \mathbb{Q}Id$. Theorem 5.5 thus tells us that if there is a surface $\Sigma \subset X$ such that $\mathrm{CH}_0(\Sigma) \to \mathrm{CH}_0(X)$ is universally surjective, then $\mathrm{CH}_0(X)$ is universally trivial. The existence of a special Hodge class σ provides us with an algebraic cycle Z on X of class σ, by the Hodge conjecture for integral Hodge classes proved in [32]. Adding to σ a high multiple of h^2, we can even assume that Z is the class of a smooth surface Σ in general position. Indeed, as we are working with codimension 2 cycles, their classes are generated by c_2 of vector bundles on X. For any vector bundle E of rank r, a twist of E is very ample and its c_2 is represented by a rank locus associated to a morphism $\phi : \mathcal{O}^{r-1} \to E$, and this rank locus is smooth in codimension 5. Let now Σ be as above. Recall the rational map

$$\Phi' : X^2 \dashrightarrow X,\ \Phi'(x,x') = x'',\ x + x' + x'' = \langle x, x' \rangle \cap X.$$

We now have

Lemma 5.7. *Assume the restriction of Φ' to Σ^2 is dominant of degree $2N$ not divisible by 4. Then the map $\mathrm{CH}_0(\Sigma) \to \mathrm{CH}_0(X)$ is universally surjective.*

Proof. Indeed, the rational map $\Phi'_{|\Sigma\times\Sigma}$ is symmetric, that is factors through a rational map $\psi : \Sigma^{(2)} \dashrightarrow X$, and our assumption implies that the factored map ψ has odd degree N. It follows that $\psi : \mathrm{CH}_0(\Sigma_L^{(2)}) \to \mathrm{CH}_0(X_L)$ is surjective for any field L containing $\mathbb{C}$ because its cokernel is annihilated by N, with N odd, and also by 2, since X is a cubic of dimension ≥ 2, hence admits a unirational parametrization of degree 2. On the other hand, if $z \in \mathrm{CH}_0(\Sigma_L^{(2)})$ is of degree k, z provides a 0-cycle z' on Σ_L of degree $2k$, and we obviously have

$$\psi_*(z) = k\, h^4 - j_*(z') \text{ in } \mathrm{CH}_0(X_L),$$

where $j : \Sigma \to X$ is the inclusion map. It follows that the map $j_* : \mathrm{CH}_0(\Sigma_L) \to \mathrm{CH}_0(X_L)$ is surjective as well, since the class $h^4 \in \mathrm{CH}_0(X_L)$ belongs to the image of j_*. Indeed, it belongs to the image of $\mathrm{CH}_0(X_{\mathbb{C}}) = \mathbb{Z}x_0 \to \mathrm{CH}_0(X_L)$ for any point $x_0 \in X(\mathbb{C})$, which one can take in $\Sigma(\mathbb{C})$. □

The next lemma relates the degree of $\Phi'_{|\Sigma\times\Sigma}$ to the discriminant $D(\sigma)$.

Lemma 5.8. *Let $\Sigma \subset X$ be a smooth surface in general position. Then the degree of the rational map $\Phi'_{|\Sigma\times\Sigma} : \Sigma \times \Sigma \dashrightarrow X$ is congruent to $D(\sigma)$ modulo 4, where $\sigma = [\Sigma]$.*

Proof. Let $x \in X$ be a general point of X and let

$$\pi_x : X \dashrightarrow \mathbb{P}^4$$

be the linear projection from x. To say that $(z, z') \in \Sigma^2$ satisfies $\Phi'(z, z') = x$ is equivalent to say that z, z' and x are collinear, or that $\pi_x(z) = \pi_x(z')$. As Σ is in general position,the restriction of π_x to Σ maps Σ to a surface Σ' which is smooth apart from finitely many double points corresponding to pairs $\{z, z'\}$ as above. It follows that the degree of $\Phi'_{|\Sigma\times\Sigma}$ is equal to twice the number N of these double points (this argument appears in [16, 7.2]). We now compare the geometry of the two immersions

$$\Sigma \subset X,\ \pi_{x,\Sigma} := \pi_{x|\Sigma} : \Sigma \to \mathbb{P}^4.$$

