Syzygies of Prym and paracanonical curves of genus 8

Elisabetta Colombo, Gavril Farkas, Alessandro Verra, and Claire Voisin

Abstract. By analogy with Green’s Conjecture on syzygies of canonical curves, the Prym-Green conjecture predicts that the resolution of a general level $p$ paracanonical curve of genus $g$ is natural. The Prym-Green Conjecture is known to hold in odd genus for almost all levels. Probabilistic arguments strongly suggested that the conjecture might fail for level 2 and genus 8 or 16. In this paper, we present three geometric proofs of the surprising failure of the Prym-Green Conjecture in genus 8, hoping that the methods introduced here will shed light on all the exceptions to the Prym-Green Conjecture for genera with high divisibility by 2.

Keywords. Paracanonical curve; syzygy; genus 8; moduli of Prym varieties

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[Français]

Titre. Syzygies de Prym et courbes paracanoniques de genre 8

Résumé. Par analogie avec la conjecture de Green sur les syzygies des courbes canoniques, la conjecture de Prym-Green prédit que la résolution d’une courbe générale, paracanonical, de genre $g$ et de niveau $p$ est naturelle. Cette conjecture est connue en genre impair pour presque tout niveau. Des arguments probabilistes ont fortement suggéré qu’elle pourrait s’avérer fausse pour le niveau 2 en genre 8 et 16. Dans cet article, nous présentons trois démonstrations géométriques de la surprenante non-validité de la conjecture de Prym-Green en genre 8, en espérant que les méthodes introduites apporteront un éclairage nouveau sur toutes les exceptions à la conjecture de Prym-Green pour des genres divisibles par une grande puissance de 2.
1. Introduction

By analogy with Green’s Conjecture on the syzygies of a general canonical curve [18], [19], the Prym-Green Conjecture, formulated in [10] and [3], predicts that the resolution of a paracanonical curve

\[ \phi_{K_C \otimes \eta} : C \hookrightarrow \mathbb{P}^{g-2}, \]

where \( C \) is a general curve of genus \( g \) and \( \eta \in \text{Pic}^0(C)[\ell] \) is an \( \ell \)-torsion point is natural. For even genus \( g = 2i + 6 \), the Prym-Green Conjecture amounts to the vanishing statement

\[ K_{i,2}(C, K_C \otimes \eta) = K_{i+1,1}(C, K_C \otimes \eta) = 0, \] (1.1)

in terms of Koszul cohomology groups. Equivalently, the genus \( g \) paracanonical level \( \ell \) curve \( C \subseteq \mathbb{P}^{g-2} \) satisfies the Green-Lazarsfeld property \( (N_i) \). The Prym-Green Conjecture has been proved for all odd genera \( g \) when \( \ell = 2 \), see [8], or \( \ell \geq \sqrt{\frac{g+2}{2}} \), see [9]. For even genus, the Prym-Green Conjecture has been established by degeneration and using computer algebra tools in [3] and [4], for all \( \ell \leq 5 \) and \( g \leq 18 \), with two possible mysterious exceptions in level 2 and genus \( g = 8, 16 \) respectively. The last section of [3] provides various pieces of evidence, including a probabilistic argument, strongly suggesting that for \( g = 8 \), one has \( \dim K_{i,2}(C, K_C \otimes \eta) = 1 \), and thus the vanishing (1.1) fails in this case. It is tempting to believe that the exceptions \( g = 8, 16 \) can be extrapolated to higher genus, and that for genera \( g \) with high divisibility by 2, there are genuinely novel ways of constructing syzygies of Prym-canonical curves waiting to be discovered. It would be very interesting to test experimentally the next relevant case \( g = 24 \). Unfortunately, due to memory and running time constraints, this is currently completely out of reach, see [3] and [7].

The aim of this paper is to confirm the expectation formulated in [3] and offer several geometric explanations for the surprising failure of the Prym-Green Conjecture in genus 8, hoping that the geometric methods described here for constructing syzygies of Prym-canonical curves will eventually shed light on all the exceptions to the Prym-Green Conjecture. We choose a general Prym-canonical curve of genus 8

\[ \phi_{K_C \otimes \eta} : C \hookrightarrow \mathbb{P}^6, \]

with \( \eta^{\otimes 2} = O_C \). Set \( L := K_C \otimes \eta \) and denote \( I_{C,L}(k) := \text{Ker}\{\text{Sym}^k H^0(C, L) \rightarrow H^0(C, L^\otimes k)\} \) for all \( k \geq 2 \). Observe that \( \dim I_{C,L}(2) = \dim K_{1,1}(C, L) = 7 \) and \( \dim I_{C,L}(3) = 49 \), therefore as \( [C, \eta] \) varies in moduli, the multiplication map

\[ \mu_{C,L} : I_{C,L}(2) \otimes H^0(C, L) \rightarrow I_{C,L}(3) \]

globalizes to a morphism of vector bundles of the same rank over the stack \( \mathcal{R}_8 \) classifying pairs \( [C, \eta] \), where \( C \) is a smooth curve of genus 8 and \( \eta \in \text{Pic}^0[2] \setminus \{O_C\} \).
Theorem 1. For a general Prym curve $[C,\eta] \in \mathcal{R}_8$, one has $K_{1,2}(C,L) \neq 0$. Equivalently the multiplication map $\mu_{C,L} : I_{C,L}(2) \otimes H^0(C,L) \to I_{C,L}(3)$ is not an isomorphism.

We present three different proofs of Theorem 1. The first proof, presented in Section 3 uses the structure theorem already pointed out in [3] for degenerate syzygies of paracanonical curves in $\mathbb{P}^6$. Precisely, if a paracanonical genus 8 curve $\phi : C \to \mathbb{P}^6$, where $\eta \neq \mathcal{O}_C$, has a syzygy $0 \neq \gamma \in K_{1,2}(C,K_C \otimes \eta)$ of sub-maximal rank (see Section 2 for a precise definition), then the syzygy scheme of $\gamma$ consists of an isolated point $p \in \mathbb{P}^6 \setminus C$ and a residual septic elliptic curve $E \subseteq \mathbb{P}^6$ meeting $C$ transversally along a divisor $e$ of degree 14, such that if $e$ is viewed as a divisor on $C$ and $E$ respectively, then

$$e_C \in |K_C \otimes \eta \otimes 2| \quad \text{and} \quad e_E \in |\mathcal{O}_E(2)|.$$  \hfill (1.2)

The union $D := C \cup E \to \mathbb{P}^6$, endowed with the line bundle $\mathcal{O}_D(1)$ is a degenerate spin curve of genus 22 in the sense of [5]. The locus of stable spin structures with at least 7 sections defines a subvariety of codimension 21 = $\left(\begin{array}{c} 22 \\ 7 \end{array}\right)$ inside the moduli space $\overline{S}_{22}$ of stable odd spin curves of genus 22. By restricting this condition to the locus of spin structures having $D := C \cup E$ as underlying curve, it turns out that one has enough parameters to realize this condition for a general $C \subseteq \mathbb{P}^6$ if and only if

$$\dim|K_C \otimes \eta \otimes 2| = 7,$$

which happens precisely when $\eta \otimes 2 \cong \mathcal{O}_C$. Therefore for each Prym-canonical curve $C \subseteq \mathbb{P}^6$ of genus 8 there exists a corresponding elliptic curve $E \subseteq \mathbb{P}^6$ such that the intersection divisor $E \cdot C$ verifies (1.2), which forces $K_{1,2}(C,K_C \otimes \eta) \neq 0$.

The second and the third proofs involve the reformulation given in Section 2.B (see Proposition 5) of the condition that a paracanonical curve $\phi : C \to \mathbb{P}^6$ have a non-trivial syzygy. Precisely, if $\phi_L(C)$ is scheme-theoretically generated by quadrics, then $K_{1,2}(C,L) \neq 0$, if and only if there exists a quartic hypersurface in $\mathbb{P}^6$ singular along $C \subseteq \mathbb{P}^6$, which is not a quadratic polynomial in quadrics vanishing along $C$, that is, it does not belong to the image of the multiplication map

$$\text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4).$$

Equivalently, one has $H^1(\mathbb{P}^6, I_{C,\mathbb{P}^6}^2(4)) \neq 0$.

The second proof presented in Section 4 uses intersection theory on the stack $\mathcal{R}_8$. The virtual Koszul divisor of Prym curves $[C,\eta] \in \mathcal{R}_8$ having $K_{1,2}(C,K_C \otimes \eta) \neq 0$, splits into two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, corresponding to the case whether $C \subseteq \mathbb{P}^6$ is not scheme-theoretically cut out by quadrics, or $H^1(\mathbb{P}^6, I_{C,\mathbb{P}^6}^2(4)) \neq 0$ respectively. We determine the virtual classes of both closures $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$. Using an explicit uniruled parametrization of $\overline{\mathcal{R}}_8$ constructed in [11], we conclude that the class $[\overline{\mathcal{D}}_2] \in CH^1(\overline{\mathcal{R}}_8)$ cannot possibly be effective (see Theorem 20). Therefore, again $K_{2,1}(C,K_C \otimes \eta) \neq 0$, for every Prym curve $[C,\eta] \in \mathcal{R}_8$.

The third proof given in Section 5 even though subject to a plausible, but still unproved transversality assumption, is constructive and potentially the most useful, for we feel it might offer hints to the case $g = 16$ and further. The idea is to consider rank 2 vector bundles $E$ on $C$ with canonical determinant and $h^0(C, E) = h^0(C, E(\eta)) = 4$. (Note that the condition that $\eta$ is 2-torsion is equivalent to the fact that $E(\eta)$ also has canonical determinant, which is essential for the existence of such nonsplit vector bundles, cf. [15].) By pulling back to $C$ the determinantal quartic hypersurface consisting of rank 3 tensors in

$$\mathbb{P}\left( H^0(C,E)^\vee \otimes H^0(C,E(\eta))^\vee \right) \cong \mathbb{P}^{15}$$

under the natural map $H^0(C,K_C \otimes \eta)^\vee \to H^0(C,E)^\vee \otimes H^0(C,E(\eta))^\vee$, we obtain explicit quartic hypersurfaces singular along the curve $C \subseteq \mathbb{P}^6$. Our proof that these are not quadratic polynomials
into quadrics vanishing along the curve, that is, they do not lie in the image of Sym^2 I_{C,L}(2) remains incomplete, but there is a lot of evidence for this.

The methods of Section 5 suggests the following analogy in the next case $g = 16$. If $[C, \eta] \in R_{16}$ is a Prym curve of genus 16, there exist vector bundles $E$ on $C$ with det $E \cong K_C$ and satisfying $h^0(C, E) = h^0(C, E(\eta)) = 6$. Potentially they could be used to prove that $K_{5,2}(C, K_C \otimes \eta) \neq 0$ and thus confirm the next exception to the Prym-Green Conjecture.

2. Syzygies of paracanonical curves of genus 8

Let $C$ be a general smooth projective curve of genus 8. For a non-trivial line bundle $\eta \in \text{Pic}^0(C)$, we shall study the paracanonical line bundle $L := K_C \otimes \eta$. When $\eta$ is a 2-torsion point, we speak of the Prym-canonical line bundle $L$. For each paracanonical bundle $L$, we have $h^0(C, L) = 7$ and an induced embedding

$$\phi_L : C \to \mathbb{P}^6.$$ 

The goal is to understand the reasons for the non-vanishing of the Koszul group $K_{1,2}(C, L)$ of a Prym-canonical bundle $L$, as suggested experimentally by the results of [3], [4].

