# SYZYGIES OF PRYM AND PARACANONICAL CURVES OF GENUS 8

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## 1. Introduction

By analogy with Green's Conjecture on the syzygies of a general canonical curve [18], [19], the Prym-Green Conjecture, formulated in [10] and [3], predicts that the resolution of a *paracanonical* curve

$$\phi_{K_C\otimes\eta}: C \hookrightarrow \mathbf{P}^{g-2},$$

where $C$ is a general curve of genus $g$ and $\eta \in \mathrm{Pic}^0(C)[\ell]$ is an $\ell$-torsion point is natural. For even genus $g = 2i+6$, the Prym-Green Conjecture amounts to the vanishing statement

$$K_{i,2}(C, K_C\otimes\eta) = K_{i+1,1}(C, K_C\otimes\eta) = 0, \tag{1}$$

in terms of Koszul cohomology groups. Equivalently, the genus $g$ paracanonical level $\ell$ curve $C \subseteq \mathbf{P}^{g-2}$ satisfies the Green-Lazarsfeld property $(N_i)$. The Prym-Green Conjecture has been proved for all *odd* genera $g$ when $\ell = 2$, see [8], or $\ell \geq \sqrt{\frac{g+2}{2}}$, see [9]. For even genus, the Prym-Green Conjecture has been established by degeneration and using computer algebra tools in [3] and [4], for all $\ell \leq 5$ and $g \leq 18$, with two possible mysterious exceptions in level 2 and genus $g = 8, 16$ respectively. The last section of [3] provides various pieces of evidence, including a probabilistic argument, strongly suggesting that for $g = 8$, one has $\dim K_{1,2}(C, K_C \otimes \eta) = 1$, and thus the vanishing (1) fails in this case. It is tempting to believe that the exceptions $g = 8, 16$ can be extrapolated to higher genus, and that for genera $g$ with high divisibility by 2, there are genuinely novel ways of constructing syzygies of Prym-canonical curves waiting to be discovered. It would be very interesting to test experimentally the next relevant case $g = 24$. Unfortunately, due to memory and running time constraints, this is currently completely out of reach, see [3] and [7].

The aim of this paper is to confirm the expectation formulated in [3] and offer several geometric explanations for the surprising failure of the Prym-Green Conjecture in genus 8, hoping that the geometric methods described here for constructing syzygies of Prym-canonical curves will eventually shed light on all the exceptions to the Prym-Green Conjecture. We choose a general Prym-canonical curve of genus 8

$$\phi_{K_C\otimes\eta}: C \hookrightarrow \mathbf{P}^6,$$

with $\eta^{\otimes 2} = \mathcal{O}_C$. Set $L := K_C\otimes\eta$ and denote $I_{C,L}(k) := \mathrm{Ker}\big\{\mathrm{Sym}^k H^0(C,L) \to H^0(C, L^{\otimes k})\big\}$ for all $k \geq 2$. Observe that $\dim I_{C,L}(2) = \dim K_{1,1}(C,L) = 7$ and $\dim I_{C,L}(3) = 49$, therefore as $[C,\eta]$ varies in moduli, the multiplication map

$$\mu_{C,L}: I_{C,L}(2)\otimes H^0(C,L) \to I_{C,L}(3)$$

globalizes to a morphism of vector bundles of the same rank over the stack $\mathcal{R}_8$ classifying pairs $[C,\eta]$, where $C$ is a smooth curve of genus 8 and $\eta \in \mathrm{Pic}^0[2] \setminus \{\mathcal{O}_C\}$.

**Theorem 1.** *For a general Prym curve $[C,\eta] \in \mathcal{R}_8$, one has $K_{1,2}(C,L) \neq 0$. Equivalently the multiplication map $\mu_{C,L}: I_{C,L}(2)\otimes H^0(C,L) \to I_{C,L}(3)$ is not an isomorphism.*

We present three different proofs of Theorem 1. The first proof, presented in Section 3 uses the structure theorem already pointed out in [3] for degenerate syzygies of paracanonical curves in $\mathbf{P}^6$. Precisely, if a paracanonical genus 8 curve $\phi_{K_C\otimes\eta}: C \hookrightarrow \mathbf{P}^6$, where $\eta \neq \mathcal{O}_C$, has a syzygy $0 \neq \gamma \in K_{1,2}(C, K_C\otimes\eta)$ of *sub-maximal rank* (see Section 2 for a precise definition), then the syzygy scheme of $\gamma$ consists of an isolated point $p \in \mathbf{P}^6 \setminus C$ and a residual septic elliptic curve $E \subseteq \mathbf{P}^6$ meeting $C$ transversally along a divisor $e$ of degree 14, such that if $e$ is viewed as a divisor on $C$ and $E$ respectively, then

$$e_C \in |K_C \otimes \eta^{\otimes 2}| \quad \text{and} \quad e_E \in |\mathcal{O}_E(2)|. \tag{2}$$

The union $D := C \cup E \hookrightarrow \mathbf{P}^6$, endowed with the line bundle $\mathcal{O}_D(1)$ is a degenerate spin curve of genus 22 in the sense of [5]. The locus of stable spin structures with at least 7 sections defines a subvariety of codimension $21 = \binom{7}{2}$ inside the moduli space $\overline{\mathcal{S}}_{22}^-$ of stable odd spin curves of genus 22. By restricting this condition to the locus of spin structures having $D := C \cup_e E$ as underlying curve, it turns out that one has enough parameters to realize this condition for a general $C \subseteq \mathbf{P}^6$ if and only if

$$\dim|K_C \otimes \eta^{\otimes 2}| = 7,$$

which happens precisely when $\eta^{\otimes 2} \cong \mathcal{O}_C$. Therefore for each Prym-canonical curve $C \subseteq \mathbf{P}^6$ of genus 8 there exists a corresponding elliptic curve $E \subseteq \mathbf{P}^6$ such that the intersection divisor $E \cdot C$ verifies (2), which forces $K_{1,2}(C, K_C \otimes \eta) \neq 0$.

The second and the third proofs involve the reformulation given in Section 2.2 (see Proposition 5) of the condition that a paracanonical curve $\phi_L : C \hookrightarrow \mathbf{P}^6$ have a non-trivial syzygy. Precisely, if $\phi_L(C)$ is scheme-theoretically generated by quadrics, then $K_{1,2}(C, L) \neq 0$, if and only if there exists a quartic hypersurface in $\mathbf{P}^6$ singular along $C \subseteq \mathbf{P}^6$, which is not a quadratic polynomial in quadrics vanishing along $C$, that is, it does not belong to the image of the multiplication map

$$\mathrm{Sym}^2 I_{C,L}(2) \to I_{C,L}(4).$$

Equivalently, one has $H^1(\mathbf{P}^6, \mathcal{I}^2_{C/\mathbf{P}^6}(4)) \neq 0$.

The second proof presented in Section 4 uses intersection theory on the stack $\overline{\mathcal{R}}_8$. The virtual Koszul divisor of Prym curves $[C, \eta] \in \mathcal{R}_8$ having $K_{1,2}(C, K_C \otimes \eta) \neq 0$, splits into two divisors $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively, corresponding to the case whether $C \subseteq \mathbf{P}^6$ is not scheme-theoretically cut out by quadrics, or $H^1(\mathbf{P}^6, \mathcal{I}^2_{C/\mathbf{P}^6}(4)) \neq 0$ respectively. We determine the virtual classes of both closures $\overline{\mathfrak{D}}_1$ and $\overline{\mathfrak{D}}_2$. Using an explicit uniruled parametrization of $\overline{\mathcal{R}}_8$ constructed in [11], we conclude that the class $[\overline{\mathfrak{D}}_2] \in CH^1(\overline{\mathcal{R}}_8)$ cannot possibly be effective (see Theorem 20). Therefore, again $K_{2,1}(C, K_C \otimes \eta) \neq 0$, for every Prym curve $[C, \eta] \in \mathcal{R}_8$.

The third proof given in Section 5 even though subject to a plausible, but still unproved transversality assumption, is constructive and potentially the most useful, for we feel it might offer hints to the case $g = 16$ and further. The idea is to consider rank 2 vector bundles $E$ on $C$ with canonical determinant and $h^0(C, E) = h^0(C, E(\eta)) = 4$. (Note that the condition that $\eta$ is 2-torsion is equivalent to the fact that $E(\eta)$ also has canonical determinant, which is essential for the existence of such nonsplit vector bundles, cf. [15].) By pulling back to $C$ the determinantal quartic hypersurface consisting of rank 3 tensors in

$$\mathbf{P}\Big(H^0(C, E)^\vee \otimes H^0(C, E(\eta))^\vee\Big) \cong \mathbf{P}^{15}$$

under the natural map $H^0(C, K_C \otimes \eta)^\vee \to H^0(C, E)^\vee \otimes H^0(C, E(\eta))^\vee$, we obtain explicit quartic hypersurfaces singular along the curve $C \subseteq \mathbf{P}^6$. Our proof that these are not quadratic polynomials into quadrics vanishing along the curve, that is, they do not lie in the image of $\mathrm{Sym}^2 I_{C,L}(2)$ remains incomplete, but there is a lot of evidence for this.

The methods of Section 5 suggests the following analogy in the next case $g = 16$. If $[C, \eta] \in \mathcal{R}_{16}$ is a Prym curve of genus 16, there exist vector bundles $E$ on $C$ with $\det E \cong K_C$ and satisfying $h^0(C, E) = h^0(C, E(\eta)) = 6$. Potentially they could be used to prove that $K_{5,2}(C, K_C \otimes \eta) \neq 0$ and thus confirm the next exception to the Prym-Green Conjecture.

## 2. Syzygies of paracanonical curves of genus 8

Let $C$ be a general smooth projective curve of genus 8. For a non-trivial line bundle $\eta \in \mathrm{Pic}^0(C)$, we shall study the *paracanonical* line bundle $L := K_C \otimes \eta$. When $\eta$ is a 2-torsion point, we speak of the *Prym-canonical* line bundle $L$. For each paracanonical bundle

$L$, we have $h^0(C,L)=7$ and an induced embedding

$$\phi_L : C \hookrightarrow \mathbf{P}^6.$$

The goal is to understand the reasons for the non-vanishing of the Koszul group $K_{1,2}(C,L)$ of a Prym-canonical bundle $L$, as suggested experimentally by the results of [3], [4].

Let $I_C(2)=I_{C,L}(2)\subseteq H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(2))$, respectively $I_C(3)=I_{C,L}(3)\subseteq H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(3))$ be the ideal of quadrics, respectively cubics, vanishing on $\phi_L(C)$. It is well-known that whenever $L$ is projectively normal, the non-vanishing of the Koszul cohomology group $K_{1,2}(C,L)$ is equivalent to the non-surjectivity of the multiplication map

$$\mu_{C,L} : H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))\otimes I_C(2)\to I_C(3). \tag{3}$$

Note that

$$\dim\ I_C(2)=\binom{8}{2}-21=7,\quad \text{and}\quad \dim I_C(3)=\binom{9}{3}-3\cdot 14+7=49,$$

respectively, so that the two spaces appearing in the map (3) have the same dimension. Denote by $P_8^{14}$ the universal degree 14 Picard variety over $\mathcal{M}_8$ consisting of pairs $[C,L]$, where $[C]\in\mathcal{M}_8$ and $L\neq K_C$. The jumping locus

$$\mathfrak{Kosz} := \Big\{[C,L]\in P_8^{14} : K_{1,2}(C,L)\neq 0\Big\}$$

is a divisor. It turns out, cf. Theorem 5.3 of [3] and Proposition 8, that $\mathfrak{Kosz}$ splits into two components depending on the *rank* of the corresponding non-zero syzygy from $K_{1,2}(C,L)$.

