On the coniveau of rationally connected threefolds

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Abstract

We prove that the *integral* cohomology modulo torsion of a rationally connected threefold comes from the integral cohomology of a smooth curve via the cylinder homomorphism associated to a family of 1-cycles. Equivalently, it is of *strong* coniveau 1. More generally, for a rationally connected manifold X of dimension n, we show that the strong coniveau $\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})$ and coniveau $N^{n-2}H^{2n-3}(X,\mathbb{Z})$ coincide for cohomology modulo torsion.

0 Introduction

We work over $\mathbb{C}$ and cohomology is Betti cohomology. Given an abelian group A, recall that a cohomology class $\alpha \in H^k(X,A)$ has coniveau $\geq c$ if $\alpha_{|U} = 0$ for some Zariski open set $U = X \setminus Y$, with $\operatorname{codim} Y \geq c$. Equivalently, α comes from the relative cohomology $H^k(X,U,A)$. If X is smooth projective of dimension n and $A = \mathbb{Z}$, using Poincaré duality, $\alpha \in H_{2n-k}(X,\mathbb{Z})$ comes from a homology class on Y

$$\alpha = j_*\beta \text{ in } H_{2n-k}(X,\mathbb{Z}), \tag{1}$$

for some $\beta \in H_{2n-k}(Y,\mathbb{Z})$. In the situation above, the closed algebraic subset Y cannot in general taken to be smooth. Take for example a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ with n odd, $n \geq 3$. Then $\rho(X) = 1$ and by the Lefschetz theorem on hyperplane sections, for any smooth hypersurface $Y \subset X$, $H^{n-2}(Y,\mathbb{Z}) = 0$, so no degree n cohomology class on X is supported on a smooth hypersurface. One can wonder however if, in the situation above, after taking a desingularization $\tau : \widetilde{Y} \to Y$ of Y, with composite map $\tilde{j} = j \circ \tau : \widetilde{Y} \to X$, one can rewrite (1) in the form

$$\alpha = \tilde{j}_*\tilde{\beta} \text{ in } H_{2n-k}(X,A). \tag{2}$$

In the situation described above, when X is smooth projective, Deligne [7] shows that, with $\mathbb{Q}$-coefficients,

$$\operatorname{Im}(\tilde{j}_* : H_{2n-k}(\widetilde{Y},\mathbb{Q}) \to H_{2n-k}(X,\mathbb{Q})) = \operatorname{Im}(j_* : H_{2n-k}(Y,\mathbb{Q}) \to H_{2n-k}(X,\mathbb{Q})),$$

so that the answer is yes with $\mathbb{Q}$-coefficients. With $\mathbb{Z}$-coefficients, this is wrong, as shows the following simple example: Let $j' : \widetilde{C} \hookrightarrow A$ be a smooth genus 2 curve in an abelian surface. Let $\mu_2 : A \to A$ be the multiplication by 2 and let $C = \mu_2(\widetilde{C}) \subset A$, with inclusion map $j : C \to A$. As $j(C)$ is an ample curve, the Lefschetz theorem on hyperplane sections says that $j_* : H_1(C,\mathbb{Z}) \to H_1(A,\mathbb{Z})$ is surjective. However, C admits $\tilde{j} := \mu_2 \circ j' : \widetilde{C} \to A$ as normalization and the map $\tilde{j}_* : H_1(\widetilde{C},\mathbb{Z}) \to H_1(A,\mathbb{Z})$ is not surjective as $\tilde{j}_* = 2j'_*$ so $\operatorname{Im}\tilde{j}_*$ is contained in $2H_1(A,\mathbb{Z})$.

In this example, the degree 1 homology of A (or degree 3 cohomology) is however supported on smooth curves. To follow the terminology introduced by Benoist and Ottem in [2], let us say that a cohomology class $\alpha \in H^k(X,\mathbb{Z})$ on a smooth projective complex manifold X is of strong coniveau $\geq c$ if there exists a *smooth* projective manifold of dimension $n-c$, and a morphism $f : Y \to X$ such that $\alpha = j_*\beta$ for some cohomology class $\beta \in H^{k-2c}(Y,\mathbb{Z})$. ($Y$ being smooth, we can apply Poincaré duality and use the Gysin morphism in cohomology.) Benoist and Ottem prove the following result.

Theorem 0.1. *(Benoist-Ottem, [2]) If $c \geq 1$ and $k \geq 2c+1$, there exist complex projective manifolds X and integral cohomology classes of degree k on X which are of coniveau $\geq c$ but not of strong coniveau $\geq c$.*

Their construction however imposes restrictions on the dimension of X and for example, the case where $k = 3$, $c = 1$, $\dim X = 3$ remains open. For $c = 1$, the examples constructed in [2] are varieties of general type.

We study in this paper the case of rationally connected 3-folds (and more generally degree 3 homology of rationally connected manifolds). As we will recall in Section 3, the integral cohomology of degree > 0 of a smooth complex projective rationally connected manifold is of coniveau ≥ 1. However, except in specific cases, there are no general available results for the strong coniveau. Our main result is the following.

Theorem 0.2. *Let X be a smooth projective rationally connected threefold over $\mathbb{C}$. Then the cohomology $H^3(X, \mathbb{Z})$ modulo torsion has strong coniveau 1.*

It turns out that an equivalent formulation is the following

Corollary 0.3. *If X is a rationally connected threefold, there exist a smooth curve C and a family of 1-cycles $\mathcal{Z} \in \mathrm{CH}^2(C \times X)$ such that the cylinder homomorphism $[\mathcal{Z}]_* : H^1(C, \mathbb{Z}) \to H^3(X, \mathbb{Z})_{\mathrm{tf}}$ is surjective.*

Here and in the sequel, we denote $\Gamma_{\mathrm{tf}} := \Gamma/\mathrm{Torsion}$ for any abelian group Γ.

Proof. (See more generally Proposition 1.3.) Theorem 0.2 says that there exist a smooth projective surface Σ and a morphism $f : \Sigma \to X$ such that $f_* : H^1(\Sigma, \mathbb{Z}) \to H^3(X, \mathbb{Z})_{\mathrm{tf}}$ is surjective. The existence of a Poincaré divisor $\mathcal{D} \in \mathrm{CH}^1(\mathrm{Pic}^0(\Sigma) \times \Sigma)$, satisfying the property that $[\mathcal{D}]_* : H_1(\mathrm{Pic}^0(\Sigma), \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z})$ is an isomorphism, provides a codimension 2-cycle

$$\mathcal{Z} = (Id, f)_*(\mathcal{D}) \in \mathrm{CH}^2(\mathrm{Pic}^0(\Sigma) \times X)$$

such that

$$[\mathcal{Z}]_* : H_1(\mathrm{Pic}^0(\Sigma), \mathbb{Z}) \to H^3(X, \mathbb{Z})_{\mathrm{tf}}$$

is surjective. We finally choose any smooth curve C complete intersection of ample hypersurfaces in $\mathrm{Pic}^0(\Sigma)$ and restrict $\mathcal{Z}$ to C. The corollary then follows by the Lefschetz hyperplane section theorem applied to $C \subset \mathrm{Pic}^0(\Sigma)$. □

This theorem will be proved in Section 2.3. We will prove in fact more generally (see Theorem 2.19)

Theorem 0.4. *For any rationally connected smooth projective variety of dimension n, one has the equality*

$$N^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}} = \widetilde{N}^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}.$$

Furthermore, the equality

$$H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}} = \widetilde{N}^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}$$

holds assuming that the Abel-Jacobi map $\Phi_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J^{2n-3}(X)$ is injective on torsion.

This last assumption, which is automatically satisfied when $n = 3$, is related to the following question mentioned in [17, 1.3.3]).

Question 0.5. *Let X be a rationally connected manifold. Is the Abel-Jacobi map $\Phi_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J^{2n-3}(X)$ injective on torsion cycles?*

Note that, as explained in *loc. cit.*, the group

$$\mathrm{Tors}(\mathrm{Ker}\,(\Phi_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J^{2n-3}(X)))$$

is a stable birational invariant of projective complex manifolds X, which is trivial when X admits a Chow decomposition of the diagonal. Results of Suzuki [15] give a complete understanding of this birational invariant in terms of coniveau (see also Section 2.1).

In Section 1, we discuss various notions of coniveau in relation to rationality or stable rationality questions, which we will need to split the statement of the main theorems into two different statements. In particular we introduce the "cylinder homomorphism filtration" $N_{c,\mathrm{cyl}}$, which has a strong version $\widetilde{N}_{c,\mathrm{cyl}}$. The cylinder homomorphism filtration $N_{c,\mathrm{cyl}}H_{k+2c}(X,\mathbb{Z})$ on the homology (or cohomology) of a smooth projective manifold X uses proper flat families $\mathcal{Z} \to Z$ of subschemes of X of dimension c, and the associated cylinder map $H_k(Z,\mathbb{Z}) \to H_{k+2c}(X,\mathbb{Z})$, which by flatness can be defined without any smoothness assumption on Z. The strong version $\widetilde{N}_{c,\mathrm{cyl}}H_{k+2c}(X,\mathbb{Z})$ is similar but imposes the smoothness assumption to Z (so flatness is not needed anymore). When $c = 1$, it is better to use the stable-cylinder filtration $N_{1,\mathrm{cyl,st}}H_{k+2}(X,\mathbb{Z})$ (where X is mooth projective of dimension n), which is generated by the cylinder homomorphisms

$$H_k(Z,\mathbb{Z}) \to H_{k+2}(X,\mathbb{Z})$$

for all families of semi-stable maps from curves to X, without smoothness assumption on Z (but we will assume that $\dim Z \leq k$). These various filtrations and their inclusions are discussed in Section 1. We prove in Section 2.2 Theorem 2.5, which is the first step towards the proof of Theorem 0.2, and in dimension 3 says the following.

Theorem 0.6. *(Cf. Corollary 2.7) Restricting to the torsion free part of cohomology, one has the equality*

$$N_{1,\mathrm{cyl,st}}H^3(X,\mathbb{Z})_{\mathrm{tf}} = N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}$$

for any smooth projective complex threefold X.

The second step of the proof of Theorem 0.2 is the following result, now valid for rationally connected manifolds of any dimension and also for the torsion part of homology.

Theorem 0.7. *(Cf. Theorem 2.17) Let X be rationally connected smooth projective over $\mathbb{C}$. Then*

$$N_{1,\mathrm{cyl,st}}H^{2n-3}(X,\mathbb{Z}) = \widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z}). \tag{3}$$

Equivalently, $N_{1,\mathrm{cyl,st}}H^{2n-3}(X,\mathbb{Z}) = \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})$.

Thanks. *I thank Fumiaki Suzuki for reminding me his results for* 1*-cycles in [15], which improved Theorem 0.4 and removed an assumption in Theorem 2.19.*

1 Various notions of niveau and coniveau

We are going to discuss in this section another filtration on cohomology, namely the (strong) cylinder homomorphism filtration (which is better understood in homology, so that we will speak of niveau) with emphasis on the niveau 1. It is particularly interesting in the case of niveau 1 because we will be able in this case to extract from this definition further stable birational invariants, which is not the case for higher niveau. We will work with Betti cohomology with integral coefficients and our varieties X will be smooth projective of dimension n over $\mathbb{C}$. We already mentioned in the introduction the coniveau filtration $N^cH^k(X,\mathbb{Z})$ and the strong coniveau filtration $\widetilde{N}^cH^k(X,\mathbb{Z})$. By definition, $\widetilde{N}^cH^k(X,\mathbb{Z})$ is generated by the images $\Gamma_*H^{k-2c}(Y,\mathbb{Z})$, for all smooth projective varieties Y of dimension

$n-c$ and all morphisms $\Gamma : Y \to X$, or more generally codimension n correspondences $\Gamma \in \mathrm{CH}^n(Y \times X)$.

We now introduce a different filtration,

$$\widetilde{N}_{c,\mathrm{cyl}} H^k(X, \mathbb{Z}) \subset H^k(X, \mathbb{Z}), \tag{4}$$

that we will call the strong cylinder homomorphism filtration (see [13]).

Definition 1.1. *We denote by $\widetilde{N}_{c,\mathrm{cyl}} H^k(X, \mathbb{Z}) \subset H^k(X, \mathbb{Z})$ the subgroup of $H^k(X, \mathbb{Z})$ generated by the images of the cylinder homomorphisms*

$$\Gamma_* : H_{2n-k-2c}(Z, \mathbb{Z}) \to H_{2n-k}(X, \mathbb{Z}) = H^k(X, \mathbb{Z}), \tag{5}$$

for all smooth projective varieties Z and correspondences $\Gamma \in \mathrm{CH}^{n-c}(Z \times X)$.

We will occasionally use the notation $\widetilde{N}_{c,\mathrm{cyl}} H_k(X, \mathbb{Z}) \subset H_k(X, \mathbb{Z})$ for the corresponding filtration on homology, which is in fact more natural. We can think to Γ as a family of cycles of dimension c in X parameterized by Z.

Lemma 1.2. *We have $\widetilde{N}_{c,\mathrm{cyl}} H^k(X, \mathbb{Z}) \subset \widetilde{N}^{k+c-n} H^k(X, \mathbb{Z})$. In particular, for $k = n$, we have $\widetilde{N}_{1,\mathrm{cyl}} H^n(X, \mathbb{Z}) \subset \widetilde{N}^1 H^n(X, \mathbb{Z})$.*

Proof. In the definition 1.1, we observe that, as Z is smooth, by the Lefschetz theorem on hyperplane section, its homology of degree $2n-k-2c$ is supported on smooth subvarieties Z' of Z of dimension $\leq 2n-k-2c$. It follows that we can restrict in (5) to the case where $\dim Z \leq 2n-k-2c$. The inclusion $\widetilde{N}_{c,\mathrm{cyl}} H^k(X, \mathbb{Z}) \subset \widetilde{N}^{k+c-n} H^k(X, \mathbb{Z})$ then follows from the fact that, by desingularization, cycles $\Gamma \in \mathrm{CH}^{n-c}(Z \times X)$ can be chosen to be represented by combinations with integral coefficients of smooth projective varieties Γ_i mapping to $Z \times X$, so that

$$\mathrm{Im}\, \Gamma_* \subset \sum_i \mathrm{Im}\, \Gamma_{i*}.$$

As $\dim Z \leq 2n-k-2c$ and $\mathrm{codim}\, (\Gamma_i / Z \times X) = n-c$, we have $\dim \Gamma_i \leq 2n-k-c$, so that, by definition,

$$\mathrm{Im}\, \Gamma_{i*} \subset \widetilde{N}^{k+c-n} H^k(X, \mathbb{Z}).$$

□

With $\mathbb{Q}$-coefficients, the definition (4) appears in [16]. For $k = n$ and $\mathbb{Q}$-coefficients, the Lefschetz standard conjecture for smooth projective varieties Y of dimension $n-c$ and for degree $n-2c$ predicts that

$$\widetilde{N}_{c,\mathrm{cyl}} H^n(X, \mathbb{Q}) = \widetilde{N}^c H^n(X, \mathbb{Q}). \tag{6}$$

Indeed, the hard Lefschetz theorem gives for any smooth projective variety Y of dimension $n-c$ the hard Lefschetz isomorphism

$$L^c : H^{n-2c}(Y, \mathbb{Q}) \cong H^n(Y, \mathbb{Q})$$

where the Lefschetz operator L is the cup-product operator with the class $c_1(H)$ for some very ample divisor H on Y, and the Lefschetz standard conjecture predicts the existence of a codimension-$n-2c$ cycle $\mathcal{Z}_{\mathrm{Lef}} \in \mathrm{CH}^{n-2c}(Y \times Y)_{\mathbb{Q}}$ such that

$$[\mathcal{Z}_{\mathrm{Lef}}]_* \circ L^c = Id : H^{n-2c}(Y, \mathbb{Q}) \to H^{n-2c}(Y, \mathbb{Q}).$$

Restricting $\mathcal{Z}_{\mathrm{Lef}}$ to $Z \times Y$, where $Z \subset Y$ is a smooth complete intersection of c ample hypersurfaces in $|H|$, we get a cycle

$$\mathcal{Z}'_{\mathrm{Lef}} \in \mathrm{CH}^{n-2c}(Z \times Y)_{\mathbb{Q}}$$

such that

$$[\mathcal{Z}'_{\mathrm{Lef}}]_* : H^{n-2c}(Z,\mathbb{Q}) \to H^{n-2c}(Y,\mathbb{Q})$$

is surjective. In other words, the Lefschetz standard conjecture predicts that

$$H^{n-2c}(Y,\mathbb{Q}) = \widetilde{N}_{c,\mathrm{cyl}} H^{n-2c}(Y,\mathbb{Q})$$

for Y smooth projective of dimension $n-c$. Coming back to X, any class α in $\widetilde{N}^c H^n(X,\mathbb{Q})$ is of the form $\Gamma_*\beta$ for some class $\beta \in H^{n-2c}(Y,\mathbb{Q})$, for some smooth (nonnecessarily connected) projective variety Y of dimension $n-c$, and the previous construction shows that, assuming the Lefschetz standard conjecture for Y, one has

$$\alpha = [\Gamma \circ \mathcal{Z}'_{\mathrm{Lef}}]_* \gamma$$

for some $\gamma \in H^{n-2c}(Z,\mathbb{Q})$, where Z is constructed as above. As $\Gamma \circ \mathcal{Z}'_{\mathrm{Lef}} \in \mathrm{CH}^{n-c}(Z \times X)_{\mathbb{Q}}$, where Z is smooth projective of dimension $\dim n - 2c$, this proves the equality (6).