Let $I_C(2) = I_{C,L}(2) \subseteq H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$, respectively $I_C(3) = I_{C,L}(3) \subseteq H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(3))$ be the ideal of quadrics, respectively cubics, vanishing on $\phi_L(C)$. It is well-known that whenever $L$ is projectively normal, the non-vanishing of the Koszul cohomology group $K_{1,2}(C, L)$ is equivalent to the non-surjectivity of the multiplication map

$$\mu_{C,L} : H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)) \otimes I_C(2) \to I_C(3). \quad (2.3)$$

Note that

$$\dim I_C(2) = \binom{8}{2} - 21 = 7, \quad \text{and} \quad \dim I_C(3) = \binom{9}{3} - 3 \cdot 14 + 7 = 49,$$

respectively, so that the two spaces appearing in the map (2.3) have the same dimension. Denote by $P_{14}^8$ the universal degree 14 Picard variety over $\mathcal{M}_8$ consisting of pairs $[C, L]$, where $[C] \in \mathcal{M}_8$ and $L \neq K_C$. The jumping locus

$$\mathcal{R}_{033} := \{[C, L] \in P_{14}^8 : K_{1,2}(C, L) \neq 0\}$$

is a divisor. It turns out, cf. Theorem 5.3 of [3] and Proposition 8, that $\mathcal{R}_{033}$ splits into two components depending on the rank of the corresponding non-zero syzygy from $K_{1,2}(C, L)$.

Definition 2. The rank of a non-zero syzygy $\gamma = \sum_{i=0}^6 \ell_i \otimes q_i \in \text{Ker}(\mu_{C,L})$ is the dimension of the subspace $\langle \ell_0, \ldots, \ell_6 \rangle \subseteq H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$. The syzygy scheme $\text{Syz}(\gamma)$ of $\gamma$ is the largest subscheme $Y \subseteq \mathbb{P}^6$ such that $\gamma \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)) \otimes I_Y(2)$.

It is shown in [3], that $\mathcal{R}_{033}$ splits into divisors $\mathcal{R}_{0336}$ and $\mathcal{R}_{0337}$, depending on whether the syzygy $0 \neq \gamma \in \text{Ker}(\mu_{C,L})$ has rank 6 or 7 respectively. By a specialization argument to irreducible nodal curves, it follows from [3] that $\mathcal{R}_{8} \not\subseteq \mathcal{R}_{0337}$. A direct, more transparent proof of this fact will be given in Proposition 13.

2.A. Paracanonical curves of genus 8 with special syzygies and elliptic curves

We summarize a few facts already stated or recalled in Section 5 of [3] concerning rank 6 syzygies of paracanonical curves in $\mathbb{P}^6$. Very generally, let

$$\gamma = \sum_{i=1}^6 \ell_i \otimes q_i \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)) \otimes H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$$
be a rank 6 linear syzygy among quadrics in \( \mathbb{P}^6 \). The linear forms \( \ell_1, \ldots, \ell_6 \) define a point \( p \in \mathbb{P}^6 \). Following Lemma 6.3 of [16], there exists a skew-symmetric matrix of linear forms \( A := (a_{ij})_{i,j=1,\ldots,6} \), such that
\[
q_i = \sum_{j=1}^6 \ell_j a_{ij}.
\]
In the space \( \mathbb{P}^{20} \) with coordinates \( \ell_1, \ldots, \ell_6 \) and \( a_{ij} \) for \( 1 \leq i < j \leq 6 \), one considers the 15-dimensional variety \( X_6 \) defined by the 6 quadratic equations \( \sum_{j=1}^6 \ell_j a_{ij} = 0 \), where \( i = 1, \ldots, 6 \) and by the cubic equation \( \text{Pfaff}(A) = 0 \) in the variables \( a_{ij} \). The original space \( \mathbb{P}^6 \) embeds in \( \mathbb{P}^{20} \) via evaluation. The syzygy scheme \( \text{Syz}(\gamma) \) is the union of the point \( p \) and of the intersection \( D \) of \( \mathbb{P}^6 \) with the variety \( X_6 \). It follows from Theorem 4.4 of [6], that for a general rank 6 syzygy \( \gamma \) as above, \( D \subseteq \mathbb{P}^6 \) is a smooth curve of genus 22 and degree 21 such that \( O_D(1) \) is a theta characteristic.

In the case at hand, that is, when \( [C, L] \in \mathfrak{Roz}_{36} \), the curve \( D \) must be reducible, for it has \( C \) as a component. More precisely:

**Lemma 3.** For a general paracanonical curve \( C \subseteq \mathbb{P}^6 \) having a rank 6 syzygy, the curve \( D \) is nodal and consists of two components \( C \cup E \), where \( E \subseteq \mathbb{P}^6 \) is an elliptic septic curve. Furthermore, \( O_D(2) = \omega_D \). The intersection \( e := C \cdot E \), viewed as a divisor on \( C \), satisfies \( e_C \in |O_C(2) \otimes K_C^2| \), and as a divisor on \( E \), satisfies \( e_E \in |O_E(2)| \).

**Remark 4.** Note that \( C \) is Prym-canonical or canonical if and only if \( e_C \in |K_C| \).

The construction above is reversible. Firstly, general element \( [C, L] \in \mathfrak{Roz}_{36} \) can be reconstructed as the residual curve of a reducible spin curve \( D \subseteq \mathbb{P}^6 \) of genus 22 containing an elliptic curve \( E \subseteq \mathbb{P}^6 \) with \( \text{deg}(E) = 7 \) as a component such that the union of \( D \) and some point \( p \in \mathbb{P}^6 \setminus E \) is the syzygy scheme of a rank 6 linear syzygy among quadrics in \( \mathbb{P}^6 \).

Furthermore, given a reducible spin curve \( D = C \cup E \subseteq \mathbb{P}^6 \) of genus 22 as above, that is, with \( \omega_D \cong O_D(2) \), the genus 8 component \( C \) has a nontrivial syzygy of rank 6 involving the quadrics in the 6-dimensional subspace \( I_D(2) \subseteq I_C(2) \), see Lemma 29 for a proof of this fact.

### 2.B. Syzygies and quartics singular along paracanonical curves

We first discuss an alternative characterization of the non-surjectivity of the map \( \mu_{C,L} \):

**Proposition 5.** Assume the paracanonical curve \( \phi_L(C) \) is projectively normal and scheme-theoretically cut out by quadrics. Then \( K_{1,2}(C,L) \neq 0 \) if and only if there exists a degree 4 homogeneous polynomial on \( \mathbb{P}^6 \), which vanishes to order at least 2 along \( C \) but does not belong to the image of the multiplication map \( \text{Sym}^2 I_{C,L}(2) \rightarrow I_{C,L}(4) \).

**Proof.** We work on the variety \( X \xrightarrow{\tau} \mathbb{P}^6 \) defined as the blow-up of \( \mathbb{P}^6 \) along \( \phi_L(C) \). Let \( E \) be the exceptional divisor of the blow-up, and consider the line bundle \( H := \tau^*O_{\mathbb{P}^6}(2)(-E) \) on \( X \). Its space of sections identifies to \( I_C(2) \), and our assumption that \( C \) is scheme-theoretically cut out by quadrics says equivalently that \( H \) is a globally generated line bundle on \( X \). The nonvanishing of \( K_{1,2}(C,L) \) is equivalent to the non-surjectivity of the multiplication map
\[
H^0(X, H) \otimes H^0(X, \tau^*O(1)) \rightarrow H^0(X, H \otimes \tau^*O(1)),
\]
where we use the identification
\[
H^0(X, H \otimes \tau^*O(1)) = H^0(X, \tau^*O(3)(-E)) = I_C(3).
\]
As $H$ is globally generated by its space $W := I_C(2)$ of global sections, the Koszul complex

$$0 \to \bigwedge^7 W \otimes \mathcal{O}_X(-7H) \to \ldots \to \bigwedge^2 W \otimes \mathcal{O}_X(-2H) \to W \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X \to 0$$  \hspace{1cm} (2.5)

is exact. We now twist this complex by $\tau^*\mathcal{O}_{\mathbb{P}^6}(1)(H)$ and take global sections. The last map is then the multiplication map (2.4). The successive terms of this twisted complex are

$$\bigwedge^i W \otimes \mathcal{O}_X((\tau^*\mathcal{O}(1))(i-1+H)),$n

for $0 \leq i \leq 7$. The spectral sequence abutting to the hypercohomology of this complex, that is 0, has

$$E_{2}^{0,0} = \text{coker} \left\{ W \otimes H^0(X, \tau^*\mathcal{O}(1)) \to H^0(X, H \otimes \tau^*\mathcal{O}(1)) \right\}$$  \hspace{1cm} (2.6)

and the terms $E_{1}^{i,-i-1}$ for $i < -1$ are equal to $\bigwedge^{-i} W \otimes H^{-i-1}(X, \tau^*\mathcal{O}(1)((i+1)H))$. Similarly, we have

$$E_{1}^{i,-i} = \bigwedge^{-i} W \otimes H^{-i}(X, \tau^*\mathcal{O}(1)((i+1)H)).$$

Lemma 6. (i) We have

$$E_{1}^{i,-i-1} = \bigwedge^{-i} W \otimes H^{-i-1}(X, \tau^*\mathcal{O}(1)((i+1)H)) = 0,$$ \hspace{1cm} (2.7)

for $-i - 1 = 5, \ldots, 1$.

(ii) For $-i - 1 = 6$, that is, $i = -7$, we have

$$E_{1}^{7,-6} = \bigwedge^7 W \otimes H^6(X, \tau^*\mathcal{O}(1)(-6H)) = \bigwedge^7 W \otimes I_C(4), \hspace{1cm} (2.8)$$

where $I_C(4) \subseteq I_C(4)$ is the set of quartic polynomials vanishing at order at least 2 along $C$, and

$$E_{1}^{6,-6} = \bigwedge^6 W \otimes H^6(X, \tau^*\mathcal{O}(1)(-5H)) = \bigwedge^6 W \otimes I_C(2). \hspace{1cm} (2.9)$$

(iii) We have $E_{1}^{i,-i} = 0$, for $-6 < i < 0$.

Proof of Lemma 6. (i) We want equivalently to show that

$$H^\ell(X, \tau^*\mathcal{O}(1)(-\ell H)) = 0,$$ \hspace{1cm} when $\ell = 5, \ldots, 1$.

Recall that $H = \tau^*\mathcal{O}(2)(-E)$. Furthermore,

$$K_X = \tau^*\mathcal{O}_{\mathbb{P}^6}(-7)(4E).$$  \hspace{1cm} (2.10)

So we have to prove that

$$H^\ell(X, \tau^*\mathcal{O}(-2\ell + 1)(\ell E)) = 0, \hspace{1cm} \text{for } \ell = 5, \ldots, 1.$$ \hspace{1cm} (2.11)

Examining the spectral sequence induced by $\tau$, and using the fact that

$$R^s\tau_* (\mathcal{O}_X(tE)) = 0$$
for $s \neq 0, 4$ and also for $s = 4, t \leq 4$, we see that for $1 \leq \ell \leq 4$,
\[ H^\ell(X, \tau^*\mathcal{O}(-2\ell + 1)(\ell E)) = H^\ell(\mathbb{P}^6, \mathcal{O}(-2\ell + 1) \otimes R^0\tau_*\mathcal{O}_X(\ell E)). \]

For $1 \leq \ell \leq 4$, the right hand side is zero, because it is equal to $H^\ell(\mathbb{P}^6, \mathcal{O}(-2\ell + 1))$.

