

# Compact Kähler manifolds with no projective specialization

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## Abstract

We show the existence of a compact Kähler manifold which does not fit in a proper flat family over an irreducible base with one projective (possibly singular) fiber. We also give a topological version of this statement. This strengthens our earlier counterexamples to the Kodaira algebraic approximation problem.

## 0 Introduction

In the paper [3], Kodaira proved that any compact Kähler surface admits “algebraic approximations”, that is, becomes projective by an arbitrarily small deformation. This result, obtained as a consequence of the classification of surfaces, has been reproved in [1] by an infinitesimal method. A similar statement has been obtained recently by Lin [5] for threefolds. However, starting from dimension 4, we constructed in [7] compact Kähler manifolds not homeomorphic to, and in particular not deformation equivalent to, a complex projective manifold. In the paper [8], we also exhibited examples of compact Kähler manifolds, no smooth bimeromorphic models of which is homeomorphic to a complex projective manifold.

The manifolds  $X$  constructed in [7] thus have the property that there is no smooth proper holomorphic map  $\mathcal{X} \rightarrow B$ , where  $B$  is a connected analytic space, with two points  $b, b' \in B$  such that  $\mathcal{X}_b \cong X$  and the fiber  $\mathcal{X}_{b'}$  is projective. This does not say anything however on the singular specializations  $\mathcal{X}_{b'}$  of such proper flat holomorphic maps  $\mathcal{X} \rightarrow B$  with one fiber  $\mathcal{X}_b$  isomorphic to  $X$ .

**Definition 0.1.** *A compact complex manifold has a projective specialization if there exists a proper flat holomorphic map  $\mathcal{X} \rightarrow B$ , where  $B$  is an irreducible analytic space, with one fiber  $\mathcal{X}_b$  isomorphic to  $X$  and one projective fiber  $\mathcal{X}_0$ .*

The main result of this paper is the following theorem, which addresses a question asked to us by A. Kazhymurat.

**Theorem 0.2.** *There exists a compact Kähler manifold  $X$  with no projective specialization. Such manifolds exist starting from dimension 5.*

Note that the result presumably holds in dimension 4 as well. Compact Kähler manifolds  $X$  satisfying this property are obtained using a variant of the main constructions in [7], [8]. The constructions are recalled in Section 1. That the examples constructed in Section 1.1 satisfy the conclusion of Theorem 0.2 will be obtained as a consequence of the following criterion (see Theorem 2.4 for a slightly stronger version)

**Theorem 0.3.** *Let  $X$  be a compact Kähler manifold. Assume*

(i) *The Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$  is rigid under small deformations of  $X$ .*

(ii) *The endomorphisms of the Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$  are semi-simple.*

*Then, if  $X$  admits a projective specialization, the cohomology algebra  $H^*(X, \mathbb{Q})$  admits a polarizable Hodge structure.*

Here we use the following definition (see [9])

**Definition 0.4.** *A Hodge structure on the cohomology algebra of a compact complex manifold  $X$  is the data of a (effective) weight  $k$  Hodge structure on each cohomology group  $H^k(X, \mathbb{Q})$ , such that the cup-product map*

$$H^k(X, \mathbb{Q}) \otimes H^l(X, \mathbb{Q}) \rightarrow H^{k+l}(X, \mathbb{Q})$$

*is a morphism of Hodge structures for each pair of integers  $k, l$ . The Hodge structure is said polarizable if for each  $k$ , the weight  $k$  Hodge structure on  $H^k(X, \mathbb{Q})$  is polarizable.*

We will also present in Section 3.1 a variant of these results, using stronger assumptions on the cohomology algebra of the manifold  $X$ . This will allow us to strengthen the theorem above by making it into a topological statement. We will prove indeed in Section 3.1 the following result.

**Theorem 0.5.** *There exists a compact Kähler manifold  $X_1$  satisfying the following property: There is no proper flat holomorphic map  $\mathcal{X} \rightarrow B$ , where  $B$  is an irreducible analytic space, with one smooth fiber  $\mathcal{X}_b$  homeomorphic to  $X_1$  and one projective fiber  $\mathcal{X}_0$ .*

**Remark 0.6.** Even if the statement concerns the homeomorphism class of  $X$ , one cannot replace “irreducible” by “connected” in such a statement, because, in the presence of singular fibers, it is a priori not true that the smooth fibers are all homeomorphic if  $B$  is not irreducible.

As one can notice, an important difference between Theorems 0.3 and 0.7 is that the later does not assume the existence of a Kähler fiber in the family  $\mathcal{X}$ . We discuss in Section 3.2 some consequences of the existence of one singular projective fiber.

A compact Kähler manifold  $X_1$  satisfying this property is constructed in Section 1.2. Theorem 0.5 will be obtained as a consequence of the following

**Theorem 0.7.** *Let  $X$  be a compact complex manifold. Assume the endomorphisms of the cohomology algebra of  $X$  are semi-simple. Then, if  $X$  admits a projective specialization, the cohomology algebra of  $X$  carries a Hodge structure which is polarizable.*

Note that, in Theorem 0.7, the assumptions on the cohomology algebra are much stronger than in Theorem 0.3, since in the former we are considering all the automorphisms of the cohomology algebra, while in the later, only the automorphisms preserving the given Hodge structure are considered.

We will prove Theorem 0.2 in Section 2.2 and Theorem 0.5 in Section 3.1. We will also discuss in Section 3.2 some further generalities on compact complex manifolds admitting a projective degeneration.