Coming back to $\mathbb{Z}$-coefficients, there is one case where $\widetilde{N}_{\mathrm{cyl}} H^k(X,\mathbb{Z})$ and $\widetilde{N} H^k(X,\mathbb{Z})$ exactly compare, namely

Proposition 1.3. *We have, for any c and any smooth projective variety X of dimension n,*

$$\widetilde{N}_{n-c,\mathrm{cyl}} H^{2c-1}(X,\mathbb{Z}) = \widetilde{N}^{c-1} H^{2c-1}(X,\mathbb{Z}) \tag{7}$$

Proof. The inclusion $\subset$ follows from Lemma 1.2. For the reverse inclusion, $\widetilde{N}^{c-1} H^{2c-1}(X,\mathbb{Z})$ is by definition generated by the groups $\Gamma_* H^1(Y,\mathbb{Z})$, for all smooth projective Y of dimension $n-c+1$ and all correspondences $\Gamma \in \mathrm{CH}^n(Y \times X)$. For each such Y, there exists a Poincaré (or universal) divisor

$$\mathcal{D} \in \mathrm{CH}^1(\mathrm{Pic}^0(Y) \times Y)$$

such that

$$[\mathcal{D}]_* : H_1(\mathrm{Pic}^0(Y),\mathbb{Z}) \to H^1(Y,\mathbb{Z})$$

is the natural isomorphism. (We identify here $\mathrm{Pic}^0(Y)$ with the intermediate Jacobian $J^1(Y) = H^{0,1}(Y)/H^1(Y,\mathbb{Z})$ via the Abel map.) Let now

$$\mathcal{Z} := (Id,\Gamma)_* \mathcal{D} \in \mathrm{CH}^c(\mathrm{Pic}^0(Y) \times X).$$

We have

$$[\mathcal{Z}]_* = [\Gamma]_* \circ [\mathcal{D}]_* : H_1(\mathrm{Pic}^0(Y),\mathbb{Z}) \to H^{2c-1}(X,\mathbb{Z})$$

and it has the same image as $[\Gamma]_*$. Thus we proved that $\widetilde{N}^{c-1} H^{2c-1}(X,\mathbb{Z})$ is generated by cylinder homomorphisms associated to families of cycles in X of dimension $n-c$ parameterized by a smooth basis. □

Note that for $c = n-1$, Proposition 1.3 applies to degree $2n-3$ cohomology, that is, degree 3 homology, which we will be considering in next section.

The niveau 1 of the cylinder filtration produces stable birational invariants. The following result strengthens the corresponding statement for strong coniveau in [2].

Proposition 1.4. *The quotient $H_k(X,\mathbb{Z})/\widetilde{N}_{1,\mathrm{cyl}} H_k(X,\mathbb{Z})$ is a stable birational invariant of a smooth projective variety X.*

Proof. The invariance under the relation $X \sim X \times \mathbb{P}^r$ is obvious by the projective bundle formula which shows that $H_k(X \times \mathbb{P}^r,\mathbb{Z}) = H_k(X,\mathbb{Z}) + \widetilde{N}_{1,\mathrm{cyl}} H_k(X \times \mathbb{P}^r,\mathbb{Z})$, so that $\mathrm{pr}_{X*} : H_k(X \times \mathbb{P}^r,\mathbb{Z}) \to H_k(X,\mathbb{Z})$ is an isomorphism modulo $\widetilde{N}_{1,\mathrm{cyl}}$. It remains to prove the invariance under birational maps. In fact, it suffices to prove the invariance under blow-ups along smooth centers, because the considered groups admit both contravariant functorialities under pull-backs and covariant functoriality under proper push-forwards for generically finite maps (see [17, Lemma 1.9]). For a blow-up $\tau : \widetilde{X} \to X$ the standard formulas show that $H_k(\widetilde{X},\mathbb{Z}) = \tau^* H_k(X,\mathbb{Z}) + \widetilde{N}_{1,\mathrm{cyl}} H_k(\widetilde{X},\mathbb{Z})$, so that $\tau_* : H_k(\widetilde{X},\mathbb{Z}) \to H_k(X,\mathbb{Z})$ is an isomorphism modulo $\widetilde{N}_{1,\mathrm{cyl}}$. □

The following result is a motivation for introducing Definition 1.1.

Proposition 1.5. *Let X be a smooth projective variety admitting a cohomological decomposition of the diagonal. Then for any k such that $2n > k > 0$,*

$$\widetilde{N}_{1,\mathrm{cyl}}H^k(X,\mathbb{Z}) = H^k(X,\mathbb{Z}) = \widetilde{N}^1H^k(X,\mathbb{Z}).$$

In particular, these equalities hold if X is stably rational.

Proof. The second equality already appears in [2]. Both equalities follow from [20], where the following result is proved.

Theorem 1.6. *If a smooth projective variety X of dimension n admits a cohomological decomposition of the diagonal, there exist smooth projective varieties Z_i of dimension $n-2$, integers n_i, and correspondences $\Gamma_i \in \mathrm{CH}^{n-1}(Z_i \times X)$ such that, choosing a point $x \in X$,*

$$[\Delta_X - x \times X - X \times x] = \sum_i n_i(\Gamma_i, \Gamma_i)_*[\Delta_{Z_i}] \text{ in } H^{2n}(X \times X, \mathbb{Z}). \tag{8}$$

In (8), the correspondence (Γ_i, Γ_i) between $Z_i \times Z_i$ and $X \times X$ is defined as $\mathrm{pr}_1^*\Gamma_i \cdot \mathrm{pr}_2^*\Gamma_i$, where we identify $Z_i \times Z_i \times X \times X$ with $Z_i \times X \times Z_i \times X$, which defines the two projections

$$\mathrm{pr}_1,\ \ \mathrm{pr}_2 : Z_i \times Z_i \times X \times X \to Z_i \times X.$$

Another way to formulate (8) is obtained by introducing the transpose ${}^t\Gamma_i \in \mathrm{CH}^{n-1}(X \times Z_i)$, which satisfies ${}^t\Gamma_{i*} = \Gamma_i^*$. Then (8) is equivalent to the equality of cohomological self-correspondences of X

$$[\Delta_X - X \times x - x \times X] = \sum_i n_i[\Gamma_i \circ {}^t\Gamma_i] \text{ in } H^{2n}(X \times X, \mathbb{Z}). \tag{9}$$

Applying both sides of (9) to any $\alpha \in H^{0<*<2n}(X,\mathbb{Z})$, we get

$$\alpha = \sum_i n_i[\Gamma_i]_* \circ [\Gamma_i]^*\alpha \text{ in } H^*(X,\mathbb{Z}),$$

With $[\Gamma_i]^*\alpha \in H^{*-2}(Z_i,\mathbb{Z})$. As $\dim Z_i = n-2$ and $\dim \Gamma_i = n-1$, this proves that $\alpha \in \widetilde{N}_{1,\mathrm{cyl}}H^*(X,\mathbb{Z})$ and $\alpha \in \widetilde{N}^1H^*(X,\mathbb{Z})$. □

Remark 1.7. Although Theorem 1.6 is stated in [20] only in the cohomological setting, it is true as well, with the same proof, in the Chow setting, see [14]. The same proof as above thus gives the following result.

Theorem 1.8. *If X admits a Chow decomposition of the diagonal, there exist correspondences $\Gamma_i \in \mathrm{CH}^{n-1}(Z_i \times X)$ and integers n_i, such that*

$$\Gamma_i^* : \mathrm{CH}^{n>*>0}(X) \to \oplus_i \mathrm{CH}^{*-1}(Z_i)$$

has for left inverse $\sum_i n_i\Gamma_{i*}$. In particular $\sum_i n_i\Gamma_{i*} : \oplus\mathrm{CH}^*(Z_i) \to \mathrm{CH}^{*+1}(X)$ is surjective for $n-2 \geq * \geq 0$.

Corollary 1.9. *If X admits a Chow decomposition of the diagonal, the Chow groups $\mathrm{CH}^i(X)$ for $0 < i < n$ satisfy*

$$\widetilde{N}_{1,\mathrm{cyl}}\mathrm{CH}^i(X) = \mathrm{CH}^i(X) = \widetilde{N}^1\mathrm{CH}^i(X),$$

where the definition of strong coniveau and cylinder niveau is extended to Chow groups in the obvious way.

Proposition 1.5 works as well with $\mathbb{Q}$-coefficients, so we get in this case:

Proposition 1.10. *Let X be a smooth projective variety admitting a cohomological decomposition of the diagonal with rational coefficients. Then for any k such that $2n > k > 0$,*

$$\widetilde{N}_{1,\mathrm{cyl}}H^k(X,\mathbb{Q}) = H^k(X,\mathbb{Q}) = \widetilde{N}^1H^k(X,\mathbb{Q}).$$

In particular, these equalities hold if X is rationally connected.

We now introduce a weaker notion, namely the cylinder homomorphism filtration. Let X be a smooth projective manifold of dimension n. We have $H_k(X,\mathbb{Z}) \cong H^{2n-k}(X,\mathbb{Z})$ by Poincaré duality.

Definition 1.11. *We define $N_{c,\mathrm{cyl}}H^k(X,\mathbb{Z})$ as the group generated by the cylinder homomorphisms*

$$f_* \circ p^* : H_{2n-k-2c}(Z,\mathbb{Z}) \to H_{2n-k}(X,\mathbb{Z}) \cong H^k(X,\mathbb{Z}),$$

for all morphisms $f : Y \to X$, and flat projective morphisms $p : Y \to Z$ of relative dimension c, where $\dim Z \leq 2n-k-2c$.

In this definition, the morphism $p^* : H_{2n-k-2c}(Z,\mathbb{Z}) \to H_{2n-k}(Y,\mathbb{Z})$ is obtained at the chain level by taking the inverse image p^{-1} under the flat map p. Note that we do not ask here smoothness of Z, and this is the main difference with Definition 1.1. It is obvious that

$$N_{c,\mathrm{cyl}}H^k(X,\mathbb{Z}) \subset N^{k+c-n}H^k(X,\mathbb{Z})$$

because with the above notation, one has $\dim Y \leq 2n-k-c$. Restricting to the case where Z is smooth, we claim that

$$\widetilde{N}_{c,\mathrm{cyl}}H^k(X,\mathbb{Z}) \subset N_{c,\mathrm{cyl}}H^k(X,\mathbb{Z}).$$

Indeed, $\widetilde{N}_{c,\mathrm{cyl}}H^k(X,\mathbb{Z})$ is generated by images of correspondences $f_* \circ p^* : H^{k-2c}(Z,\mathbb{Z}) \to H^k(X,\mathbb{Z})$ for all morphisms $f : Y \to Z$, where Y is smooth and projective, and morphisms $p : Y \to Z$ of relative dimension c, where $\dim Z = n-2c$. By flattening, there exists a commutative diagram

$$\begin{array}{ccc} Y' & \stackrel{\tau_Y}{\to} & Y \\ p' \downarrow & & p \downarrow, \\ Z' & \stackrel{\tau_Z}{\to} & Z \end{array}$$

where $\tau_Y : Y' \to Y$ is proper birational, Z' is smooth and $\tau_Z : Z' \to Z$ is proper birational, and $p' : Y' \to Z'$ is flat. Then we have, denoting $f' := f \circ \tau_Y$

$$f'_* \circ p'^* = f_* \circ p^* \circ \tau_{Z*} : H_{2n-2c-k}(Z',\mathbb{Z}) \to H_{2n-k}(X,\mathbb{Z}).$$

The map $\tau_{Z*} : H_{2n-2c-k}(Z',\mathbb{Z}) \to H_{2n-2c-k}(Z,\mathbb{Z}) = H^{k-2c}(Z,\mathbb{Z})$ is surjective since Z is smooth and τ_Z is proper birational, hence we conclude that $\mathrm{Im}\, f_* \circ p^* \subset .\mathrm{Im}\, f'_* \circ p'^*$, proving the claim.

In conclusion we have the chain of inclusions

$$\widetilde{N}_{c,\mathrm{cyl}}H^k(X,\mathbb{Z}) \subset N_{c,\mathrm{cyl}}H^k(X,\mathbb{Z}) \subset N^{k+c-n}H^k(X,\mathbb{Z}). \tag{10}$$

We are concerned in the paper with the niveau 1 of the cylinder filtration, which is parameterized by curves. In this case, we can use the following variant of the cylinder homomorphism filtration. It has the advantage that we can apply to it the beautiful results we know about the deformation theory of morphisms from semistable curves (see [10]), while the local study of the Hilbert scheme, even for curves on threefolds, is hard.

Definition 1.12. *We define $N_{1,\mathrm{cyl,st}}H^k(X,\mathbb{Z})$ as the group generated by the cylinder homomorphisms*

$$f_* \circ p^* : H_{2n-k-2}(Z,\mathbb{Z}) \to H_{2n-k}(X,\mathbb{Z}) \cong H^k(X,\mathbb{Z}),$$

for all morphisms $f : Y \to X$, and projective flat semi-stable morphisms $p : Y \to Z$ of relative dimension 1, where $\dim Z \leq 2n-k-2$.

The relationships between the definitions 1.11 and 1.12 is not straightforward, since semi-stable reduction of a general flat morphism $f : Y \to Z$ of relative dimension 1 will not exist on Z but after base change, which will change the homology of Z. One may expect however that the two definitions coincide.