For $\ell = 5$, we have to compute the space $H^5(X, \tau^*\mathcal{O}(-9)(5E))$, which by Serre duality and by (2.10), is dual to the space
\[ H^1(X, \tau^*\mathcal{O}(2)(-E)) = H^1(\mathbb{P}^6, \mathcal{O}(2) \otimes \mathcal{I}_C) = 0. \]

(ii) We have to compute the spaces $H^6(X, \tau^*\mathcal{O}(1)(-6H))$ and $H^6(X, \tau^*\mathcal{O}(1)(-5H))$. As $H := \tau^*\mathcal{O}(2)(-E)$, this is rewritten as $H^6(X, \tau^*\mathcal{O}(-11)(6E))$ and $H^6(X, \tau^*\mathcal{O}(-9)(5E))$ respectively. If we dualize using (2.10), we get
\[ H^6(X, \tau^*\mathcal{O}(-11)(6E))^\vee = H^0(X, \tau^*\mathcal{O}(4)(-2E)) = I_C(4)_2, \]
\[ H^6(X, \tau^*\mathcal{O}(-9)(5E))^\vee = H^0(X, \tau^*\mathcal{O}(2)(-E)) = I_C(2). \]

(iii) We have
\[ E_{i}^{1,-i} = E_{1}^{-6,6} = \bigwedge^i W \otimes H^{-i}(X, \tau^*\mathcal{O}(1)((i + 1)H)) = \bigwedge^i W \otimes H^{-i}(X, \tau^*\mathcal{O}(2i + 3)((-i - 1)E)). \]

For $1 \leq -i \leq 5$, we have $R^s\tau_*\mathcal{O}_X((-i - 1)E) = 0$ unless $s = 0$. Furthermore, we have
\[ R^0\tau_*\mathcal{O}_X((-i - 1)E) = \mathcal{O}_{\mathbb{P}^6}, \]
so that
\[ H^{-i}(X, \tau^*\mathcal{O}(2i + 3)((-i - 1)E)) = H^{-i}(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2i + 3)) = 0. \]

\[ \square \]

**Corollary 7.** Only one $E_2^{p,q}$-terms of this spectral sequence is possibly nonzero in degree $-1$, namely
\[ E_2^{-7,6} = \text{Ker}\left\{ \bigwedge^7 W \otimes I_C(4)_2 \rightarrow \bigwedge^6 W \otimes I_C(2)^\vee \right\}. \tag{2.12} \]

Furthermore, all the differentials $d_r$ starting from $E_2^{-7,6}$ vanish for $2 \leq r < 7$.

Note that the map
\[ \bigwedge^7 W \otimes I_C(4)_2 \rightarrow \bigwedge^6 W \otimes I_C(2)^\vee \]
is nothing but the transpose of the multiplication map
\[ W \otimes I_C(2) \rightarrow I_C(4)_2, \]
up to trivialization of $\bigwedge^7 W$. It follows that
\[ (E_2^{-7,6})^\vee = \text{Coker}\left\{ W \otimes I_C(2) \rightarrow I_C(4)_2 \right\}. \tag{2.13} \]

Corollary 7 concludes the proof of the proposition since it implies that we have an isomorphism given by $d_7$ between (2.12) and (2.6), or a perfect duality between (2.12) and the cokernel (2.13).

Proposition 5 has the following consequence. Recall that $\mathcal{P}_8^{14}$ is the moduli space of pairs $[C, L]$, with $C$ being a smooth curve of genus $8$ and $L \neq K_C$ a paracanonical line bundle.
Proposition 8. The Koszul divisor $\mathfrak{K}_{53}$ of $P_{8}^{14}$ is the union of two divisors, one of them being the set of pairs $[C,L]$ such that $\phi_{L}(C)$ is not scheme-theoretically cut out by quadrics, the other being the set of pairs $[C,L]$ such that $H^{1}(P^{5},I_{C}(4)) \neq 0$, or equivalently, such that there exists a quartic which is singular along $\phi_{L}(C)$ but does not lie in $\text{Sym}^{2}I_{C}(2)$.

Proof. We first have to prove that the locus of pairs $[C,L]$ such that $\phi_{L}(C)$ is not scheme-theoretically cut-out by quadrics is contained in the divisor $\mathfrak{K}_{53}$. This is a consequence of the following lemmas:

Lemma 9. If $L \neq K_{C}$ is a projectively normal paracanonical line bundle on a curve of genus $8$, then $\phi_{L}(C)$ is scheme-theoretically cut out by quadrics.

Proof of Lemma 9. We observe that the twisted ideal sheaf $I_{C}(3)$ is regular in Castelnuovo-Mumford sense. Indeed, we have

$$H^{i}(P^{5},I_{C}(3-i)) = H^{i-1}(C, L^{\otimes(3-i)})$$

for $i \geq 2$, and the right hand side is obviously 0 for $i-1 \geq 2$, and also 0 for $i-1 = 1$ since $H^{1}(C,L) = 0$ because $L \neq K_{C}$ and $\deg L = 2g - 2$. For $i = 1$, we have

$$H^{1}(P^{5},I_{C}(2)) = 0$$

by projective normality. Being regular, the sheaf $I_{C}(3)$ is generated by global sections. \qed

Corollary 10. If $C, L$ are as above, and $C$ is not scheme-theoretically cut out by quadrics, then the multiplication map

$$I_{C}(2) \otimes H^{0}(P^{5},O_{P^{5}}(1)) \to I_{C}(3)$$

is not surjective.

To conclude the proof of the proposition, we just have to show that the sublocus of $P_{8}^{14}$ where $L$ is not projectively normal is not a divisor, since the statement of the proposition will be then an immediate consequence of Proposition 5. We argue along the lines of [12]. First of all, a line bundle $L$ of degree $14$ is not generated by sections if and only if $L = K_{C}(-x + y)$ for some points $x, y \in C$. This determines a codimension 6 locus of $P_{8}^{14}$. Similarly $L$ is not very ample if and only if $L = K_{C}(-x - y + z + t)$, for some points $x, y, z, t \in C$, which is satisfied in a codimension 4 locus of $P_{8}^{14}$. Finally, assume $L$ is very ample but $\phi_{L}(C)$ is not projectively normal. Equivalently

$$\text{Sym}^{2}H^{0}(C,L) \to H^{0}(C,L^{\otimes 2})$$

is not surjective, which means that there exists a rank 2 vector bundle $F$ on $C$ which is a nontrivial extension

$$0 \to K_{C} \otimes L^{\vee} \to F \to L \to 0,$$

such that $h^{0}(C,F) = 7$. If $x, y, z \in C$, there exists a nonzero section $\sigma \in H^{0}(C,F)$ vanishing on $x, y$ and $z$, and thus $F$ is also an extension

$$0 \to D \to F \to K_{C} \otimes D^{\vee} \to 0,$$  \hspace{1cm} (2.14)

where $D$ is a line bundle such that $h^{0}(C,D(-x - y - z)) \neq 0$, and $h^{0}(C,L \otimes D^{\vee}) \neq 0$. We thus have $h^{0}(C,D) + h^{0}(C,K_{C} \otimes D^{\vee}) \geq 7$ and $\text{Cliff}(D) \leq 2$. As $D$ is effective of degree at least 3, one has the following possibilities:

a) $h^{0}(C,K_{C} \otimes D^{\vee}) = 0$, and then $D = L$, which contradicts the fact that the extension (2.14) is not split;

b) $h^{0}(C,K_{C} \otimes D^{\vee}) = 1$ and $h^{0}(C,D) \geq 6$, and then $D = L(-x)$ and $h^{0}(K_{C} \otimes L^{\vee}(x)) \neq 0$, so $L = K_{C}(x - y)$, which happens in a locus of codimension at least 6 in $P_{8}^{14}$;

c) $D$ contributes to the Clifford index of $C$. As the locus of curves $[C] \in \mathcal{M}_{8}$ with $\text{Cliff}(C) \leq 2$ is of codimension 2 in $\mathcal{M}_{8}$, this situation does not occur in codimension 1. \qed
We shall need later on the following result:

**Lemma 11.** Let \( \phi_L : C \hookrightarrow \mathbb{P}^6 \) be a projectively normal paracanonical curve of genus 8. If \( C \) is scheme-theoretically cut out by quadrics, the multiplication map

\[ \text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4) \quad (2.15) \]

is injective.

**Proof.** As the restriction map \( \phi_L^* : H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) \to H^0(C, L \otimes 2) \) is surjective, its kernel \( I_{C,L}(2) \) is of dimension 7. Let as before \( \tau : X \to \mathbb{P}^6 \) be the blow-up of \( \mathbb{P}^6 \) along \( \phi_L(C) \), and let \( E \) be its exceptional divisor. We view \( I_{C,L}(2) \) as \( H^0(X, \tau^* \mathcal{O}(2)(-E)) \) and our assumption is that \( I_{C,L}(2) \) generates the line bundle \( H := \tau^* \mathcal{O}(2)(-E) \) everywhere on \( X \). Thus \( I_{C,L}(2) \) provides a morphism

\[ \psi : X \to \mathbb{P}(I_{C,L}(2)). \quad (2.16) \]

Now we have \( \deg c_1(H)^6 \neq 0 \) by Sublemma 12 below, and thus the morphism \( \psi \) has to be generically finite, hence dominant since both spaces have dimension 6. It is then clear that the pull-back map

\[ \psi^* : H^0(\mathbb{P}(I_{C,L}(2)), \mathcal{O}(2)) \to H^0(X, H \otimes 2) \]

is injective. On the other hand, this morphism is nothing but the map (2.15). \( \square \)

**Sublemma 12.** With the same notation as above, we have

\[ \deg c_1(H)^6 = 8. \quad (2.17) \]

**Proof.** We have

\[ c_1(H)^6 = \sum_i \binom{6}{i} (-2)^i h^i \cdot E^{6-i}, \]

where \( h := \tau^* c_1(\mathcal{O}_{\mathbb{P}^6}(1)) \), and

\[ h^i \cdot E^{6-i} = 0 \]

for \( i \neq 6, 1, 0 \). Furthermore

\[ h^6 = 1, \quad h \cdot E^5 = \deg \phi_L(C) = 14 \]

and \( E^6 = c_1(N_C) \). By adjunction formula

\[ \deg c_1(N_C) = 7 \deg \phi_L(C) + \deg K_C = 8 \cdot 14. \]

It follows that

\[ \deg c_1(H)^6 = 64 - 6 \cdot 28 + 8 \cdot 14 = 8, \]

which proves (2.17). \( \square \)

Proposition 5 and Lemma 11 describe precisely the splitting of the Koszul divisor \( \text{Kosz} \) into the divisors \( \text{Kosz}^6 \) and \( \text{Kosz}^7 \) corresponding to paracanonical curves \( [C, L] \in P_5^{14} \) having a non-zero syzygy \( \gamma \in K_{1,2}(C, L) \) of rank 6 or respectively 7. Precisely, \( \text{Kosz}^6 \) is a unirational divisor (cf. [3] Theorem 5.3) consisting of those paracanonical curves \( C \subseteq \mathbb{P}^6 \) for which \( H^1(\mathbb{P}^6, T^2_C(4)) \neq 0 \). The divisor \( \text{Kosz}^7 \) consists of paracanonical curves \( C \subseteq \mathbb{P}^6 \) which are not scheme-theoretically cut out by quadrics.
3. First proof: reducible spin curves

3.A. The syzygy is degenerate

The first observation is the following result (already observed experimentally in [3]), which turns out to be useful for the description given below of the general paracanonical curve of genus 8 with nontrivial syzygies.