**Thanks.** *I thank Aknazar Kazhymurat for asking me this question and Radu Laza for helpful discussion.*

## 1 The examples

### 1.1 The manifold $X$

We will discuss here the construction in dimension 7 for simplicity. The case of dimension  $> 7$  will follow by taking a product with projective space and applying similar arguments. In dimension 7, a manifold  $X$  satisfying the assumptions of Theorem 0.3 will be obtained by a small variant of [8]. We start from a 2-dimensional complex torus  $T$  admitting an endomorphism  $\phi$  which is an isogeny and whose action  $\psi = \phi_*$  on  $H_1(T, \mathbb{Q})$  has the property that the Galois group of the field  $\mathbb{Q}[\psi]$  is as big as possible, that is, acts on the eigenvalues of  $\psi$  as the symmetric group of 4 elements. It is clear that such a complex torus equipped with

an endomorphisms is rigid, and it is proved in [7] that such a torus is not projective (this does not contradict [2] which considers a finite group action). Let  $\widehat{T} := \text{Pic}^0(T)$ . Denote by  $\mathcal{L}$  the Poincaré divisor on  $T \times \widehat{T}$ , and let  $\mathcal{L}_\phi := (\phi, Id_{\widehat{T}})^* \mathcal{L}$ . Let  $t \in T$ ,  $\hat{t} \in \widehat{T}$  be two points. Let  $W = \mathbb{P}^1 \times T \times \widehat{T}$ . We choose two distinct points  $t_1, t_2 \in \mathbb{P}^1$ .  $W$  contains then two disjoint complex tori, namely  $t_1 \times T \times \hat{t}$  and  $t_2 \times t \times \widehat{T}$ . We first blow them up in  $W$ , getting a compact Kähler manifold  $\widetilde{W}$ . Next, using the same notation  $\mathcal{L}, \mathcal{L}_\phi$  for the line bundles pulled-back to  $\widetilde{W}$ , we construct the  $\mathbb{P}^1$ -bundles

$$P_1 := \mathbb{P}(\mathcal{E}_1), \quad \mathcal{E}_1 := \mathcal{O}_{\widetilde{W}} \oplus \mathcal{L}, \quad P_2 := \mathbb{P}(\mathcal{E}_2), \quad \mathcal{E}_2 := \mathcal{O}_{\widetilde{W}} \oplus \mathcal{L}_\phi \quad (1)$$

over  $\widetilde{W}$ . Note that the  $\mathbb{P}^1$ -bundles  $P_1, P_2$  over  $\widetilde{W}$  have sections  $\sigma_1$ , resp.  $\sigma_2$ , given by the surjective morphisms  $\mathcal{O}_{\widetilde{W}} \oplus \mathcal{L} \rightarrow \mathcal{L}$ , resp.  $\mathcal{O}_{\widetilde{W}} \oplus \mathcal{L}_\phi \rightarrow \mathcal{L}_\phi$ . It follows that the fibered product  $P_1 \times_{\widetilde{W}} P_2 \rightarrow \widetilde{W}$  has a section  $\sigma = (\sigma_1, \sigma_2)$ . The compact Kähler manifold  $X$  will be defined as the blow-up of the fibered product of  $P_1$  and  $P_2$  along  $\Sigma := \sigma(\widetilde{W})$ :

$$X := P_1 \widetilde{\times}_{\widetilde{W}} P_2 \overset{\Sigma}{\text{blow-up}}. \quad (2)$$

We now prove a few properties that will be needed in the next sections.

**Lemma 1.1.** *The Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$  is rigid under small deformations of  $X$ .*

By “rigid under small deformations of  $X$ ”, we mean that any flat holomorphic proper holomorphic map  $\mathcal{X} \rightarrow B$  to a pointed analytic space  $(B, b_0)$ , with  $\mathcal{X}_{b_0} \cong X$  induces near  $b_0$  the trivial variation of Hodge structures. (As  $X$  is Kähler, the fibers  $\mathcal{X}_b$  are also Kähler for  $b$  close to  $b_0$ .)

*Proof of Lemma 1.1.* We observe that the cohomology algebra of  $X$  is generated in degree  $\leq 2$ , so it suffices to prove rigidity for the induced Hodge structures on  $H^1(X, \mathbb{Q})$  and  $H^2(X, \mathbb{Q})$ . We have

$$H^2(X, \mathbb{Q}) = \bigwedge^2 H^1(X, \mathbb{Q}) \oplus \mathbb{Q}d \oplus \mathbb{Q}\hat{d} \oplus \mathbb{Q}h \oplus \mathbb{Q}H_1 \oplus \mathbb{Q}H_2 \oplus \mathbb{Q}e, \quad (3)$$

where  $e$  is the class of the exceptional divisor  $E$  over  $\Sigma$ ,  $d, \hat{d}$  are the pull-backs to  $X$  of the classes of the exceptional divisors of  $\widetilde{W}$ ,  $h$  is the pull-back to  $X$  of the class  $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$  via the composite morphism

$$X \rightarrow P_1 \times_{\widetilde{W}} P_2 \rightarrow \widetilde{W} \rightarrow W \xrightarrow{\text{pr}_1} \mathbb{P}^1,$$

and  $H_1, H_2$  are the pull-backs to  $X$  of the first Chern classes  $c_1(\mathcal{O}_{P_i}(1))$  of the projective bundles  $P_i \rightarrow \widetilde{W}$ .

Given a deformation  $\mathcal{X} \rightarrow B$  of  $X \cong \mathcal{X}_0$ , we can apply the Kodaira stability theorems [4]. They say that a small deformation of a compact complex manifold which is the blow-up  $\widetilde{Y} \rightarrow Y$  of a manifold  $Y$  along a submanifold  $\Sigma \subset Y$  is the blow-up of a deformation  $Y_t$  of  $Y$  along a deformation  $\Sigma_t \subset Y_t$  of  $\Sigma$ ; furthermore a small deformation of a projective bundle fibration  $P \rightarrow Y$  is a projective bundle fibration  $P_t \rightarrow Y_t$ . This implies in our case that for  $b$  close to  $b_0$ , there is a divisor  $E_b \subset \mathcal{X}_b$  which is a deformation of  $E$  and is the exceptional divisor of a contraction  $\mathcal{X}_b \rightarrow Q_b$ , where  $Q_b$  is a deformation of  $Q_{b_0} = P_1 \times_{\widetilde{W}} P_2$ . Furthermore, still by [4], the  $\mathbb{P}^1 \times \mathbb{P}^1$ -fibration on  $Q_{b_0}$  deforms to a  $\mathbb{P}^1 \times \mathbb{P}^1$ -fibration on  $Q_b$ , hence we get a deformation  $Q_b \rightarrow \widetilde{W}_b$  of the morphism  $P_1 \times_{\widetilde{W}} P_2 \rightarrow \widetilde{W}$ . Finally and again by Kodaira stability theorem, the manifold  $\widetilde{W}_b$  for  $b$  close to  $b_0$  contains contractible divisors which are deformations of the exceptional divisors of  $\widetilde{W}$  over  $W$ . Thus for  $b$  close to  $b_0$ ,  $\widetilde{W}_b$  contracts to a compact complex manifold which is a deformation  $W_b$  of the complex manifold  $\mathbb{P}^1 \times T \times \widehat{T}$ , hence is a  $\mathbb{P}^1$ -fibration over a complex torus  $K_b$  which is itself a

