ON THE SMOOTHABILITY PROBLEM WITH RATIONAL COEFFICIENTS

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Abstract. We consider the problem of smoothing algebraic cycles with rational coefficients on smooth projective complex varieties up to homological equivalence. We show that a solution to this problem would be incompatible with the validity of the Hartshorne conjecture on complete intersections in projective space. We also solve unconditionally a symplectic variant of this problem.

Introduction

0.1. **Smoothing cycles.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. For $c \geq 0$, consider the subgroup $H^{2c}(X,\mathbb{Z})_{\mathrm{alg}}$ of $H^{2c}(X,\mathbb{Z})$ generated by fundamental classes of (closed irreducible algebraic) subvarieties of codimension c in X. Borel and Haefliger have asked in [BH61, Section 5.17] whether $H^{2c}(X,\mathbb{Z})_{\mathrm{alg}}$ is generated by classes of *smooth* subvarieties. In other words, is it possible to smooth algebraic cycles up to homological equivalence?

The strongest result in that direction, due to Kollár and Voisin [KV23, Theorem 1.2] (following an earlier work of Hironaka [Hir68, Theorem, Section 5, p. 50]), states that the answer to this question is positive in the range $c > \frac{n}{2}$. Additional positive results have been obtained by Kleiman [Kle69, Theorem 5.8] when $c = 2$ and $n \in \{4, 5\}$.

It was however discovered by Hartshorne, Rees and Thomas [HRT74, Theorem 1] that the answer to the question of Borel and Haefliger is negative in general. Further counterexamples have appeared in [Deb95, Théorème 6], in [Ben20, Theorem 0.3] (which includes examples in the threshold case $c = \frac{n}{2}$), and in [BD23, Theorem 1.2].

0.2. **Rational coefficients.** We shall focus on the rational analogue of this question.

Question 0.1. *Let X be a smooth projective variety over $\mathbb{C}$. Fix $c \geq 0$. Is the $\mathbb{Q}$-vector space $H^{2c}(X,\mathbb{Q})_{\mathrm{alg}}$ generated by classes of smooth subvarieties of codimension c in X?*

The above question was investigated by Kleiman in [Kle69] by exploiting the formula $c_c(\mathcal{O}_Z) = (-1)^c(c-1)!\,[Z]$, a well-known consequence of the Grothendieck–Riemann–Roch theorem valid for all codimension c subvarieties $Z \subset X$. Using this formula, one can reduce Question 0.1 to the case of Chern classes of vector bundles, which come by pull-back from Grassmannians (see the discussion in [Kle69, p. 282]). On the one hand, this reduces Question 0.1 to the case where X is a Grassmannian. On the other hand, combined with an analysis of the singularities of the Schubert subvarieties of Grassmannians, this line of reasoning allowed Kleiman to show that Question 0.1 has a positive answer when $c > \frac{n}{2} - 1$ (see [Kle69, Theorem 5.8]).

In constrast with the integral case, no counterexample to Question 0.1 has been discovered to date. It is not even known whether the original counterexample of Hartshorne, Rees and Thomas to the integral question – the second Chern class of the tautological bundle on the Grassmannian $G(3,6)$ – is a $\mathbb{Q}$-linear combination of classes of smooth subvarieties.

0.3. **Relation with Harthorne's conjecture.** In the influential article [Har74], Hartshorne introduced the following conjecture (in a more precise form: there, the explicit bound $n > 3c$ is suggested).

Conjecture 0.2. *Let X be a smooth subvariety of codimension c in $\mathbb{P}^n$. If $n \gg c$, then X is a complete intersection.*

Our first main result, proven in Section 1, is as follows.

Theorem 0.3. *Question 0.1 and Conjecture 0.2 cannot both have positive answers (even for $c = 2$).*

As both Question 0.1 and Conjecture 0.2 are in some sense tameness statements concerning algebraic cycles, we find it surprising that they cannot hold simultaneously.