**Proposition 13.** Let $C \subseteq \mathbb{P}^6$ be a smooth paracanonical curve of genus 8 and degree 14, scheme-theoretically generated by quadrics. Then a nontrivial syzygy

$$\gamma \in \text{Ker}\{I_C(2) \otimes H^0(\mathcal{O}_{\mathbb{P}^6}(1)) \rightarrow I_C(3)\}$$

must be degenerate, that is of rank at most 6.

**Proof.** We use the morphism

$$\psi : X \rightarrow \mathbb{P}(I_C(2))$$

introduced in (2.16), where $\tau : X \rightarrow \mathbb{P}^6$ is the blow-up of $C$ with exceptional divisor $E$, and $H := \tau^*\mathcal{O}_{\mathbb{P}^6}(-2E)$. This gives us a morphism

$$(\tau, \psi) : X \rightarrow \mathbb{P}^6 \times \mathbb{P}^6$$

which is of degree 1 on its image, and the syzygy $\gamma$ induces a hypersurface $Y$ of bidegree $(1, 1)$ in $\mathbb{P}^6 \times \mathbb{P}^6$ containing the 6-dimensional variety $(\tau, \psi)(X)$. Assume to the contrary that $\gamma$ has maximal rank 7, or equivalently that $Y$ is smooth. Then by the Lefschetz Hyperplane Restriction Theorem, the restriction map $H^{10}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z}) \rightarrow H^{10}(Y, \mathbb{Z})$ is surjective, so that $[(\tau, \psi)(X)]_Y \in H^{10}(Y, \mathbb{Z})$ is the restriction of a class $\beta \in H^{10}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z})$, which implies that

$$[(\tau, \psi)(X)] = \beta \cdot [Y] \text{ in } H^{12}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z}),$$

(3.18)

where $[Y] \in H^2(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z})$ is the class of $Y$, that is $h_1 + h_2$, with $h_i$ for $i = 1, 2$ being the pull-backs of the hyperplane classes on each factor. Note that $H^{12}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z})$ is the set of degree 6 homogeneous polynomials with integral coefficients in $h_1$ and $h_2$. We now have:

**Lemma 14.** An element $\alpha \in H^{12}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z})$ is of the form $(h_1 + h_2) \cdot \beta$ if and only if it satisfies the condition

$$\sum_{i=0}^6 (-1)^i h_1^i \cdot h_2^{6-i} \cdot \alpha = 0 \text{ in } H^{24}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z}) = \mathbb{Z}. $$

(3.19)

**Proof of Lemma 14.** We have $(h_1 + h_2) \cdot (\sum_i (-1)^i h_1^i \cdot h_2^{6-i}) = 0$ in $H^{14}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z})$, so one implication is obvious. That the two conditions are equivalent then follows from the fact that both conditions determine a saturated corank 1 sublattice of $H^{12}(\mathbb{P}^6 \times \mathbb{P}^6, \mathbb{Z})$. □

To conclude that $\gamma$ has to be degenerate, in view of Lemma 14, it suffices to prove that the class $[(\tau, \psi)(X)]$ does not satisfy (3.19). Since $(\tau, \psi)^*h_1 = c_1(H)$ and $(\tau, \psi)^*h_2 = 2c_1(H) - E$, it is enough to prove that

$$\sum_{i=0}^6 (-1)^i c_1(H)^i \cdot (2c_1(H) - E)^{6-i} \neq 0,$$

which follows from the computations made in the proof of Sublemma 12. □
3.B. Syzygies and spin curves of genus 22 in $\mathbb{P}^6$

Recall that $\mathfrak{S}_g$ denotes the moduli stack of odd stable spin curves of genus $g$, see [5] for details. We start with a nodal genus 22 spin curve of the form $[D := C \cup E, \varrho] \in \mathfrak{S}_{22}$, where $C$ is a smooth genus 8 curve, $E$ is a smooth elliptic curve and $e := C \cap E$ consists of 14 distinct points, thus $p_a(D) = 22$. Assume $\varrho \in \text{Pic}^{21}(D)$ verifies $\varrho \otimes 2 \cong \omega_D$, hence the restricted line bundles $\varrho_E$ and $\varrho_C$ have degrees 7 and 14 respectively. Furthermore, $h^0(E, \varrho_E) = 7$, whereas $h^0(C, \varrho_C) = 7$ if and only if $\varrho_C \not\cong K_C$.

The intersection divisor $e$ on the two components of $D$ is characterized by

$$e_C \in |\varrho_C \otimes K_C|^r \quad \text{and} \quad e_E \in |\varrho_E \otimes 2|.$$

Note in particular that $e_C \in |K_C|$ if and only if $\varrho_C \otimes 2 = K_C^2$, that is $(C, \varrho_C)$ is canonical or Prym canonical.

The line bundle $\varrho$ on $D$ fits into the Mayer-Vietoris exact sequence:

$$0 \rightarrow \varrho \rightarrow \varrho_C \oplus \varrho_E \xrightarrow{r} \mathcal{O}_e(\varrho) \rightarrow 0,$$

where $r$ is defined by the isomorphisms on the fibers of $\varrho_C$ and $\varrho_E$ over the points in $e$. Given $\varrho_C \in \text{Pic}^{14}(C)$ with $\varrho_C \otimes 2 = K_C(e)$ and $\varrho_E \in \text{Pic}^{7}(E)$ with $\varrho_E \otimes 2 = \mathcal{O}_C(e)$, there is a finite number of stable spin curves $[D, \varrho] \in \mathfrak{S}_{22}$ such that the restrictions of $\varrho$ to $C$ and $E$ are isomorphic to $\varrho_C$ and $\varrho_E$ respectively. Passing to global sections in the Mayer-Vietoris sequence, we obtain the exact sequence:

$$0 \rightarrow H^0(D, \varrho) \rightarrow H^0(C, \varrho_C) \oplus H^0(E, \varrho_E) \xrightarrow{r} H^0(\mathcal{O}_e(\varrho)) \rightarrow \cdots. \quad (3.20)$$

Note that $r$ is represented by a $14 \times 14$ matrix and $h^0(D, \varrho) = 14 - \text{rk}(r)$. In the case of a reducible spin curve coming from the syzygy of a paracanonical genus 8 curve in $\mathfrak{R}_{8,6}$, one has $h^0(D, \varrho) = \text{rk}(r) = 7$.

3.C. Proof of Theorem 1 via reducible spin curves

Theorem 1 states that every Prym canonical curve of genus 8 has a syzygy of rank 6. First we observe the existence of such a curve having the generic behavior described in Lemma 3.

**Lemma 15.** There exists a curve $[C, \eta] \in \mathcal{R}_8$, whose Prym canonical model is scheme theoretically cut out by quadrics, and $K_{2,1}(C, K_C \otimes \eta)$ is 1-dimensional, generated by a syzygy $\gamma$ of rank 6. The syzygy scheme of $\gamma$ is the union of a point $p$ and a nodal curve $D = C \cup E$, such that $E$ is a smooth elliptic curve of degree 7 and $e := C \cdot E \in |K_C|$ consists of 14 mutually distinct points. Moreover, no cubic polynomial on $\mathbb{P}^6$ vanishes with multiplicity 2 along $C$.

**Proof.** Examples of singular Prym canonical curves having all these properties have been produced in [3] Proposition 4.4 or [4]. A generic deformation in $\mathcal{R}_8$ of these singular examples will provide the required smooth Prym canonical curve. \(\square\)

(First) proof of Theorem 1. We denote by $X$ the moduli space of elements $[C, \eta, x_1, \ldots, x_{14}]$, where $[C, \eta] \in \mathcal{R}_8$ is a smooth Prym curve of genus 8 and $x_i \in C$ are pairwise distinct points such that $x_1 + \cdots + x_{14} \in |K_C| \cong \mathbb{P}^7$. Since the fibres of the forgetful map $X \rightarrow \mathcal{R}_8$ are 7-dimensional, it follows that $X$ is an irreducible variety of dimension 28.

Let $T$ be the locally closed parameter space of odd genus 22 spin curves having the form

$$\left([D := C \cup \{x_1, \ldots, x_{14}\} \ E, \varrho] \ : \ [C] \in \mathcal{M}_8, \sum_{i=1}^{14} x_i \in |K_C|, \ [E, x_1, \ldots, x_{14}] \in \mathcal{M}_{1,14}, \ \varrho \otimes 2 = \omega_D\right).$$
Observe that points in $T$, apart from the spin structure $[D, \vartheta] \in \mathcal{S}_{22}$ also carry an underlying Prym structure $[C, \eta := K_C \otimes \vartheta_C] \in \mathcal{R}_8$, for $\vartheta_C^{\otimes 2} \cong K_C(x_1 + \cdots + x_{14}) \cong K_C^2$. One has an induced finite morphism $T \to X \times \mathcal{M}_{1,14}$, as well as a map $\mu : T \to \mathcal{R}_8$ forgetting the 14-pointed elliptic curve. It follows that $\dim T = \dim X + \dim \mathcal{M}_{1,14} = 42$. The locus

$$T_7 := \{ [D, \vartheta] \in T : h^0(D, \vartheta) \geq 7 \}$$

has the structure of a skew-symmetric degeneracy locus. Applying [13] Theorem 1.10, each component of $T_7$ has codimension at most $\binom{14}{2} = 21$ inside $T$, that is, $\dim(T_7) \geq \dim(\mathcal{R}_8)$.

By passing to a general 8-nodal Prym canonical curve $[C, \eta]$, following [3] Proposition 4.5, as well as Lemma 15, we have that $\dim K_{1,2}(C, K_C \otimes \eta) = 1$. In particular, the fibre $\mu^{-1}([C, \eta])$ contains an isolated point, which shows that $T_7$ is non-empty and has a component which maps dominantly under $\mu$ onto $\mathcal{R}_8$. Theorem 1 now follows.