deformation of  $K_{b_0} = T \times \widehat{T}$ . Furthermore, the complex manifold  $W_b$  contains the images of the two contracted divisors, which are complex submanifolds obtained as small deformations of  $T \times \widehat{t} \subset W$  and  $t \times \widehat{T} \subset W$ . They are thus 2-dimensional complex tori  $T_b, T'_b$  contained in  $W_b$ , which map to 2-dimensional subtori  $T_b, T'_b$  of  $K_b$  intersecting in one point. We thus conclude that  $K_b \cong T_b \oplus T'_b$ .

Moreover the image  $\Sigma \subset P_1 \times_{\widetilde{W}} P_2$  of the contracted divisor  $E$  deforms to the image  $\Sigma_b \subset Q_b$  of the contracted divisor  $E_b$ , hence the  $\mathbb{P}^1 \times \mathbb{P}^1$ -fibration  $Q_b \rightarrow \widetilde{W}'_b$  has a section for  $b$  close to  $b_0$ . It follows that there are vector bundles  $\mathcal{E}_{1,b}, \mathcal{E}_{2,b}$  on  $\widetilde{W}_b$  obtained as deformations of the vector bundles  $\mathcal{E}_1, \mathcal{E}_2$  of (1) on  $\widetilde{W}$ , such that

$$Q_b \cong \mathbb{P}(\mathcal{E}_{1,b}) \times_{\widetilde{W}_b} \mathbb{P}(\mathcal{E}_{2,b}).$$

Finally, by taking the determinants of  $\mathcal{E}_{1,b}, \mathcal{E}_{2,b}$ , we conclude that there are holomorphic line bundles  $\mathcal{L}_b, \mathcal{L}_{\phi,b}$  on  $\widetilde{W}_b$  which are respective deformations of  $\mathcal{L}, \mathcal{L}_\phi$  on  $\widetilde{W}$ . As these line bundles are pulled-back at  $b_0$  from line bundles on  $K_{b_0}$ , the same is true of the line bundles  $\mathcal{L}_b, \mathcal{L}_{\phi,b}$  on  $\widetilde{W}_b$ . In conclusion we proved that the complex torus  $K_b \cong T_b \oplus T'_b$  carries two holomorphic line bundles  $\mathcal{L}_b, \mathcal{L}_{\phi,b}$  deforming  $\mathcal{L}$  and  $\mathcal{L}_\phi$  respectively. Using  $\mathcal{L}_b$ , one concludes that  $T'_b \cong \widehat{T}_b$ , and using  $\mathcal{L}_{\phi,b}$ , one concludes that  $T_b$  carries an endomorphism deforming  $\phi$ . It follows that  $T_b \cong T$  since the pair  $(T, \phi)$  is rigid. We thus proved that the Hodge structure on  $H^1(\mathcal{X}_b, \mathbb{Q})$  is isomorphic to the Hodge structure on  $H^1(X, \mathbb{Q})$ . Furthermore, by the above arguments, all the degree 2 Hodge classes on  $X$  listed in (3) deform to degree 2 Hodge classes on  $\mathcal{X}_b$ , so (3) shows that the Hodge structure on  $H^2(\mathcal{X}_b, \mathbb{Q})$  is isomorphic to the Hodge structure on  $H^2(X, \mathbb{Q})$ .  $\square$

**Lemma 1.2.** *The automorphisms of the Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$  are semisimple.*

*Proof.* The automorphisms of the Hodge structure on  $H^1(X, \mathbb{Q}) = H^1(T, \mathbb{Q}) \oplus H^1(T, \mathbb{Q})^*$  are of the form

$$\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix},$$

where  $\psi_1$  resp.  $\psi_2$  are automorphisms of the Hodge structure on  $H^1(T, \mathbb{Q})$ , resp.  $H^1(T, \mathbb{Q})^*$ . Indeed, the two tori  $T, \widehat{T}$  are simple and not isogenous. Next, using the fact that  ${}^t\psi$ , resp.  $\psi$  acts in an irreducible way as an automorphism of the Hodge structure on  $H^1(T, \mathbb{Q})$ , resp.  $H^1(T, \mathbb{Q})^*$ , we get that  $\psi_1 \in \mathbb{Q}[{}^t\psi]$ , resp.  $\psi_2 \in \mathbb{Q}[\psi]$ . As the endomorphism  $\psi$  is semisimple, it follows that any automorphism of the Hodge structure on  $H^1(X, \mathbb{Q})$  is semisimple.