We conclude this section with the case of the smooth Fano complete intersections

$$X = \cap_{i=1}^{N-n} Y_i \subset \mathbb{P}^N,$$

with $\deg Y_i = d_i$ and $\sum_i d_i \leq N$. Given such a smooth n-dimensional variety X, let $F(X) \subset G(2, N+1)$ be its Fano variety of lines. Being the zero-locus of a general section of a globally generated vector bundle on the Grassmannian of lines $G(2, N+1)$, $F(X)$ is smooth for general X. The universal family of lines

$$\begin{array}{ccc} P & \stackrel{q}{\longrightarrow} & X \\ p \downarrow & & \\ F(X) & & \end{array}$$

provides a "cylinder homomorphism"

$$P_* = q_* \circ p^* : H_{n-2}(F(X), \mathbb{Z}) \to H_n(X, \mathbb{Z}) = H^n(X, \mathbb{Z}). \tag{11}$$

When $F(X)$ is smooth, we can choose a dimension $n-2$ smooth complete intersection $Z \stackrel{j}{\hookrightarrow} F(X)$ of ample hypersurfaces. Then by Lefschetz theorem on hyperplane sections,

$$j_* : H_{n-2}(Z, \mathbb{Z}) \to H_{n-2}(F(X), \mathbb{Z})$$

is surjective and thus $\operatorname{Im} P_* = \operatorname{Im} P_* \circ j_*$. By smoothness of Z, we can write $(P \circ j)_*$ in cohomology

$$(P \circ j)_* : H^{n-2}(Z, \mathbb{Z}) \to H^n(X, \mathbb{Z}).$$

It is then clear that $\operatorname{Im} P_*$ is contained in $\widetilde{N}_{1,\mathrm{cyl}} H^n(X, \mathbb{Z})$.

Theorem 1.13. *(i) For any smooth Fano complete intersection $X \subset \mathbb{P}^N$ of dimension n of hypersurfaces of degrees $d_1, \ldots, d_{N-n}$, the morphism P_* of (11) is surjective.*

(ii) We have $N_{1,\mathrm{cyl,st}} H^n(X, \mathbb{Z}) = H^n(X, \mathbb{Z})$.

(iii) If either $F(X)$ has the expected dimension $2N - 2 - \sum_i (d_1 + 1)$ and $\operatorname{Sing} F(X)$ is of codimension $\geq n-2$ in $F(X)$, or $\dim X = 3$, we have

$$H^n(X, \mathbb{Z}) = \widetilde{N}_{1,\mathrm{cyl}} H^n(X, \mathbb{Z}) = \widetilde{N}^1 H^n(X, \mathbb{Z}).$$

Note that (ii) is not directly implied by (i) when $F(X)$ is singular, because $\dim F(X)$ can be $> n-2$ and we cannot apply Lefschetz hard section theorem to reduce to a $Z \subset F(X)$ of dimension $n-2$.

Proof of Theorem 1.13. (i) We first prove

Claim 1.14. *It suffices to prove the surjectivity statement of (i) for a general smooth X for which the variety of lines $F(X)$ is smooth (or equivalently any such X).*

Proof. Indeed, let X_0 be a smooth complete intersection as above and choose a family $\mathcal{X} \to \Delta$ of smooth deformations X_t of X_0 parameterized by the disk, so that the general fiber $\mathcal{X}_t$ has its variety of lines $F(\mathcal{X}_t)$ smooth of the expected dimension. Then we can consider the corresponding family $\mathcal{F} \to \Delta$ of Fano varieties of lines, and we have the family of cylinder homomorphisms

$$P_* : H_{n-2}(\mathcal{F}_t, \mathbb{Z}) \to H_n(\mathcal{X}_t, \mathbb{Z}).$$

Now we observe that we can assume that we have a topological retraction $r_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}_0$, compatible via P with a topological retraction $r_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_0$. By smoothness, r_X induces a homeomorphism $\mathcal{X}_t \cong \mathcal{X}_0$, hence an isomorphism

$$r_{\mathcal{X}*} : H_n(\mathcal{X}_t, \mathbb{Z}) \cong H_n(\mathcal{X}_0, \mathbb{Z}).$$

As we have

$$r_{\mathcal{X}*} \circ P_* = P_* \circ r_{\mathcal{F}*} : H_{n-2}(\mathcal{F}_t, \mathbb{Z}) \to H_n(\mathcal{X}_0, \mathbb{Z}),$$

we see that the surjectivity of $P_* : H_{n-2}(\mathcal{F}_t, \mathbb{Z}) \to H_n(\mathcal{X}_t, \mathbb{Z})$ implies the surjectivity of $P_* : H_{n-2}(\mathcal{F}_0, \mathbb{Z}) \to H_n(\mathcal{X}_0, \mathbb{Z})$. □

The claim being proved, we now assume that $F(X)$ is smooth and we show that $P_* : H_{n-2}(F(X), \mathbb{Z}) \to H_n(X, \mathbb{Z})$ is surjective. We now claim that it suffices to prove that the primitive homology of X is in the image of P_*. If n is odd, the homology and primitive homology coincide so there is nothing to prove. If $n = 2m$, we observe that some special X which are smooth and with variety of lines smooth of the expected dimension, contain m-cycles W which are of degree 1 and whose class is in $\operatorname{Im} P_*$. For example, we choose X to have a m-dimensional linear sections which are the union of two cones over complete intersections in $\mathbb{P}^{N-m-1}$. Each component of the cone has its class contained in $\operatorname{Im} P_*$ so it suffices that the various degrees are coprime. The class $[W] \in H_n(X, \mathbb{Z})$ then maps via j_* to the generator of $H_n(\mathbb{P}^N, \mathbb{Z})$, where j is the inclusion map of X in $\mathbb{P}^N$, and by definition $\operatorname{Ker} j_* =: H_n(X, \mathbb{Z})_{\text{prim}}$. It is clear that $[\mathbb{P}^m]$ is in the image of P_*, so if the image of P_* contains $\operatorname{Ker} j_*$, it contains the whole of $H_n(X, \mathbb{Z})$, which proves the claim.

We next restrict as above the cylinder homomorphism to a smooth $Z \subset F(X)$ of dimension $n - 2$. We will now show that the image of $P_{Z*} : H_{n-2}(Z, \mathbb{Z}) \to H_n(X, \mathbb{Z})$ contains $H_n(X, \mathbb{Z})_{\text{prim}}$. By the theory of vanishing cycles [18, 2.1], it suffices to show that $\operatorname{Im} P_{Z*}$ contains one vanishing cycle, since they are all conjugate and generate $H_n(X, \mathbb{Z})_{\text{prim}}$. Let $Y \subset \mathbb{P}^{N+1}$ be a general smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{N-n}$, so that $\dim Y = n + 1$, Y is smooth, $F(Y)$ is smooth and Y is covered by lines. We choose a general complete intersection Z_Y of ample hypersurfaces $Z_Y \subset F(Y)$ with the following properties: one has $\dim Z_Y = n$, the restricted family of lines gives a dominating (generically finite) morphism $q_Y : P_Y \to Y$, and, letting $X \subset Y$ be a general hyperplane section, $F(X)$ is smooth of the expected dimension, and $Z_Y \cap F(X) =: Z$ is a smooth complete intersection in $F(X)$ as above. As X is chosen to be a general hyperplane section of Y, $X' := q_Y^{-1}(X) \subset P_Y$ is by Bertini a smooth hypersurface $X' \subset P_Y$. Furthermore the image of $q_{Y,X*} : H_n(X', \mathbb{Z}) \to H_n(X, \mathbb{Z})$ contains a vanishing cycle since when X has a nodal degeneration at a generic point y of Y, X' also acquiers a nodal degeneration at all the preimages of y in P_Y, assuming q_Y is étale over a neighborhood of y. (This argument appears in [4, p 2.14].) Finally, we observe that, via $p_Y : P_Y \to Z_Y$, X' identifies naturally with the blow-up of Z_Y along Z so that $H_n(X', \mathbb{Z}) = H_n(Z_Y, \mathbb{Z}) \oplus H_{n-2}(Z, \mathbb{Z})$, and that the image of the map $P_{Y*} : H_n(Z_Y, \mathbb{Z}) \to H_n(X, \mathbb{Z})$ is contained in the image of the restriction map $H_{n+2}(Y, \mathbb{Z}) \to H_n(X, \mathbb{Z})$ which is equal to $\mathbb{Z}h^m$ by the Lefschetz theorem on hyperplane sections. The fact that the image of $q_{Y,X*} : H_n(X', \mathbb{Z}) \to H_n(X, \mathbb{Z})$ contains a vanishing cycle thus implies that the image of $P_{Z*} : H_{n-2}(Z, \mathbb{Z}) \to H_n(X, \mathbb{Z})$ contains a vanishing cycle. Thus (i) is proved.

(ii) We modify the construction above as follows : first of all we replace $\mathcal{F} \to \Delta$ by a family $\mathcal{Z} \subset \mathcal{F}$ whose fiber over $t \in \Delta^*$ is a $n-2$-dimensional complete intersection $\mathcal{Z}_t \subset F(\mathcal{X}_t)$ of ample hypersurfaces. Finally, we replace $\mathcal{Z}$ by the union $\mathcal{Z}'$ of irreducible components of $\mathcal{Z}$ which dominate Δ. Then the central fiber $\mathcal{Z}'_0$ has dimension $n - 2$, and for the general fiber, we know by (i), by smoothness of $F(\mathcal{X}_t)$, and by the lefschetz theorem on hyperplane sections that the restriction P' of P to $\mathcal{Z}'$ has the property that $P'_{t*} : H_{n-2}(\mathcal{Z}'_t, \mathbb{Z}) \to H_n(\mathcal{X}_t, \mathbb{Z})$ is surjective. We then conclude as in the proof of Claim 1.14 that $P'_{0*} : H_{n-2}(\mathcal{Z}'_0, \mathbb{Z}) \to H_n(\mathcal{X}_0, \mathbb{Z})$ is surjective, and as $\dim \mathcal{Z}'_0 = n - 2$ and the fibers of $P'_0 \to \mathcal{Z}'_0$ are smooth, (ii) is proved.

(iii) The case where $\dim X = 3$ is a consequence of (ii) and of Theorem 2.17. Indeed, (ii) says that $H_3(X,\mathbb{Z}) = N_{1,\mathrm{cyl,st}}H_3(X,\mathbb{Z})$ and by Theorem 2.17, we thus have $H_3(X,\mathbb{Z}) = \widetilde{N}^1_{\mathrm{cyl}}H_3(X,\mathbb{Z})$, hence a fortiori $H_3(X,\mathbb{Z}) = \widetilde{N}^1 H_3(X,\mathbb{Z})$.

We now conclude the proof when $F(X)$ has the right dimension and $\mathrm{Sing}\, F(X)$ is of codimension $\geq n-2$ in $F(X)$. As the Fano variety of lines $F(X)$ has the right dimension, we know already by the proof of (ii) that if $Z \subset X$ is a general complete intersection of ample hypersurfaces which is of dimension $n-2$, the cylinder homomorphism $[P]_* : H_{n-2}(Z,\mathbb{Z}) \to H_n(X,\mathbb{Z})$ is surjective. Furthermore, the assumption on $\mathrm{Sing}\, F(X)$ implies that Z has isolated singularities. We now apply Proposition 3.4 proved in section 2.3, which says that $\mathrm{Im}\,([P]_* : H_{n-2}(Z,\mathbb{Z}) \to H_n(X,\mathbb{Z}))$ is contained in $\widetilde{N}_{1,\mathrm{cyl}}H^n(X,\mathbb{Z})$. Thus $\widetilde{N}_{1,\mathrm{cyl}}H^n(X,\mathbb{Z}) = H^n(X,\mathbb{Z})$ and a fortiori $\widetilde{N}^1 H^n(X,\mathbb{Z}) = H^n(X,\mathbb{Z})$ by Lemma 1.2. $\square$

Remark 1.15. Theorem 1.13 (i) is proved in [13] with $\mathbb{Q}$-coefficients.

2 Proof of Theorem 0.2

2.1 Abel-Jacobi map for 1-cycles

Let X be a smooth complex projective manifold of dimension n. For any smooth connected projective curve C and cycle $\mathcal{Z} \in \mathrm{CH}^{n-1}(C \times X)$, one has an Abel-Jacobi map

$$\Phi_{\mathcal{Z}} : J(C) \to J^{2n-3}(X), \tag{12}$$

$$z \mapsto \Phi_X(\mathcal{Z}_*(z)),$$

where $J^{2n-3}(X) = H^{2n-3}(X,\mathbb{C})/(F^{n-1}H^{2n-3}(X,\mathbb{C}) \oplus H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$. The morphism $\Phi_{\mathcal{Z}}$ is the morphism of complex tori associated with the morphism of Hodge structures

$$[\mathcal{Z}]_* : H^1(C,\mathbb{Z}) \to H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}. \tag{13}$$

By definition, the images of all morphisms $[\mathcal{Z}]_*$ as above generate $\widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$, and applying Proposition 1.3, we find that they generate as well $\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} \subset N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$.

Consider first the case of a general smooth projective threefold. As proved in [6], the group $H^3(X,\mathbb{Z})/N^1H^3(X,\mathbb{Z})$ has no torsion, as it injects into the unramified cohomology group $H^0(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z}))$, and the sheaf $\mathcal{H}^3(\mathbb{Z})$ has no torsion. It follows that the group $H^3(X,\mathbb{Z})_{\mathrm{tf}}/N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}$ has no torsion. The inclusion of lattices

$$N^1H^3(X,\mathbb{Z})_{\mathrm{tf}} \subset H^3(X,\mathbb{Z})_{\mathrm{tf}}$$

is a morphism of integral Hodge structures of weight 3 which, thanks to the fact that $H^3X,\mathbb{Z})_{\mathrm{tf}}/N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}$ has no torsion, induces an injection of the corresponding intermediate Jacobians

$$J(N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}) \hookrightarrow J(H^3(X,\mathbb{Z})_{\mathrm{tf}}) = J^3(X).$$

In higher dimension, it is observed by Walker [22] that the Abel-Jacobi map for 1-cycles

$$\Phi_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J^{2n-3}(X)$$

factors through a surjective morphism

$$\widetilde{\Phi}_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}) \tag{14}$$

where the intermediate Jacobian $J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$ is not in general a subtorus of $J(H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$. The point is that it is not necessarily the case for higher coniveau $n-2>1$ that

$$N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} \subset H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$$

is a saturated sublattice. We refer to [15] for the discussion of such phenomena. There is a related stable birational invariant, which is the torsion of the group

$$H^{2n-3}(X,\mathbb{Z})_{\rm tf}/N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}.$$

Concerning the Walker lift (**??**), Suzuki proves

Theorem 2.1. *[15] Let X be a rationally connected manifold of dimension n. Then the Walker Abel-Jacobi map $\widetilde{\Phi}_X : {\rm CH}_1(X)_{\rm alg} \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$ is injective on torsion.*

Let us come back to a general surjective morphism $\phi : A \to B$ of complex tori A, B that we represent as quotients

$$A = A_{0,1}/A_{\mathbb{Z}},\ \ B = B_{0,1}/B_{\mathbb{Z}},$$

of complex vector spaces by lattices, with induced morphisms

$$\phi_{\mathbb{Z}} = \phi_* : A_{\mathbb{Z}} \to B_{\mathbb{Z}},$$

$$\phi_{0,1} = \phi_* : A_{0,1} \to B_{0,1}$$

respectively on integral homology $H_1(\ ,\mathbb{Z})$ and on $H_{0,1}$-groups. The subgroup ${\rm Ker}\,\phi$ is a finite union of translates of the subtorus

$$K := {\rm Ker}\,\phi_{0,1}/{\rm Ker}\,\phi_{\mathbb{Z}}.$$

More precisely,

Lemma 2.2. *Let*

$$D_\phi := \{\alpha \in A_{\mathbb{Q}},\, \phi_{\mathbb{Q}}(\alpha) \in B_{\mathbb{Z}}\}. \tag{15}$$

Then (i) the group $T_\phi = D_\phi/A_{\mathbb{Z}}$ is isomorphic to the torsion subgroup of ${\rm Ker}\,\phi$ and

$$ {\rm Ker}\,\phi = K + T_\phi. \tag{16}$$

(ii) The group $T_\phi/{\rm Ker}\,\phi_{\mathbb{Q}}$ is isomorphic to the group of connected components of ${\rm Ker}\,\phi$.