**Definition 2.** The rank of a non-zero syzygy $\gamma=\sum_{i=0}^6 \ell_i\otimes q_i\in \mathrm{Ker}(\mu_{C,L})$ is the dimension of the subspace $\langle \ell_0,\ldots,\ell_6\rangle\subseteq H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))$. The syzygy scheme $\mathrm{Syz}(\gamma)$ of $\gamma$ is the largest subscheme $Y\subseteq \mathbf{P}^6$ such that $\gamma\in H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))\otimes I_Y(2)$.

It is shown in [3], that $\mathfrak{Kosz}$ splits into divisors $\mathfrak{Kosz}_6$ and $\mathfrak{Kosz}_7$, depending on whether the syzygy $0\neq\gamma\in \mathrm{Ker}(\mu_{C,L})$ has rank 6 or 7 respectively. By a specialization argument to irreducible nodal curves, it follows from [3] that $\mathcal{R}_8\nsubseteq \mathfrak{Kosz}_7$. A direct, more transparent proof of this fact will be given in Proposition 13.

2.1. **Paracanonical curves of genus 8 with special syzygies and elliptic curves.** We summarize a few facts already stated or recalled in Section 5 of [3] concerning rank 6 syzygies of paracanonical curves in $\mathbf{P}^6$. Very generally, let

$$\gamma=\sum_{i=1}^6 \ell_i\otimes q_i\in H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))\otimes H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(2))$$

be a rank 6 linear syzygy among quadrics in $\mathbf{P}^6$. The linear forms $\ell_1,\ldots,\ell_6$ define a point $p\in\mathbf{P}^6$. Following Lemma 6.3 of [16], there exists a skew-symmetric matrix of linear forms $A:=(a_{ij})_{i,j=1,\ldots,6}$, such that

$$q_i=\sum_{j=1}^6 \ell_j a_{ij}.$$

In the space $\mathbf{P}^{20}$ with coordinates $\ell_1,\ldots,\ell_6$ and $a_{ij}$ for $1\le i<j\le 6$, one considers the 15-dimensional variety $X_6$ defined by the 6 quadratic equations $\sum_{j=1}^6 \ell_j a_{ij}=0$, where $i=1,\ldots,6$ and by the cubic equation $\mathrm{Pfaff}(A)=0$ in the variables $a_{ij}$. The original space $\mathbf{P}^6$ embeds in $\mathbf{P}^{20}$ via evaluation. The syzygy scheme $\mathrm{Syz}(\gamma)$ is the union of the point $p$ and of the intersection $D$ of $\mathbf{P}^6$ with the variety $X_6$. It follows from Theorem 4.4 of [6], that for a general rank 6 syzygy $\gamma$ as above, $D\subseteq\mathbf{P}^6$ is a smooth curve of genus 22 and degree 21 such that $\mathcal{O}_D(1)$ is a theta characteristic.

In the case at hand, that is, when $[C,L]\in\mathfrak{Kosz}_6$, the curve $D$ must be reducible, for it has $C$ as a component. More precisely:

**Lemma 3.** *For a general paracanonical curve $C \subseteq \boldsymbol{P}^6$ having a rank 6 syzygy, the curve $D$ is nodal and consists of two components $C \cup E$, where $E \subseteq \boldsymbol{P}^6$ is an elliptic septic curve. Furthermore, $\mathcal{O}_D(2) = \omega_D$. The intersection $e := C \cdot E$, viewed as a divisor on $C$ satisfies $e_C \in |\mathcal{O}_C(2) \otimes K_C^{\vee}|$, and as a divisor on $E$, satisfies $e_E \in |\mathcal{O}_E(2)|$.*

**Remark 4.** Note that $C$ is Prym-canonical or canonical if and only if $e_C \in |K_C|$.

The construction above is reversible. Firstly, general element $[C, L] \in \mathfrak{Kosz}_6$ can be reconstructed as the residual curve of a reducible spin curve $D \subseteq \mathbf{P}^6$ of genus 22 containing an elliptic curve $E \subseteq \mathbf{P}^6$ with $\deg(E) = 7$ as a component such that the union of $D$ and some point $p \in \mathbf{P}^6 \setminus E$ is the syzygy scheme of a rank 6 linear syzygy among quadrics in $\mathbf{P}^6$.

Furthermore, given a reducible spin curve $D = C \cup_e E \subseteq \mathbf{P}^6$ of genus 22 as above, that is, with $\omega_D \cong \mathcal{O}_D(2)$, the genus 8 component $C$ has a nontrivial syzygy of rank 6 involving the quadrics in the 6-dimensional subspace $I_D(2) \subseteq I_C(2)$, see Lemma 29 for a proof of this fact.

2.2. **Syzygies and quartics singular along paracanonical curves.** We first discuss an alternative characterization of the non-surjectivity of the map $\mu_{C,L}$:

**Proposition 5.** *Assume the paracanonical curve $\phi_L(C)$ is projectively normal and scheme-theoretically cut out by quadrics. Then $K_{1,2}(C, L) \neq 0$ if and only if there exists a degree 4 homogeneous polynomial on $\boldsymbol{P}^6$, which vanishes to order at least 2 along $C$ but does not belong to the image of the multiplication map $\mathrm{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)$.*

*Proof.* We work on the variety $X \xrightarrow{\tau} \mathbf{P}^6$ defined as the blow-up of $\mathbf{P}^6$ along $\phi_L(C)$. Let $E$ be the exceptional divisor of the blow-up, and consider the line bundle $H := \tau^*\mathcal{O}_{\mathbf{P}^6}(2)(-E)$ on $X$. Its space of sections identifies to $I_C(2)$, and our assumption that $C$ is scheme-theoretically cut out by quadrics says equivalently that $H$ is a globally generated line bundle on $X$. The nonvanishing of $K_{1,2}(C, L)$ is equivalent to the non-surjectivity of the multiplication map

$$H^0(X, H) \otimes H^0(X, \tau^*\mathcal{O}(1)) \to H^0(X, H \otimes \tau^*\mathcal{O}(1)), \tag{4}$$

where we use the identification

$$H^0\big(X, H \otimes \tau^*\mathcal{O}(1)\big) = H^0\big(X, \tau^*\mathcal{O}(3)(-E)\big) = I_C(3).$$

As $H$ is globally generated by its space $W := I_C(2)$ of global sections, the Koszul complex

$$0 \to \bigwedge^7 W \otimes \mathcal{O}_X(-7H) \to \ldots \to \bigwedge^2 W \otimes \mathcal{O}_X(-2H) \to W \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X \to 0 \tag{5}$$

is exact. We now twist this complex by $\tau^*\mathcal{O}_{\mathbf{P}^6}(1)(H)$ and take global sections. The last map is then the multiplication map (4). The successive terms of this twisted complex are

$$\bigwedge^i W \otimes \mathcal{O}_X\big(\tau^*\mathcal{O}(1)\big)((-i+1)H),$$

for $0 \leq i \leq 7$. The spectral sequence abutting to the hypercohomology of this complex, that is 0, has

$$E_2^{0,0} = \mathrm{coker}\Big\{W \otimes H^0(X, \tau^*\mathcal{O}(1)) \to H^0(X, H \otimes \tau^*\mathcal{O}(1))\Big\} \tag{6}$$

and the terms $E_1^{i,-i-1}$ for $i < -1$ are equal to $\bigwedge^{-i} W \otimes H^{-i-1}\big(X, \tau^*\mathcal{O}(1)((i+1)H)\big)$. Similarly, we have

$$E_1^{i,-i} = \bigwedge^{-i} W \otimes H^{-i}\big(X, \tau^*\mathcal{O}(1)((i+1)H)\big).$$

**Lemma 6.** *(i) We have*

$$E_1^{i,-i-1} = \bigwedge^{-i} W \otimes H^{-i-1}\big(X, \tau^*\mathcal{O}(1)((i+1)H)\big) = 0, \tag{7}$$

*for $-i-1 = 5, \ldots, 1$.*

*(ii) For $-i-1=6$, that is, $i=-7$, we have*

$$E_1^{-7,6} = \bigwedge^7 W \otimes H^6\big(X, \tau^*\mathcal{O}(1)(-6H)\big) = \bigwedge^7 W \otimes I_C(4)_2^\vee, \tag{8}$$

*where $I_C(4)_2 \subseteq I_C(4)$ is the set of quartic polynomials vanishing at order at least 2 along $C$, and*

$$E_1^{-6,6} = \bigwedge^6 W \otimes H^6\big(X, \tau^*\mathcal{O}(1)(-5H)\big) = \bigwedge^6 W \otimes I_C(2)^\vee. \tag{9}$$

*(iii) We have $E_1^{i,-i} = 0$, for $-6 < i < 0$.*

*Proof.* (i) We want equivalently to show that

$$H^\ell(X, \tau^*\mathcal{O}(1)(-\ell H)) = 0, \quad \text{when } \ell = 5, \ldots, 1.$$

Recall that $H = \tau^*\mathcal{O}(2)(-E)$. Furthermore,

$$K_X = \tau^*\mathcal{O}_{\mathbf{P}^6}(-7)(4E). \tag{10}$$

So we have to prove that

$$H^\ell\big(X, \tau^*\mathcal{O}(-2\ell+1)(\ell E)\big) = 0, \ \text{ for } \ell = 5, \ldots, 1. \tag{11}$$

Examining the spectral sequence induced by $\tau$, and using the fact that

$$R^s\tau_*(\mathcal{O}_X(tE)) = 0$$

for $s \neq 0$, 4 and also for $s = 4$, $t \leq 4$, we see that for $1 \leq \ell \leq 4$,

$$H^\ell\big(X, \tau^*\mathcal{O}(-2\ell+1)(\ell E)\big) = H^\ell\big(\mathbf{P}^6, \mathcal{O}(-2\ell+1) \otimes R^0\tau_*\mathcal{O}_X(\ell E)\big).$$

For $1 \leq \ell \leq 4$, the right hand side is zero, because it is equal to $H^\ell\big(\mathbf{P}^6, \mathcal{O}(-2\ell+1)\big)$.