**Remark 16.** The same construction can be carried out at the level of general paracanonical curves $[C, L] \in P^4_8$, where $L \in \text{Pic}^{14}(C) \setminus \{K_C\}$. The key difference is that we replace $T$ by a variety $T'$ parametrizing objects

$$\left( [D := C \cup \{x_1, \ldots, x_{14}\}, E, \vartheta, L] : [C, x_1, \ldots, x_{14}] \in \mathcal{M}_{14,8}, L \in \text{Pic}^{14}(C) \setminus \{K_C\}, \sum_{i=1}^{14} x_i \in |L^{\otimes 2} \otimes K_C^2|, [E, x_1, \ldots, x_{14}] \in \mathcal{M}_{14,14}, \vartheta^{\otimes 2} = \omega_D \right).$$

Similarly, we have a morphism $\mu' : T' \to P^4_8$ retaining the pair $[C, L]$ alone. The main difference compared to the Prym canonical case is that now

$$\dim |L^{\otimes 2} \otimes K_C^2| = 6,$$

therefore $\dim(T') = \dim(P^4_8) + \dim(\mathcal{M}_{1,14}) + 6 = 49$. The degeneracy locus $T'_7 \subseteq T'$ defined by the condition $h^0(D, \vartheta) \geq 7$ has codimension 21 inside $T'$, that is,

$$\dim(T'_7) = 28 = \dim(P^4_8) - 1.$$

It follows that the image $\mu'(T'_7) \subseteq P^4_8$ has codimension 1, which is in accordance with $\mathcal{R}_{0536}$ being a divisor in $P^4_8$.

4. **Second proof: Divisor class calculations on $\overline{R}_g$**

Recall [10] that $\overline{R}_g$ is the Deligne-Mumford moduli space of Prym curves of genus $g$, whose geometric points are triples $[X, \eta, \beta]$, where $X$ is a quasi-stable curve of genus $g$, $\eta \in \text{Pic}(X)$ is a line bundle of total degree 0 such that $\eta_E = \mathcal{O}_E(1)$ for each smooth rational component $E \subseteq X$ with $|E \cap X - E| = 2$ (such a component is said to be *exceptional*), and $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component is an isomorphism. If $\pi : \overline{R}_g \to \mathcal{M}_g$ is the map dropping the Prym structure, one has the formula

$$\pi^*(\delta_0) = \delta'_0 + \delta''_0 + 2\delta_{0,\text{ram}} \in CH^1(\overline{R}_g),$$

where $\delta'_0 := [\Delta'_0]$, $\delta''_0 := [\Delta''_0]$, and $\delta_{0,\text{ram}} := [\Delta_{0,\text{ram}}]$ are irreducible boundary divisor classes on $\overline{R}_g$, which we describe by specifying their respective general points.

We choose a general point $[C_{xy}] \in \Delta_0 \subseteq \mathcal{M}_g$ corresponding to a smooth 2-pointed curve $(C, x, y)$ of genus $g - 1$ and consider the normalization map $\nu : C \to C_{xy}$, where $\nu(x) = \nu(y)$. A general point
of $\Delta_0'$ (respectively of $\Delta_0''$) corresponds to a pair $[C_{xy}, \eta]$, where $\eta \in \text{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \text{Pic}^0(C)$ is non-trivial (respectively, $\nu^*(\eta) = \mathcal{O}_C$). A general point of $\Delta_0^\text{ram}$ is a Prym curve of the form $(X, \eta)$, where $X := C \cup \{x,y\} \mathbb{P}^1$ is a quasi-stable curve with $p_a(X) = g$ and $\eta \in \text{Pic}^0(X)$ is a line bundle such that $\eta_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1)$ and $\eta_{\mathbb{P}^1}^2 = \mathcal{O}_C(-x-y)$. In this case, the choice of the homomorphism $\beta$ is uniquely determined by $X$ and $\eta$. In what follows, we work on the partial compactification $\mathcal{R}_g \subseteq \widehat{\mathcal{R}}_g$ of $\mathcal{R}_g$ obtained by removing the boundary components $\pi^{-1}(\Delta_j)$ for $j = 1, \ldots, \lfloor g/2 \rfloor$, as well as $\Delta''_0$. In particular, $CH^1(\mathcal{R}_g) = \mathbb{Q}(\lambda, \delta'_0, \delta''_0)$. For a stable Prym curve $[X, \eta] \in \widehat{\mathcal{R}}_g$, set $L := \omega_X \otimes \eta \in \text{Pic}^{2g-2}(X)$ to be the paracanonical bundle. For $i \geq 1$, we introduce the vector bundle $\mathcal{N}_k$ over $\mathcal{R}_g$, having fibres

$$\mathcal{N}_k[X, \eta] = H^0(X, L^\otimes k).$$

The first Chern class of $\mathcal{N}_k$ is computed in [10] Proposition 1.7:

$$c_1(\mathcal{N}_k) = \left(\frac{k}{2}\right)(12\lambda - \delta'_0 - 2\delta''_0) + \lambda - \frac{k^2}{4}\delta''_0. \tag{4.22}$$

Then we define the locally free sheaves $\mathcal{G}_k$ on $\widehat{\mathcal{R}}_g$ via the exact sequences

$$0 \rightarrow \mathcal{G}_k \rightarrow \text{Sym}^k \mathcal{N}_1 \rightarrow \mathcal{N}_k \rightarrow 0,$$

that is, satisfying $\mathcal{G}_k[X, \eta] := I_{X,L}(k) \subseteq \text{Sym}^k H^0(X, L)$. Using (4.22) one computes $c_1(\mathcal{G}_k)$.

We also need the class of the vector bundle $\mathcal{G}$ with fibres

$$\mathcal{G}[X, \eta] = H^0(X, \omega_X^5 \otimes \eta^\otimes 4) = H^0(X, \omega_X \otimes L^\otimes 4).$$

Lemma 17. One has $c_1(\mathcal{G}) = 121\lambda - 10\delta'_0 - 24\delta''_0 \in CH^1(\mathcal{R}_g)$.

Proof. We apply Grothendieck-Riemann-Roch to the universal Prym curve $f : \mathcal{C} \rightarrow \mathcal{R}_g$. Denote by $L \in \text{Pic}(\mathcal{C})$ the universal Prym bundle, whose restriction to each Prym curve is the corresponding 2-torsion point, that is, $L|_{f^{-1}([X, \eta])} = \eta$, for each point $[X, \eta] \in \mathcal{R}_g$. Since $R^1 f_* (\omega_{\mathcal{C}}^\otimes \otimes L^\otimes 4) = 0$, we write

$$c_1(\mathcal{G}) = f_* \left[ \left( 1 + 5c_1(\omega_f) + 4c_1(L) + \frac{5c_1(\omega_f) + 4c_1(L)}{2} \right) \cdot \left( 1 - \frac{c_1(\omega_f)}{2} + \frac{c_1(\omega_f)^2}{12} + \overline{\text{Sing}(f)} \right) \right].$$

We use then the formulas $f_*(c_1^2(L)) = -2\delta''_0$ and $f_*(c_1(L) \cdot c_1(\omega_f)) = 0$ (see [10], Proposition 1.6) coupled with Mumford’s formula $f_*(c_1^2(\Omega_f^1) + \overline{\text{Sing}(f)}) = 12\lambda$ as well with the identity

$$\kappa_1 := f_*(c_1^2(\omega_f)) = 12\lambda - \delta'_0 - 2\delta''_0,$$

in order to conclude. □

The Koszul locus

$$\mathcal{Z}_8 := \mathfrak{kos}_3 \cap \mathcal{R}_8 = \left\{ [C, \eta] \in \mathcal{R}_8 : K_{1,2}(C, K_C \otimes \eta) \neq 0 \right\}$$

is a virtual divisor on $\mathcal{R}_8$, that is, the degeneracy locus of a map between vector bundles of the same rank over $\mathcal{R}_8$. If it is a genuine divisor (which we aim to rule out), the class of its closure in $\mathcal{R}_8$ is given by [3] Theorem F:

$$[\mathcal{Z}_8] = 27\lambda - 4\delta'_0 - 6\delta''_0 \in CH^1(\mathcal{R}_8).$$
Remark 18. Some of the considerations above can be extended to higher order torsion points. We recall that \( \mathcal{R}_{g,\ell} \) is the moduli space of pairs \( [C, \eta] \), where \( C \) is a smooth curve of genus \( g \) and \( \eta \in \text{Pic}^0(C) \) is a non-trivial \( \ell \)-torsion point. It is then shown in [3] that the locus \( \mathcal{Z}_{8,\ell} := \mathfrak{Ros}_{7} \cap \mathcal{R}_{8,\ell} \subseteq P^1 \) is a divisor on \( \mathcal{R}_{8,\ell} \) for each other level \( \ell \geq 3 \). The class of the compactification of \( \mathcal{Z}_{8,\ell} \) is given by the following formula, see [3] Theorem F:

\[
[\mathcal{Z}_{8,\ell}] = 27\lambda - 4\delta\prime_0 - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{4(a^2 - a\ell + \ell^2)}{\ell} \delta_0^{(a)} \in C H^1(\tilde{\mathcal{R}}_{8,\ell}).
\]

We refer to [3] Section 1.4, for the definition of the boundary divisor classes \( \delta_0^{(a)} \), where \( a = 1, \ldots, \lfloor \frac{\ell}{2} \rfloor \). If \( \pi : \tilde{\mathcal{R}}_{g,\ell} \rightarrow \overline{\mathcal{M}}_g \) is the map forgetting the level \( \ell \) structure, then

\[
\pi^*(\delta_0) = \delta\prime_0 + \delta\prime_0 + \ell \sum_{\ell=1}^{\lfloor \frac{\ell}{2} \rfloor} \delta_0^{(a)}.
\]

We fix now a genus 8 Prym-canonically embedded curve \( \phi_L : C \hookrightarrow \mathbb{P}^6 \). As usual, we denote the kernel bundle by \( M_L := \Omega^1_{\mathbb{P}^6}(1) \), hence we have the exact sequence

\[
0 \longrightarrow N_C^\vee \otimes L^{\otimes 4} \longrightarrow M_L \otimes L^{\otimes 3} \longrightarrow K_C \otimes L^{\otimes 4} \longrightarrow 0. \tag{4.23}
\]

This can be interpreted as an exact sequence of vector bundles over \( \tilde{\mathcal{R}}_8 \). Denoting by \( \mathcal{H} \) the vector bundle over \( \tilde{\mathcal{R}}_8 \) with fibres \( H^0(C, N_C^\vee \otimes L^{\otimes 4}) \), we compute using the previous formulas and the fact that \( \text{rk}(N_1) = h^0(C, L) = 7 \) and \( \text{rk}(N_3) = h^0(C, L^{\otimes 3}) = 35 \):

\[
c_1(\mathcal{H}) = 35c_1(N_1) + 7c_1(N_3) - c_1(N_3) - c_1(G) = 100\lambda - 5\delta\prime_0 - \frac{53}{2} \delta_0^\text{ram}. \tag{4.24}
\]

Thus \( \mathcal{D}_1 = \mathfrak{Ros}_{7} \cap \mathcal{R}_8 \) and \( \mathcal{D}_2 = \mathfrak{Ros}_{6} \cap \mathcal{R}_8 \). We have already seen in Proposition 5 that \( K_{1,2}(C, L) \neq 0 \) if and only if either \( \phi_L(C) \subseteq \mathbb{P}^6 \) is not scheme-theoretically cut out by quadrics, or else, \( H^1(\mathbb{P}^6, \mathcal{I}_C^2(4)) \neq 0 \). We write

\[
\mathcal{Z}_8 = \mathcal{D}_1 + \mathcal{D}_2, \quad \text{where}
\]

\[
\mathcal{D}_1 := \left\{ [C, \eta] \in \mathcal{R}_8 : \phi_L(C) \subseteq \mathbb{P}^6 \text{ is scheme-theoretically not cut out by quadrics} \right\}
\]

and

\[
\mathcal{D}_2 := \left\{ [C, \eta] \in \mathcal{R}_8 : H^1(\mathbb{P}^6, \mathcal{I}_C^2(4)) \neq 0 \right\}.
\]