It remains to prove semisimplicity of the action on the higher degree cohomology groups. As the cohomology algebra of  $X$  is generated in degree  $\leq 2$ , it suffices to prove the simplicity on degree 2 cohomology. As the automorphisms of the Hodge structure on  $H^2(X, \mathbb{Q})$  preserve the space of Hodge classes, using the decomposition (3), we only need to prove

**Claim 1.3.** *An automorphism  $\gamma$  of the Hodge structure on the cohomology algebra of  $X$  has a power  $\gamma^r$ ,  $r > 0$  which preserves the space  $\langle d, \widehat{d}, h, H_1, H_2, e \rangle \subset H^2(X, \mathbb{Q})$  and acts in a semisimple way on it.*

*Proof.* The automorphism  $\gamma$  acts on  $H^*(X, \mathbb{Q})$  preserving the algebra structure and the Hodge structure. In particular, it preserves the subalgebra generated by  $H^1(X, \mathbb{Q})$  and the subspace  $\text{Hdg}^2(X)$ , hence it preserves  $\text{Hdg}^2(X) \cap \bigwedge^2 H^1(X, \mathbb{Q})$ . We claim that a power of  $\gamma$  preserves up to a scalar multiple the classes  $h, H_1, H_2$ . For the class  $h$ , the reason is that  $h^2 = 0$  in  $H^4(X, \mathbb{Q})$ , and  $h$  is an isolated solution of this set of equations in  $\mathbb{P}(\text{Hdg}^2(X))$ ; moreover  $\gamma$  acts on  $\mathbb{P}(\text{Hdg}^2(X))$  preserving the algebraic subset defined by these equations, since it preserves the algebra structure of  $H^*(X, \mathbb{Q})$ . For the classes  $H_1, H_2$ , one observes that they satisfy an equation of the form  $H_i^2 = H_i l_i$ , for some classes  $l_i \in \bigwedge^2 H^1(X, \mathbb{Q}) \cap \text{Hdg}^2(X)$ . These equations come from the fact that the vector bundles  $\mathcal{E}_i$  of (1) have  $c_2 = 0$ . Again the classes  $H_i$  are isolated solutions of these equations in  $\mathbb{P}(\text{Hdg}^2(X))$  so that a

power of  $\gamma$  will fix them. Similarly, a power of  $\gamma$  preserves up to a scalar the classes  $d$  and  $\hat{d}$ , because they have the property that the cup-product map  $d \cup, \hat{d} \cup : H^1(X, \mathbb{Q}) \rightarrow H^3(X, \mathbb{Q})$  is not injective, and they are the only elements of  $\mathbb{P}(\text{Hdg}^2(X))$  which do not belong to  $\mathbb{P}(\bigwedge^2 H^1(X, \mathbb{Q}) \cap \text{Hdg}^2(X))$  and satisfy this condition. We conclude that a power  $\gamma^r$ ,  $r > 0$ , of  $\gamma$  preserves the subalgebra generated by  $H^1(X, \mathbb{Q})$  and  $h, H_1, H_2, d, \hat{d}$ , that is, the cohomology subalgebra

$$H^*(P_1 \times_{\widehat{W}} P_2, \mathbb{Q}) \xrightarrow{\tau} H^*(X, \mathbb{Q}).$$

But then, using the fact that the morphism  $\tau : X \rightarrow P_1 \times_{\widehat{W}} P_2$  is birational,  $\gamma^r$  also preserves the class  $e$ , since it generates the kernel of the morphism  $\tau_* : H^2(X, \mathbb{Q}) \rightarrow H^2(P_1 \times_{\widehat{W}} P_2, \mathbb{Q})$ . One uses here the fact that  $\tau_*$  is determined by  $\tau^*$  and the algebra structure, by Poincaré duality. Thus a power  $\gamma^r$  preserves  $\langle h, H_1, H_2, d, \hat{d}, e \rangle$  and acts in a diagonalized way on it. The claim is proved.  $\square$

We thus proved that a power of  $\gamma$  acts in a semisimple way on  $H^*(X, \mathbb{Q})$ , which implies that  $\gamma$  acts in a semisimple way since  $\gamma$  is an automorphism.  $\square$

## 1.2 The manifold $X_1$

We will discuss here the construction in dimension 5 for simplicity. The case of dimension  $> 5$  will follow by taking a product with projective space and applying similar arguments. In dimension 5, a manifold  $X_1$  satisfying the assumptions of Theorem 0.7 will be obtained by a small variant of [7]. We start as in the previous section from a 2-dimensional complex torus  $T$  admitting an endomorphism  $\phi$  which is an isogeny and whose action  $\psi = \phi_*$  on  $H_1(T, \mathbb{Q})$  has the property that the Galois group of the field  $\mathbb{Q}[\psi]$  acts on the eigenvalues of  $\psi$  as the symmetric group of 4 elements.

We start with  $W := T \times T \times \mathbb{P}^1$  and choose four distinct elements  $t_1, \dots, t_4$  in  $\mathbb{P}^1$ . Let  $t \in T$  be chosen.  $W$  contains the following four disjoint complex submanifolds, all isomorphic to  $T$ :

$$T \times t \times t_1, \quad t \times T \times t_2, \quad \Delta_T \times t_3, \quad \Gamma_\phi \times t_4, \quad (4)$$

where as usual  $\Delta_T$  is the diagonal of  $T$  and  $\Gamma_\phi$  is the graph of  $\phi$ .

We will denote by  $X_1$  the compact Kähler manifold obtained by blowing-up  $W$  along these four submanifolds. The tiny difference with the construction of [7] is that the product with  $\mathbb{P}^1$  and the choice of four distinct points  $t_i$  in  $\mathbb{P}^1$  makes the four centers (4) of blow-up disjoint, while their images in  $T \times T$  are not disjoint. This forced us in [7] to blow-up first their intersection points in order to separate them before blow-up, making the cohomological computations heavier.

We will need the following statement concerning the Hodge structures on the cohomology algebra of  $X_1$ .

**Proposition 1.4.** *The automorphisms of the cohomology algebra  $H^*(X_1, \mathbb{Q})$  are semi-simple.*

*Proof.* As the cohomology algebra  $H^*(X_1, \mathbb{Q})$  is generated in degree  $\leq 2$ , it suffices to prove that the action of an automorphism  $\gamma$  of  $H^*(X_1, \mathbb{Q})$  is semi-simple on  $H^1(X_1, \mathbb{Q})$  and  $H^2(X_1, \mathbb{Q})$ . We have  $H^2(X_1, \mathbb{Q}) = \text{Hdg}^2(X_1) \oplus \bigwedge^2 H^1(X_1, \mathbb{Q})$ , where the space  $\text{Hdg}^2(X_1)$  of Hodge classes on  $X_1$  is generated by the class  $h$  which is the pull-back to  $X_1$  of the class  $\text{pr}_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$  on  $W$  and the classes  $d_i$  of the four exceptional divisors of  $X_1$  over  $W$ . Clearly the subspace  $\bigwedge^2 H^1(X_1, \mathbb{Q})$  is preserved by any automorphism of the cohomology algebra. We first prove

**Lemma 1.5.** *Any endomorphism  $\gamma$  of the cohomology algebra  $H^*(X_1, \mathbb{Q})$  preserves the subspace*

$$\text{Hdg}^2(X_1) \subset H^2(X_1, \mathbb{Q})$$

and acts in a semi-simple way on it.