For $\ell = 5$, we have to compute the space $H^5(X, \tau^*\mathcal{O}(-9)(5E))$, which by Serre duality and by (10), is dual to the space

$$H^1(X, \tau^*\mathcal{O}(2)(-E)) = H^1(\mathbf{P}^6, \mathcal{O}(2) \otimes \mathcal{I}_C) = 0.$$

(ii) We have to compute the spaces $H^6(X, \tau^*\mathcal{O}(1)(-6H))$ and $H^6(X, \tau^*\mathcal{O}(1)(-5H))$. As $H := \tau^*\mathcal{O}(2)(-E)$, this is rewritten as $H^6(X, \tau^*\mathcal{O}(-11)(6E))$ and $H^6(X, \tau^*\mathcal{O}(-9)(5E))$ respectively. If we dualize using (10), we get

$$H^6\big(X, \tau^*\mathcal{O}(-11)(6E)\big)^\vee = H^0\big(X, \tau^*\mathcal{O}(4)(-2E)\big) = I_C(4)_2,$$

$$H^6\big(X, \tau^*\mathcal{O}(-9)(5E)\big)^\vee = H^0\big(X, \tau^*\mathcal{O}(2)(-E)\big) = I_C(2).$$

(iii) We have

$$E_1^{i,-i} = E_1^{-6,6} = \bigwedge^{-i} W \otimes H^{-i}\big(X, \tau^*\mathcal{O}(1)((i+1)H)\big) = \bigwedge^{-i} W \otimes H^{-i}\big(X, \tau^*\mathcal{O}(2i+3)((-i-1)E)\big).$$

For $1 \leq -i \leq 5$, we have $R^s\tau_*\mathcal{O}_X((-i-1)E) = 0$ unless $s = 0$. Furthermore, we have $R^0\tau_*\mathcal{O}_X((-i-1)E) = \mathcal{O}_{\mathbf{P}^6}$, so that

$$H^{-i}\big(X, \tau^*\mathcal{O}(2i+3)((-i-1)E)\big) = H^{-i}\big(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(2i+3)\big) = 0.$$

□

**Corollary 7.** *Only one $E_2^{p,q}$-terms of this spectral sequence is possibly nonzero in degree $-1$, namely*

$$E_2^{-7,6} = \mathrm{Ker}\Big\{\bigwedge^7 W \otimes I_C(4)_2^\vee \to \bigwedge^6 W \otimes I_C(2)^\vee\Big\}. \tag{12}$$

*Furthermore, all the differentials $d_r$ starting from $E_2^{-7,6}$ vanish for $2 \leq r < 7$.*

Note that the map

$$\bigwedge^7 W \otimes I_C(4)_2^\vee \to \bigwedge^6 W \otimes I_C(2)^\vee$$

is nothing but the transpose of the multiplication map

$$W \otimes I_C(2) \to I_C(4)_2,$$

up to trivialization of $\bigwedge^7 W$. It follows that

$$(E_2^{-7,6})^\vee = \mathrm{Coker}\Big\{W \otimes I_C(2) \to I_C(4)_2\Big\}. \tag{13}$$

Corollary 7 concludes the proof of the proposition since it implies that we have an isomorphism given by $d_7$ between (12) and (6), or a perfect duality between (12) and the cokernel (13). □

Proposition 5 has the following consequence. Recall that $P_8^{14}$ is the moduli space of pairs $[C, L]$, with $C$ being a smooth curve of genus 8 and $L \neq K_C$ a paracanonical line bundle.

**Proposition 8.** *The Koszul divisor $\mathfrak{Kosz}$ of $P_8^{14}$ is the union of two divisors, one of them being the set of pairs $[C, L]$ such that $\phi_L(C)$ is not scheme-theoretically cut out by quadrics, the other being the set of pairs $[C, L]$ such that $H^1(\boldsymbol{P}^6, \mathcal{I}_C^2(4)) \neq 0$, or equivalently, such that there exists a quartic which is singular along $\phi_L(C)$ but does not lie in* $\mathrm{Sym}^2 I_C(2)$.

*Proof.* We first have to prove that the locus of pairs $[C, L]$ such that $\phi_L(C)$ is not scheme-theoretically cut-out by quadrics is contained in the divisor $\mathfrak{Kosz}$. This is a consequence of the following lemmas:

**Lemma 9.** *If $L \neq K_C$ is a projectively normal paracanonical line bundle on a curve of genus 8, then $\phi_L(C)$ is scheme-theoretically cut out by cubics.*

*Proof.* We observe that the twisted ideal sheaf $\mathcal{I}_C(3)$ is regular in Castelnuovo-Mumford sense. Indeed, we have

$$H^i(\mathbf{P}^6, \mathcal{I}_C(3-i)) = H^{i-1}(C, L^{\otimes(3-i)})$$

for $i \geq 2$, and the right hand side is obviously 0 for $i-1 \geq 2$, and also 0 for $i-1 = 1$ since $H^1(C, L) = 0$ because $L \neq K_C$ and $\deg L = 2g-2$. For $i = 1$, we have

$$H^1(\mathbf{P}^6, \mathcal{I}_C(2)) = 0$$

by projective normality. Being regular, the sheaf $\mathcal{I}_C(3)$ is generated by global sections. □

**Corollary 10.** *If $C$, $L$ are as above, and $C$ is not scheme-theoretically cut out by quadrics, then the multiplication map*

$$I_C(2) \otimes H^0(\boldsymbol{P}^6, \mathcal{O}_{\boldsymbol{P}^6}(1)) \to I_C(3)$$

*is not surjective.*

To conclude the proof of the proposition, we just have to show that the sublocus of $P_8^{14}$ where $L$ is not projectively normal is not a divisor, since the statement of the proposition will be then an immediate consequence of Proposition 5. We argue along the lines of [12]. First of all, a line bundle $L$ of degree 14 is not generated by sections if and only if $L = K_C(-x+y)$ for some points $x, y \in C$. This determines a codimension 6 locus of $P_8^{14}$. Similarly $L$ is not very ample if and only if $L = K_C(-x-y+z+t)$, for some points $x$, $y$, $z$, $t$ of $C$, which is satisfied in a codimension 4 locus of $P_8^{14}$. Finally, assume $L$ is very ample but $\phi_L(C)$ is not projectively normal. Equivalently

$$\mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is not surjective, which means that there exists a rank 2 vector bundle $F$ on $C$ which is a nontrivial extension

$$0 \longrightarrow K_C \otimes L^\vee \longrightarrow F \longrightarrow L \longrightarrow 0,$$

such that $h^0(C,F)=7$. If $x$, $y$, $z\in C$, there exists a nonzero section $\sigma\in H^0(C,F)$ vanishing on $x$, $y$ and $z$, and thus $F$ is also an extension

$$0\longrightarrow D\longrightarrow F\longrightarrow K_C\otimes D^\vee\longrightarrow 0, \tag{14}$$

where $D$ is a line bundle such that $h^0(C,D(-x-y-z))\neq 0$, and $h^0(C,L\otimes D^\vee)\neq 0$. We thus have $h^0(C,D)+h^0(C,K_C\otimes D^\vee)\geq 7$ and $\mathrm{Cliff}(D)\leq 2$. As $D$ is effective of degree at least 3, one has the following possibilities:

a) $h^0(C,K_C\otimes D^\vee)=0$, and then $D=L$, which contradicts the fact that the extension (14) is not split;

b) $h^0(C,K_C\otimes D^\vee)=1$ and $h^0(C,D)\geq 6$, and then $D=L(-x)$ and $h^0(K_C\otimes L^\vee(x))\neq 0$, so $L=K_C(x-y)$, which happens in a locus of codimension at least 6 in $P_8^{14}$;

c) $D$ contributes to the Clifford index of $C$. As the locus of curves $[C]\in\mathcal{M}_8$ with $\mathrm{Cliff}(C)\leq 2$ is of codimension 2 in $\mathcal{M}_8$, this situation does not occur in codimension 1. □

We shall need later on the following result:

**Lemma 11.** *Let $\phi_L: C\hookrightarrow \boldsymbol{P}^6$ be a projectively normal paracanonical curve of genus 8. If $C$ is scheme-theoretically cut out by quadrics, the multiplication map*

$$\mathrm{Sym}^2\ I_{C,L}(2)\to I_{C,L}(4) \tag{15}$$

*is injective.*

*Proof.* As the restriction map $\phi_L^*: H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(2))\to H^0(C,L^{\otimes 2})$ is surjective, its kernel $I_{C,L}(2)$ is of dimension 7. Let as before $\tau: X\to\mathbf{P}^6$ be the blow-up of $\mathbf{P}^6$ along $\phi_L(C)$, and let $E$ be its exceptional divisor. We view $I_{C,L}(2)$ as $H^0(X,\tau^*\mathcal{O}(2)(-E))$ and our assumption is that $I_{C,L}(2)$ generates the line bundle $H:=\tau^*\mathcal{O}(2)(-E)$ everywhere on $X$. Thus $I_{C,L}(2)$ provides a morphism

$$\psi: X\to\mathbf{P}(I_{C,L}(2)). \tag{16}$$

Now we have $\deg c_1(H)^6\neq 0$ by Sublemma 12 below, and thus the morphism $\psi$ has to be generically finite, hence dominant since both spaces have dimension 6. It is then clear that the pull-back map

$$\psi^*: H^0\big(\mathbf{P}(I_{C,L}(2)),\mathcal{O}(2)\big)\to H^0(X,H^{\otimes 2})$$

is injective. On the other hand, this morphism is nothing but the map (15). □

**Sublemma 12.** *With the same notation as above, we have*

$$\deg c_1(H)^6=8. \tag{17}$$

*Proof.* We have

$$c_1(H)^6=\sum_i\binom{6}{i}(-2)^i h^i\cdot E^{6-i},$$

where $h:=\tau^*c_1(\mathcal{O}_{\mathbf{P}^6}(1))$, and

$$h^i\cdot E^{6-i}=0$$

for $i\neq 6$, 1, 0. Furthermore

$$h^6=1,\ \text{and}\ h\cdot E^5=\deg\phi_L(C)=14$$

and $E^6=c_1(N_C)$. By adjunction formula

$$\deg c_1(N_C)=7\deg\phi_L(C)+\deg K_C=8\cdot 14.$$

It follows that

$$\deg c_1(H)^6=64-6\cdot 28+8\cdot 14=8,$$

which proves (17). □

Proposition 5 and Lemma 11 describe precisely the splitting of the Koszul divisor $\mathfrak{Kosz}$ into the divisors $\mathfrak{Kosz}_6$ and $\mathfrak{Kosz}_7$ corresponding to paracanonical curves $[C, L] \in P_8^{14}$ having a non-zero syzygy $\gamma \in K_{1,2}(C, L)$ of rank 6 or respectively 7. Precisely, $\mathfrak{Kosz}_6$ is a unirational divisor (cf. [3] Theorem 5.3) consisting of those paracanonical curves $C \subseteq \mathbf{P}^6$ for which $H^1(\mathbf{P}^6, \mathcal{I}_C^2(4)) \neq 0$. The divisor $\mathfrak{Kosz}_7$ consists of paracanonical curves $C \subseteq \mathbf{P}^6$ which are not scheme-theoretically cut out by quadrics.

## 3. First proof: reducible spin curves

3.1. **The syzygy is degenerate.** The first observation is the following result (already observed experimentally in [3]), which turns out to be useful for the description given below of the general paracanonical curve of genus 8 with nontrivial syzygies.