The two corresponding normal bundle exact sequences give

$$\begin{aligned} 0 \to T_\Sigma \to T_{X|\Sigma} \to N_{\Sigma/X} \to 0, \\ 0 \to T_\Sigma \to \pi_{x,\Sigma}^* T_{\mathbb{P}^4} \to N_{\Sigma/\mathbb{P}^4} \to 0, \end{aligned} \tag{60}$$

which by Whitney formula provides

$$\begin{aligned} c_2(T_\Sigma) = c_2(T_{X|\Sigma}) - c_2(N_{\Sigma/X}) + K_\Sigma \cdot c_1(N_{\Sigma/X}), \\ c_2(T_\Sigma) = c_2(\pi_{x,\Sigma}^* T_{\mathbb{P}^4}) - c_2(N_{\Sigma/\mathbb{P}^4}) + K_\Sigma \cdot c_1(N_{\Sigma/\mathbb{P}^4}). \end{aligned} \tag{61}$$

We now use the equalities

$$c_2(T_{X|\Sigma}) = 6h_{|\Sigma}^2,\ c_2(\pi_{x,\Sigma}^* T_{\mathbb{P}^4}) = 10h_{|\Sigma}^2,$$

$$c_1(N_{\Sigma/X}) = K_\Sigma + 3h_{|\Sigma},\ c_1(N_{\Sigma/\mathbb{P}^4}) = K_\Sigma + 5h_{|\Sigma}$$

together with

$$c_2(N_{\Sigma/X}) = \sigma^2,\ \sigma = [\Sigma] \in H^4(X, \mathbb{Z}),$$

$$c_2(N_{\Sigma/\mathbb{P}^4}) = (h_{|\Sigma}^2)^2 - 2N = (\sigma \cdot h^2)^2 - 2N.$$

Thus (61) becomes

$$c_2(T_\Sigma) = 6h^2 \cdot \sigma - \sigma^2 + K_\Sigma \cdot (K_\Sigma + 3h), \tag{62}$$
$$c_2(T_\Sigma) = 10h^2 \cdot \sigma - (\sigma \cdot h^2)^2 + 2N + K_\Sigma \cdot (K_\Sigma + 5h).$$

We now add these two equalities and consider the result modulo 4, which gives

$$2c_2(T_\Sigma) = -\sigma^2 - (\sigma \cdot h^2)^2 + 2N + 2K_\Sigma^2 \bmod 4. \tag{63}$$

As $2c_2(T_\Sigma) - 2K_\Sigma^2$ is divisible by 4 by Noether's formula, we conclude that

$$\sigma^2 + (\sigma \cdot h^2)^2 = 2N \bmod 4. \tag{64}$$

As $D(\sigma) = 3\sigma^2 - (\sigma \cdot h^2)^2$ is equal to $-\sigma^2 - (\sigma \cdot h^2)^2$ modulo 4, we proved that $2N = -D(\sigma) = D(\sigma) \bmod 4$. □

Combining Lemmas 5.8 and 5.7, we conclude that the map $\mathrm{CH}_0(\Sigma) \to \mathrm{CH}_0(X)$ is universally surjective, hence that $\mathrm{CH}_0(X)$ is universally trivial. □

Remark 5.9. If one looks at the proof of the integral Hodge conjecture for cubic fourfolds given in [32], one easily sees that it gives more, namely: the group of Hodge classes of degree 4 on a cubic fourfold is generated by classes of rational surfaces. Thus the surface Σ above can be chosen rational. However, Lemma 5.8 does not allow us to conclude that if $D(\sigma)$ is not divisible by 4, X admits a unirational parametrization of odd degree. Indeed, the rational surface produced by the construction of [32] will be presumably singular, and not in general position.

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