More precisely, we will show that if Conjecture 0.2 holds for $c = 2$, then Question 0.1 fails to hold for codimension 2 cycles on $X = G(k, n)$ whenever $k, n-k \gg 0$ (see Corollary 1.3). Our strategy of proof is as follows. A smooth codimension 2 subvariety $Z \subset G(k,n)$ is the zero locus of a section of a rank 2 vector bundle E on $G(k,n)$ (by Serre's construction and a Barth–Lefschetz-type theorem). We embed both $\mathbb{P}^k$ and $\mathbb{P}^{n-k}$ in $G(k,n)$, in many different ways. According to an equivalent formulation of Conjecture 0.2 for $c = 2$ (see Conjecture 1.1 below), the restrictions of E to all these copies of projective space should split as direct sums of line bundles. The heart of the proof, to which §1.2 is devoted, is to deduce that E itself splits as a direct sum of line bundles. This information constrains $c_2(E)$, hence the cycle class of Z, and this contradicts the validity of Question 0.1.

The important particular case of unstable vector bundles is considered in §1.3. Weaker results concerning the splitting of vector bundles of rank ≥ 3 as direct sums of line bundles, obtained following the same strategy, appear in §1.4.

0.4. **A symplectic analogue.** Our second main result answers positively the symplectic counterpart of Question 0.1.

Theorem 0.4. *Let (M, ω) be a compact symplectic $\mathcal{C}^\infty$ manifold. For all $c \geq 0$, the $\mathbb{Q}$-vector space $H^{2c}(M, \mathbb{Q})$ is generated by fundamental classes of symplectic $\mathcal{C}^\infty$ submanifolds of M of (real) codimension $2c$.*

The proof of Theorem 0.4 is given in Section 2. We construct topological complex vector bundles on M by means of Sullivan's rational homotopy theory (in §2.1), and we apply the Auroux–Donaldson theorem to find sections of these bundles whose zero loci are symplectic submanifolds (in §2.2).

We claim that Theorem 0.4 shows that there are no purely topological obstructions to the validity of Question 0.1, in a very strong sense (and therefore, in the light of Theorem 0.3, that it can be viewed as a negative indication concerning the Hartshorne conjecture). To justify this claim, consider Question 0.1 in the crucial case (as we explained above) where $X = G(k, n)$ is a Grassmannian.

As $H^*(X, \mathbb{Z})$ is then entirely algebraic, Question 0.1 predicts that $H^*(X, \mathbb{Q})$ is generated as a $\mathbb{Q}$-vector space by classes of smooth algebraic subvarieties. For this to be true, it is necessary that $H^*(X, \mathbb{Q})$ be generated by classes of orientable $\mathcal{C}^\infty$ submanifolds. With integral coefficients, this is not always true, and this was the original topological obstruction to the question of Borel and Haefliger exploited by Harthorne, Rees and Thomas in [HRT74]. In contrast, this necessary condition always holds with rational coefficients, as was shown by Thom [Tho54, Corollaire II.30].

One can devise finer possible topological obstructions to the validity of Question 0.1. If $H^*(X, \mathbb{Q})$ were generated by classes of smooth algebraic subvarieties, then it would a fortiori be generated by classes of orientable $\mathcal{C}^\infty$ submanifolds $Y \subset X$ whose normal bundle $N_{Y/X}$ has a complex structure (with integral coefficients, this obstruction is considered in [HRT74, §3]), whose tangent bundle T_Y also has a complex structure, and such that moreover there is an isomorphism of complex topological vector bundles $N_{Y/X} \oplus T_Y \simeq T_X|_Y$.