**Proposition 13.** *Let $C \subseteq \boldsymbol{P}^6$ be a smooth paracanonical curve of genus 8 and degree 14, scheme-theoretically generated by quadrics. Then a nontrivial syzygy*

$$\gamma \in \operatorname{Ker}\left\{I_C(2) \otimes H^0(\mathcal{O}_{\boldsymbol{P}^6}(1)) \to I_C(3)\right\}$$

*must be degenerate, that is of rank at most 6.*

*Proof.* We use the morphism

$$\psi : X \to \mathbf{P}(I_C(2))$$

introduced in (16), where $\tau : X \to \mathbf{P}^6$ is the blow-up of $C$ with exceptional divisor $E$, and $H := \tau^*\mathcal{O}_{\mathbf{P}^6}(-2E)$. This gives us a morphism

$$(\tau, \psi) : X \to \mathbf{P}^6 \times \mathbf{P}^6$$

which is of degree 1 on its image, and the syzygy $\gamma$ induces a hypersurface $Y$ of bidegree $(1, 1)$ in $\mathbf{P}^6 \times \mathbf{P}^6$ containing the 6-dimensional variety $(\tau, \psi)(X)$. Assume to the contrary that $\gamma$ has maximal rank 7, or equivalently that $Y$ is smooth. Then by the Lefschetz Hyperplane Restriction Theorem, the restriction map $H^{10}(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z}) \to H^{10}(Y, \mathbb{Z})$ is surjective, so that $[(\tau, \psi)(X)]_Y \in H^{10}(Y, \mathbb{Z})$ is the restriction of a class $\beta \in H^{10}(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z})$, which implies that

$$[(\tau, \psi)(X)] = \beta \cdot [Y] \text{ in } H^{12}(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z}), \tag{18}$$

where $[Y] \in H^2(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z})$ is the class of $Y$, that is $h_1 + h_2$, with $h_i$ for $i = 1, 2$ being the pull-backs of the hyperplane classes on each factor. Note that $H^{12}(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z})$ is the set of degree 6 homogeneous polynomials with integral coefficients in $h_1$ and $h_2$. We now have:

**Lemma 14.** *An element $\alpha \in H^{12}(\boldsymbol{P}^6 \times \boldsymbol{P}^6, \mathbb{Z})$ is of the form $(h_1 + h_2) \cdot \beta$ if and only if it satisfies the condition*

$$\sum_{i=0}^{6} (-1)^i h_1^i \cdot h_2^{6-i} \cdot \alpha = 0 \text{ in } H^{24}(\boldsymbol{P}^6 \times \boldsymbol{P}^6, \mathbb{Z}) = \mathbb{Z}. \tag{19}$$

*Proof.* We have $(h_1 + h_2) \cdot \left(\sum_i (-1)^i h_1^i \cdot h_2^{6-i}\right) = 0$ in $H^{14}(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z})$, so one implication is obvious. That the two conditions are equivalent then follows from the fact that both conditions determine a saturated corank 1 sublattice of $H^{12}(\mathbf{P}^6 \times \mathbf{P}^6, \mathbb{Z})$. □

To conclude that $\gamma$ has to be degenerate, in view of Lemma 14, it suffices to prove that the class $[(\tau, \psi)(X)]$ does not satisfy (19). Since $(\tau, \psi)^*h_1 = c_1(H)$ and $(\tau, \psi)^*h_2 = 2c_1(H) - E$, it is enough to prove that

$$\sum_{i=0}^{6} (-1)^i c_1(H)^i \cdot (2c_1(H) - E)^{6-i} \neq 0,$$

which follows from the computations made in the proof of Sublemma 12. □

3.2. **Syzygies and spin curves of genus 22 in $\mathbf{P}^6$.** Recall that $\overline{\mathcal{S}}_g^-$ denotes the moduli stack of odd stable spin curves of genus $g$, see [5] for details. We start with a nodal genus 22 spin curve of the form $[D := C \cup E, \vartheta] \in \overline{\mathcal{S}}_{22}^-$, where $C$ is a smooth genus 8 curve, $E$ is a smooth elliptic curve and $e := C \cap E$ consists of 14 distinct points, thus $p_a(D) = 22$. Assume $\vartheta \in \mathrm{Pic}^{21}(D)$ verifies $\vartheta^{\otimes 2} \cong \omega_D$, hence the restricted line bundles $\vartheta_E$ and $\vartheta_C$ have degrees 7 and 14 respectively. Furthermore, $h^0(E, \vartheta_E) = 7$, whereas $h^0(C, \vartheta_C) = 7$ if and only if $\vartheta_C \not\cong K_C$. The intersection divisor $e$ on the two components of $D$ is characterized by

$$e_C \in |\vartheta_C^{\otimes 2} \otimes K_C^\vee| \ \text{ and } \ e_E \in |\vartheta_E^{\otimes 2}|.$$

Note in particular that $e_C \in |K_C|$ if and only if $\vartheta_C^{\otimes 2} = K_C^{\otimes 2}$, that is $(C, \vartheta_C)$ is canonical or Prym canonical.

The line bundle $\vartheta$ on $D$ fits into the Mayer-Vietoris exact sequence:

$$0 \longrightarrow \vartheta \longrightarrow \vartheta_C \oplus \vartheta_E \stackrel{r}{\longrightarrow} \mathcal{O}_e(\vartheta) \longrightarrow 0,$$

where $r$ is defined by the isomorphisms on the fibers of $\vartheta_C$ and $\vartheta_E$ over the points in $e$. Given $\vartheta_C \in \mathrm{Pic}^{14}(C)$ with $\vartheta_C^{\otimes 2} = K_C(e)$ and $\vartheta_E \in \mathrm{Pic}^7(E)$ with $\vartheta_E^{\otimes 2} = \mathcal{O}_E(e)$, there is a finite number of stable spin curves $[D, \theta] \in \overline{\mathcal{S}}_{22}^-$ such that the restrictions of $\vartheta$ to $C$ and $E$ are isomorphic to $\vartheta_C$ and $\vartheta_E$ respectively. Passing to global sections in the Mayer-Vietoris sequence, we obtain the exact sequence:

$$0 \longrightarrow H^0(D, \vartheta) \longrightarrow H^0(C, \vartheta_C) \oplus H^0(E, \vartheta_E) \stackrel{r}{\longrightarrow} H^0(\mathcal{O}_e(\vartheta)) \longrightarrow \cdots . \tag{20}$$

Note that $r$ is represented by a $14 \times 14$ matrix and $h^0(D, \vartheta) = 14 - \mathrm{rk}(r)$. In the case of a reducible spin curve coming from the syzygy of a paracanonical genus 8 curve in $\mathfrak{Kosz}_6$, one has $h^0(D, \vartheta) = \mathrm{rk}(r) = 7$.

3.3. **Proof of Theorem 1 via reducible spin curves.** Theorem 1 states that every Prym canonical curve of genus 8 has a syzygy of rank 6. First we observe the existence of such a curve having the generic behavior described in Lemma 3.

**Lemma 15.** *There exists a curve $[C, \eta] \in \mathcal{R}_8$, whose Prym canonical model is scheme theoretically cut out by quadrics, and $K_{2,1}(C, K_C \otimes \eta)$ is 1-dimensional, generated by a syzygy $\gamma$ of rank 6. The syzygy scheme of $\gamma$ is the union of a point $p$ and a nodal curve $D = C \cup E$, such that $E$ is a smooth elliptic curve of degree 7 and $e := C \cdot E \in |K_C|$ consists of 14 mutually distinct points. Moreover, no cubic polynomial on $\boldsymbol{P}^6$ vanishes with multiplicity 2 along $C$.*

*Proof.* Examples of *singular* Prym canonical curves having all these properties have been produced in [3] Proposition 4.4 or [4]. A generic deformation in $\overline{\mathcal{R}}_8$ of these singular examples will provide the required smooth Prym canonical curve. □

*(First) proof of Theorem 1.* We denote by $X$ the moduli space of elements $[C, \eta, x_1, \ldots, x_{14}]$, where $[C, \eta] \in \mathcal{R}_8$ is a Prym curve of genus 8 and $x_i \in C$ are pairwise distinct points with $x_1 + \cdots + x_{14} \in |K_C| \cong \mathbf{P}^7$. Since the fibres of the forgetful map $X \to \mathcal{R}_8$ are 7-dimensional, it follows that $X$ is an irreducible variety of dimension 28.

Let $T$ be the locally closed parameter space of odd genus 22 spin curves having the form

$$\Bigl([D := C \cup_{\{x_1, \ldots, x_{14}\}} E, \vartheta] : [C] \in \mathcal{M}_8, \ \sum_{i=1}^{14} x_i \in |K_C|, \ [E, x_1, \ldots, x_{14}] \in \mathcal{M}_{1,14}, \ \vartheta^{\otimes 2} = \omega_D\Bigr).$$

Observe that points in $T$, apart from the spin structure $[D, \vartheta] \in \overline{\mathcal{S}}_{22}^-$ also carry an underlying Prym structure $[C, \eta := K_C \otimes \vartheta_C^\vee] \in \mathcal{R}_8$, for $\vartheta_C^{\otimes 2} \cong K_C(x_1 + \cdots + x_{14}) \cong K_C^{\otimes 2}$. One has an induced finite morphism $T \to X \times \mathcal{M}_{1,14}$, as well as a map $\mu : T \to \mathcal{R}_8$ forgetting the 14-pointed elliptic curve. It follows that $\dim T = \dim X + \dim \mathcal{M}_{1,14} = 42$. The locus

$$T_7 := \bigl\{[D, \vartheta] \in T : h^0(D, \vartheta) \geq 7\bigr\}$$

has the structure of a skew-symmetric degeneracy locus. Applying [13] Theorem 1.10, each component of $T_7$ has codimension at most $\binom{7}{2} = 21$ inside $T$, that is, $\dim(T_7) \geq \dim(\mathcal{R}_8)$.

By passing to a general 8-nodal Prym canonical curve $[C,\eta]$, following [3] Proposition 4.5, as well as Lemma 15, we have that $\dim K_{1,2}(C, K_C\otimes \eta)=1$. In particular, the fibre $\mu^{-1}([C,\eta])$ contains an isolated point, which shows that $T_7$ is non-empty and has a component which maps dominantly under $\mu$ onto $\mathcal{R}_8$. Theorem 1 now follows. □

**Remark 16.** The same construction can be carried out at the level of general paracanonical curves $[C,L]\in P_8^{14}$, where $L\in \mathrm{Pic}^{14}(C)-\{K_C\}$. The key difference is that we replace $T$ by a variety $T'$ parametrizing objects

$$\Bigl(\bigl[D:=C\cup_{\{x_1,\ldots,x_{14}\}} E, \vartheta, L\bigr] : [C,x_1,\ldots,x_{14}]\in \mathcal{M}_{14,8},\ L\in \mathrm{Pic}^{14}(C)-\{K_C\},$$

$$\sum_{i=1}^{14} x_i \in |L^{\otimes 2}\otimes K_C^{\vee}|,\ [E,x_1,\ldots,x_{14}]\in \mathcal{M}_{1,14}, \vartheta^{\otimes 2}=\omega_D\Bigr).$$

Similarly, we have a morphism $\mu' : T'\rightarrow P_8^{14}$ retaining the pair $[C,L]$ alone. The main difference compared to the Prym canonical case is that now

$$\dim |L^{\otimes 2}\otimes K_C^{\vee}|=6,$$

therefore $\dim(T')=\dim(P_8^{14})+\dim(\mathcal{M}_{1,14})+6=49$. The degeneracy locus $T_7'\subseteq T'$ defined by the condition $h^0(D,\vartheta)\geq 7$) has codimension 21 inside $T'$, that is,

$$\dim(T_7')=28=\dim(P_8^{14})-1.$$

It follows that the image $\mu'(T_7')\subseteq P_8^{14}$ has codimension 1, which is in accordance with $\mathfrak{Kos}_{36}$ being a divisor in $P_8^{14}$.

## 4. Second proof: Divisor class calculations on $\overline{\mathcal{R}}_g$

Recall [10] that $\overline{\mathcal{R}}_g$ is the Deligne-Mumford moduli space of Prym curves of genus $g$, whose geometric points are triples $[X,\eta,\beta]$, where $X$ is a quasi-stable curve of genus $g$, $\eta\in \mathrm{Pic}(X)$ is a line bundle of total degree 0 such that $\eta_E=\mathcal{O}_E(1)$ for each smooth rational component $E\subseteq X$ with $|E\cap \overline{X-E}|=2$ (such a component is said to be *exceptional*), and $\beta:\eta^{\otimes 2}\rightarrow \mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component is an isomorphism. If $\pi:\overline{\mathcal{R}}_g\rightarrow \overline{\mathcal{M}}_g$ is the map dropping the Prym structure, one has the formula

$$\pi^*(\delta_0)=\delta_0^{'}+\delta_0^{''}+2\delta_0^{\mathrm{ram}}\in CH^1(\overline{\mathcal{R}}_g), \tag{21}$$

where $\delta_0^{'}:=[\Delta_0^{'}]$, $\delta_0^{''}:=[\Delta_0^{''}]$, and $\delta_0^{\mathrm{ram}}:=[\Delta_0^{\mathrm{ram}}]$ are irreducible boundary divisor classes on $\overline{\mathcal{R}}_g$, which we describe by specifying their respective general points.