We have already observed that \( \dim I_{C,L}(2) = 7 \) and \( \chi(\mathbb{P}^6, \mathcal{I}_C^2(4)) = 28 \). If \( \mathcal{Z}_8 \) is a divisor, then \( \mathcal{D}_2 \) is a divisor as well and for \( [C, \eta] \in \mathcal{R}_8 \setminus \mathcal{D}_2 \), we have that

\[
\dim \text{Sym}^2 I_{C,L}(2) = \dim I_{C,L}(4)_2 = 28.
\]

Paying some attention to its definition, the divisor \( \mathcal{D}_1 \) can be thought as the degeneracy locus

\[
\left\{ [C, \eta] \in \mathcal{R}_8 : \text{Sym}^2 I_{C,L}(2) \not\hookrightarrow I_{C,L}(4)_2 \right\},
\]

which is an effective divisor on \( \tilde{\mathcal{R}}_8 \). We compute the class of this divisor:
Theorem 19. We have the following formulas:

\[ [\mathfrak{D}_1] = 7\lambda - \frac{1}{2}\delta' - \frac{3}{4}\delta_0^{\text{ram}} \in CH^1(\tilde{R}_8) \]

and

\[ [\mathfrak{D}_2] = 20\lambda - \frac{7}{2}\delta_0 - \frac{21}{4}\delta_0^{\text{ram}} \in CH^1(\tilde{R}_8). \]

Proof. We first globalization over \( \tilde{R}_8 \) the following exact sequence:

\[ 0 \to I_{C,L}(4)_2 \to I_{C,L}(4) \to H^0(C, N_C^\vee \otimes L^{\otimes 4}) \to H^1(P^6, I_C^2(4)) \to 0. \]

Denote by \( \mathcal{A} \) the sheaf on \( \tilde{R}_8 \) supported along the divisor \( \mathfrak{D}_2 \), whose fibre over a general point of that divisor is equal to to \( H^1(P^6, I_C^2(4)) \). There is a surjective morphism of sheaves

\[ \mathcal{H} \to \mathcal{A} \]

and denote by \( \mathcal{G}'_4 \) its kernel. Since \( \mathcal{A} \) is locally free along \( \mathfrak{D}_2 \) and \( \tilde{R}_8 \) is a smooth stack, using the Auslander-Buchsbaum formula we find that \( \mathcal{G}'_4 \) is a locally free sheaf of rank equal to \( \text{rk}(\mathcal{H}) = \chi(C, N_C^\vee(4L)) = 19 \cdot 7 \). Precisely, \( \mathcal{G}'_4 \) is an elementary transformation of \( \mathcal{H} \) along the divisor \( \mathfrak{D}_2 \). Furthermore, \( c_1(\mathcal{G}'_4) = c_1(\mathcal{H}) - [\mathfrak{D}_2]. \)

The morphism \( \mathcal{G}_4 \to \mathcal{H} \) globalizing the maps \( I_{C,L}(4) \to H^0(C, N_C^\vee \otimes L^{\otimes 4}) \) factors through the subsheaf \( \mathcal{G}'_4 \) and we form the exact sequence:

\[ 0 \to \mathcal{G}_4^2 \to \mathcal{G}_4 \to \mathcal{G}'_4 \to 0. \]

The multiplication maps \( \text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)_2 \) globalize to a sheaf morphism

\[ \nu : \text{Sym}^2(\mathcal{G}_2) \to \mathcal{G}_4^2 \]

between locally free sheaves of the same rank 28 over the stack \( \tilde{R}_8 \). The degeneration locus of \( \nu \) is precisely the divisor \( \mathfrak{D}_1 \). We compute:

\[ c_1(\text{Sym}^2(\mathcal{G}_2)) = 8c_1(\mathcal{G}_2) = 8(8c_1(N_1) - c_1(N_2)) = -40\lambda + 8(\delta' + \delta_0^{\text{ram}}), \]

and

\[ c_1(\mathcal{G}_4^2) = 120c_1(N_1) - c_1(N_4) - c_1(\mathcal{H}) + [\mathfrak{D}_2] = -53\lambda + 11\delta' + \frac{25}{2}\delta_0^{\text{ram}} + [\mathfrak{D}_2]. \]

We obtain the relation \( [\mathfrak{D}_1] - [\mathfrak{D}_2] = -13\lambda + 3\delta' + \frac{9}{2}\delta_0^{\text{ram}}. \) Since at the same time

\[ [\mathfrak{D}_1] + [\mathfrak{D}_2] = [Z_8] = 27\lambda - 4\delta_0' - 6\delta_0^{\text{ram}}, \]

we solve the system and conclude. \( \square \)

We are now in a position to give a second proof of Theorem 1:

Theorem 20. The class \( [\mathfrak{D}_2] \) cannot be effective. It follows that \( Z_8 = R_8 \) and \( K_{1,2}(C, K_C \otimes \eta) \neq 0, \) for every Prym curve \( [C, \eta] \in R_8. \)

Proof. We use the sweeping curve of the boundary divisor \( \Delta'_0 \) of \( \tilde{R}_8 \) constructed via Nikulin surfaces in [11] Lemma 3.2: Precisely, through the general point of \( \Delta'_0 \) there passes a rational curve \( \Gamma \subseteq \Delta'_0 \), entirely contained in \( \tilde{R}_8 \), having the following numerical characters:

\[ \Gamma \cdot \lambda = 8, \quad \Gamma \cdot \delta' = 42, \quad \text{and} \quad \Gamma \cdot \delta_0^{\text{ram}} = 8. \]

We note that \( \Gamma \cdot \mathfrak{D}_2 < 0. \) Writing \( \mathfrak{D}_2 = \alpha \cdot \delta_0' + E, \) where \( \alpha \geq 0 \) and \( E \) is an effective divisor whose support is disjoint from \( \Delta'_0 \), we immediately obtain a contradiction. \( \square \)
The divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ can be defined in an identical manner at the level of each moduli space $\mathcal{R}_{8,\ell}$ of twisted level $\ell$ curves of genus $g$. As already pointed out, in the case $\ell \geq 3$ it follows from [3] Proposition 4.4 that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are actual divisors. Repeating the same calculations as for $\ell = 2$, we obtain the following formula on the partial compactification $\mathcal{R}_{8,\ell}$ of $\mathcal{R}_{8,\ell}$:

\[
[\mathcal{D}_2] = 20\lambda - \frac{7}{2} \delta_0' - \sum_{a=1}^{[\frac{4}{\ell}]} \frac{1}{2\ell}(7a^2 - 7a\ell + 17\ell^2 - 20\ell)\delta_0^{(a)} \in CH^1(\mathcal{R}_{8,\ell}). \tag{4.25}
\]

As an application, we mention a different proof of one of the main results from [1]:

**Theorem 21.** The canonical class of $\mathcal{R}_{8,\ell}$ is big for $\ell \geq 3$. It follows that $\mathcal{R}_{8,\ell}$ is a variety of general type for $\ell = 3, 4, 6$.

**Proof.** Using formula (4.25), it is a routine exercise to check that for $\ell \geq 3$ the canonical class computed in [3] Proposition 1.5

\[
K_{\mathcal{R}_{8,\ell}} = 13\lambda - 2\delta_0' - (\ell + 1) \sum_{a=1}^{[\frac{4}{\ell}]} \delta_0^{(a)}
\]

can be written as a positive combination of the big class $\lambda$ and the effective class $[\mathcal{D}_2]$, hence it is big. Arguing along the lines of [3] Remark 3.5, it is easy to extend this result to the full compactification $\mathcal{R}_{8,\ell}$ and deduce that $K_{\mathcal{R}_{8,\ell}}$ is big.

To conclude that $\mathcal{R}_{8,\ell}$ is of general type, one needs, apart from the bigness of the canonical class $K_{\mathcal{R}_{8,\ell}}$ of the moduli stack, a result that the singularities of the coarse moduli space $\mathcal{R}_{8,\ell}$ impose no adunction conditions. This is only known for $2 \leq \ell \leq 6, \ell \neq 5$, see [2].

---

5. Rank 2 vector bundles and singular quartics

Our goal in this section is to propose a construction of syzygies of Prym canonical curves of genus 8. We also sketch the proof of the fact that these syzygies are nontrivial. We fix again a general element $[C, \eta] \in \mathcal{R}_8$ and set $L := K_C \otimes \eta$. According to Proposition 5, in order to prove that $K_{2,1}(C, L) \neq 0$, we have to produce quartic hypersurfaces in $\mathbb{P}^5$ which vanish at order at least 2 along $\phi_L(C)$, but do not lie in the image of the map $\text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)$. The goal of this section is to produce such quartics from rank 2 vector bundles on $C$. The (incomplete) proof that the quartics we construct are not in the image of $\text{Sym}^2 I_{C,L}(2)$ depends on an unproved general position statement (*), but there might be other approaches exploiting the fact that the hypersurfaces in question are determinantal.

The following construction produces quartics vanishing at order 2 along $C$. Let $E$ be a rank 2 vector bundle on $C$, with determinant $K_C$. Assume

\[
h^0(C, E) = 4, \quad h^0(C, E(\eta)) = 4. \tag{5.26}
\]

Setting $V_0 := H^0(C, E)$ and $V_1 := H^0(C, E(\eta))$, we have a natural map

\[
V_0 \otimes V_1 \to H^0(C, L),
\]

defined using evaluation and the following composite map:

\[
H^0(E) \otimes H^0(E(\eta)) \to H^0(E \otimes E(\eta)) \cong H^0(\mathcal{E}nd E \otimes L) \xrightarrow{\text{Tr}} H^0(C, L). \tag{5.27}
\]

This map gives dually a morphism

\[
H^0(C, L)^\vee \to V_0^\vee \otimes V_1^\vee,
\]

(which will be proved below to be injective for a general choice of $E$). We consider the quartic hypersurface $D_4$ on $\mathbb{P}(V_0^\vee \otimes V_1^\vee)$ parametrizing tensors of rank at most 3.
Lemma 22. The restriction $D_{4,E}$ of this quartic to $P(H^0(C,L)^\vee) \subseteq P(V_0^\vee \otimes V_1^\vee)$ is singular along the curve $C$.

Proof. The quartic $D_4$ is singular along the set $T_2 \subseteq P(V_0^\vee \otimes V_1^\vee)$ of tensors of rank at most 2. The quartic $D_{4,E}$ in $P(H^0(C,L)^\vee)$ is thus singular along $T_2 \cap P(H^0(C,L)^\vee)$, which obviously contains $C \subseteq P(H^0(C,L)^\vee)$, since at a point $p \in C$, the map $V_0 \otimes V_1 \to H^0(C,L)$ composed with the evaluation at $p$ factors through $E_{|p} \otimes E(\eta)_{|p}$. □

By Brill-Noether theory, the variety $W_7^4(C)$ of degree 7 pencils on $C$ is 4-dimensional. There should thus exist finitely many elements $D \in W_7^4(C)$ with the property that

$$h^0(C,D) \geq 2, \ h^0(C,D \otimes \eta) \geq 2.$$  \hfill (5.28)

We now have the following lemma:

Lemma 23. Let $[C,\eta] \in \mathcal{R}_8$ be as above and $D \in W_7^4(C)$ satisfying (5.28). Then

(i) $h^0(C,D) = 2$ and $h^0(C,D \otimes \eta) = 2$. The multiplication map

$$\left( H^0(C,D) \otimes H^0(C,KC \otimes D^\vee) \right) \oplus \left( H^0(C,D \otimes \eta) \otimes H^0(C,KC \otimes D^\vee \otimes \eta) \right) \to H^0(C,KC)$$

is surjective (in fact, an isomorphism).