Proof. (i) A torsion point of A is an element of $A_{\mathbb{Q}}/A_{\mathbb{Z}}$ and it is in ${\rm Ker}\,\phi$ when any of its lifts α in $A_{\mathbb{Q}}$ maps to $B_{\mathbb{Z}}$ via $\phi_{\mathbb{Q}}$. This proves the first statement. For the equality (15), as $K \subset {\rm Ker}\,\phi$ and $T_\phi \subset {\rm Ker}\,\phi$, we just have to show that ${\rm Ker}\,\phi \subset K + T_\phi$. The result has nothing to do with complex tori, as we can work as well with the corresponding real tori $A_{\mathbb{R}}/A_{\mathbb{Z}}$, $B_{\mathbb{R}}/B_{\mathbb{Z}}$ which are naturally isomorphic as real tori to A and B respectively. Let $t \in {\rm Ker}\,\phi$, and let $t_{\mathbb{R}}$ be a lift of t in $A_{\mathbb{R}}$. Then $\phi_{\mathbb{R}}(t) \in B_{\mathbb{Z}}$. Let $b_t = \phi_{\mathbb{R}}(t) \in B_{\mathbb{Z}}$ and let

$$K_{\mathbb{R},t} = \{v \in A_{\mathbb{R}},\, \phi_{\mathbb{R}}(v) = b_t\} \subset A_{\mathbb{R}}.$$

Then $K_{\mathbb{R},t}$ is affine, modeled on the vector space ${\rm Ker}\,\phi_{\mathbb{R}}$, contains $t_{\mathbb{R}}$, and is defined over $\mathbb{Q}$. Hence it has a rational point $t_{\mathbb{Q}}$ which belongs to D_ϕ and thus

$$t_{\mathbb{R}} = t_{\mathbb{Q}} + t'$$

with $t' \in {\rm Ker}\,\phi_{\mathbb{R}}$, which proves that $t \in K + T_\phi$ by projection modulo $A_{\mathbb{Z}}$ since $K = {\rm Ker}\,\phi_{\mathbb{R}}/{\rm Ker}\,\phi_{\mathbb{Z}}$.

(ii) We have ${\rm Tors}\,K = {\rm Ker}\,\phi_{\mathbb{Q}}/{\rm Ker}\,\phi_{\mathbb{Z}}$, so $T_\phi/{\rm Ker}\,\phi_{\mathbb{Q}}$ is isomorphic to ${\rm Tors}\,({\rm Ker}\,\phi)/{\rm Tors}\,K$. Using the fact that ${\rm Ker}\,\phi$ is a group which is a finite union of translates of the divisible group K, it is immediate to see that ${\rm Tors}\,({\rm Ker}\,\phi)/{\rm Tors}\,K$ is isomorphic to the group of connected components of ${\rm Ker}\,\phi$. ∎

Remark 2.3. By (ii) the group T_ϕ is finite if ϕ is an isogeny, and in general it is finite modulo the torsion points of A contained in the connected component K of 0 of $\operatorname{Ker}\phi$. It follows that, in the formula (15), we can replace T_ϕ by a finite subgroup of T_ϕ.

We will also use the following property of the group T_ϕ.

Lemma 2.4. *Let as above $\phi : A \to B$ be a surjective morphism of tori. Then, with notation as above, the group T_ϕ maps surjectively, via*

$$\phi_{\mathbb{Q}} = \phi_* : A_{\mathbb{Q}} \to B_{\mathbb{Q}},$$

to $B_{\mathbb{Z}}/\phi_{\mathbb{Z}}(A_{\mathbb{Z}})$. The kernel of the map $\overline{\phi}$ so defined is $\operatorname{Ker}\phi_{\mathbb{Q}}/\operatorname{Ker}\phi_{\mathbb{Z}}$ (that is, the torsion subgroup of K). The image of $\overline{\phi}$ is isomorphic to $\operatorname{Tors}(B_{\mathbb{Z}}/\operatorname{Im}\phi_{\mathbb{Z}})$. In particular, $\operatorname{Im}\overline{\phi}$ is isomorphic to the group of connected components of $\operatorname{Ker}\phi$.

Proof. We have indeed by definition $T_\phi = D_\phi/A_{\mathbb{Z}}$, where $D_\phi = \phi_{\mathbb{Q}}^{-1}(B_{\mathbb{Z}})$ by (14). Using the fact that $\phi_{\mathbb{Q}} : A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ is surjective, we get that $\phi_{\mathbb{Q}} : D_\phi \to B_{\mathbb{Z}}$ is surjective. The kernel of the induced surjective map

$$\overline{\phi_{\mathbb{Q}}} : D_\phi \to B_{\mathbb{Z}}/\phi_{\mathbb{Z}}(A_{\mathbb{Z}})$$

is clearly $\operatorname{Ker}\phi_{\mathbb{Q}} + A_{\mathbb{Z}}$, hence $\overline{\phi_{\mathbb{Q}}}$ factors through T_ϕ, and the induced map $\overline{\phi} : T_\phi \to B_{\mathbb{Z}}/\phi_{\mathbb{Z}}(A_{\mathbb{Z}})$ has for kernel the image of $\operatorname{Ker}\phi_{\mathbb{Q}}$ in T_ϕ. For the last point, as T_ϕ is of torsion, $\operatorname{Im}\overline{\phi}$ is of torsion, and conversely, a torsion element of $B_{\mathbb{Z}}/\phi_{\mathbb{Z}}(A_{\mathbb{Z}})$ lifts to an element of $A_{\mathbb{Q}}$. □

Coming back to the morphisms induced by the Abel-Jacobi map, the inclusion of the finite index sublattice

$$\widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} \to N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$$

induces an isogeny of intermediate Jacobians

$$J(\widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}) \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}). \tag{17}$$

By definition of $\widetilde{N}_{1,\mathrm{cyl}}$, for any smooth projective curve C and codimension-$n-1$ cycle $\mathcal{Z} \in \mathrm{CH}^{n-1}(C \times X)$, the morphism $[\mathcal{Z}]^* : H^1(C,\mathbb{Z}) \to H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$ takes value in

$$\widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} \subset N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}.$$

It follows that the morphism $\Phi_{\mathcal{Z}}$ of (12), or rather its Walker lift $\widetilde{\Phi}_{\mathcal{Z}}$, factors through a morphism

$$\widetilde{\widetilde{\Phi}}_{\mathcal{Z}} : J(C) \to J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}). \tag{18}$$

Let us clarify one point. One could naively believe that these liftings provide a further lift of the Walker Abel-Jacobi map

$$\widetilde{\Phi}_X : \mathrm{CH}^{n-1}(X)_{\mathrm{alg}} \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}) \tag{19}$$

defined on cycles algebraically equivalent to 0, to a morphism

$$\widetilde{\widetilde{\Phi}}_X : \mathrm{CH}^{n-1}(X)_{\mathrm{alg}} \to J(\widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}) = J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}). \tag{20}$$

For $n = 3$, the existence of such a lifting would imply the equality $\widetilde{N}^1H^3(X,\mathbb{Z})_{\mathrm{tf}} = N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}$ which is the content of Theorem 0.2 and that we prove only for rationally connected threefolds. Indeed, by [12], the Abel-Jacobi map (18) is the universal regular homomorphism for codimension 2 cycles, so such a factorization is possible only if the natural map (16) between the two intermediate Jacobians is an isomorphism. The reason why the various liftings (17) do not allow to construct a lift of (18) to a morphism (19) is the fact that a 1-cycle $Z \in \mathrm{CH}_1(X)_{\mathrm{alg}}$ does not come canonically from a family of 1-cycles parameterized by a smooth curve C as above. Two different such representations could lead to two different lifts of $\widetilde{\Phi}_X(Z)$ in $J(\widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$. A first lift allows to write $Z = \partial\Gamma_1$ for some 3-chain supported on a smooth projective surface S_1 mapping to X, and a second lift will allow to write $Z = \partial\Gamma_2$ for some 3-chain supported on a smooth projective surface S_2 mapping to X. Then $\Gamma_1 - \Gamma_2$ has no boundary, hence provides a priori a homology class γ in $H_3(X,\mathbb{Z}) \cong H^{2n-3}(X,\mathbb{Z})$ which is in $N^{n-2}H^{2n-3}(X,\mathbb{Z})$ but is not supported on a smooth surface and has no reason to be in $\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})$. Due to the ambiguity of the choice, the Abel-Jacobi image of Z will be well-defined only modulo these cycles γ. Note that this argument also explains the existence of the Walker lift.

Coming back to the case where X is a rationally connected 3-fold, Theorem 0.2 is equivalent to the fact that

$$\widetilde{N}^1_{\mathrm{cyl}}H^3(X,\mathbb{Z})_{\mathrm{tf}} = H^3(X,\mathbb{Z})_{\mathrm{tf}}.$$

Equivalently, for some smooth projective curve C, and cycle $\mathcal{Z}$ as above, the morphism (13) is surjective. If we consider the corresponding morphism (12) of intermediate Jacobians, its surjectivity holds once the morphism (13) becomes surjective after passing to $\mathbb{Q}$-coefficients, and the surjectivity of (13) is equivalent to the fact that $\mathrm{Ker}\,\Phi_{\mathbb{Z}}$ is connected.

2.2 Cylinder homomorphism filtration on degree 3 homology

Recall the definition of the cylinder homomorphism and, for niveau 1, stable cylinder homomorphism filtrations (Definitions 1.11 and 1.12). The proof of Theorem 0.2 has two independent steps. The first one is the following statement that works without any rational connectedness assumption. Here we recall that, in higher dimension, the Abel-Jacobi map for 1-cycles has the Walker factorization through

$$\widetilde{\Phi}_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}).$$

Theorem 2.5. *Let X be a complex projective manifold of dimension n. Then, if the Walker Abel-Jacobi map $\widetilde{\Phi}_X : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$ is injective on torsion, one has*

$$N_{1,\mathrm{cyl},\mathrm{st}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}. \quad (21)$$

In dimension 3, $N^1H^3(X,\mathbb{Z})_{\mathrm{tf}} \subset H^3(X,\mathbb{Z})_{\mathrm{tf}}$ has torsion free cokernel so $J(N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}) \to J(H^3(X,\mathbb{Z})_{\mathrm{tf}})$ is injective and $\Phi_X = \widetilde{\Phi}_X$. Furthermore, we can apply the following theorem due to Bloch (see [3], [12]).

Theorem 2.6. *Let X be a smooth projective variety over $\mathbb{C}$. The Abel-Jacobi map Φ_X : $\mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X)$ is injective on torsion cycles.*

Theorem 2.5 thus gives in this case

Corollary 2.7. *(Cf. Theorem 0.6) Let X be a complex projective threefold. Then*

$$N_{1,\mathrm{cyl},\mathrm{st}}H^3(X,\mathbb{Z})_{\mathrm{tf}} = N^1H^3(X,\mathbb{Z})_{\mathrm{tf}}. \quad (22)$$

For rationally connected manifolds of any dimension, we can apply Suzuki's theorem 2.1. Theorem 2.5 thus gives in this case

Corollary 2.8. *Let X be a rationally connected complex projective manifold of dimension n. Then*

$$N_{1,\mathrm{cyl},\mathrm{st}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}. \quad (23)$$

We do not know if these statements hold true for the whole group $H^{2n-3}(X,\mathbb{Z})$ (instead of its torsion free part). By definition, they say that if the Abel-Jacobi map for 1-cycles is injective on torsion, the torsion free part of coniveau-$n-2$, degree-$2n-3$ cohomology of X is generated by cylinder homomorphisms

$$f_* \circ p^* : H_1(C,\mathbb{Z}) \to H_3(X,\mathbb{Z})_{\rm tf}$$

for all diagrams

$$\begin{array}{ccc} Y & \stackrel{f}{\to} & X \\ p\downarrow & & \\ C & & , \end{array} \tag{24}$$

where p is flat semi-stable projective of relative dimension 1, and C is *any* reduced curve (possibly singular, and not necessarily projective).

Proof of Theorem 2.5. We first choose a smooth connected projective curve C and a cycle $\mathcal{Z} \in {\rm CH}^{n-1}(C\times X)$ with the property that

$$[\mathcal{Z}]_* : H^1(C,\mathbb{Z}) \to \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf} \tag{25}$$

is surjective. We have a lot of freedom in choosing this curve. The cycle $\mathcal{Z}$ induces a Walker Abel-Jacobi morphism $\widetilde{\Phi}_{\mathcal{Z}} = \widetilde{\Phi}_X \circ \mathcal{Z}_* : J(C) \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$ with lift

$$\widetilde{\widetilde{\Phi}}_{\mathcal{Z}} : J(C) \to J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$$

as explained in (17), which is induced by the morphism of Hodge structures (24). Choosing a reference point $0 \in C$, we get an embedding $C \to J(C)$, hence a restricted Abel-Jacobi map

$$\widetilde{\Phi}_{\mathcal{Z},C,0} : C \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$$

with lift

$$\widetilde{\widetilde{\Phi}}_{\mathcal{Z},C,0} : C \to J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}).$$

Lemma 2.9. *Choosing C and 0 in an adequate way, we can assume the following:*

(i) Let $\alpha : J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}) \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$ be the natural isogeny with torsion kernel T_α. Then there are points $x_i \in C$, (say with $x_0 = 0$) such that the set of points

$$\{\widetilde{\widetilde{\Phi}}_{\mathcal{Z},C,0}(x_i)\} \subset J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$$

is equal to T_α.

(ii) The cycles $\mathcal{Z}_{x_i} - \mathcal{Z}_{x_0}$ are of torsion in ${\rm CH}_1(X)$.

Proof. (i) We first start with any curve C_0 and cycle $\mathcal{Z}_0$ with surjective $[\mathcal{Z}_0]_*$ as in (24). Then we will replace C_0 by a general complete intersection curve C in $J(C_0)$ whose image in $J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$ passes through all the points in T_α. We observe that the cycle $\mathcal{Z}_0 \in {\rm CH}^{n-1}(C_0 \times X)$ induces a cycle $\mathcal{Z}_{0,J(C_0)} \in {\rm CH}^{n-1}(J(C_0)\times X)$ with the property that

$$[\mathcal{Z}_{0,J(C_0)}]_* : H_1(J(C_0),\mathbb{Z}) \to \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}$$

is surjective, so we can take for $\mathcal{Z}$ the restriction to $C\times X$ of $\mathcal{Z}_{0,J(C_0)}$, and the surjectivity of $[\mathcal{Z}]_* : H^1(C,\mathbb{Z}) \to \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}$ follows from the Lefschetz theorem on hyperplane sections which gives the surjectivity of the map $H^1(C,\mathbb{Z}) \to H_1(J(C_0),\mathbb{Z})$.