We choose a general point $[C_{xy}]\in \Delta_0\subset \overline{\mathcal{M}}_g$ corresponding to a smooth 2-pointed curve $(C,x,y)$ of genus $g-1$ and consider the normalization map $\nu: C\rightarrow C_{xy}$, where $\nu(x)=\nu(y)$. A general point of $\Delta_0^{'}$ (respectively of $\Delta_0^{''}$) corresponds to a pair $[C_{xy},\eta]$, where $\eta\in \mathrm{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta)\in \mathrm{Pic}^0(C)$ is non-trivial (respectively, $\nu^*(\eta)=\mathcal{O}_C$). A general point of $\Delta_0^{\mathrm{ram}}$ is a Prym curve of the form $(X,\eta)$, where $X:=C\cup_{\{x,y\}}\mathbf{P}^1$ is a quasi-stable curve with $p_a(X)=g$ and $\eta\in \mathrm{Pic}^0(X)$ is a line bundle such that $\eta_{\mathbf{P}^1}=\mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^2=\mathcal{O}_C(-x-y)$. In this case, the choice of the homomorphism $\beta$ is uniquely determined by $X$ and $\eta$. In what follows, we work on the partial compactification $\widetilde{\mathcal{R}}_g\subseteq \overline{\mathcal{R}}_g$ of $\mathcal{R}_g$ obtained by removing the boundary components $\pi^{-1}(\Delta_j)$ for $j=1,\ldots,\lfloor \frac{g}{2}\rfloor$, as well as $\Delta_0^{''}$. In particular, $CH^1(\widetilde{\mathcal{R}}_g)=\mathbb{Q}\langle \lambda, \delta_0^{'}, \delta_0^{\mathrm{ram}}\rangle$.

For a stable Prym curve $[X,\eta]\in \widetilde{\mathcal{R}}_g$, set $L:=\omega_X\otimes \eta\in \mathrm{Pic}^{2g-2}(X)$ to be the paracanonical bundle. For $i\geq 1$, we introduce the vector bundle $\mathcal{N}_k$ over $\widetilde{\mathcal{R}}_g$, having fibres

$$\mathcal{N}_k[X,\eta]=H^0(X,L^{\otimes k}).$$

The first Chern class of $\mathcal{N}_k$ is computed in [10] Proposition 1.7:

$$c_1(\mathcal{N}_k)=\binom{k}{2}\Bigl(12\lambda-\delta_0^{'}-2\delta_0^{\mathrm{ram}}\Bigr)+\lambda-\frac{k^2}{4}\delta_0^{\mathrm{ram}}. \tag{22}$$

Then we define the locally free sheaves $\mathcal{G}_k$ on $\widetilde{\mathcal{R}}_g$ via the exact sequences

$$0 \longrightarrow \mathcal{G}_k \longrightarrow \mathrm{Sym}^k \mathcal{N}_1 \longrightarrow \mathcal{N}_k \longrightarrow 0,$$

that is, satisfying $\mathcal{G}_k[X,\eta] := I_{X,L}(k) \subseteq \mathrm{Sym}^k H^0(X,L)$. Using (22) one computes $c_1(\mathcal{G}_k)$.

We also need the class of the vector bundle $\mathcal{G}$ with fibres

$$\mathcal{G}[X,\eta] = H^0(X, \omega_X^{\otimes 5} \otimes \eta^{\otimes 4}) = H^0(X, \omega_X \otimes L^{\otimes 4}).$$

**Lemma 17.** *One has* $c_1(\mathcal{G}) = 121\lambda - 10\delta_0^{'} - 24\delta_0^{\mathrm{ram}} \in CH^1(\widetilde{\mathcal{R}}_g)$.

*Proof.* We apply Grothendieck-Riemann-Roch to the universal Prym curve $f : \mathcal{C} \to \widetilde{\mathcal{R}}_g$. Denote by $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ the universal *Prym bundle*, whose restriction to each Prym curve is the corresponding 2-torsion point, that is, $\mathcal{L}_{|f^{-1}([X,\eta])} = \eta$, for each point $[X,\eta] \in \widetilde{\mathcal{R}}_g$. Since $R^1 f_*(\omega_f^{\otimes 5} \otimes \mathcal{L}^{\otimes 4}) = 0$, we write

$$c_1(\mathcal{G}) = f_*\Big[\Big(1+5c_1(\omega_f)+4c_1(\mathcal{L})+\frac{(5c_1(\omega_f)+4c_1(\mathcal{L}))^2}{2}\Big)\cdot\Big(1-\frac{c_1(\omega_f)}{2}+\frac{c_1^2(\Omega_f^1)+[\mathrm{Sing}(f)]}{12}\Big)\Big]_2.$$

We use then the formulas $f_*(c_1^2(\mathcal{L})) = -\delta_0^{\mathrm{ram}}/2$ and $f_*(c_1(\mathcal{L})\cdot c_1(\omega_f)) = 0$ (see [10], Proposition 1.6) coupled with Mumford's formula $f_*(c_1^2(\Omega_f^1) + [\mathrm{Sing}(f)]) = 12\lambda$ as well with the identity

$$\kappa_1 := f_*(c_1^2(\omega_f)) = 12\lambda - \delta_0^{'} - 2\delta_0^{\mathrm{ram}},$$

in order to conclude. □

The Koszul locus

$$\mathcal{Z}_8 := \mathfrak{Kosz} \cap \mathcal{R}_8 = \Big\{[C,\eta] \in \mathcal{R}_8 : K_{1,2}(C, K_C \otimes \eta) \neq 0\Big\}$$

is a virtual divisor on $\mathcal{R}_8$, that is, the degeneracy locus of a map between vector bundles of the same rank over $\widetilde{\mathcal{R}}_8$. If it is a genuine divisor (which we aim to rule out), the class of its closure in $\widetilde{\mathcal{R}}_8$ is given by [3] Theorem F:

$$[\overline{\mathcal{Z}}_8] = 27\lambda - 4\delta_0^{'} - 6\delta_0^{\mathrm{ram}} \in CH^1(\widetilde{\mathcal{R}}_8).$$

**Remark 18.** Some of the considerations above can be extended to higher order torsion points. We recall that $\mathcal{R}_{g,\ell}$ is the moduli space of pairs $[C,\eta]$, where $C$ is a smooth curve of genus $g$ and $\eta \in \mathrm{Pic}^0(C)$ is a non-trivial $\ell$-torsion point. It is then shown in [3] that the locus $\mathcal{Z}_{8,\ell} := \mathfrak{Kosz} \cap \mathcal{R}_{8,\ell} \subseteq P_8^{14}$ is a divisor on $\mathcal{R}_{8,\ell}$ for each other level $\ell \geq 3$. The class of the compactification of $\mathcal{Z}_{8,\ell}$ is given by the following formula, see [3] Theorem F:

$$[\overline{\mathcal{Z}}_{8,\ell}] = 27\lambda - 4\delta_0^{'} - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{4(a^2 - a\ell + \ell^2)}{\ell}\delta_0^{(a)} \in CH^1(\widetilde{\mathcal{R}}_{8,\ell}).$$

We refer to [3] Section 1.4, for the definition of the boundary divisor classes $\delta_0^{(a)}$, where $a = 1, \ldots, \lfloor \frac{\ell}{2} \rfloor$. If $\pi : \overline{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{M}}_g$ is the map forgetting the level $\ell$ structure, then

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + \ell \sum_{\ell=1}^{\lfloor \frac{\ell}{2} \rfloor} \delta_0^{(a)}.$$

We fix now a genus 8 Prym-canonically embedded curve $\phi_L : C \hookrightarrow \mathbf{P}^6$. As usual, we denote the kernel bundle by $M_L := \Omega^1_{\mathbf{P}^6|C}(1)$, hence we have the exact sequence

$$0 \longrightarrow N_C^{\vee}(4L) \longrightarrow M_L \otimes L^{\otimes 3} \longrightarrow K_C \otimes L^{\otimes 4} \longrightarrow 0. \qquad (23)$$

This can be interpreted as an exact sequence of vector bundles over $\widetilde{\mathcal{R}}_8$. Denoting by $\mathcal{H}$ the vector bundle over $\widetilde{\mathcal{R}}_8$ with fibres $H^0(C, N_C^{\vee} \otimes L^{\otimes 4})$, we compute using the previous formulas and the fact that $\mathrm{rk}(\mathcal{N}_1) = h^0(C,L) = 7$ and $\mathrm{rk}(\mathcal{N}_3) = h^0(C, L^{\otimes 3}) = 35$:

$$c_1(\mathcal{H}) = 35c_1(\mathcal{N}_1) + 7c_1(\mathcal{N}_3) - c_1(\mathcal{N}_4) - c_1(\mathcal{G}) = 100\lambda - 5\delta_0^{'} - \frac{53}{2}\delta_0^{\mathrm{ram}}. \tag{24}$$

Thus $\mathfrak{D}_1 = \mathfrak{Kosz}_7 \cap \mathcal{R}_8$ and $\mathfrak{D}_2 = \mathfrak{Kosz}_6 \cap \mathcal{R}_8$. We have already seen in Proposition 5 that $K_{1,2}(C, L) \neq 0$ if and only if either $\phi_L(C) \subseteq \mathbf{P}^6$ is not scheme-theoretically cut out by quadrics, or else, $H^1(\mathbf{P}^6, \mathcal{I}_C^2(4)) \neq 0$. We write

$$\mathcal{Z}_8 = \mathfrak{D}_1 + \mathfrak{D}_2, \quad \text{where}$$

$$\mathfrak{D}_1 := \Big\{[C, \eta] \in \mathcal{R}_8 : \phi_L(C) \subseteq \mathbf{P}^6 \text{ is scheme-theoretically not cut out by quadrics}\Big\}$$

and

$$\mathfrak{D}_2 := \Big\{[C, \eta] \in \mathcal{R}_8 : H^1(\mathbf{P}^6, \mathcal{I}_C^2(4)) \neq 0\Big\}.$$

We have already observed that dim $I_{C,L}(2) = 7$ and $\chi(\mathbf{P}^6, \mathcal{I}_C^2(4)) = 28$. If $\mathcal{Z}_8$ is a divisor, then $\mathfrak{D}_2$ is a divisor as well and for $[C, \eta] \in \mathcal{R}_8 \setminus \mathfrak{D}_2$, we have that

$$\dim \mathrm{Sym}^2 I_{C,L}(2) = \dim I_{C,L}(4)_2 = 28.$$

Paying some attention to its definition, the divisor $\mathfrak{D}_1$ can be thought as the degeneracy locus

$$\Big\{[C, \eta] \in \mathcal{R}_8 : \mathrm{Sym}^2 I_{C,L}(2) \stackrel{\neq}{\longrightarrow} I_{C,L}(4)_2\Big\},$$

which is an effective divisor on $\widetilde{\mathcal{R}}_8$. We compute the class of this divisor:

**Theorem 19.** *We have the following formulas:*

$$[\overline{\mathfrak{D}}_1] = 7\lambda - \frac{1}{2}\delta_0^{'} - \frac{3}{4}\delta_0^{\mathrm{ram}} \in CH^1(\widetilde{\mathcal{R}}_8)$$