(ii) The multiplication map

$$\left( H^0(C,D) \otimes H^0(C,KC \otimes D^\vee \otimes \eta) \right) \oplus \left( H^0(C,D \otimes \eta) \otimes H^0(C,KC \otimes D^\vee) \right) \to H^0(C,KC(\eta))$$

is surjective.

Proof. This can be proved by a degeneration argument, for example by degenerating $C$ to the union of two curves of genus 4 meeting at one point. □

By Brill-Noether theory, the following corollary follows from (i) above:

Corollary 24. For $[C,\eta]$ as above, the set of pencils $D \in W_7^4(C)$ satisfying (5.28) is finite.

Given such a $D$, we form the rank 2 vector bundle

$$E = D \oplus (KC \otimes D^\vee)$$

on $C$ which satisfies the conditions (5.26). The associated quartic is however not interesting for our purpose, due to the following fact:

Lemma 25. The quartic on $P(H^0(C,L)^\vee)$ associated to the vector bundle $D \oplus (KC \otimes D^\vee)$ is the union of the two quadrics $Q_0$ and $Q_1$ associated respectively with the multiplication maps

$$H^0(D) \otimes H^0((KC \otimes D^\vee)(\eta)) \to H^0(KC(\eta))$$

and $H^0(D(\eta)) \otimes H^0(KC \otimes D^\vee) \to H^0(KC(\eta))$.

Both these quadrics contain $C$.

Proof. Indeed we have in this case

$$V_0 = H^0(C,E) = H^0(C,D) \oplus H^0(C,KC \otimes D^\vee),$$

and

$$V_1 = H^0(C,E(\eta)) = H^0(C,D \otimes \eta) \oplus H^0(C,KC \otimes D^\vee \otimes \eta).$$
Furthermore, it is clear that the map of (5.27) factors through the projection
\[ V_0 \otimes V_1 \to \left( H^0(C, D) \otimes H^0(C, K_C \otimes D^\vee \otimes \eta) \right) \oplus \left( H^0(C, K_C \otimes D^\vee) \otimes H^0(D, D \otimes \eta) \right) \]
and induces on each summand the multiplication map. The quadric \( Q_0 \) is by definition associated with the multiplication map
\[ \mu_0 : H^0(C, D) \otimes H^0(C, K_C \otimes D^\vee \otimes \eta) \to H^0(C, K_C \otimes \eta), \]
and is the set of elements \( f \) in \( \mathbf{P}(H^0(K_C \otimes \eta))^{\vee} \) such that \( \mu_0(f) \) is a tensor of rank \( \leq 1 \). Similarly for \( Q_1 \), with \( D \) being replaced with \( D(\eta) \). Finally we use the fact that a tensor
\[ (\mu_0^* f, \mu_1^* f) \in \left( H^0(C, D) \otimes H^0(C, K_C \otimes D^\vee \otimes \eta) \right) \oplus \left( H^0(C, K_C \otimes D^\vee) \otimes H^0(D, D \otimes \eta) \right) \]
has rank at most 3 if and only if one of \( \mu_0^* f \) and \( \mu_1^* f \) has rank at most 1. \( \square \)

We now sketch the proof of the fact that for \( C \) general of genus 8 and \( D \in W^4_2(C) \) satisfying (5.28), for a general deformation \( E \) of the vector bundle \( D \oplus (K_C \otimes D^\vee) \) satisfying \( \det E \cong K_C \) and \( h^0(C, E) = 4 \), the associated quartic \( D_{4,E} \) singular along \( C \) is not defined by an element of \( \text{Sym}^2 I_C(2) \). Combined with Proposition 5, this provides a third approach to Theorem 1. The proof of this fact rests on an unproven general position statement \((*)\), so it is incomplete.

**Sketch of proof of the nontriviality of the syzygy.** The vector bundle \( E \) is generated by sections, as it is a general section-preserving deformation of the vector bundle
\[ D \oplus (K_C \otimes D^\vee) \]
which is generated by global sections, and similarly for \( E(\eta) \). Along \( C \subseteq \mathbf{P}(H^0(C, L)^{\vee}) \), then the rational map
\[ \mathbf{P}(H^0(C, L)^{\vee}) \dashrightarrow \mathbf{P}(H^0(E)^{\vee} \otimes H^0(E(\eta))^{\vee}) \]
is well-defined and the image of \( C \) is contained in the locus \( T_{2,E} \) of tensors of rank exactly 2. In fact, the case of \( D \oplus (K_C \otimes D^\vee) \) shows that this map is a morphism for general \( E \) (one just needs to know that \( H^0(C, K_C \otimes \eta) \) is generated by the two vector spaces \( H^0(D) \otimes H^0(K_C \otimes D^\vee \otimes \eta) \) and \( H^0(D \otimes \eta) \otimes H^0(K_C \otimes D^\vee) \) respectively, or rather their images under the multiplication map. Note that on \( T_{2,E} \), there is a rank 2 vector bundle \( M \) which restricts to \( E \) on \( C \).

In the case of the split vector bundle \( E_{sp} = D \oplus (K_C \otimes D^\vee) \), Lemma 25 shows that the Zariski closure \( \overline{T_{2,E_{sp}}} \) parameterizing tensors of rank \( \leq 2 \) in \( \mathbf{P}(H^0(C, L)^{\vee}) \subseteq \mathbf{P}(V_0^{\vee} \otimes V_1^{\vee}) \) is equal to the singular locus of \( D_{4,E_{sp}} \) and consists of the union of the two planes \( P_0, P_1 \) defined as the singular loci of the quadrics \( Q_0, Q_1 \) respectively, and the intersection \( Q_0 \cap Q_1 \). The locus \( T_{2,E_{sp}} \setminus T_{2,E_{sp}} \) is the locus where the tensor has rank 1, and this happens exactly along the two conics \( P_0 \cap Q_1 \) and \( P_1 \cap Q_0 \). The curve \( C \) is contained in \( Q_0 \cap Q_1 \) and does not intersect \( P_0 \cup P_1 \). In particular, the rational map \( \phi : \mathbf{P}^9 \dashrightarrow \mathbf{P}^9 \) given by the linear system \( I_C(2) \) is well defined along \( P_0 \cup P_1 \). We believe that the following general position statement concerning the two planes \( P_i \) is true for general \( C \) and \( D, \eta \) as above.
The surfaces $\phi(P_i)$ are projectively normal Veronese surfaces, generating a hyperplane $\langle \phi(P_i) \rangle \subseteq \mathbb{P}^6$. Furthermore, the surface $\phi(P_0) \cup \phi(P_1) \subseteq \mathbb{P}^6$ is contained in a unique quadric in $\mathbb{P}^6$, namely the union of the two hyperplanes $\langle \phi(P_0) \rangle$ and $\langle \phi(P_1) \rangle$.

We now prove that, assuming $(*)$, for a general vector bundle $E$ as above, the associated quartic $D_{4,E}$ singular along $C$ is not defined by an element of $\text{Sym}^2 I_C(2)$. As $P_0, P_1$ are 2-dimensional reduced components of $T_{2,E_{sp}}$, hence of the right dimension, the theory of determinantal hypersurfaces shows that for general $E$ as above, there is a reduced surface $\Sigma_E \subseteq T_{2,E_{sp}}$ whose specialization when $E = E_{sp}$ contains $P_0 \cup P_1$. Let $E \to C \times B$ be a family of vector bundles on $C$ parameterized by a smooth curve $B$, with general fiber $E$ and special fiber $E_{sp}$. Denote by $E_b$ the restriction of $E$ to $C \times \{b\}$. Property $(*)$ then implies that $\phi(\Sigma_{E_b})$ for general $b \in B$ is contained in at most one quadric $Q_{E_b}$ in $\mathbb{P}^6$. We argue by contradiction and assume that the quartic $D_{4,E_b}$ is a pull-back $\phi^{-1}(Q)$ for general $b$. One thus must have $Q = Q_{E_b}$. Next, the determinantal quartic $D_{4,E_b}$ is singular along $T_{2,E_b}$, hence along $\Sigma_{E_b}$. Let $b \mapsto q_{E_b} \in \text{Sym}^2 I_C(2)$, where $q_{E_b}$ is a defining equation for the quadric $Q_{E_b}$. Then we find that the first order derivative of the family $\phi^* q_{E_b}$ at $b_0$ also vanishes along $\Sigma_{E_{b_0}}$, hence it must be proportional to $\phi^* q_{E_{b_0}}$. We then conclude that the quadric $Q_{E_b}$ is in fact constant, and thus must be equal to the quadric $Q_{E_{sp}}$. We now reach a contradiction by proving the following lemma.

**Lemma 26.** If the determinantal quartic $D_{4,E_b}$ is constant, equal to $D_{sp} = Q_0 \cup Q_1$, then the vector bundle $E_b$ on $C$ does not deform with $b \in B$.

**Proof.** Denoting $V_{0,b} := H^0(C, E_b)$, $V_{1,b} := H^0(C, \mathcal{E}_b(\eta))$, we have the multiplication map

$$V_{0,b} \otimes V_{1,b} \to H^0(C, K_C \otimes \eta)$$

which is surjective for generic $b$ since it is surjective for $E_0 = D \oplus (K_C \otimes D^\vee)$ (see Lemma 23). The determinantal quartic $D_{4,E_b}$ is the vanishing locus of the determinant of the corresponding bundle map

$$\sigma_b : V_{0,b} \otimes \mathcal{O}_{P(H^0(C, K_C \otimes \eta)^\vee)} \to V_{1,b} \otimes \mathcal{O}_{P(H^0(C, K_C \otimes \eta)^\vee)}$$

on $P(H^0(C, K_C \otimes \eta)^\vee)$. We know that $D_{4,E_b} = Q_0 \cup Q_1$ for any $b \in B$, where the quadrics $Q_i$ are singular (of rank 4), but with singular locus $P_i$ not intersecting $C \subseteq Q_0 \cap Q_1$. The morphism $\sigma_b$ has rank exactly 1 generically along each $Q_i$ and the kernel of $\sigma_{D_{4,b}}$ determines a line bundle $K_{i,b}$ on its smooth locus $Q_i \setminus P_i$. This line bundle is independent of $b$ since Pic($Q_i \setminus P_i$) has no continuous part. The restriction of $K_{i,b}$ to $C$ is thus constant. Finally, on the smooth part of $(Q_0 \cap Q_1)_{\text{reg}}$, the kernel $\text{Ker}(\sigma)$ contains the two line bundles $K_{i,b}(Q_0 \cap Q_1)$. Restricting to $C \subseteq (Q_0 \cap Q_1)_{\text{reg}}$, we conclude that $\text{Ker} \sigma_{b|C}$ contains $K_{i,0|C}$ for $i = 0, 1$. For $b = 0$, one has

$$\text{Ker} \sigma_{0|C} = K_{0,0|C} \oplus K_{1,0|C}$$

and this thus remains true for general $b$. Finally, it follows from the construction and the fact that $E_b$ is generated by its sections that $\text{Ker} \sigma_{b|C} = \mathcal{E}_b^\vee$, which finishes the proof.