(ii) We do the same construction as above, except that we first choose torsion elements $\beta_i \in J(C_0)$ over each $\alpha_i \in J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$. We then ask that $C \subset J(C_0)$ passes through the points β_i at x_i.

As β_i is a torsion point of $J(C_0)$, the 0-cycle $\{\beta_i\} - \{0\}$ is of torsion in $\mathrm{CH}_0(J(C_0))$, hence the cycle

$$\mathcal{Z}_{x_i} - \mathcal{Z}_{x_0} = \mathcal{Z}_{0,J(C_0)*}(\{\beta_i\} - \{0\})$$

is of torsion in $\mathrm{CH}_1(X)$, which proves (ii). □

We also note that we can assume the cycle $\mathcal{Z} \in \mathrm{CH}^{n-1}(C\times X)$ to be effective, represented by a surface mapping to X and C with smooth fibers over the points x_i (and semistable fibers otherwise).This follows indeed from the definition of $\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}$ as coming from the degree 3 homology of a smooth projective surface Y mapping to X. The statement thus follows from the corresponding assertion for any universal divisor on $\mathrm{Pic}^0(Y)\times Y$ restricted to $C \times Y$ for an adequate choice of curve $C \subset \mathrm{Pic}^0(Y)$, to which we can apply Bertini type theorems by adding ample divisors coming from C and Y. We assume now that we are in the situation of Lemma 2.9. The cycles $\mathcal{Z}_{x_i} - \mathcal{Z}_{x_0} \in \mathrm{CH}^{n-1}(X)_{\rm alg}$ are thus of torsion by Lemma 2.9, (ii), and annihilated by $\widetilde{\Phi}_X$ since we have

$$\widetilde{\widetilde{\Phi}}_{\mathcal{Z}}(x_i - x_0) = \beta_i,\ \alpha \circ \widetilde{\widetilde{\Phi}}_{\mathcal{Z}} = \widetilde{\Phi}_{\mathcal{Z}},$$

and $\alpha(\beta_i) = 0$ by Lemma 2.9,(i). By assumption, the Walker Abel-Jacobi map $\widetilde{\Phi}_X$ is injective on torsion cycles, hence the cycles $\mathcal{Z}_{x_i} - \mathcal{Z}_{x_0} \in \mathrm{CH}_1(X)_{\rm alg}$ are rationally equivalent to 0, which means that there exist smooth (not necessarily connected) projective surfaces Σ_i, and morphisms

$$f_i : \Sigma_i \to X,\ p_i : \Sigma_i \to \mathbb{P}^1,$$

such that

$$f_{i*}(p_i^{-1}(0) - p_i^{-1}(\infty)) = \mathcal{Z}_{x_i} - \mathcal{Z}_{x_0}. \tag{26}$$

Let γ_i be a continuous path from x_0 to x_i on C. We thus get a real 3-chain

$$\Gamma_i = (p_X)_*\mathcal{Z}_{\gamma_i}$$

in X satisfying

$$\partial\Gamma_i = \mathcal{Z}_{x_i} - \mathcal{Z}_{x_0}.$$

Next, let γ be a continuous path from 0 to ∞ on $\mathbb{CP}^1$. Then we get a real 3-chain $\Gamma_i' = f_{i*}(p_i^{-1}(\gamma))$ in X also satisfying

$$\partial\Gamma_i' = \mathcal{Z}_{x_i} - \mathcal{Z}_{x_0}.$$

It follows that $\Gamma_i - \Gamma_i'$ satisfies $\partial(\Gamma_i - \Gamma_i') = 0$, hence has a homology class

$$\eta_i \in H_3(X,\mathbb{Z}), \tag{27}$$

which belongs to $N^{n-2}H_3(X,\mathbb{Z})$ since the chains Γ_i, Γ_i' are supported on surfaces in X.

We now apply to the isogeny

$$\alpha : J(\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf}) \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\rm tf})$$

the results of Section 2.1. For the clarity of the argument, it will be more convenient to use the homology groups $H_3(X,\mathbb{Z})$ instead of the cohomology groups $H^{2n-3}(X,\mathbb{Z})$ (they are isomorphic by Poincaré duality). We thus have the group isomorphism

$$\overline{\alpha} : T_\alpha \to N^{n-2}H_3(X,\mathbb{Z})_{\rm tf}/\widetilde{N}^{n-2}H_3(X,\mathbb{Z})_{\rm tf}$$

discussed in Lemma 2.4.

Lemma 2.10. *For any i, the class η_i of (26) satisfies*

$$\eta_i = \overline{\alpha}(\beta_i) \text{ in } N^{n-2}H_3(X,\mathbb{Z})_{\rm tf}/\widetilde{N}^{n-2}H_3(X,\mathbb{Z})_{\rm tf}. \tag{28}$$

Remark 2.11. The construction of η_i depends on the choice of γ_i, so it is in fact naturally defined only modulo a class coming from $H_1(C,\mathbb{Z})$, hence modulo $\widetilde{N}^{n-2}H_3(X,\mathbb{Z})_{\rm tf}$.

Proof of Lemma 2.10. As we are working with the torsion free part $H_3(X,\mathbb{Z})_{\rm tf}$, which embeds in the complex vector space $F^2H^3(X)^*$, it suffices to check the result after integration of classes in $F^2H^3(X)$. These classes are represented by closed forms ν of type $(3,0)+(2,1)$ on X. When pulling-back these forms on Σ_i via f_i and push-forward to $\mathbb{P}^1$ via p_i we get 0 since there are no nonzero holomorphic forms on $\mathbb{P}^1$. We thus conclude that $\int_{\Gamma'_i}\nu = 0$, hence

$$\int_{\eta_i}\nu = \int_{\Gamma_i}\nu, \tag{29}$$

for any closed form ν of type $(3,0)+(2,1)$ on X.

It remains to understand why (28) is equivalent to (27). In fact, consider the general case of an isogeny $\phi : A = A_{\mathbb{R}}/A_{\mathbb{Z}} \to B_{\mathbb{R}}/B_{\mathbb{Z}}$ of real tori, with induced morphisms

$$\phi_{\mathbb{Z}} : H_1(A,\mathbb{Z}) \to H_1(B,\mathbb{Z}),$$

$$\phi_{\mathbb{R}} : H_1(A,\mathbb{R}) \to H_1(B,\mathbb{R})$$

on degree 1 homology. Then, referring to the proof of Lemma 2.2 for the notation, the isomorphism $\overline{\phi} : T_\phi \to B_{\mathbb{Z}}/\phi_{\mathbb{Z}}(A_{\mathbb{Z}})$ is obtained by passing to the quotient from the natural map $\phi_{\mathbb{R}}^{-1}(H_1(B,\mathbb{Z})) =: D_\phi \to H_1(B,\mathbb{Z}))$ given by restricting $\phi_{\mathbb{R}}$ to D_ϕ.

In our case, the map $\alpha_{\mathbb{R}}$ is induced by the cylinder map associated with the cycle $\mathcal{Z}$, and the choice of a path γ_i from x_0 to x_i determines a class in $H_1(J(C),\mathbb{R})$ whose image in $J(C)$ is the Abel-Jacobi image of $x_i - x_0$. The image of this class under $\alpha_{\mathbb{R}}$ is the element $\int_{\Gamma_i} \in F^2H^3(X)^* \cong H^3(X,\mathbb{R})^*$. Hence the equality (28) exactly says that $\overline{\alpha}(\beta_i) = \eta_i$ modulo torsion and $\operatorname{Im}\alpha_{\mathbb{Z}}$. □

We now conclude the proof of Theorem 2.5. We start from a smooth projective curve C and cycle $\mathcal{Z} \in \mathrm{CH}^{n-1}(C\times X)$ satisfying the properties stated in Lemma 2.9 and such that

$$[\mathcal{Z}]_*(H_1(C,\mathbb{Z})) = \widetilde{N}^{n-2}H_3(X,\mathbb{Z})_{\rm tf}.$$

We then get as in (25) the surfaces Σ_i and the morphisms

$$f_i : \Sigma_i \to X,\ p_i : \Sigma_i \to \mathbb{P}^1 \tag{30}$$

with the property that

$$f_{i*}(p_i^{-1}(0) - p_i^{-1}(\infty)) = \mathcal{Z}_{x_i} - \mathcal{Z}_{x_0}. \tag{31}$$

Let us first explain the proof in a simplified case. Assume that there is a single index $i = 1$, f_1 is an embedding along the curves $p_1^{-1}(0)$ and $p_1^{-1}(\infty)$ which are smooth curves in Σ_1, and we have identifications of smooth curves in X

$$f_1(p_1^{-1}(0)) = \mathcal{Z}_{x_1},\ f_1(p_1^{-1}(\infty)) = \mathcal{Z}_{x_0}. \tag{32}$$

In this case, we construct the singular curve C' as the union of C and a copy of $\mathbb{P}^1$ glued by two points to C, with $0 \in \mathbb{P}^1$ identified to $x_1 \in C$, $\infty \in \mathbb{P}^1$ identified to $x_0 \in C$. Over C', we put the family $\mathcal{Z}' \to C'$ of curves in X which over C is $f : \mathcal{Z} \to X$, $p : \mathcal{Z} \to C$ and over $\mathbb{P}^1$ is $f_1 : \Sigma_1 \to X$, $p_1 : \Sigma_1 \to \mathbb{P}^1$. They glue by assumption over the intersection points using the identifications (31). Flatness is easy to check in this case. For semi-stability, it suffices to restrict to the Zariski open set C'_0 of C' (which contains all the singular points of C') parameterizing semi-stable fibers.

If we now look at the cylinder homomorphism

$$\mathcal{Z}'_* : H_1(C'_0, \mathbb{Z}) \to H_3(X, \mathbb{Z})_{\rm tf},$$

its image contains $\mathcal{Z}_* H_1(C, \mathbb{Z}) = \widetilde{N}^{n-2} H_3(X, \mathbb{Z})$ and an extra generator over the loop in C'_0 made of the pathes γ on $\mathbb{P}^1$ and γ_1 on C (which we can assume to avoid the points with non-semistable fibers). Lemma 2.10 tells us that the image of this path under $\mathcal{Z}'_*$ is the element η_1 of $N^{n-2} H_3(X, \mathbb{Z})_{\rm tf}$ which, together with $\widetilde{N}^{n-2} H_3(X, \mathbb{Z})_{\rm tf}$, generates $N^{n-2} H_3(X, \mathbb{Z})_{\rm tf}$. As ${\rm Im}\, \mathcal{Z}'_* \subset N_{1,{\rm cyl},{\rm st}} H_3(X, \mathbb{Z})$, we proved the theorem in this case.

Remark 2.12. The reason why the above argument does not cover the general case is the fact that rational equivalence of two curves in X does not in general take the simple form described above.

Let us now prove the general case. Out of the data (29), (30), we shall construct a modified family over a singular curve as above. Up to now, we have not been really using the fact that we are working with 1-cycles, but we will use it now. We fix i and prove the following:

Claim 2.13. *After replacing X by $X \times \mathbb{P}^r$ and modifying the family $p : \mathcal{Z} \to C$, $f : \mathcal{Z} \to X$ by gluing components $\mathcal{Z}'_l \to C$, $\mathcal{Z}'_l \to X$ with trivial Abel-Jacobi map, we can choose the rational equivalence relation (29), (30) so that it takes the following form: There exist a chain $C_1, \ldots, C_m$ of smooth curves with two marked points $s_j, t_j \in C_j$, glued by $t_j = s_{j+1}$, and surfaces Σ_j, $j = 1, \ldots, m$ with two maps*

$$f_j : \Sigma_j \to X, \; p_j : \Sigma_j \to C_j, \tag{33}$$

satisfying the conditions:

(i) For each $j = 1, \ldots, m$, f_j is an embedding, and the relationp$_j$ is flat with semistable fibers (so (32) is a family of stable maps to X parameterized by C_j).

(ii) For $1 \leq j \leq m-1$, the stable map $f_j : p_j^{-1}(t_j) \to X$ is isomorphic to the stable map $f_{j+1} : p_{j+1}^{-1}(s_{j+1}) \to X$.

(iii) We have equalities of stable maps

$$(f_1 : p_1^{-1}(s_1) \to X) = (f_{|\mathcal{Z}_{x_i}} : \mathcal{Z}_{x_i} \to X), \; (f_m : p_m^{-1}(t_m) \to X) = (f_{|\mathcal{Z}_{x_0}} : \mathcal{Z}_{x_0} \to X).$$

(iv) The Abel-Jacobi map $C_j \to J^{2n-3}(X)$ is trivial for each family of curves $p_j : \Sigma_j \to C_j$, $f_j : \Sigma_j \to X$.

Furthermore, we can choose the surfaces Σ_j to be unions of smooth surfaces with normal crossings.

Claim 2.13 concludes the proof of Theorem 2.5 by the same argument as before, except that the loop $\gamma \cup \gamma_1$ on $\mathbb{P}^1 \cup C$ is replaced by the continuous path $\gamma \cup \gamma_1$ on $C' = \cup_j C_j \cup C$ constructed as follows: let C' be the curve which is the union $\cup_j C_j \cup C$, with the points t_j, s_{j+1} identified for $j \leq m-1$ and the points s_1 identified with x_0, the point t_m identified with x_i. We choose the continuous path γ on $\cup_j C_j \subset C'$ to be the union of arbitrarily chosen pathes from s_j to t_j on C_j. We thus have a closed 1-chain $\gamma \cup \gamma_1$ on C'. There is a family of semistable maps

$$f' : \Sigma' \to X, \, p' : \Sigma' \to C', \tag{34}$$

constructed from Claim 2.13 by gluing the various pieces

$$f_j : \Sigma_j \to X, \, p_j : \Sigma_j \to C_j$$

using the identifications (ii) and (iii). Using the fact that the Abel-Jacobi map associated with the families $\Sigma_j \to C_j$, $\Sigma_j \to X$ is trivial on C_j (assumption (iv)), we conclude as in Lemma 2.10 that the element

$$\eta_i \in N^{n-2}H_3(X,\mathbb{Z})_{\rm tf}/\widetilde{N}^{n-2}H_3(X,\mathbb{Z})_{\rm tf}$$

is the image of an element of $N_{1,{\rm cyl,st}}H_3(X,\mathbb{Z})_{\rm tf}$, namely the image of the class of $\gamma\cup\gamma_1$ in $H_1(C',\mathbb{Z})$ under the cylinder homomorphism associated to the family (33). Doing this for every i, we conclude that $N_{1,{\rm cyl,st}}H_3(X,\mathbb{Z})_{\rm tf} = N^{n-2}H_3(X,\mathbb{Z})_{\rm tf}$. □

Proof of Claim 2.13. Starting from the data (29), (30), where we fix i and write Σ_i as a disjoint union of smooth connected surfaces Σ_j mapping to X via f_j and to $\mathbb{P}^1$ via p_j, we can choose embeddings $i_j : \Sigma_j \hookrightarrow \mathbb{P}^r$ and then $f'_j = (f_j, i_j) : \Sigma_j \to X\times\mathbb{P}^r$ is an embedding. We can even assume that the surfaces $f'_j(\Sigma_j)$ are disjoint. Next we observe that, by resolution of singularities, for each surface Σ_j, the group of nonzero rational functions on Σ_j is generated by those rational functions $\phi : \Sigma_j \dashrightarrow \mathbb{P}^1$ with the following property: after replacing Σ_j by a blowing-up $\widetilde{\Sigma}_{j,\phi}$, ϕ induces a surjective map $\Sigma_j \to \mathbb{P}^1$ which is a Lefschetz pencil. Furthermore, the divisor of ϕ is up to sign of the form $A-B-C$, where A, B, C are smooth irreducible curves.