*and*

$$[\overline{\mathfrak{D}}_2] = 20\lambda - \frac{7}{2}\delta_0^{'} - \frac{21}{4}\delta_0^{\mathrm{ram}} \in CH^1(\widetilde{\mathcal{R}}_8).$$

*Proof.* We first globalize over $\widetilde{\mathcal{R}}_8$ the following exact sequence:

$$0 \longrightarrow I_{C,L}(4)_2 \longrightarrow I_{C,L}(4) \longrightarrow H^0(C, N_C^{\vee}(4L)) \longrightarrow H^1(\mathbf{P}^6, \mathcal{I}_C^2(4)) \longrightarrow 0.$$

Denote by $\mathcal{A}$ the sheaf on $\widetilde{\mathcal{R}}_8$ supported along the divisor $\mathfrak{D}_2$, whose fibre over a general point of that divisor is equal to to $H^1(\mathbf{P}^6, \mathcal{I}_C^2(4))$. There is a surjective morphism of sheaves

$$\mathcal{H} \to \mathcal{A}$$

and denote by $\mathcal{G}_4'$ its kernel. Since $\mathcal{A}$ is locally free along $\mathfrak{D}_2$ and $\widetilde{\mathcal{R}}_8$ is a smooth stack, using the Auslander-Buchsbaum formula we find that $\mathcal{G}_4'$ is a locally free sheaf of rank equal to $\mathrm{rk}(\mathcal{H}) = \chi(C, N_C^{\vee}(4L)) = 19 \cdot 7$. Precisely, $\mathcal{G}_4'$ is an elementary transformation of $\mathcal{H}$ along the divisor $\mathfrak{D}_2$. Furthermore, $c_1(\mathcal{G}_4') = c_1(\mathcal{H}) - [\overline{\mathfrak{D}}_2]$.

The morphism $\mathcal{G}_4 \to \mathcal{H}$ globalizing the maps $I_{C,L}(4) \to H^0(C, N_C^{\vee}(4L))$ factors through the subsheaf $\mathcal{G}_4'$ and we form the exact sequence:

$$0 \longrightarrow \mathcal{G}_4^2 \longrightarrow \mathcal{G}_4 \longrightarrow \mathcal{G}_4' \longrightarrow 0.$$

The multiplication maps $\mathrm{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)_2$ globalize to a sheaf morphism

$$\nu : \mathrm{Sym}^2(\mathcal{G}_2) \to \mathcal{G}_4^2$$

between locally free sheaves of the same rank 28 over the stack $\widetilde{\mathcal{R}}_8$. The degeneration locus of $\nu$ is precisely the divisor $\overline{\mathfrak{D}}_1$. We compute:

$$c_1(\mathrm{Sym}^2(\mathcal{G}_2)) = 8c_1(\mathcal{G}_2) = 8(8c_1(\mathcal{N}_1) - c_1(\mathcal{N}_2)) = -40\lambda + 8(\delta_0^{'} + \delta_0^{\mathrm{ram}}),$$

and

$$c_1(\mathcal{G}_4^2) = 120c_1(\mathcal{N}_1) - c_1(\mathcal{N}_4) - c_1(\mathcal{H}) + [\overline{\mathfrak{D}}_2] = -53\lambda + 11\delta_0^{'} + \frac{25}{2}\delta_0^{\mathrm{ram}} + [\overline{\mathfrak{D}}_2].$$

We obtain the relation $[\overline{\mathfrak{D}}_1]-[\overline{\mathfrak{D}}_2]=-13\lambda+3\delta_0^{'}+\frac{9}{2}\delta_0^{\mathrm{ram}}$. Since at the same time

$$[\overline{\mathfrak{D}}_1]+[\overline{\mathfrak{D}}_2]=[\mathcal{Z}_8]=27\lambda-4\delta_0^{'}-6\delta_0^{\mathrm{ram}},$$

we solve the system and conclude. □

We are now in a position to give a second proof of Theorem 1:

**Theorem 20.** *The divisor class $[\overline{\mathfrak{D}}_2]$ cannot be effective. It follows that $\mathcal{Z}_8=\mathcal{R}_8$ and $K_{1,2}(C,K_C\otimes\eta)\neq 0$ for every Prym curve $[C,\eta]\in\mathcal{R}_8$.*

*Proof.* We use the sweeping curve of the boundary divisor $\Delta_0^{'}$ of $\widetilde{\mathcal{R}}_8$ constructed via Nikulin surfaces in [11] Lemma 3.2: Precisely, through the general point of $\Delta_0^{'}$ there passes a rational curve $\Gamma\subset\Delta_0^{'}$, entirely contained in $\widetilde{\mathcal{R}}_8$, having the following numerical characters:

$$\Gamma\cdot\lambda=8,\quad \Gamma\cdot\delta_0^{'}=42,\quad \text{and}\quad \Gamma\cdot\delta_0^{\mathrm{ram}}=8.$$

We note that $\Gamma\cdot\overline{\mathfrak{D}}_2<0$. Writing $\overline{\mathcal{D}}_2\equiv\alpha\cdot\delta_0^{'}+E$, where $\alpha\geq 0$ and $E$ is an effective divisor whose support is disjoint from $\Delta_0^{'}$, we immediately obtain a contradiction. □

The divisors $\mathfrak{D}_1$ and $\mathfrak{D}_2$ can be defined in an identical manner at the level of each moduli space $\overline{\mathcal{R}}_{8,\ell}$ of twisted level $\ell$ curves of genus $g$. As already pointed out, in the case $\ell\geq 3$ it follows from [3] Proposition 4.4 that both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are actual divisors. Repeating the same calculations as for $\ell=2$, we obtain the following formula on the partial compactification $\widetilde{\mathcal{R}}_{8,\ell}$ of $\mathcal{R}_{8,\ell}$:

$$[\overline{\mathfrak{D}}_2]=20\lambda-\frac{7}{2}\delta_0^{'}-\sum_{a=1}^{\lfloor\frac{\ell}{2}\rfloor}\frac{1}{2\ell}(7a^2-7a\ell+17\ell^2-20\ell)\delta_0^{(a)}\in CH^1(\widetilde{\mathcal{R}}_{8,\ell}).\tag{25}$$

As an application, we mention a different proof of one of the main results from [1]:

**Theorem 21.** *The canonical class of $\overline{\mathcal{R}}_{8,\ell}$ is big for $\ell\geq 3$. It follows that $\overline{\mathcal{R}}_{8,\ell}$ is a variety of general type for $\ell=3,4,6$.*

*Proof.* Using formula (25), it is a routine exercise to check that for $\ell\geq 3$ the canonical class computed in [3] Proposition 1.5

$$K_{\widetilde{\mathcal{R}}_{8,\ell}}=13\lambda-2\delta_0^{'}-(\ell+1)\sum_{a=1}^{\lfloor\frac{\ell}{2}\rfloor}\delta_0^{(a)}$$

can be written as a *positive* combination of the big class $\lambda$ and the effective class $[\overline{\mathfrak{D}}_2]$, hence it is big. Arguing along the lines of [3] Remark 3.5, it is easy to extend this result to the full compactification $\overline{\mathcal{R}}_{8,\ell}$ and deduce that $K_{\overline{\mathcal{R}}_{8,\ell}}$ is big.

To conclude that $\overline{\mathcal{R}}_{8,\ell}$ is of general type, one needs, apart from the bigness of the canonical class $K_{\widetilde{\mathcal{R}}_{8,\ell}}$ of the moduli stack, a result that the singularities of the coarse moduli space $\overline{\mathcal{R}}_{8,\ell}$ impose no adjunction conditions. This is only known for $2\leq\ell\leq 6,\ell\neq 5$, see [2]. □

## 5. Rank 2 vector bundles and singular quartics

Our goal in this section is to propose a construction of syzygies of Prym canonical curves of genus 8. We also sketch the proof of the fact that these syzygies are nontrivial. We fix again a general element $[C,\eta]\in\mathcal{R}_8$ and set $L:=K_C\otimes\eta$. According to Proposition 5, in order to prove that $K_{2,1}(C,L)\neq 0$, we have to produce quartic hypersurfaces in $\mathbf{P}^6$ which vanish at order at least 2 along $\phi_L(C)$, but do not lie in the image of the map $\mathrm{Sym}^2 I_{C,L}(2)\to I_{C,L}(4)$. The goal of this section is to produce such quartics from rank 2 vector bundles on $C$. The (incomplete) proof that the quartics we construct are not in the image of $\mathrm{Sym}^2 I_{C,L}(2)$ depends on an unproved general position statement (*), but there might be other approaches exploiting the fact that the hypersurfaces in question are determinantal.

The following construction produces quartics vanishing at order 2 along $C$. Let $E$ be a rank 2 vector bundle on $C$, with determinant $K_C$. Assume

$$h^0(C,E)=4,\ \ h^0(C,E(\eta))=4. \tag{26}$$

Setting $V_0:=H^0(C,E)$ and $V_1:=H^0(C,E(\eta))$, we have a natural map

$$V_0\otimes V_1\to H^0(C,L),$$

defined using evaluation and the following composite map:

$$H^0(E)\otimes H^0(E(\eta))\to H^0(E\otimes E(\eta))\cong H^0(\mathcal{E}nd\ E\otimes L)\xrightarrow{\mathrm{Tr}} H^0(C,L). \tag{27}$$

This map gives dually a morphism

$$H^0(C,L)^\vee\to V_0^\vee\otimes V_1^\vee,$$

(which will be proved below to be injective for a general choice of $E$). We consider the quartic hypersurface $D_4$ on $\mathbf{P}(V_0^\vee\otimes V_1^\vee)$ parametrizing tensors of rank at most 3.

**Lemma 22.** *The restriction $D_{4,E}$ of this quartic to $\boldsymbol{P}\big(H^0(C,L)^\vee\big)\subseteq\boldsymbol{P}\big(V_0^\vee\otimes V_1^\vee\big)$ is singular along the curve $C$.*

*Proof.* The quartic $D_4$ is singular along the set $T_2\subseteq\mathbf{P}(V_0^\vee\otimes V_1^\vee)$ of tensors of rank at most 2. The quartic $D_{4,E}$ in $\mathbf{P}(H^0(C,L)^\vee)$ is thus singular along $T_2\cap\mathbf{P}(H^0(C,L)^\vee)$, which obviously contains $C\subseteq\mathbf{P}\big(H^0(C,L)^\vee\big)$, since at a point $p\in C$, the map $V_0\otimes V_1\to H^0(C,L)$ composed with the evaluation at $p$ factors through $E_{|p}\otimes E(\eta)_{|p}$. □

By Brill-Noether theory, the variety $W^1_7(C)$ of degree 7 pencils on $C$ is 4-dimensional. There should thus exist finitely many elements $D\in W^1_7(C)$ with the property that

$$h^0(C,D)\geq 2,\ h^0(C,D\otimes\eta)\geq 2. \tag{28}$$

We now have the following lemma:

**Lemma 23.** *Let $[C,\eta]\in\mathcal{R}_8$ be as above and $D\in W^1_7(C)$ satisfying (28). Then*

*(i) $h^0(C,D)=2$ and $h^0(C,D\otimes\eta)=2$. The multiplication map*

$$\Big(H^0(C,D)\otimes H^0(C,K_C\otimes D^\vee)\Big)\oplus\Big(H^0(C,D\otimes\eta)\otimes H^0(C,K_C\otimes D^\vee\otimes\eta)\Big)\to H^0(C,K_C)$$