*Proof.* The subspace  $\mathbb{Q}h$  is preserved by  $\gamma$  because, up to a coefficient,  $h$  is the only class in  $H^2(X_1, \mathbb{Q})$  which is not in  $\bigwedge^2 H^1(X_1, \mathbb{Q})$  and satisfies  $h^2 = 0$ . Next we claim that the  $\mathbb{Q}$ -vector subspace  $\langle d_i \rangle \subset \text{Hdg}^2(X_1)$  is stable under  $\gamma$ . This follows indeed from the structure of the cohomology algebra: Looking at the cup-product map

$$\alpha \cup : H^1(X_1, \mathbb{Q}) \rightarrow H^3(X_1, \mathbb{Q})$$

for  $\alpha \in H^2(X_1, \mathbb{Q})$ , we see that the  $d_i$ 's are up to a scalar multiple the only classes not in  $\bigwedge^2 H^1(X_1, \mathbb{Q})$  for which the cup product map  $d_i \cup : H^1(X_1, \mathbb{Q}) \rightarrow H^3(X_1, \mathbb{Q})$  has a kernel of dimension  $\geq 4$ . Hence  $\gamma$  preserves the subspace  $\langle d_i \rangle \subset H^2(X_1, \mathbb{Q})$  and thus preserves  $\text{Hdg}^2(X_1)$ . Finally, exactly for the same reason,  $\gamma$  acts projectively on the space  $\mathbb{P}(\langle d_i \rangle)$  by permuting the  $d_i$ 's, so that a power of  $\gamma$  acts on the projective space  $\mathbb{P}(\langle d_i \rangle)$  fixing the  $d_i$ 's. We thus proved that the action of a power of  $\gamma$  on the vector space  $\text{Hdg}^2(X_1)$  is diagonalizable, which implies semisimplicity of the action of  $\gamma$  on  $\text{Hdg}^2(X_1)$ , since  $\gamma$  is an automorphism.  $\square$

For the action of  $\gamma$  on degree 1 cohomology, we next prove

**Lemma 1.6.** *Any automorphism  $\gamma$  of the cohomology algebra  $H^*(X_1, \mathbb{Q})$  has a power  $\gamma^r$ ,  $r > 0$ , acting on  $H^1(X_1, \mathbb{Q}) = H^1(T, \mathbb{Q})^2$  via a matrix*

$$\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_1 \end{pmatrix}, \quad (5)$$

where  $\psi_1 \in \mathbb{Q}[{}^t\psi]$ .

*Proof.* The proof of Lemma 1.5 shows that  $\gamma$  preserves the class  $h$  and the  $\mathbb{Q}$ -vector space  $K := \langle d_i \rangle$ . We also saw that a power  $\gamma^r$  for some  $r > 0$  acts on  $\mathbb{P}(K)$  fixing the  $d_i$ . It then follows that the action of  $\gamma^r$  on  $H^1(X_1, \mathbb{Q})$  preserves the four subspaces  $L_i := \text{Ker}(d_i \cup : H^1(X_1, \mathbb{Q}) \rightarrow H^3(X_1, \mathbb{Q}))$ . We have

$$H^1(X_1, \mathbb{Q}) = L_1 \oplus L_2,$$

and this decomposition is preserved by  $\gamma^r$ . Furthermore the subspace  $L_3 \subset L_1 \oplus L_2$  defines an isomorphism  $L_1 \cong L_2$ , so we can see  $L_4 \subset L_1 \oplus L_2 \cong L_1 \oplus L_1$  as the graph of an automorphism  $\psi'$  of  $L_1$ , that is easily seen to be conjugate to  ${}^t\psi$ . As  $\gamma^r$  preserves the  $L_i$ , it follows from the above analysis that  $\gamma^r$  acts on  $L_1$  and  $L_2$  via the same automorphism  $\psi_1$ , that has the property that  $\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_1 \end{pmatrix}$  maps the graph of  ${}^t\psi$  to itself. Equivalently,  $\psi_1$  commutes with  ${}^t\psi$ . As  $\psi$  has only simple eigenvalues, it follows that  $\psi_1 \in \mathbb{Q}[{}^t\psi]$ .  $\square$

As  $\psi_1$  is semisimple, Lemma 1.6 implies that the action of a power  $\gamma^r$ ,  $r > 0$  of  $\gamma$  on  $H^1(X_1, \mathbb{Q})$  is semisimple, and it follows that the action of  $\gamma^r$  on  $H^1(X_1, \mathbb{Q})$  and  $\bigwedge^2 H^1(X_1, \mathbb{Q})$  is semisimple. Using Lemma 1.5, we conclude that the action of  $\gamma^r$  on  $H^2(X_1, \mathbb{Q})$  is semisimple. This concludes the proof of Proposition 1.4.  $\square$

To conclude this section, we note for future use that the compact Kähler manifolds  $X$  and  $X_1$  constructed above satisfy the following property (see [7], [8]).

**Theorem 1.7.** *For any Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$ , resp.  $H^*(X_1, \mathbb{Q})$ , the induced weight 1 Hodge structure on  $H^1(X, \mathbb{Q})$ , resp.  $H^1(X_1, \mathbb{Q})$  is not polarizable.*

The existence of a polarizable Hodge structure on the cohomology algebra is the necessary criterion used in [7] for a compact Kähler manifold to be homeomorphic to a complex projective manifold. Theorem 1.7 is proved in [7] for the manifold  $X_1$ . The case of  $X$  is treated by similar arguments. In both cases, the idea is that the structure of the cohomology algebra forces a Hodge structure on the cohomology algebra to have its degree 1 part isomorphic to a direct sum  $H_1 \oplus H_2$  of weight 1 Hodge structures admitting an endomorphism conjugate to  $\psi$  or  ${}^t\psi$ , which makes them not polarizable.