Theorem 0.4 dashes any hope that such topological obstructions might lead to a counterexample to Question 0.1. To see it, fix a (Kähler) symplectic form ω on X, and let $Y \subset X$ be a symplectic $\mathcal{C}^\infty$ submanifold. Identify $N_{Y/X}$ with the ω-orthogonal complement of T_Y in $TX|_Y$. Choose ω-compatible complex structures on $N_{Y/X}$ and T_Y (they exist by [MS74, Proposition 2.6.4 (i)]). The uniqueness of ω-compatible complex structures up to homotopy (apply [MS74, Proposition 2.6.4 (i)] again) shows that $N_{Y/X} \oplus T_Y$ and $T_X|_Y$ are isomorphic as topological complex vector bundles.

1. Relation between the Borel–Haefliger question and the Hartshorne conjecture

We work over the field $\mathbb{C}$ of complex numbers. Let $G(k, V)$ be the Grassmanniann of vector subspaces of dimension k of a complex vector space V, and set $G(k, n) := G(k, \mathbb{C}^n)$. A vector bundle on an algebraic variety is said to be *decomposable* if it is a direct sum of line bundles.

1.1. Statement.

Thanks to the Barth–Lefschetz theorem [Bar70, Lar73] and Serre's construction, the particular case $c = 2$ of Hartshorne's Conjecture 0.2 is equivalent to the following conjecture (see [Har74, Conjecture 6.3] and the discussion surrounding it).

Conjecture 1.1. *There exists an integer $n_0 \geq 5$ such that any rank 2 vector bundle on $\mathbb{P}^n$, with $n \geq n_0$, is decomposable.*

Hartshorne's original conjecture predicts that one can choose $n_0 = 7$, but no counterexample for $n_0 = 5$ is known. We prove the following implication.

Proposition 1.2. *Assuming Conjecture 1.1, if $k, n-k \geq n_0$, any rank 2 vector bundle on $G(k, n)$ is decomposable.*

Let $\mathcal{L}$ be the Plücker line bundle on $G(k,n)$.

Corollary 1.3. *If Conjecture 1.1 holds true, any smooth codimension* 2 *subvariety* Z *of the Grassmannian* $G(k,n)$ *with* $k, n-k \geq n_0$ *is a complete intersection and its cohomology class* $[Z]$ *is thus proportional to* $c_1(\mathcal{L})^2$*. In particular, classes of smooth codimension* 2 *subvarieties do not generate over* $\mathbb{Q}$ *the space* $H^4(G(k,n),\mathbb{Q})$*.*

Proof. Since $n_0 \geq 5$, one has $n \geq 10$. A Barth–Lefschetz-type theorem due to Sommese (see [Som82, Proposition 3.4 and (3.6.3)]) therefore shows that if Z as in the corollary, then the restriction maps $H^l(G(k,n),\mathbb{Z}) \to H^l(Z,\mathbb{Z})$ are bijective for $l \leq 2$, and hence that $\mathrm{Pic}(Z)$ generated by $\mathcal{L}_{|Z}$. In particular Z is subcanonical and, as $H^2(G(k,n),\mathcal{L}^i) = 0$ for all i, the Serre construction applies, providing a rank 2 vector bundle E such that Z is the zero-locus of a section of E. The class of Z is then equal to $c_2(E)$ so the corollary follows from Proposition 1.2. ☐

1.2. **Proof of Proposition 1.2.** The Grassmannian $G(k,n)$ is swept out by $\mathbb{P}^k$'s and by $\mathbb{P}^{n-k}$'s. We get a $\mathbb{P}^k \subset G(k,n)$ by fixing a vector subspace $W \subset \mathbb{C}^n$ of dimension $k+1$ and by considering the set of its hyperplanes; we will denote it by $\mathbb{P}^k_W$. We next get a $\mathbb{P}^{n-k} \subset G(k,n)$ by fixing a vector subspace $W' \subset \mathbb{C}^n$ of dimension $k-1$ and by considering the set of vector subspaces of $\mathbb{C}^n$ of dimension k containing W'; we will denote it by $\mathbb{P}^{n-k}_{W'}$. We note that the intersection $\mathbb{P}^k_W \cap \mathbb{P}^{n-k}_{W'}$ is a line when $W' \subset W$.