The p_j are given by rational funtions ϕ_j on Σ_j, that we factor as above

$$\phi_j = \phi_{j1}\ldots\phi_{j2}\ldots\phi_{js_j}$$

on Σ_j, with corresponding blown-up surfaces $\Sigma_{jl} := \widetilde{\Sigma}_{j,\phi_l}$ and morphisms p_{jl} to $\mathbb{P}^1$. We choose disjoint embeddings i_{jl}, $l = 1,\ldots, s_j$ of the surfaces Σ_{jl} in $\mathbb{P}^r$ and do the same trick as before. After performing these operations, we get surfaces $\Sigma_{jl} \overset{f'_{jl}}{\hookrightarrow} X\times\mathbb{P}^r$ with morphisms $p_{jl} : \Sigma_{jl} \to \mathbb{P}^1$, satisfying condition (i).

Let $\pi : X\times\mathbb{P}^r \to X$ be the first projection. The data above satisfy the equality of cycles

$$\pi_*(\sum_{jl} f'_{jl*}({\rm div}\,\phi_{jl})) = \mathcal{Z}_{x_i} - \mathcal{Z}_{x_0}. \tag{35}$$

Note that the curves $\mathcal{Z}_{x_i}$ and $\mathcal{Z}_{x_0}$ can be assumed to be smooth connected. We can also assume, by removing finitely many points of X and working with the complement X^0 if necessary, that all the irreducible curves in the support of $f_{jl}({\rm div}\,\phi_{jl})$ map to smooth curves D in X^0. Denote by $D_{\alpha,0}$, (resp. $D_{\beta,i}$) the curves in ${\rm Supp}\,(\sum_{jl} f'_{jl*}({\rm div}\,\phi_{jl}))$ mapping to $\mathcal{Z}_{x_0}$ (resp. to $\mathcal{Z}_{x_i}$) via π, and, for any other curve $D\subset X^0$, by $D_{\gamma,D}$ the curves in ${\rm Supp}\,(\sum_{jl} f'_{jl*}({\rm div}\,\phi_{jl}))$ mapping to D. Then it follows from (34) that

$$\begin{aligned}\sum_\alpha n_{\alpha,0}{\rm deg}\,(D_{\alpha,0}/\mathcal{Z}_{x_0}) &= 1 \\ \sum_\beta n_{\beta,i}{\rm deg}\,(D_{\beta,i}/\mathcal{Z}_{x_i}) &= -1 \\ \sum_\gamma n_{\gamma,D}{\rm deg}\,(D_{\gamma,D}/D) &= 0.\end{aligned} \tag{36}$$

where $n_{\alpha,0} = \pm 1$ is the multiplicity of $D_{\alpha,0}$ in the cycle $\sum_{jl} f'_{jl*}({\rm div}\,\phi_{jl})$, and similarly for $n_{\beta,i}$ and $n_{\gamma,D}$. Assuming $r = 1$ for simplicity, each curve $D_{\alpha,0}$ is rationally equivalent in the surface $\mathcal{Z}_{x_0}\times\mathbb{P}^1$ to a disjoint union of ${\rm deg}\,(D_{\alpha,0}/\mathcal{Z}_{x_0})$ sections $\mathcal{Z}_{x_0}\times t_{\alpha,0,s}$, $t_{\alpha,0,s}\in\mathbb{P}^1$, modulo vertical curves $x\times\mathbb{P}^1$, which provides a rational function $\psi_{\alpha,0}$ on $\mathcal{Z}_{x_0}\times\mathbb{P}^1$. Similarly for $\mathcal{Z}_{x_i}$ and D, providing rational functions $\psi_{\alpha,0}$ on $\mathcal{Z}_{x_0}\times\mathbb{P}^1$, resp. $\psi_{\beta,i}$ on $\mathcal{Z}_{x_i}\times\mathbb{P}^1$ and $\psi_{\gamma,D}$ on $D\times\mathbb{P}^1$. Using (35) and choosing another point $t_0\in\mathbb{P}^1$, we finally have a rational function ψ_0 on $\mathcal{Z}_{x_0}\times\mathbb{P}^1$, resp. ψ_i on $\mathcal{Z}_{x_i}\times\mathbb{P}^1$, resp. ψ_D on $D\times\mathbb{P}^1$ for each curve $D\neq\mathcal{Z}_{x_0}$, $\mathcal{Z}_{x_i}$, such that the following equalities of 1-cycles in $X^0 \times \mathbb{P}^1$

$$\begin{aligned} \operatorname{div} \psi_0 &= \sum_{\alpha,s} n_{\alpha,0} \mathcal{Z}_{x_0} \times t_{\alpha,0,s} - \mathcal{Z}_{x_0} \times t_0 \\ \operatorname{div} \psi_i &= \sum_{\beta,s'} n_{\beta,i} \mathcal{Z}_{x_i} \times t_{\beta,i,s'} + \mathcal{Z}_{x_i} \times t_0 \\ \operatorname{div} \psi_D &= \sum_{\gamma,s''} n_{\gamma,D} D \times t_{\gamma,D,s''}. \end{aligned} \tag{37}$$

hold modulo vertical cycles $z \times \mathbb{P}^1$, $z \in \mathcal{Z}_0(X^0)$. Equivalently, these equalities hold in $\mathcal{Z}_1(X^{00} \times \mathbb{P}^1)$, where $X^{00} \subset X^0$ is the complement in X of finitely many points.

Adding to the previous surfaces $\Sigma_{jl} \stackrel{f'_{jl}}{\to} X \times \mathbb{P}^1$ and rational functions ϕ_{jl} the surfaces

$$\mathcal{Z}_{x_0} \times \mathbb{P}^1,\, \mathcal{Z}_{x_i} \times \mathbb{P}^1,\, D \times \mathbb{P}^1$$

naturally contained in $X \times \mathbb{P}^1$, with the rational functions $\psi_{j,0}$ and ψ_0, the surface $\mathcal{Z}_{x_i} \times \mathbb{P}^1$ with the rational functions $\psi_{j,i}$ and ψ_i, and the surfaces $D \times \mathbb{P}^1$, with the rational functions $\psi_{j,D}$ and ψ_D, we arrived now at a situation where we have surfaces $\Sigma'_l \subset X^{00} \times \mathbb{P}^1$, and rational functions χ_l on Σ'_l such that the 1-cycles $\operatorname{div} \chi_l$ of $X^{00} \times \mathbb{P}^1$ have the property that each irreducible curve in $\cup_l \mathrm{Supp}\,(\operatorname{div} \chi_l)$ appears twice with opposite multiplicities ± 1 in $\sum_l \operatorname{div} \chi_l$, except for $\mathcal{Z}_{x_0} \times t_0$ and $\mathcal{Z}_{x_1} \times t_0$ which appear only once, the first one with multiplicity 1, the second one with multiplicity -1. Working now over the whole of X, and taking the Zariski closure $\overline{\Sigma'_l} \subset X \times \mathbb{P}^1$ of these surfaces in $X \times \mathbb{P}^1$, we find that the 1-cycle $\sum_l \operatorname{div} \chi_l$ of $X \times \mathbb{P}^1$ is the sum of a vertical 1-cycle $z \times \mathbb{P}^1$ and $\mathcal{Z}_{x_0} \times t_0 - \mathcal{Z}_{x_i} \times t_0$. Recalling that the supports of the divisors of the original rational functions on the original surfaces in $X \times \mathbb{P}^1$ were normal crossing divisors, we can arrange by looking more closely at the surfaces $D \times \mathbb{P}^1$ (and in particular by normalizing the curves D) that this is still true for the supports of the divisors $\operatorname{div} \chi_l$ in Σ'_l. The cycle $z \times \mathbb{P}^1$ is rationally equivalent to 0 in $X \times \mathbb{P}^1$, because the cycle

$$\sum_l \operatorname{div} \chi_l = \mathcal{Z}_{x_0} \times t_0 - \mathcal{Z}_{x_i} \times t_0 + z \times \mathbb{P}^1$$

is rationally equivalent to 0, and the cycle $\mathcal{Z}_{x,0} \times t_0 - \mathcal{Z}_{x,i} \times t_0$ is rationally equivalent to 0. It follows that z is rationally equivalent to 0 in X. Writing $z = \sum_\beta \epsilon_\beta x_\beta$, with $x_\beta \in X$, $\epsilon_\beta = \pm 1$, this provides us with a curve E in X, and a rational function ψ_E on E with divisor $\sum_\beta \epsilon_\beta x_\beta$, hence also a surface $E \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$ with rational function $\tilde{\psi}_E$ with divisor $\sum_\beta \epsilon_\beta x_\beta \times \mathbb{P}^1$ in $X \times \mathbb{P}^1$. As the function $\psi_E : E \to \mathbb{P}^1$ has nonreduced fibers (corresponding to ramification), this last rational function $\tilde{\psi}_E$ does not provide a semistable family but this is not a serious issue. Indeed, we did not ask in Claim 2.13 that the curves are projective, so we can simply remove the points parameterizing nonreduced fibers, assuming they are not gluing points.

Consider the disjoint union Σ'' of all the surfaces above, with morphisms

$$p'' : \Sigma'' \to \mathbb{P}^1,\ f'' : \Sigma'' \to X \times \mathbb{P}^1. \tag{38}$$

We find that

$$f''_*(p''^*(0 - \infty)) = \mathcal{Z}_{x_0} \times t_0 - \mathcal{Z}_{x_i} \times t_0$$

and more precisely that we have

$$f''(p''^{-1}(0)) = \mathcal{Z}_{x_0} \times t_0 \cup A,\ f''(p''^{-1}(\infty)) = \mathcal{Z}_{x_i} \times t_0 \cup A, \tag{39}$$

where both curves $\mathcal{Z}_{x_0} \times t_0 \cup A$ and $\mathcal{Z}_{x_i} \times t_0 \cup A$ have ordinary double points and both maps

$$f''_0 : p''^{-1}(0) \to \mathcal{Z}_{x_0} \times t_0 \cup A,\ f''_\infty : p''^{-1}(\infty) \to \mathcal{Z}_{x_i} \times t_0 \cup A$$

are partial normalizations. In other words, we almost achieved the previous situation of (31), but with the curve A glued to $\mathcal{Z}_{x_0} \times t_0$ and $\mathcal{Z}_{x_i} \times t_0$. From now on, we write X for $X \times \mathbb{P}^1$ and $\mathcal{Z}_{x_0}$ for $\mathcal{Z}_{x_0} \times t_0$, $\mathcal{Z}_{x_i}$ for $\mathcal{Z}_{x_i} \times t_0$. The two fibers

$$f''_0 : p''^{-1}(0) \to X, \text{ resp. } f''_\infty : p''^{-1}(\infty) \to X \tag{40}$$

are obtained by gluing to the curve $\mathcal{Z}_{x_0}$, resp. $\mathcal{Z}_{x_i}$, a curve $A_0 \to X$, resp. $A_\infty \to X$, partially normalizing A. In other words

$$p''^{-1}(0) = \mathcal{Z}_{x_0} \cup A_0, \;\; p''^{-1}(\infty) = \mathcal{Z}_{x_i} \cup A_\infty. \tag{41}$$

Unfortunately, the two stable maps $A_0 \to X$ and $A_\infty \to X$ a priori are different, and it not even clear that we glue them respectively to $\mathcal{Z}_{x_0}$ and $\mathcal{Z}_{x_i}$ by the same number of points. What we know however is that the genera of $\mathcal{Z}_{x_0}$ and $\mathcal{Z}_{x_i}$ are equal, and the genera of the fibers $\mathcal{Z}_{x_0} \cup A_0$ and $\mathcal{Z}_{x_i} \cup A_\infty$ appearing in (**??**) are equal, because in both cases, by construction, these curves are deformations of each other. It follows that the total numbers of gluing points in the unions $\mathcal{Z}_{x_0} \cup A_0$ and $\mathcal{Z}_{x_i} \cup A_\infty$ (including those between the components of A_0 or A_∞) are the same. Let W_0 be the set of gluing points in $p''^{-1}(0) = \mathcal{Z}_{x_0} \cup_j A_j$. The set W_0 splits into a union $W_0 = W_{00} \sqcup W_{0A}$, where W_{00} is the set of gluing points of $\mathcal{Z}_{x_0}$ with the components A_j and W_{0A} is the set of gluing points between the components A_j (thus determining the curve A_0). We have similarly a set $W_\infty = W_{\infty i} \sqcup W_{\infty A}$. Although we know that W_0 and W_∞ have the same cardinality, we do not know that the sets W_{00} and $W_{\infty i}$ have the same cardinality, and neither that the curves A_0 and A_∞ have the same topology. To circumvent this problem, we will use the following

Lemma 2.14. *Let Y be a complex projective manifold and let C, A_j be smooth curves in Y meeting transversally in distinct points $z_1, \ldots, z_M$. Then for a smooth complete intersection curve R meeting the A_j and C in sufficiently many points, and for any two subsets $\{z_{i_1}, \ldots, z_{i_N}\}$, $\{z_{j_1}, \ldots, z_{j_N}\}$ of N points, there exists a family of stable maps*

$$m : \mathcal{C} \to X, \;\; \psi : \mathcal{C} \to D \tag{42}$$

parameterized by a smooth connected quasiprojective curve D, and two points $0_1, 0_2 \in D$ with the following properties:

(i) the stable curve

$$m_{0_1} : \mathcal{C}_{0_1} \to X$$

over 0_1 is the normalization of $C \cup R \cup_i A_i$ at the points $z_{i_1}, \ldots, z_{i_N}$, and the stable curve

$$m_{0_2} : \mathcal{C}_{0_2} \to X$$

over 0_2 is the normalization of $C \cup R \cup_i A_i$ at the points $z_{j_1}, \ldots, z_{j_N}$;

(ii) the family of curves (40) has trivial Abel-Jacobi map.

Remark 2.15. The curve R is necessary in this statement, as it adds to connectivity of the curves. Lemma 2.14 is wrong without it, for topological reasons. For example, consider the union C of three curves C_1, C_2, C_3 isomorphic to $\mathbb{P}^1$, and glued as follows: C_2 is glued to C_1 in two points x, y, and C_3 is glued to C_1 in two points z, w. Then the curves obtained by normalizing C in x, y and x, z are not deformations of each other, since one is disconnected, not the other.