## 2 Proof of Theorem 0.2

### 2.1 Preliminary results

We start with a general statement concerning proper holomorphic fibrations with one Kähler fiber.

**Proposition 2.1.** *Let  $\pi : \mathcal{X} \rightarrow B$  be a proper flat holomorphic map, where  $B$  is an irreducible analytic space. Assume one smooth fiber  $\mathcal{X}_t$  is Kähler. Then there exists a dense open subset  $U \subset B$  which is the complement of a closed real analytic subset of  $B$ , such that the fiber  $\mathcal{X}_b$ , for  $b \in U$ , carries a Hodge structure on its cohomology algebra.*

*Proof.* For any complex manifold  $X$  and for any integer  $k$ , there is a ‘‘Hodge filtration’’ on the cohomology  $H^k(X, \mathbb{C})$  defined as

$$F^r H^k(X, \mathbb{C}) = \text{Im}(\mathbb{H}^k(X, \Omega_X^{\bullet \geq r}) \rightarrow \mathbb{H}^k(X, \Omega_X^{\bullet})). \quad (6)$$

These filtrations for various  $k$  are obviously compatible with the cup-product map in the sense that  $F^r H^k \cup F^s H^l \subset F^{r+s} H^{k+l}$ . In order to prove that there is a Hodge structure on the cohomology algebra of  $X$ , it thus suffices to show that these Hodge filtrations define Hodge structures, which is equivalent to saying that they satisfy the opposite condition for all  $k, r$

$$F^r H^k(X, \mathbb{C}) \oplus \overline{F^{k-r+1} H^k(X, \mathbb{C})} \cong H^k(X, \mathbb{C}). \quad (7)$$

In the situation of the proposition, we know that one fiber is Kähler. It follows that its Frölicher spectral sequence degenerates at  $E_1$ . This implies that the Frölicher spectral sequence of  $\mathcal{X}_b$  degenerates at  $E_1$  for  $b$  in an analytic-Zariski open neighborhood  $V$  of  $b$  in  $B$  (contained in the open subset  $B^0$  of  $B$  parameterizing smooth fibers), and as  $B$  is irreducible, this open set is dense. Next, the degeneracy at  $E_1$  of the Frölicher spectral sequence of the fibers  $\mathcal{X}_b$  over  $V$  implies that, denoting  $\pi_V : \mathcal{X}_V \rightarrow V$  the restriction of  $\pi$  to  $\mathcal{X}_V := \pi^{-1}(V)$ , the subsheaves

$$F^r \mathcal{H}^k \subset \mathcal{H}^k, \quad \mathcal{H}^k = \mathbb{R}^k \pi_{V*} \Omega_{\mathcal{X}^0/B^0}^{\bullet} = \mathbb{R}^k \pi_{V*} \mathbb{C} \otimes \mathcal{O}_{B^0}$$

are locally free subbundles satisfying the base change property, because the sheaf  $\mathcal{H}^k$  is locally free. It then follows from the constancy of the rank of  $F^r H^k(\mathcal{X}_b, \mathbb{C})$ ,  $b \in V$ , that the locus  $U$  of points  $b \in V$  where the opposite conditions (7) are satisfied is the complement of a closed real analytic open set in  $V$ . This locus is not empty by assumption and, as  $B$  is irreducible,  $U$  is dense in  $B$ .  $\square$

**Corollary 2.2.** *In the situation above, assume the compact Kähler fiber  $\mathcal{X}_t$  has the property that the Hodge structure on its cohomology algebra is rigid under small deformations of  $\mathcal{X}_t$ . Then, for any integer  $k$ , the monodromy action*

$$\pi_1(B^0, t) \rightarrow \text{Aut } H^k(\mathcal{X}_t, \mathbb{Q}) \quad (8)$$

*acts via automorphisms of Hodge structures.*

*Proof.* The Hodge structure of the cohomology algebra of  $\mathcal{X}_t$  being rigid under small deformations of  $\mathcal{X}_t$ , for any integer  $k$ , the Hodge structure on the degree  $k$  cohomology of the fibers  $\mathcal{X}_b$  for  $b \in U$  close to  $b$  is locally constant. It follows by analytic continuation that the Hodge filtration, which (unlike the Hodge decomposition) varies holomorphically and is defined on the analytic Zariski dense open set  $V \subset B^0$ , is locally constant on  $V$ . As the natural map  $\pi_1(V, t) \rightarrow \pi_1(B^0, t)$  is surjective, the monodromy action (8) on  $H^k(\mathcal{X}_t, \mathbb{Q})$  preserves the Hodge filtration, hence the Hodge structure.  $\square$

**Remark 2.3.** The proof also shows that for any  $b \in V$ , the Hodge filtration on  $H^k(\mathcal{X}_b, \mathbb{C})$  induces a Hodge structure on  $H^k(\mathcal{X}_b, \mathbb{Q})$  which is isomorphic to that of  $H^k(\mathcal{X}_t, \mathbb{C})$ .

## 2.2 Projective specializations of compact Kähler manifolds

Our goal in this section is to prove Theorem 0.2, as a consequence of the following result on projective specializations of compact Kähler manifolds. This is a slight improvement of Theorem 0.3.

**Theorem 2.4.** *Let  $X$  be a compact Kähler manifold. Assume*

(i) *The Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$  is rigid under small deformations of  $X$ .*

(ii) *The endomorphisms of the Hodge structure on the cohomology algebra  $H^*(X, \mathbb{Q})$  preserving a nonzero element of  $H^2(X, \mathbb{Q})$  with nonzero top self-intersection are semi-simple.*

*Then, if  $X$  admits a projective specialization, the cohomology algebra  $H^*(X, \mathbb{Q})$  admits a polarizable Hodge structure.*

Before giving the proof, we prove Theorem 0.2.