*is surjective (in fact, an isomorphism).*

*(ii) The multiplication map*

$$\Big(H^0(C,D)\otimes H^0(C,K_C\otimes D^\vee\otimes\eta)\Big)\oplus\Big(H^0(C,D\otimes\eta)\otimes H^0(C,K_C\otimes D^\vee)\Big)\to H^0(C,K_C(\eta))$$

*is surjective.*

*Proof.* This can be proved by a degeneration argument, for example by degenerating $C$ to the union of two curves of genus 4 meeting at one point. □

By Brill-Noether theory, the following corollary follows from (i) above:

**Corollary 24.** *For $[C,\eta]$ as above, the set of pencils $D\in W^1_7(C)$ satisfying (28) is finite.*

Given such a $D$, we form the rank 2 vector bundle

$$E=D\oplus(K_C\otimes D^\vee)$$

on $C$ which satisfies the conditions (26). The associated quartic is however not interesting for our purpose, due to the following fact:

**Lemma 25.** *The quartic on $\boldsymbol{P}(H^0(C,L)^\vee)$ associated to the vector bundle $D\oplus(K_C\otimes D^\vee)$ is the union of the two quadrics $Q_0$ and $Q_1$ associated respectively with the multiplication maps*

$$H^0(D)\otimes H^0((K_C\otimes D^\vee)(\eta))\to H^0(K_C(\eta))\ \textit{and}\ H^0(D(\eta))\otimes H^0(K_C\otimes D^\vee)\to H^0(K_C(\eta)).$$

*Both these quadrics contain $C$.*

*Proof.* Indeed we have in this case

$$V_0 = H^0(C,E) = H^0(C,D) \oplus H^0(C, K_C \otimes D^\vee), \text{ respectively}$$

$$V_1 = H^0(C,E(\eta)) = H^0(C, D\otimes \eta) \oplus H^0(C, K_C \otimes D^\vee \otimes \eta).$$

Furthermore, it is clear that the map of (27) factors through the projection

$$V_0 \otimes V_1 \to \Big(H^0(C,D)\otimes H^0\big(C, K_C\otimes D^\vee \otimes \eta)\Big) \oplus \Big(H^0(C,K_C\otimes D^\vee)\otimes H^0(C, D\otimes\eta)\Big)$$

and induces on each summand the multiplication map. The quadric $Q_0$ is by definition associated with the the multiplication map

$$\mu_0 : H^0(C,D)\otimes H^0(C,K_C\otimes D^\vee\otimes\eta) \to H^0(C, K_C\otimes \eta),$$

and is the set of elements $f$ in $\mathbf{P}(H^0(K_C\otimes\eta))^\vee$ such that $\mu_0^*(f)$ is a tensor of rank $\leq 1$. Similarly for $Q_1$, with $D$ being replaced with $D(\eta)$. Finally we use the fact that a tensor

$$(\mu_0^* f, \mu_1^* f) \in \Big(H^0(C,D)\otimes H^0(C,K_C\otimes D^\vee\otimes\eta)\Big) \oplus \Big(H^0(C,K_C\otimes D^\vee)\otimes H^0(C,D\otimes\eta)\Big)$$

has rank at most 3 if and only if one of $\mu_0^* f$ and $\mu_1^* f$ has rank at most 1. $\square$

We recall from [15] or [14] that the Brill-Noether condition $h^0(C,E)\geq 4$ imposes only $10 = \binom{5}{2}$ equations on the parameter space of rank 2 vector bundles $E$ with determinant $K_C$. As $\det E(\eta) \cong K_C$ as well, we conclude that the equations (26) impose only 20 conditions. As the moduli space $\mathcal{SU}_C(2,K_C)$ of semistable rank 2 vector bundles on $C$ having determinant $K_C$ has dimension $3g-3=21$, in our case we conclude that there is a positive dimensional family of such vector bundles on $C$ satisfying (26).

We now sketch the proof of the fact that for $C$ general of genus 8 and $D \in W^1_7(C)$ satisfying (28), for a general deformation $E$ of the vector bundle $D\oplus(K_C\otimes D^\vee)$ satisfying $\det E \cong K_C$ and $h^0(C,E)=4$, the associated quartic $D_{4,E}$ singular along $C$ is not defined by an element of $\mathrm{Sym}^2 I_C(2)$. Combined with Proposition 5, this provides a third approach to Theorem 1. The proof of this fact rests on an unproven general position statement (*), so it is incomplete.

*Sketch of proof of the nontriviality of the syzygy.* The vector bundle $E$ is generated by sections, as it is a general section-preserving deformation of the vector bundle

$$D\oplus (K_C\otimes D^\vee)$$

which is generated by global sections, and similarly for $E(\eta)$. Along $C \subseteq \mathbf{P}(H^0(C,L)^\vee)$, then the rational map

$$\mathbf{P}\big(H^0(C,L)^\vee\big) \dashrightarrow \mathbf{P}\big(H^0(E)^\vee \otimes H^0(E(\eta))^\vee\big)$$

is well-defined and the image of $C$ is contained in the locus $T_{2,E}$ of tensors of rank exactly 2. In fact, the case of $D\oplus(K_C\otimes D^\vee)$ shows that this map is a morphism for general $E$ (one just needs to know that $H^0(C,K_C\otimes\eta)$ is generated by the two vector spaces $H^0(D)\otimes H^0(K_C\otimes D^\vee\otimes\eta)$ and $H^0(D\otimes\eta)\otimes H^0(K_C\otimes D^\vee)$ respectively, or rather their images under the multiplication map. Note that on $T_{2,E}$, there is a rank 2 vector bundle $M$ which restricts to $E$ on $C$.

In the case of the split vector bundle $E_{\mathrm{sp}} = D\oplus(K_C\otimes D^\vee)$, Lemma 25 shows that the Zariski closure $\overline{T_{2,E_{\mathrm{sp}}}}$ parameterizing tensors of rank $\leq 2$ in $\mathbf{P}(H^0(C,L)^\vee)\subseteq \mathbf{P}(V_0^\vee\otimes V_1^\vee)$ is equal to the singular locus of $D_{4,E_{\mathrm{sp}}}$ and consists of the union of the two planes $P_0$, $P_1$ defined as the singular loci of the quadrics $Q_0$, $Q_1$ respectively, and the intersection $Q_0\cap Q_1$. The locus $\overline{T_{2,E_{\mathrm{sp}}}}\setminus T_{2,E_{\mathrm{sp}}}$ is the locus where the tensor has rank 1, and this happens exactly along the two conics $P_0\cap Q_1$ and $P_1\cap Q_0$. The curve $C$ is contained in $Q_0\cap Q_1$ and does not intersect $P_0\cup P_1$. In particular, the rational map $\phi : \widetilde{\mathbf{P}^6} \dashrightarrow \mathbf{P}^6$ given by the linear system $I_C(2)$ is well defined along $P_0\cup P_1$. We believe that the following general position statement concerning the two planes $P_i$ is true for general $C$ and $D$, $\eta$ as above.

(*) *The surfaces $\phi(P_i)$ are projectively normal Veronese surfaces, generating a hyperplane $\langle\phi(P_i)\rangle \subseteq \boldsymbol{P}^6$. Furthermore, the surface $\phi(P_0)\cup\phi(P_1) \subseteq \boldsymbol{P}^6$ is contained in a unique quadric in $\boldsymbol{P}^6$, namely the union of the two hyperplanes $\langle\phi(P_0)\rangle$ and $\langle\phi(P_1)\rangle$.*

We now prove that, assuming (*), for a general vector bundle $E$ as above, the associated quartic $D_{4,E}$ singular along $C$ is not defined by an element of $\mathrm{Sym}^2 I_C(2)$. As $P_0$, $P_1$ are 2-dimensional reduced components of $\overline{T_{2,E_{\rm sp}}}$, hence of the right dimension, the theory of determinantal hypersurfaces shows that for general $E$ as above, there is a reduced surface $\Sigma_E \subseteq \overline{T_{2,E}}$ whose specialization when $E = E_{\rm sp}$ contains $P_0 \cup P_1$. Let $\mathcal{E} \to C \times B$ be a family of vector bundles on $C$ parameterized by a smooth curve $B$, with general fiber $E$ and special fiber $E_{\rm sp}$. Denote by $\mathcal{E}_b$ the restriction of $\mathcal{E}$ to $C \times \{b\}$. Property (*) then implies that $\phi(\Sigma_{\mathcal{E}_b})$ for general $b \in B$ is contained in at most one quadric $Q_{\mathcal{E}_b}$ in $\mathbf{P}^6$. We argue by contradiction and assume that the quartic $D_{4,\mathcal{E}_b}$ is a pull-back $\phi^{-1}(Q)$ for general $b$. One thus must have $Q = Q_{\mathcal{E}_b}$. Next, the determinantal quartic $D_{4,\mathcal{E}_b}$ is singular along $T_{2,\mathcal{E}_b}$, hence along $\Sigma_{\mathcal{E}_b}$. Let $b \mapsto q_{\mathcal{E}_b} \in \mathrm{Sym}^2 I_C(2)$, where $q_{\mathcal{E}_b}$ is a defining equation for the quadric $Q_{\mathcal{E}_b}$. Then we find that the first order derivative of the family $\phi^* q_{\mathcal{E}_b}$ at $b_0$ also vanishes along $\Sigma_{\mathcal{E}_{b_0}}$, hence it must be proportional to $\phi^* q_{\mathcal{E}_{b_0}}$. We then conclude that the quadric $Q_{\mathcal{E}_b}$ is in fact constant, and thus must be equal to the quadric $Q_{E_{\rm sp}}$. We now reach a contradiction by proving the following lemma:

**Lemma 26.** *If the determinantal quartic $D_{4,\mathcal{E}_b}$ is constant, equal to $D_{\rm sp} = Q_0 \cup Q_1$, then the vector bundle $\mathcal{E}_b$ on $C$ does not deform with $b \in B$.*

*Proof.* Denoting $V_{0,b} := H^0(C, \mathcal{E}_b)$, $V_{1,b} := H^0(C, \mathcal{E}_b(\eta))$, we have the multiplication map

$$V_{0,b} \otimes V_{1,b} \to H^0(C, K_C \otimes \eta)$$

which is surjective for generic $b$ since it is surjective for $\mathcal{E}_0 = D \oplus (K_C \otimes D^\vee)$ (see Lemma 23). The determinantal quartic $D_{4,\mathcal{E}_b}$ is the vanishing locus of the determinant of the corresponding bundle map

$$\sigma_b : V_{0,b} \otimes \mathcal{O}_{\mathbf{P}(H^0(C,K_C(\eta))^\vee)} \to V_{1,b}^\vee \otimes \mathcal{O}_{\mathbf{P}(H^0(C,K_C(\eta))^\vee)}(1) \tag{29}$$

on $\mathbf{P}\big(H^0(C, K_C \otimes \eta)^\vee\big)$. We know that $D_{4,\mathcal{E}_b} = Q_0 \cup Q_1$ for any $b \in B$, where the quadrics $Q_i$ are singular (of rank 4), but with singular locus $P_i$ not intersecting $C \subseteq Q_0 \cap Q_1$. The morphism $\sigma_b$ has rank exactly 1 generically along each $Q_i$ and the kernel of $\sigma_{|D_{4,b}}$ determines a line bundle $\mathcal{K}_{i,b}$ on its smooth locus $Q_i \setminus P_i$. This line bundle is independent of $b$ since $\mathrm{Pic}(Q_i \setminus P_i)$ has no continuous part. The restriction of $\mathcal{K}_{i,b}$ to $C$ is thus constant. Finally, on the smooth part of $(Q_0 \cap Q_1)_{\rm reg}$, the kernel $\mathrm{Ker}(\sigma)$ contains the two line bundles $\mathcal{K}_{i,b|Q_0\cap Q_1}$. Restricting to $C \subseteq (Q_0 \cap Q_1)_{\rm reg}$, we conclude that $\mathrm{Ker}\,\sigma_{b|C}$ contains $\mathcal{K}_{i,0|C}$ for $i = 0, 1$. For $b = 0$, one has

$$\mathrm{Ker}\,\sigma_{0|C} = \mathcal{K}_{0,0|C} \oplus \mathcal{K}_{1,0|C}$$

and this thus remains true for general $b$. Finally, it follows from the construction and the fact that $\mathcal{E}_b$ is generated by its sections that $\mathrm{Ker}\,\sigma_{b|C} = \mathcal{E}_b^\vee$, which finishes the proof. □