Proof of Lemma 2.14. We first choose a smooth surface $S \subset Y$, which is a complete intersection of ample hypersurfaces, and which contains the curves C and A_i. The curve R will be any sufficiently ample curve in S meeting C and A_i transversally. We choose R ample enough so that for any set $\{w_1, \ldots, w_N\}$ of N points in S, the set of curves in the linear system $|C + \sum_i A_i + R|$ which are singular at all the w_i's and have ordinary quadratic singularities is a Zariski open set in a projective space $\mathbb{P}_{w.}$ of the expected dimension. We next choose a curve $\overline{D}$ in $S^{[N]}$ passing through the two sets $\{z_{i_1}, \ldots, z_{i_N}\}$, $\{z_{i_1}, \ldots, z_{j_N}\}$ at points 0_1, 0_2. There exists a Zariski open set $D \subset \overline{D}$, and a section of the projective bundle above over D passing over 0_1 and 0_2 through the curve $C \cup R \cup_i A_i$. This provides a family of curves $\mathcal{C}_0$ parameterized by D, equipped with a choice of N singular points, which are ordinary quadratic. The desired family is obtained by normalizing the curves of the family $\mathcal{C}_0$ at these N points. □

Let $n_{00} := |W_{00}|$, $n_{\infty i} := |W_{\infty i}|$. We can assume that $n_{00} \geq n_{\infty i}$. Let W be the set of gluing points in the stable curve ${p''}^{-1}(0) = \mathcal{Z}_{x_0} \cup A_0$ mapping to X via f_0'' (see (39), (**??**)). Let $N := |W|$. Lemma 2.14 says that, after gluing a complete intersection curve R, we can deform the stable map

$$f_{0R}'' : \mathcal{Z}_{x_0} \cup_{W_{00}} A_0 \cup R \to X,$$

obtained by gluing to f_0'' the inclusion of R, to any other stable map

$$f_{0R}''' : \mathcal{Z}_{x_0} \cup A_0' \cup R \to X,$$

inducing the same map on normalizations, but factoring through a different gluing of the components contained in A_0 or $\mathcal{Z}_{x_0}$, assuming the total number of identification points are the same. Furthermore, according to the lemma, this deformation can be done via a family of curves with trivial Abel-Jacobi map. As we assumed that $n_{00} \geq n_{\infty i}$, we can choose $A_0' = A_\infty$ glued by $n_{\infty i}$ points to $\mathcal{Z}_{x_0}$.

Looking at the proof of Lemma 2.14 and using the same notation, we see that we can arrange that the same surface S also contains the curve $\mathcal{Z}_{x_i}$ and then, because $\mathcal{Z}_{x_0}$ and $\mathcal{Z}_{x_i}$ are algebraically equivalent, the curve R, which can be taken the same for $\mathcal{Z}_{x_0}$ and $\mathcal{Z}_{x_i}$, meets $\mathcal{Z}_{x_0}$ and $\mathcal{Z}_{x_i}$ in the same number of points.

We now have three families of semistable curves:

(1) The original family $\mathcal{Z} \to C$, $\mathcal{Z} \to X$ with respective fibers $\mathcal{Z}_{x_0}$, $\mathcal{Z}_{x_i}$ over x_0, x_i.

(2) The family $f'' : \Sigma'' \to X$, $p'' : \Sigma'' \to \mathbb{P}^1$ of (37), with respective fibers $\mathcal{Z}_{x_0} \cup A_0$, $\mathcal{Z}_{x_i} \cup A_\infty$ over 0, ∞.

(3) The family $m : \mathcal{C} \to X$, $\psi : \mathcal{C} \to D$ given by Lemma 2.14 and the arguments above, with fibers $\mathcal{Z}_{x_0} \cup A_0 \cup R$, $\mathcal{Z}_{x_0} \cup A_\infty \cup R$,

where in (3), the family has trivial Abel-Jacobi map, and the number of attachment points of $\mathcal{Z}_{x_0}$ with A_∞ is the same as the number of attachment points of $\mathcal{Z}_{x_i}$ with A_∞, and furthermore the curve R has the same number of points of attachment with $\mathcal{Z}_{x_0}$ and $\mathcal{Z}_{x_i}$.

The following lemma will allow us (after changing R if necessary) to replace the family (1) by a family over C with same Abel-Jacobi map and fibers $\mathcal{Z}_{x_0} \cup R \cup A_\infty$, $\mathcal{Z}_{x_i} \cup R \cup A_\infty$ and the family (2) by a family parameterized by $\mathbb{P}^1$, with fibers $\mathcal{Z}_{x_0} \cup R \cup A_0$, $\mathcal{Z}_{x_i} \cup R \cup A_\infty$.

Lemma 2.16. *Let*

$$f : \Sigma \to X,\, p : \Sigma \to C \tag{43}$$

be a family of semistable curves generically embedded in X and parameterized by a quasiprojective smooth curve C. Let x, y be two points of C and let S be a curve in X meeting both curves $f(\Sigma_x)$ and $f(\Sigma_y)$ transversally in M smooth points. Then, up to replacing S by a union $S' = S \cup S_1$, where S_1 is a complete intersection curve, there exists a family of stable maps

$$f_{S'} : \Sigma_{S'} \to X,\, p_{S'} : \Sigma_{S'} \to C', \tag{44}$$

where $C' \subset C$ is a Zariski dense open set of C containing x and y, with the same Abel-Jacobi map as (41), and with the following properties

(i) The curve S_1 meets $f(\Sigma_x)$ and $f(\Sigma_y)$ transversally in M' points.

(ii) The fibers of the family (42) over x and y are respectively $\Sigma_x \cup S \cup S_1$ and $\Sigma_y \cup S \cup S_1$ mapped to X via f_x, resp. f_y on the first term.

Proof. We would like to attach the curve S to the other fibers but it does not meet a priori the other fibers, so we need to vary the curve S. Let us assume for simplicity that $\dim X = 3$. We choose a smooth surface $T \subset X$ containing S and meeting the curves $f(\Sigma_x)$, $f(\Sigma_y)$ (hence the general fiber $f(\Sigma_t)$) transversally. We choose the curve S_1 in T in such a way that $S \cup S_1$ has normal crossings, is ample enough and S_1 contains the intersection points in $f(\Sigma_x) \cap T$, and $f(\Sigma_y) \cap T$ which are not on S. For a generic $t \in C$, we choose a curve S_t in the linear system $S \cup S_1$ containing the intersection $f(\Sigma_t) \cap T$, in such a way that $S_x = S_y = S \cup S_1$. Gluing S_t to Σ_t at all the intersection points $f(\Sigma_t) \cap T$ provides the desired family. □

Lemma 2.16 concludes the proof of Claim 2.13 once one observes from the proof that the same curve S_1 can be used for the families (1), (2), (3) above, providing modified families of semistable curves with fibers

(1)' $\mathcal{Z}_{x_0} \cup A_\infty \cup R \cup S_1$, $\mathcal{Z}_{x_i} \cup A_\infty \cup R \cup S_1$

(2)' $\mathcal{Z}_{x_0} \cup A_0 \cup R \cup S_1$, $\mathcal{Z}_{x_i} \cup A_\infty \cup R \cup S_1$

(3)' $\mathcal{Z}_{x_0} \cup A_0 \cup R \cup S_1$, $\mathcal{Z}_{x_0} \cup A_\infty \cup R \cup S_1$.

These three families, the first of which has the same Abel-Jacobi as the original family $\mathcal{Z} \to C$, $\mathcal{Z} \to X$, while the two others have trivial Abel-Jacobi map, provide the desired chain. □

2.3 The case of rationally connected manifolds

The second step in the proof of Theorem 0.2 is the following statement which is valid in any dimension but concerns only rationally connected projective manifolds.

Theorem 2.17. *Let X be rationally connected smooth projective of dimension n over $\mathbb{C}$. Then*

$$N_{1,\mathrm{cyl,st}}H^{2n-3}(X,\mathbb{Z}) = \widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z}) = \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z}). \tag{45}$$

Proof. The second equality is proved in Proposition 1.3. Let Z be a connected reduced curve with a family

$$p : Y \to Z,\ f : Y \to X,$$

of semistable curves in X, that is, p is flat projective of relative dimension 1 and the fibers of p are semistable curves. We can also assume these curves are imbedded in X, so the maps are stable and automorphisms free. It is enough to prove that the image of

$$f_* \circ p^* : H_1(Z,\mathbb{Z}) \to H_3(X,\mathbb{Z})$$

is contained in $\widetilde{N}_{1,\mathrm{cyl,st}}H^{2n-3}(X,\mathbb{Z})$, that is, there exists a *smooth* (but not necessarily projective) variety Z' and a family of stable curves

$$p' : Y' \to Z',\ f' : Y' \to X,$$

with

$$\mathrm{Im}\, f_* \circ p^* \subset \mathrm{Im}\,(f'_* \circ p'^* : H_1(Z',\mathbb{Z}) \to H_3(X,\mathbb{Z})).$$

Assume first that the following holds :

(*) *At each singular point of Z, the semistable map $f_z : Y_z := p^{-1}(z) \to X$ is stable, automorphism free, and has unobstructed deformations.*

Then we take for $Y' \to Z'$ the universal deformation of the general fiber f_z, or rather, its restriction to the Zariski open set Z' of the base consisting of smooth points, that is, unobstructed stable maps, which furthermore are automorphism free. By our assumption, there is a dense Zariski open set $Z^0 \overset{j}{\hookrightarrow} Z$ such that $Z \setminus Z^0$ consists of smooth points of

Z, and Z^0 maps to Z' via the classifying map j'. We thus have two commutative (in fact Cartesian) diagrams

$$\begin{array}{ccccccccc} X & \stackrel{f}{\leftarrow} & Y & \supseteq & Y^0 & \stackrel{j''}{\rightarrow} & Y' & \stackrel{f'}{\rightarrow} & X \\ & & p\downarrow & & p^0\downarrow & & p'\downarrow & & \\ & & Z & \stackrel{j}{\hookleftarrow} & Z^0 & \stackrel{j'}{\rightarrow} & Z' & & \end{array},$$

where $Y^0 = p^{-1}(Z^0)$ and

$$f' \circ j'' = f^0,\ f^0 := f_{|Y^0}.$$

We deduce from this diagram that the two maps

$$f'_* \circ p'^* : H_1(Z', \mathbb{Z}) \to H_3(X, \mathbb{Z}),\ f_* \circ p^* : H_1(Z, \mathbb{Z}) \to H_3(X, \mathbb{Z})$$

coincide on $H_1(Z^0, \mathbb{Z})$ which maps to both via the maps

$$j_* : H_1(Z^0, \mathbb{Z}) \to H_1(Z, \mathbb{Z}),\ j'_* : H_1(Z^0, \mathbb{Z}) \to H_1(Z', \mathbb{Z})$$

induced respectively by the morphisms

$$j : Z^0 \hookrightarrow Z,\ j' : Z^0 \to Z'.$$

As $Z \setminus Z^0$ consists of smooth points of Z, the map $j_* : H_1(Z^0, \mathbb{Z}) \to H_1(Z, \mathbb{Z})$ is surjective, and we conclude that

$$\mathrm{Im}\, f_* \circ p^* = \mathrm{Im}\, f^0_* \circ p^{0*} \subset \mathrm{Im}\, f'_* \circ p'^*,$$

and this finishes the proof since Z' is smooth.

It remains to show that we can achieve (*). This is proved in the following lemma. First of all, we observe that after replacing X by $X \times \mathbb{P}^r$, which does not change H_3 since, by rational connectedness, $H_1(X, \mathbb{Z}) = 0$, we can assume the map $f_z : Y_z \to X$ to be an embedding for all $z \in Z$. In particular all maps are stable. Then we have

Lemma 2.18. *Let $p : Y \to Z$, $f : Y \to X$ be a family of semi-stable curves imbedded in X, parameterized by a reduced curve Z. There exist a Zariski open set $Z^0 \stackrel{j}{\hookrightarrow} Z$ such that $Z \setminus Z^0$ consists of smooth points of Z, and a family $\tilde{p}^0 : \widetilde{Y}^0 \to Z^0$, $\tilde{f}^0 : \widetilde{Y}^0 \to X$ of semistable curves in X parameterized by Z^0 such that*

(i) the fibers $\tilde{f}_z : \widetilde{Y}^0_z \to X$ are stable maps with unobstructed deformations;

(ii) the cylinder map

$$\tilde{f}^0_* \circ (\tilde{p}^0)^* : H_1(Z^0, \mathbb{Z}) \to H_3(X, \mathbb{Z}) \tag{46}$$

coincides with the composition $f_* \circ p^* \circ j_*$.

Proof. We first choose a general sufficiently ample hypersurface W in X. There exists a Zariski open set $Z^0_1 \subset Z$, which we can assume to contain the singular points of Z, such that W meets the fibers of p only in smooth distinct points x_i, $i = 1, \ldots, N$. Furthermore attaching to the fibers Y_z a complete intersection curve C_i in X at each of these intersection points, and restricting again Z^0_1, we can assume (see [9], [10]) that the curves C_i are smooth and disjoint, the curves $Y_{z,1} = Y_z \cup C_z$ where $C_z := \cup_i C_i \subset X$ are semistable and satisfy

$$H^1(Y_z, (N_{Y_{z,1}/X})_{|Y_z}) = 0. \tag{47}$$

The family of curves

$$f^0_1 : Y^0_1 \to X,\ p^0_1 : Y^0_1 \to Z^0_1 \tag{48}$$

so constructed has the same cylinder homomorphism map (44) as the original family, since the part of the cylinder homomorphism coming from the C_i is easily seen to be trivial.

Unfortunately, the modified family does not satisfy unobstructedness because the vanishing condition (45) is satisfied only after restriction to Y_z, and not on the whole of $Y_{z,1}$. We use now rational connectedness which allows us to glue very free rational curves to the components C_i. We first do this over a dense Zariski open set M^0 of the parameter space M parameterizing the disjoint union of N complete intersection curves C_i. This construction modifies each curve $D = \cup_i C_i$ into an union $D' = \cup_i C'_i$ of C_i and very free rational curves, satisfying the property that $H^1(D', N_{D'/X}) = 0$. Then we consider the morphism

$$\eta : Z_1^0 \to M,\ z \mapsto C_z$$

appearing in the previous construction. We can assume that M^0 contains $\eta(\mathrm{Sing}\, Z_1^0)$, so that, letting $Z^0 := \eta^{-1}(M^0)$, we can construct the family

$$\tilde{p}^0 : \widetilde{Y}^0 \to Z^0,\ \tilde{f}^0 : \widetilde{Y}^0 \to X \tag{49}$$

by gluing to the curves Y_z the curves C'_i instead of C_i. The cylinder homomorphism map for the family (47) is the same as the cylinder homomorphism map (44) for the family (46) because the extra part coming from the rational legs has its cylinder map factoring through the cylinder map associated to the family of curves D' over M^0, which is trivial since M^0 is smooth and rational. □

The proof of Theorem 2.17 is now complete. □

We can now prove our main theorem

Theorem 2.19. *Let X be a rationally connected smooth projective of dimension n over $\mathbb{C}$. Then $N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$.*

When $n = 3$, one has $N^1H^3(X,\mathbb{Z}) = H^3(X,\mathbb{Z})$, so Theorem 0.2 is proved.