*Proof of Theorem 0.2.* We observe that assumptions (i) and (ii) above are satisfied by the compact Kähler manifold  $X$  constructed in Section 1.1. This is indeed the contents of Proposition 1.1 and Lemma 1.2. By Theorem 1.7, the cohomology algebra of  $X$  does not admit a polarizable Hodge structure. Theorem 2.4 thus shows that  $X$  does not admit a projective specialization.  $\square$

*Proof of Theorem 2.4.* Let  $X$  be a compact Kähler manifold satisfying the assumptions (i) and (ii) above. Let  $\pi : \mathcal{X} \rightarrow B$  be a flat proper holomorphic map between analytic spaces, where  $B$  is irreducible, with one fiber  $\mathcal{X}_b$  isomorphic to  $X$  and one fiber  $\mathcal{X}_0$  projective. We can assume by desingularization that  $B$  is smooth with discriminant divisor  $\mathcal{D} = B \setminus B^0$ , and we can choose a disc  $D \subset B$  centered at 0 and not contained in  $\mathcal{D}$ . The analytic Zariski open set  $V$  being defined as in the previous section, we can even assume that  $D$  is not contained in  $B \setminus V$ . From now on, we will only work with the restricted family

$$\pi' : \mathcal{X}' := D \times_B \mathcal{X} \rightarrow D.$$

Note that, after a finite base-change and blow-up over the central fiber, one can assume that  $\mathcal{X}'$  is smooth and the central fiber  $\mathcal{X}'_0$  is a reduced divisor with normal crossings.

For any choice of  $s \in D^* \cap V$ , one has the monodromy representation

$$\rho : \mathbb{Z} = \pi_1(D^*, s) \rightarrow \text{Aut } H^k(\mathcal{X}_s, \mathbb{Q}),$$

hence a monodromy operator  $T = \rho(\gamma)$  where  $\gamma$  is a counterclockwise loop around 0 based at  $s$ . By assumption (i), the assumptions of Corollary 2.2 are satisfied by  $X$ . Corollary 2.2 and Remark 2.3 tell that there is a Hodge structure on  $H^*(\mathcal{X}_s, \mathbb{Q})$ , isomorphic to the Hodge structure on  $H^*(X, \mathbb{Q})$ , and  $T$  acts on the cohomology algebra  $H^*(\mathcal{X}_s, \mathbb{Q})$  as an automorphism of Hodge structures, which obviously also preserves the algebra structure. We use now the fact that the central fiber is projective. There is thus a class  $H \in H^2(\mathcal{X}'_0, \mathbb{Q})$  defined as the restriction of the hyperplane class in a projective embedding of  $\mathcal{X}'_0$ . Choosing a topological retraction  $r$  of  $\mathcal{X}'$  onto  $\mathcal{X}'_0$ , the class  $r^*H|_{\mathcal{X}'_s} \in H^2(\mathcal{X}'_s, \mathbb{Q})$  is invariant under  $T$ . We have  $\int_{\mathcal{X}'_0} H^n \neq 0$ ,  $n = \dim X$ , hence  $\int_{\mathcal{X}'_s} r^*H^n \neq 0$ . By assumption (ii), the action of  $T$  is thus semi-simple. We now apply the following result (which in this context is due to Steenbrink).

**Theorem 2.5.** *(Steenbrink [6, Theorem 2.21]) The action of  $T$  on  $H^k(\mathcal{X}_s, \mathbb{Q})$  is unipotent.*

It follows that the eigenvalues of  $T$  on  $H^*(\mathcal{X}_s, \mathbb{Q})$  are equal to 1, hence as  $T$  is semi-simple, it must be the identity.

Let  $\widetilde{D}^*$  be the universal cover of  $D^*$  and let  $\widetilde{\mathcal{X}}'^* := \mathcal{X}' \times_D \widetilde{D}^*$ . Let  $H_{\text{lim}}^* := H^*(\widetilde{\mathcal{X}}'^*, \mathbb{Q})$ . The cohomology algebra  $H_{\text{lim}}^*$  is isomorphic to  $H^*(X, \mathbb{Q})$ . We now use the fact that the central fiber  $\mathcal{X}'_0$  is projective. We can then apply the following results of Steenbrink [6].

**Theorem 2.6.** (i) *There is a mixed Hodge structure on each  $H_{\text{lim}}^k$ .*

(ii) *More precisely,  $H_{\text{lim}}^k$  is computed as the abutment of a (weight) spectral sequence  ${}_W E_1^{p,q} \Rightarrow H_{\text{lim}}^{p+q}$ , and each  ${}_W E_1^{p,q}$  carries a Hodge structure. The differentials are morphisms of Hodge structures and the spectral sequence degenerates at  $E_2$ .*

(iii) *Let  $N = \log T$  (as  $Id - T$  is nilpotent, this is a polynomial in  $T$  with rational coefficients). Then for each  $r \geq 0$ ,  $N^r$  induces an isomorphism*

$$N^r : {}_W E_1^{-r, q+r} \cong {}_W E_1^{r, q-r}.$$

In our case, we proved that  $T = Id$ , so  $N = 0$ . It follows from (iii) that  ${}_W E_1^{r, q} = 0$  for  $r > 0$  or  $r < 0$ . Thus the weight filtration is trivial and the mixed Hodge structure on each  $H_{\text{lim}}^k$  is pure. Finally, the Hodge structures on  ${}_W E_1^{0, q}$  are direct sums of Hodge structures on cohomology groups of intersections of components of the normal crossing divisor  $\mathcal{X}'_0$ , assuming it has global normal crossings. As these successive intersections are projective manifolds, these Hodge structures are polarized. We thus proved that each  $H_{\text{lim}}^k$  is equipped with a polarizable Hodge structure. Looking at the construction of the Hodge filtration in [6], it is clear that these Hodge filtrations are compatible with the cup-product on cohomology, hence we get a polarizable Hodge structure on the cohomology algebra  $H_{\text{lim}}^*$ .  $\square$