□

## 6. Miscellany

6.1. **Extra remarks on the geometry of paracanonical curves of genus 8 with a nontrivial syzygy.** We now comment on an interesting rank 2 vector bundle appearing in our situation. Again, let $\phi_L : C \hookrightarrow \mathbf{P}^6$ be a paracanonical curve of genus 8. We assume $L$ is scheme-theoretically cut out by quadrics. Denoting by $N_C$ the normal bundle of $C$ in $\mathbf{P}^6$, we consider the natural map $I_C(2) \otimes \mathcal{O}_C \to N_C^\vee \otimes L^{\otimes 2}$ (which is surjective by our assumption) given by differentiation along $\phi_L(C)$, and let $F$ denote its kernel. We thus have the short exact sequence:

$$0 \longrightarrow F \longrightarrow I_C(2) \otimes \mathcal{O}_C \longrightarrow N_C^\vee \otimes L^{\otimes 2} \longrightarrow 0. \tag{30}$$

If $K_{1,2}(C,L)\neq 0$, the map $\mu : I_C(2)\otimes H^0(\mathbf{P}^6,\mathcal{O}(1))\to I_C(3)$ is not surjective, hence not injective. A fortiori, the map

$$\overline{\mu} : I_C(2)\otimes H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))\to H^0(C,N_C^\vee\otimes L^{\otimes 3})$$

induced by (30) is not injective, so that $h^0(C,F(L))\neq 0$. In fact, the equivalence between the statements $h^0(C,F(L))\neq 0$ and $K_{1,2}(C,L)\neq 0$ follows from the same argument once we know that there is no cubic polynomial on $\mathbf{P}^6$ vanishing with multiplicity 2 along $C$.

We observe now that $F$ is a vector bundle of rank 2 on $C$, with determinant equal to $\det N_C\otimes L^{\otimes(-2)}\cong K_C\otimes L^{\otimes(-3)}$. Hence if $F(L)$ has a nonzero section, assuming this section vanishes nowhere along $C$, then $F(L)$ is an extension of $K_C\otimes L^\vee$ by $\mathcal{O}_C$. This provides an extension class

$$e\in H^1(C,L\otimes K_C^\vee)=H^0(C,K_C^{\otimes 2}\otimes L^\vee)^\vee. \tag{31}$$

Assume now $L\otimes K_C^\vee=:\eta$ is a nonzero 2-torsion element of $\mathrm{Pic}^0(C)$. Then

$$e\in H^0(C,L)^\vee.$$

On the other hand, according to Theorem 20, there exists a nontrivial syzygy

$$\gamma=\sum_{i=1}^{6}\ell_i\otimes q_i\in K_{1,2}(C,L)=\mathrm{Ker}\big\{H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))\otimes I_C(2)\to I_C(3)\big\},$$

which is degenerate by Proposition 13. As we saw already, it has in fact rank 6 for generic $(C,\eta)$, hence determines a nonzero element

$$f\in H^0(\mathbf{P}^6,\mathcal{O}_{\mathbf{P}^6}(1))^\vee=H^0(C,L)^\vee=H^1(C,K_C\otimes L^\vee)=H^1(C,L\otimes K_C^\vee), \tag{32}$$

which is well-defined up to a coefficient.

**Proposition 27.** *The two elements $e$ and $f$ are proportional.*

*Proof.* Equivalently, we show that the kernels of the two linear forms $e$, $f\in H^0(C,L)^\vee$ are equal. Viewing $\gamma$ as an element of $\mathrm{Hom}\,(I_C(2)^\vee,H^0(C,L))$, we have $\mathrm{Ker}(f)=\mathrm{Im}(\gamma)$. On the other hand, the kernel of $e$ identifies with

$$\mathrm{Im}\Big\{j : H^0(C,F\otimes L^{\otimes 3}\otimes K_C^\vee)\to H^0(C,L)\Big\},$$

where the map $j$ is obtained by twisting the exact sequence $0\to\mathcal{O}_C\to F(L)\to K_C\otimes L^\vee\to 0$ by $K_C$. We have $F\otimes L^{\otimes 3}\otimes K_C^\vee\cong F^\vee$ since $\det F\cong K_C\otimes L^{\otimes(-3)}$, hence there is a natural morphism

$$i^* : I_C(2)^\vee\otimes\mathcal{O}_C\to F^\vee\cong F(L^{\otimes 3}\otimes K_C^\vee)$$

dual to the inclusion $F\hookrightarrow I_C(2)\otimes\mathcal{O}_C$ of (30). The proposition follows from the following claim:

**Claim.** *The morphism $\alpha : I_C(2)^\vee\to H^0(C,L)$ is equal to $j\circ i^*$.*

Forgetting about the last identification $F^\vee\cong F\otimes L^{\otimes 3}\otimes K_C^\vee)$, the claim amounts to the following general fact: For an evaluation exact sequence on a variety $X$

$$0\longrightarrow G\longrightarrow W\otimes\mathcal{O}_X\longrightarrow M\longrightarrow 0$$

and for a section $s\in H^0(X,G(L))=H^0(X,\mathcal{H}om(G^\vee,L))$ giving an element

$$s'\in\mathrm{Ker}\Big\{W\otimes H^0(X,L)\to H^0(X,M\otimes L)\Big\}\subseteq\mathrm{Hom}\,(W^\vee,H^0(X,L)),$$

the induced map $s : H^0(X,G^\vee)\to H^0(X,L)$ composed with the map $W^\vee\to H^0(X,G^\vee)$ equals the map $s' : W^\vee\to H^0(X,L)$. □

6.2. **Further properties.** Using the exact sequence (30) in the general case of a genus 8 paracanonical curve $[C, L] \in P_8^{14}$, we obtain:

**Lemma 28.** *A section $s \in H^0(C, F(L)) \subseteq I_{C,L}(2) \otimes H^0(C, L) = \mathrm{Hom}\big(I_{C,L}(2)^\vee, H^0(C, L)\big)$ of rank 6, determines an element $e \in |2L - K_C|$.*

*Proof.* The multiplication by $s \in H^0(F(L)) \subseteq I_{C,L}(2) \otimes H^0(C, L) = H^0(I_{C,L}(2)^\vee \otimes L)$ determines the natural maps $F^\vee \to L$ and $g_s : I_C(2)^\vee \otimes \mathcal{O}_C \to L$ sitting in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}er(g_s) & \longrightarrow & I_C(2)^\vee \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 2L - K_C & \longrightarrow & F^\vee & \longrightarrow & L & \longrightarrow & 0 \end{array},$$

where $I_C(2)^\vee \otimes \mathcal{O}_C \to F^\vee$ is the dual of the natural inclusion of (30). Passing to global sections we get the inclusion $H^0(\mathcal{K}er(g_s)) = \mathrm{Ker}\big\{I_{C,L}(2)^\vee \to H^0(C, L)\big\} \hookrightarrow H^0(2L - K_C)$, which by hypothesis in 1-dimensional hence it defines an element $e \in |2L - K_C|$. $\square$

Via the exact sequence (30) we can also show directly the following result that has been used in Section 3:

**Lemma 29.** *If there is a spin curve $D = C \cup E \hookrightarrow \boldsymbol{P}^6$ of genus 22 and degree 21 containing the genus 8 paracanonical curve $[C, L]$ as in Lemma 3 , then $H^0C, (F(L)) \neq 0$. If there is no cubic polynomial on $\boldsymbol{P}^6$ vanishing with multiplicity 2 along $C$, then $K_{1,2}(C, L) \neq 0$.*

*Proof.* Let $e = C \cap E$ and recall $c_1(F) = -3L + K_C$ and $\mathcal{O}_C(e) = 2L - K_C$. Note that $I_D(2) \subseteq I_C(2)$ is 6-dimensional. Tensor then the first vertical exact sequence of the following diagram by $L$ and pass to global sections.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^\vee & \longrightarrow & I_D(2) \otimes \mathcal{O}_C & \longrightarrow & \mathcal{I}_D/(\mathcal{I}_D \cap \mathcal{I}_C^2)(2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & I_C(2) \otimes \mathcal{O}_C & \longrightarrow & N_C^\vee(2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_C(-e) & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_{C|e} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}.$$

$\square$

6.3. **Nontrivial syzygies of paracanonical curves via vector bundles.** We return to the proof of Theorem 20 given in Section 5. Consider now a general paracanonical curve $[C, K_C \otimes \eta] \in P_8^{14}$. For a rank 2 vector bundle on $C$ of degree 14, with noncanonical determinant, the equation $h^0(C, E) \geq 4$ imposes 16 conditions. Similarly, if $\epsilon \in \mathrm{Pic}^0(C)$, the equation $h^0(C, E \otimes \epsilon) \geq 4$ imposes 16 conditions on the parameter space of $E$. Given $C$, there are $29 = 4g - 3$ parameters for $E$, and $8 = g$ parameters for $\epsilon$. It follows that we have at least a 5-dimensional family of pairs $(E, \epsilon)$, such that

$$h^0(C, E) \geq 4 \ \text{ and } \ h^0(C, E \otimes \epsilon) \geq 4. \tag{33}$$

Furthermore, the construction of Section 5 (together with Proposition 5) shows that for a general triple $(C, E, \epsilon)$ as above, one has $K_{2,1}(C, L) \neq 0$, where $L := \det E \otimes \epsilon$. Assuming the map $(E, \epsilon) \mapsto L$ is generically finite on its image, we constructed in this way a five dimensional family of paracanonical line bundles $L \in \mathrm{Pic}^{14}(C)$ with a nontrivial syzygy: $K_{1,2}(C, L) \neq 0$. This family has the following property:

**Lemma 30.** *If $L = \det E \otimes \epsilon$, where $E$ satisfies (33), the line bundle $K_C^{\otimes 2} \otimes L^\vee$ satisfies the same property. The family above, which has dimension at least five, is thus invariant under the involution $L \mapsto K_C^{\otimes 2} \otimes L^\vee$ on $P_8^{14}$, whose fixed locus is the Prym moduli space $\mathcal{R}_8$.*

*Proof.* This follows from Serre duality, replacing $E$ with $E^\vee \otimes K_C$ and $E \otimes \epsilon$ by $E^\vee \otimes \epsilon^\vee \otimes K_C$ plus the fact that $\det(E^\vee \otimes K_C) \otimes \epsilon^\vee \cong K_C^{\otimes 2} \otimes \det E^\vee \otimes \epsilon^\vee$. $\square$

One can ask in general the following question:

**Question 31.** *Is the divisor* $\mathfrak{Kosz}$ *on* $P_8^{14}$ *invariant under the involution* $L \mapsto K_C^{\otimes 2} \otimes L^\vee$*?*

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