## 6. Miscellany

### 6.A. Extra remarks on the geometry of paracanonical curves of genus 8 with a nontrivial syzygy

We now comment on an interesting rank 2 vector bundle appearing in our situation. Again, let $\phi_L : C \hookrightarrow \mathbb{P}^6$ be a paracanonical curve of genus 8. We assume $L$ is scheme-theoretically cut out by quadrics. Denoting by $N_C$ the normal bundle of $C$ in the embedding in $\mathbb{P}^6$, we consider the natural
map \( I_C(2) \otimes \mathcal{O}_C \to N_C^0 \otimes L^{\otimes 2} \) (which is surjective by our assumption) given by differentiation along \( \phi_L(C) \), and let \( F \) denote its kernel. We thus have the short exact sequence:

\[
0 \to F \to I_C(2) \otimes \mathcal{O}_C \to N_C^0 \otimes L^{\otimes 2} \to 0. \tag{6.30}
\]

If \( K_{1,2}(C, L) \neq 0 \), the map \( \mu : I_C(2) \otimes H^0(P^6, \mathcal{O}(1)) \to I_C(3) \) is not surjective, hence not injective. A fortiori, the map

\[
\pi : I_C(2) \otimes H^0(P^6, \mathcal{O}(p^e(1))) \to H^0(C, N_C^0 \otimes L^{\otimes 3})
\]

induced by (6.30) is not injective, so that \( h \) is not injective, hence not injective. A fortiori, the map

\[
\pi : I_C(2) \otimes H^0(P^6, \mathcal{O}(p^e(1))) \to H^0(C, N_C^0 \otimes L^{\otimes 3})
\]

is not injective, so that \( h \) is not injective. In fact, the equivalence between the statements \( h \neq 0 \) and \( K_{1,2}(C, L) \neq 0 \) follows from the same argument once we know that there is no cubic polynomial on \( P^6 \) vanishing with multiplicity 2 along \( C \).

We observe now that \( F \) is a vector bundle of rank 2 on the curve \( C \), with determinant equal to \( \det N_C \otimes L^{\otimes (-2)} \cong K_C \otimes L^{\otimes (-3)} \). Hence if \( F(L) \) has a nonzero section, assuming this section vanishes nowhere along \( C \), then \( F(L) \) is an extension of \( K_C \otimes L^{\vee} \) by \( \mathcal{O}_C \). This provides an extension class

\[
e \in H^1(C, L \otimes K_C^\vee) = H^0(C, K_C^{\otimes 2} \otimes L^{\otimes 2}) = H^4(C, K_C \otimes K_C^\vee), \tag{6.31}
\]

Assume now \( L \otimes K_C^\vee = \eta \) is a nonzero 2-torsion element of \( \text{Pic}^0(C) \). Then

\[
e \in H^0(C, L)^\vee.
\]

On the other hand, according to Theorem 20, there exists a nontrivial syzygy

\[
\gamma = \sum_{i=1}^6 \ell_i \otimes q_i \in K_{1,2}(C, L) = \text{Ker} \{ H^0(P^6, \mathcal{O}(p^e(1))) \otimes I_C(2) \to I_C(3) \},
\]

which is degenerate by Proposition 13. As we saw already, it has in fact rank 6 for generic \([C, \eta]\), hence determines a nonzero element

\[
f \in H^0(P^6, \mathcal{O}(p^e(1))) = H^0(C, L)^\vee = H^1(C, K_C \otimes L^\vee) = H^1(C, L \otimes K_C^\vee), \tag{6.32}
\]

which is well-defined up to a coefficient.

**Proposition 27.** The two elements \( e \) and \( f \) are proportional.

**Proof.** Equivalently, we show that the kernels of the two linear forms \( e, f \in H^0(C, L)^\vee \) are equal. Viewing \( \gamma \) as an element of \( \text{Hom}(I_C(2)^\vee, H^0(C, L)) \), we have \( \text{Ker}(f) = \text{Im}(\gamma) \). On the other hand, the kernel of \( e \) identifies with

\[
\text{Im} \left\{ j : H^0(C, F \otimes L^{\otimes 3} \otimes K_C^\vee) \to H^0(C, L) \right\},
\]

where the map \( j \) is obtained by twisting the exact sequence \( 0 \to \mathcal{O}_C \to F(L) \to K_C \otimes L^\vee \to 0 \) by \( K_C \). We have \( F \otimes L^{\otimes 3} \otimes K_C^\vee \cong F^\vee \) since \( \det F \cong K_C \otimes L^{\otimes (-3)} \), hence there is a natural morphism

\[
i^* : I_C(2)^\vee \otimes \mathcal{O}_C \to F^\vee \cong F(L^{\otimes 3} \otimes K_C^\vee)
\]

dual to the inclusion \( F \to I_C(2) \otimes \mathcal{O}_C \) of (6.30). The proposition follows from the following claim:

**Claim.** The morphism \( \alpha : I_C(2)^\vee \to H^0(C, L) \) is equal to \( j \circ i^* \).

Forgetting about the last identification \( F^\vee \cong F \otimes L^{\otimes 3} \otimes K_C^\vee \), the claim amounts to the following general fact: For an evaluation exact sequence on a variety \( X \)

\[
0 \to G \to W \otimes \mathcal{O}_X \to M \to 0
\]

and for a section \( s \in H^0(X, G(L)) = H^0(X, \mathcal{H}om(G^\vee, L)) \) giving an element

\[
s' \in \text{Ker} \left\{ W \otimes H^0(X, L) \to H^0(X, M \otimes L) \right\} \subseteq \text{Hom}(W^\vee, H^0(X, L)),
\]

the induced map \( s : H^0(X, G^\vee) \to H^0(X, L) \) composed with the map \( W^\vee \to H^0(X, G^\vee) \) equals the map \( s' : W^\vee \to H^0(X, L) \). \( \square \)
6.B. Further properties

Using the exact sequence (6.30) in the general case of a genus 8 paracanonical curve \([C, L] \in P^8_8\), we obtain:

**Lemma 28.** A section \(s \in H^0(C, F(L)) \subseteq I_{C,L}(2) \otimes H^0(C, L) = \text{Hom}(I_{C,L}(2)^\vee, H^0(C, L))\) of rank 6, determines an element \(e \in [2L - K_C]\).

*Proof.* The multiplication by \(s \in H^0(F(L)) \subseteq I_{C,L}(2) \otimes H^0(C, L) = H^0(I_{C,L}(2)^\vee \otimes L)\) determines the natural maps \(F^\vee \to L\) and \(g_s : I_{C}(2)^\vee \otimes O_C \to L\) sitting in the following diagram:

\[
\begin{array}{cccccc}
0 & \to & \text{Ker}(g_s) & \to & I_{C}(2)^\vee \otimes O_C & \to & L & \to & 0 \\
0 & \to & 2L - K_C & \to & F^\vee & \to & L & \to & 0 ,
\end{array}
\]

where \(I_{C}(2)^\vee \otimes O_C \to F^\vee\) is the dual of the natural inclusion of (6.30). Passing to global sections we get the inclusion \(H^0(\text{Ker}(g_s)) = \text{Ker}(I_{C,L}(2)^\vee \to H^0(C, L)) \to H^0(2L - K_C)\), which by hypothesis in 1-dimensional hence it defines an element \(e \in [2L - K_C]\).

Via the exact sequence (6.30) we can also show directly the following result that has been used in Section 3:

**Lemma 29.** If there is a spin curve \(D = C \cup E \to \mathbb{P}^6\) of genus 22 and degree 21 containing the genus 8 paracanonical curve \([C, L]\) as in Lemma 3, then \(H^0(C, (F(L)) \neq 0\). If there is no cubic polynomial on \(\mathbb{P}^6\) vanishing with multiplicity 2 along \(C\), then \(K_{1,2}(C, L) \neq 0\).

*Proof.* Let \(e = C \cap E\) and recall \(c_1(F) = -3L + K_C\) and \(O_C(e) = 2L - K_C\). Note that \(I_D(2) \subseteq I_{C}(2)\) is 6-dimensional. Tensor then the first vertical exact sequence of the following diagram by \(L\) and pass to global sections:

\[
\begin{array}{cccccc}
0 & \to & L^\vee & \to & I_D(2) \otimes O_C & \to & I_D/(I_D \cap I_C^2)(2) & \to & 0 \\
0 & \to & F & \to & I_C(2) \otimes O_C & \to & N_E^2(2) & \to & 0 .
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_C(-e) & \to & \mathcal{O}_C & \to & \mathcal{O}_C|_e & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

6.C. Nontrivial syzygies of paracanonical curves via vector bundles

We return to the proof of Theorem 20 given in Section 5. Consider now a general paracanonical curve \([C, K_C \otimes \eta] \in P^8_8\). For a rank 2 vector bundle on \(C\) of degree 14, with noncanonical determinant, the equation \(h^0(C, E) \geq 4\) imposes 16 conditions. Similarly, if \(\epsilon \in \text{Pic}^0(C)\), the equation \(h^0(C, E \otimes \epsilon) \geq 4\) imposes 16 conditions on the parameter space of \(E\). Given \(C\), there are 29 = 4g - 3 parameters for \(E\), and 8 = g parameters for \(\epsilon\). It follows that we have at least a 5-dimensional family of pairs \((E, \epsilon)\), such that

\[
h^0(C, E) \geq 4 \quad \text{and} \quad h^0(C, E \otimes \epsilon) \geq 4. \tag{6.33}
\]

Furthermore, the construction of Section 5 (together with Proposition 5) shows that for a general triple \((C, E, \epsilon)\) as above, one has \(K_{2,1}(C, L) \neq 0\), where \(L := \text{det} E \otimes \epsilon\). Assuming the map \((E, \epsilon) \to L\)
is generically finite on its image, we constructed in this way a five dimensional family of paracanonical line bundles \( L \in \text{Pic}^{14}(C) \) with a nontrivial syzygy: \( K_{1,2}(C, L) \neq 0 \). This family has the following property:

**Lemma 30.** If \( L = \det E \otimes \epsilon \), where \( E \) satisfies (6.33), the line bundle \( K_C^{\otimes 2} \otimes L^\vee \) satisfies the same property. The family above, which has dimension at least five, is thus invariant under the involution \( L \mapsto K_C^{\otimes 2} \otimes L^\vee \) on \( P_{14}^8 \), whose fixed locus is the Prym moduli space \( \mathcal{R}_8 \).

**Proof.** This follows from Serre duality, replacing \( E \) with \( E^\vee \otimes K_C \) and \( E \otimes \epsilon \) by \( E^\vee \otimes \epsilon^\vee \otimes K_C \) plus the fact that \( \det (E^\vee \otimes K_C) \otimes \epsilon^\vee \approx K_C^{\otimes 2} \otimes \det E^\vee \otimes \epsilon^\vee \).

One can ask in general the following question:

**Question 31.** Is the divisor \( \mathcal{R}_{093} \) on \( P_{14}^8 \) invariant under the involution \( L \mapsto K_C^{\otimes 2} \otimes L^\vee \)?

**References**