### 3 Topological variant and further remarks

#### 3.1 Proof of Theorems 0.5 and 0.7

*Proof of Theorem 0.7.* The proof uses arguments which already appeared in the previous section. First of all, we observe that the proof of Theorem 0.2 given in the previous section shows the following:

**Proposition 3.1.** *Let  $\pi : \mathcal{X} \rightarrow D$  be a flat proper holomorphic map, where  $\mathcal{X}$  is smooth, such that the central fiber is projective and the other fibers are smooth. Let  $s \in D^*$  and assume that the monodromy action  $T$  on  $H^*(\mathcal{X}_s, \mathbb{Q})$  is trivial. Then the cohomology algebra  $H^*(\mathcal{X}_s, \mathbb{Q})$  admits a polarizable Hodge structure.*

In the situation of Theorem 0.7, we have a proper flat family  $\mathcal{X} \rightarrow B$  with smooth fibers homeomorphic to  $X$  and a point  $0 \in B$  such that the fiber  $\mathcal{X}_0$  is projective. After restriction to a carefully chosen disc in  $B$  passing through 0, finite base change and semistable reduction, we get a monodromy operator  $T$  acting on  $H^*(\mathcal{X}'_t, \mathbb{Q})$  for  $t \in D^*$ . The action of  $T$  obviously preserves the structure of  $H^*(X, \mathbb{Q})$  as an algebra, hence it is semi-simple by our main assumption. Steenbrink's theorem 2.5 then says that  $T = Id$ , so that Proposition 3.1 applies.  $\square$

*Proof of Theorem 0.5.* According to Proposition 1.4, the compact Kähler manifold  $X_1$  constructed in Section 1.2 satisfies the assumptions of Theorem 0.7. Moreover, by Theorem 1.7, there is no polarizable Hodge structure on the cohomology algebra of  $X_1$ , hence  $X_1$  does not admit a projective specialization by Theorem 0.7.  $\square$

#### 3.2 Further remarks on projective specializations

Consider a proper flat holomorphic map

$$\pi : \mathcal{X} \rightarrow D$$

where  $D$  is the disc and  $\mathcal{X}$  is smooth. We can assume that only the central fiber is singular, and by Hironaka resolution and semistable reduction, we can assume after base change that the central fiber is a reduced normal crossing divisor. Let  $D^* := D \setminus \{0\}$ . We now prove

**Proposition 3.2.** *Assume the central fiber  $Y := \mathcal{X}_0$  is projective. Then the Frölicher spectral sequence of  $\mathcal{X}_t$  degenerates at  $E_1$  for  $t$  in an analytic Zariski open set of  $D^*$ .*

*Proof.* The Frölicher spectral sequence has a relative version over  $D$ , using the relative logarithmic holomorphic de Rham complex  $\Omega_{\mathcal{X}/D}^\bullet(\log Y)$ . According to Steenbrink (see [6, Theorem 2.18]), the coherent sheaf  $\mathbb{R}^k \pi_* \Omega_{\mathcal{X}/D}^\bullet(\log Y)$  on  $D$  is locally free and satisfies base change for any  $k$ . There is a relative Frölicher spectral sequence

$$E_1^{p,q} = R^q \pi_* \Omega_{\mathcal{X}/D}^p(\log Y) \Rightarrow \mathbb{R}^k \pi_* \Omega_{\mathcal{X}/D}^\bullet(\log Y). \quad (9)$$

The relative logarithmic holomorphic de Rham complex restricts on each fiber  $\mathcal{X}_t$ ,  $t \neq 0$ , to the holomorphic de Rham complex of  $\mathcal{X}_t$ . Moreover, when  $Y$  is projective, the Frölicher spectral sequence of the restricted complex  $(\Omega_{\mathcal{X}/D}^\bullet(\log Y))|_Y$  degenerates at  $E_1$  by [6, Corollary 4.20]. Uppersemicontinuity and local freeness of  $\mathbb{R}^k \pi_* \Omega_{\mathcal{X}/D}^\bullet(\log Y)$  then imply by standard arguments that the Frölicher spectral sequence (9) degenerates at  $E_1$  for  $t$  in an analytic Zariski open neighborhood of 0 in  $D$ .  $\square$

**Remark 3.3.** We also deduce from the same arguments that the sheaves  $R^q \pi_* \Omega_{\mathcal{X}/Y}^p$  are locally free and satisfy base-change in an analytic-Zariski open neighborhood of 0 in  $D$ , and that the subsheaves

$$F^r \mathbb{R}^k \pi_* \Omega_{\mathcal{X}/D}^{\bullet \geq r}(\log Y) := \text{Im}(\mathbb{R}^k \pi_* \Omega_{\mathcal{X}/D}^{\bullet \geq r}(\log Y) \rightarrow \mathbb{R}^k \pi_* \Omega_{\mathcal{X}/D}^\bullet(\log Y))$$

are locally free subsheaves and satisfy base change.

We do not know the answer to the following questions. Assume as before the central fiber  $Y := \mathcal{X}_0$  of a flat proper holomorphic map  $\mathcal{X} \rightarrow D$ , with  $\mathcal{X}$  smooth, is projective.

**Question 3.4.** *Does the Hodge filtration defined above on the cohomology of the fibers  $\mathcal{X}_t$  define Hodge structures for  $t$  close to 0?*

Assuming the answer to this question is yes, the following question remains

**Question 3.5.** *Are the Hodge structures on the cohomology of the fibers  $\mathcal{X}_t$  for  $t$  close to 0 polarizable with real coefficients?*

By this we mean precisely the following: Does there exist a class  $\alpha_t \in H^{1,1}(\mathcal{X}_t)_{\mathbb{R}}$  satisfying the hard Lefschetz property, so that the associated Lefschetz intersection pairing satisfies the Hodge-Riemann bilinear relations.

**Question 3.6.** *Does the fiber  $\mathcal{X}_t$  belong to the Fujiki class  $\mathcal{C}$ ? Are there examples where some or all fibers  $\mathcal{X}_t$  are not Kähler for  $t$  arbitrarily close to 0?*

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