Proof of Theorem 2.19. Let X be smooth projective rationally connected. By Corollary 2.8, one has

$$N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = N_{1,\mathrm{cyl,st}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}. \tag{50}$$

By Theorem 2.17, one also has

$$N_{1,\mathrm{cyl,st}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = \widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}. \tag{51}$$

Equations (48) and (49) imply that $N^{n-2}H^{2-3}(X,\mathbb{Z})_{\mathrm{tf}} = \widetilde{N}_{1,\mathrm{cyl}}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$, where the last group is also equal to $\widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$ by Proposition 1.3. The result is proved. □

3 Complements and final comments

The following important questions concerning the (strong or cylinder) coniveau for rationally connected manifolds remain completely open starting from dimension 4. As we already mentioned in the case of dimension 3, it follows from the results of [6] that for a rationally connected complex projective manifold X, one has

$$N^1H^k(X,\mathbb{Z}) = H^k(X,\mathbb{Z})$$

for any $k > 0$. Indeed, the quotient $H^k(X,\mathbb{Z})/N^1H^k(X,\mathbb{Z})$ is of torsion because X has a decomposition of the diagonal with $\mathbb{Q}$-coefficients, and on the other hand, when X is smooth quasiprojective, $H^k(X,\mathbb{Z})/N^1H^k(X,\mathbb{Z})$ is torsion free by [6].

Question 3.1. *Let X be a rationally connected complex projective manifold of dimension n. Is it true that*

$$\widetilde{N}^1H^k(X,\mathbb{Z}) = H^k(X,\mathbb{Z})$$

for $k > 0$?

Of course, this question is open only starting from $k = 3$. Our main result solves this question when $\dim X = 3$ and for the cohomology modulo torsion. In dimension 3, it leaves open the question, also mentioned in [2], whether for a rationally connected threefold X, we have the equality $H^3(X,\mathbb{Z}) = \widetilde{N}^1H^3(X,\mathbb{Z})$.

Question 3.2. *Let X be a rationally connected complex projective manifold of dimension n. Is it true that*

$$N_{1,\mathrm{cyl}}H^k(X,\mathbb{Z}) = H^k(X,\mathbb{Z})$$

for $k < 2n$?

These questions are not unrelated, due to the results of Section 1. For example, in degree $k = 3$, a positive answer to Question 3.1 even implies the much stronger statement that $\widetilde{N}_{n-2,\mathrm{cyl}}H^3(X,\mathbb{Z}) = H^3(X,\mathbb{Z})$ by Proposition 1.3. In degree $k = 2n-2$, Question 3.2 is equivalent to asking whether $H_2(X,\mathbb{Z})$ is algebraic, a question that has been studied in [21] where it is proved that it would follow from the Tate conjecture on divisor classes on surfaces over a finite field.

Another question concerns possible improvements of Theorem 2.17.

Question 3.3. *Let X be rationally connected smooth projective of dimension n over $\mathbb{C}$. Is it true that*

$$N_{1,\mathrm{cyl}}H^k(X,\mathbb{Z}) = N_{1,\mathrm{cyl,st}}H^k(X,\mathbb{Z}) = \widetilde{N}_{1,\mathrm{cyl}}H^k(X,\mathbb{Z}) \tag{52}$$

for any k?

We believe that the proof of Theorem 2.17 should work by the same smoothing argument for the cohomology of any degree. The difficulty that one meets here is that, while we had before a singular curve in the moduli space of stable maps f to X, and only needed to modify the fibers f_z in a Zariski open neighborhood of the singular points of C so as to make them unobstructed, one would need to do a similar construction for a higher dimensional variety Z with a possibly positive dimensional singular locus. In this direction, let us note the following generalization of Theorem 2.17.

Proposition 3.4. *Let X be smooth projective rationally connected of dimension n and let Z be a variety of dimension $n-2$ with isolated singularities. Let*

$$f : Y \to X,\ p : Y \to Z$$

be a family of stable maps with value in X parameterized by Z. Then for any k

$$\mathrm{Im}\,(f_* \circ p^* : H_{k-2}(Z,\mathbb{Z}) \to H_k(X,\mathbb{Z}))$$

is contained in $\widetilde{N}_{1,\mathrm{cyl}}H^{2n-k}(X,\mathbb{Z})$.

For $k = n$, $\mathrm{Im}\,(f_* \circ p^* : H_{n-2}(Z,\mathbb{Z}) \to H_n(X,\mathbb{Z}))$ is contained in $\widetilde{N}^1H^n(X,\mathbb{Z})$.

Proof. The second statement is implied by the first using Lemma 1.2. Using the fact that the singularities of Z are isolated, we apply the same construction as in the proof of Theorem 2.17 of gluing very free curves to the fiber $f_z : Y_z \to X$, getting a modified family

$$f' : Y' \to X,\ p' : Y' \to Z' \tag{53}$$

of stable maps to X parameterized by a variety $Z' \stackrel{\tau}{\to} Z$ which is birational to Z and isomorphic to Z near $\mathrm{Sing}\,Z$ with the following properties:

(a) The cylinder homomorphism ${f'}_* \circ {p'}^* : H_{k-2}(Z',\mathbb{Z}) \to H_k(X,\mathbb{Z})$ coincides with $f_* \circ p^* \circ \tau_*$.

(b) The moduli space M of stable maps to X is smooth at any point $f'_z : Y'_z \to X$, where z is a singular point of Z' (or equivalently Z), hence at the point f'_z for z general in Z'.

We now conclude as follows: first of all, we reduce to the case where the maps are embeddings (for example by replacing X by $X \times \mathbb{P}^r$), so that the stable maps are automorphism free. We then consider the universal deformation of f'_z, $z \in Z'$, given by a family of automorphism free stable maps

$$f_M : Y_M \to X,\ p_M : Y_M \to M \tag{54}$$

parameterized by M. Using the automorphism free assumption, we have a classifying morphism $g : Z' \to M$, such that the family (51) is obtained from the family (52) by base change under g. We know that M is smooth near $g(\mathrm{Sing}\, Z')$, so we can introduce a desingularization $\widetilde{M}$ of M, and a modification $\tau' : \widetilde{Z}' \to Z'$ which is an isomorphism over $\mathrm{Sing}\, Z'$, such that the rational map $g : Z' \dashrightarrow \widetilde{M}$ induces a morphism

$$\tilde{g} : \widetilde{Z}' \to \widetilde{M}.$$

Over the desingularized moduli space $\widetilde{M}$, we have the pulled-back family

$$\tilde{f}_M : Y_{\widetilde{M}} \to X,\ \tilde{p}_M : Y_{\widetilde{M}} \to \widetilde{M}, \tag{55}$$

and over $\widetilde{Z}'$, we have the family

$$\tilde{f}' : \widetilde{Y}' \to X,\ \tilde{p}' : \widetilde{Y}' \to \widetilde{Z}', \tag{56}$$

which is deduced either from (53) by base-change under $\tilde{g}$ or from (51) by base-change under τ'. We conclude that

$$\tilde{f}'_* \circ (\tilde{p}')^* = \tilde{f}_{M*} \circ \tilde{p}^*_M \circ \tilde{g}_* : H_{k-2}(\widetilde{Z}', \mathbb{Z}) \to H_k(X, \mathbb{Z}),$$

and, as $\widetilde{M}$ is smooth, we get that $\mathrm{Im}\, \tilde{f}'_* \circ (\tilde{p}')^* \subset \widetilde{N}_{1,\mathrm{cyl}} H_k(X, \mathbb{Z})$. Finally, we also have by (a)

$$\tilde{f}'_* \circ (\tilde{p}')^* = f_* \circ p^* \circ (\tau \circ \tau')_* : H_{k-2}(\widetilde{Z}', \mathbb{Z}) \to H_k(X, \mathbb{Z}),$$

and, as $\tau \circ \tau' : \widetilde{Z}' \to Z$ is proper birational and an isomorphism over $\mathrm{Sing}\, Z$, the map

$$(\tau \circ \tau')_* : H_{k-2}(\widetilde{Z}', \mathbb{Z}) \to H_{k-2}(Z, \mathbb{Z})$$

is surjective. It follows that $\mathrm{Im}\, f_* \circ p^* \subset \widetilde{N}_{1,\mathrm{cyl}} H_k(X, \mathbb{Z})$. □

Our last question concerns the representability of the Abel-Jacobi isomorphism for 1-cycles on rationally connected threefolds (we refer here to [1] for a general discussion of the motivic nature of $J^3(X)$). As discussed in Section 2.1, another way of stating Theorem 0.2 or its generalization 2.19 is to say that, if X is a rationally connected manifold of dimension n, there exist a curve C and a codimension $n-1$ cycle $\mathcal{Z} \in \mathrm{CH}^{n-1}(C \times X)$ such that the lifted Abel-Jacobi map

$$\widetilde{\Phi}_{\mathcal{Z}} : J(C) \to J(N^{n-2} H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}})$$

is surjective with *connected* fibers. (When $n = 3$, we already mentioned that $N^1 H^3(X, \mathbb{Z})_{\mathrm{tf}} = H^3(X, \mathbb{Z})_{\mathrm{tf}}$.)

Note that it was proved in [19] that, even for X rationally connected of dimension 3,there does not necessarily exist a universal codimension $n-1$ cycle

$$\mathcal{Z}_{\mathrm{univ}} \in \mathrm{CH}^{n-1}(J(N^{n-2} H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}) \times X)$$

such that the induced lifted Abel-Jacobi map

$$\widetilde{\Phi}_{\mathcal{Z}} : J(N^{n-2} H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}) \to J(N^{n-2} H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}})$$

is the identity. However, the following question remains open.

Question 3.5. *Let X be a rationally connected manifold of dimension n. Does there exist a smooth projective manifold M and a codimension $n-1$-cycle*

$$\mathcal{Z}_M \in \mathrm{CH}^{n-1}(M \times X)$$

such that the map

$$\widetilde{\Phi}_{\mathcal{Z}_M} : \mathrm{Alb}\, M \to J(N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$$

is an isomorphism?

In practice, the answer is yes for Fano threefolds, at least for generic ones. For example, one can use the Fano surface of lines for the cubic threefold (see [5]), and similarly for the quartic double solid [23]. For quartic threefolds, the surface of conics works (see [11]).

The motivation for asking this question is the following:

Proposition 3.6. *If X admits a cohomological decomposition of the diagonal, (in particular, if X is stably rational), there exist a smooth projective manifold M and a codimension-$n-1$ cycle*

$$\mathcal{Z}_M \in \mathrm{CH}^{n-1}(M \times X)$$

such that the Abel-Jacobi map

$$\widetilde{\Phi}_{\mathcal{Z}_M} : \mathrm{Alb}\, M \to J(H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$$

is an isomorphism

(Note that $N^{n-2}H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}} = H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}$ under the same assumption.)

Proof of Proposition 3.6. It follows from Theorem 1.6 that there exist a (nonnecessarily connected) smooth projective variety Z of dimension $n-2$ and a family of 1-cycles

$$\Gamma \in \mathrm{CH}^{n-1}(Z \times X)$$

such that

$$\Gamma_* : \mathrm{Alb}(Z) \to J(H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}})$$

is surjective with a right inverse $\Gamma'^* : J(H^{2n-3}(X,\mathbb{Z})_{\mathrm{tf}}) \to \mathrm{Alb}(Z)$. We now have the following lemma.

Lemma 3.7. *Let Z be a smooth projective variety of dimension $n-2$ and let $A \subset \mathrm{Alb}(Z)$ be an abelian subvariety. Then there exists a smooth projective variety Z' and a 0-correspondence $\gamma' \in \mathrm{CH}^{n-2}(Z' \times Z)$ inducing an isomorphism $\gamma'_* : \mathrm{Alb}\, Z' \cong A \subset \mathrm{Alb}\, Z$.*

Proof. Suppose first that Z is connected of dimension 1. Then for N large enough, the Abel map

$$f : Z^{(N)} \to \mathrm{Alb}(Z)$$

is a projective bundle. Let now $Z' := f^{-1}(A)$. One has $\mathrm{Alb}(Z') \cong A$, and we can take for γ' the restriction to $Z' \times Z$ of the natural incidence correspondence $I \subset Z^{(N)} \times Z$.

For the general case, we quickly reduce, using the Lefschetz theorem on hyperplane sections, to the case where Z is a connected surface. Then we consider a Lefschetz pencil $\widetilde{Z} \to \mathbb{P}^1$ of ample curves on Z. Let Z_0 be a smooth projective model of $\mathrm{Pic}^0(\widetilde{Z}/\mathbb{P}^1)$. Then Z_0 is birational to $\mathrm{Pic}^1(\widetilde{Z}/\mathbb{P}^1)$ using one of the base-points, and thus admits a natural correspondence $\gamma \in \mathrm{CH}^2(Z_0 \times \widetilde{Z})$. It is immediate to check that

$$\gamma_* : \mathrm{Alb}(Z_0) \to \mathrm{Alb}(Z)$$

is an isomorphism. Let $a : Z_0 \to \mathrm{Alb}(Z_0)$ be the Albanese map. We claim that, denoting $Z'_u := a^{-1}(A_u)$, where A_u is a generic translate of A in $\mathrm{Alb}\,(Z)$, Z'_u is smooth and we have

$$\mathrm{Alb}(Z'_u) \cong A.$$

As Z_0 is smooth and A_u is smooth, the smoothness of $a^{-1}(A_u)$ for a general translate A_u of A follows from standard transversality arguments. For the second point, we observe that, by definition, a Zariski open set of Z_0 is fibered over $\mathbb{P}^1$ into Jacobians $J(\widetilde{Z}_t)$, $t \in \mathbb{P}^1$, and that the natural map $J(\widetilde{Z}_t) \to \mathrm{Alb}(Z) = \mathrm{Alb}(\widetilde{Z})$ has connected fiber isomorphic to $J(\mathrm{Ker}\,(H^1(\widetilde{Z}_t, \mathbb{Z}) \to H^3(\widetilde{Z}, \mathbb{Z})_{\mathrm{tf}}))$. Here the connectedness of the fibers indeed follows from the Lefschetz theorem on hyperplane sections which says that the Gysin morphism

$$H^1(\widetilde{Z}_t, \mathbb{Z}) \to H^3(\widetilde{Z}, \mathbb{Z})_{\mathrm{tf}}$$

is surjective (see the discussion in Section 1). It follows that a Zariski open set of Z'_u is fibered over $\mathbb{P}^1 \times A_u$ into connected abelian varieties $J(\mathrm{Ker}\,(H^1(\widetilde{Z}_t, \mathbb{Z}) \to H^3(\widetilde{Z}, \mathbb{Z})_{\mathrm{tf}}))$. On the other hand, by the Deligne global invariant cycle theorem, there is no nonconstant morphism from $J(\mathrm{Ker}\,(H^1(\widetilde{Z}_t, \mathbb{Z}) \to H^3(\widetilde{Z}, \mathbb{Z})_{\mathrm{tf}}))$ to a fixed abelian variety. It follows that $\mathrm{Alb}\, Z_u = A$.

Finally, we have $Z'_u \subset Z_0$ and Z_0 has a natural correspondence to Z, so combining both we get a natural correspondence γ' between Z'_u and Z, inducing the morphism

$$\mathrm{Alb}(Z'_u) \cong A \subset \mathrm{Alb}\, Z.$$

Then $\Gamma \circ \gamma'$ produces the desired correspondence. □

We apply this lemma to $A := \mathrm{Im}\,(\Gamma^* : J(H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}) \to \mathrm{Alb}(Z))$. We thus get a smooth projective variety Z' with Albanese variety isomorphic to $J(H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}})$ and cycle $\Gamma' := \Gamma \circ \gamma' \in \mathrm{CH}^{n-1}(Z' \times X)$ which induces the isomorphism $\Gamma_* \circ \gamma_* : \mathrm{Alb}\, Z' \cong J(H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}})$. □

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