Let $k, n-k \geq n_0$ and let E be a rank 2 vector bundle on $G(k,n)$. For any vector subspace $W \subset \mathbb{C}^n$ of dimension $k+1$, Conjecture 1.1 gives

$$E_{|\mathbb{P}^k_W} = \mathcal{O}(a_W) \oplus \mathcal{O}(b_W), \tag{1.1}$$

for some integers a_W, b_W a priori depending on W. Similarly, for any vector subspace $W' \subset \mathbb{C}^n$ of dimension $k-1$, Conjecture 1.1 gives

$$E_{|\mathbb{P}^{n-k}_{W'}} = \mathcal{O}(a'_{W'}) \oplus \mathcal{O}(b'_{W'}), \tag{1.2}$$

for some integers $a'_{W'}$, $b'_{W'}$ a priori depending of W'.

Lemma 1.4. *The pair* $\{a_W, b_W\}$ *is in fact independent of* W*.*

Proof. If $W' \subset W$, the intersection $\mathbb{P}^k_W \cap \mathbb{P}^{n-k}_{W'}$ is a line $\Delta \cong \mathbb{P}^1$, and by restricting (1.1) and (1.2) to Δ, we conclude that

$$\{a_W, b_W\} = \{a'_{W'}, b'_{W'}\}. \tag{1.3}$$

If W_1 and W_2 are two vector subspaces of $\mathbb{C}^n$ of dimension $k+1$ with $\dim W_1 \cap W_2 = k-1$, by applying (1.3) to both inclusions $W_1 \cap W_2 \subset W_1$ and $W_1 \cap W_2 \subset W_2$, we conclude that

$$\{a_{W_1}, b_{W_1}\} = \{a_{W_2}, b_{W_2}\}. \tag{1.4}$$

We finally apply the following elementary result.

Lemma 1.5. *Let* W_0, $W_\infty \subset \mathbb{C}^n$ *be two vector subspaces of dimension* $k+1$*. There exist vector subspaces* $W_i,\ 1 \leq i \leq N$ *of* $\mathbb{C}^n$ *of dimension* $k+1$ *such that*

$$\dim W_0 \cap W_1 = k-1,\ \ldots,\ \dim W_i \cap W_{i+1} = k-1,\ \ldots,\ \dim W_N \cap W_\infty = k-1.$$

Lemma 1.5 concludes the proof of Lemma 1.4 by iterated applications of (1.4). $\square$

From now on, we denote by $\{a, b\}$ with $a \geq b$ the pair $\{a_W, b_W\}$. We now distinguish two cases.

1) **Case where $a > b$.** Consider the correspondence

$$\begin{array}{ccc} P & \xrightarrow{q} & G(k,n) \\ {\scriptstyle p}\downarrow & & \\ G(k+1,n) & & \end{array} \tag{1.5}$$

given by the universal family of $\mathbb{P}^k_W$'s, where p is a projective bundle with fiber $\mathbb{P}^k_W$ over the point $[W] \in G(k+1,n)$. We denote by $\mathcal{L}$ and $\mathcal{L}'$ the Plücker line bundles on $G(k,n)$ and $G(k+1,n)$. As we have $h^0(E_{|\mathbb{P}^k_W}(-a)) = 1$ for any $[W] \in G(k+1,n)$, the sheaf $R^0p_*(q^*(E \otimes \mathcal{L}^{-a}))$ is a line bundle on $G(k+1,n)$, hence isomorphic to $\mathcal{L}'^{\otimes l}$ for some integer l. Furthermore, the natural inclusion

$$p^*\mathcal{L}'^{\otimes l} \subset q^*(E \otimes \mathcal{L}^{-a})$$

is the inclusion of a line subbundle, and the quotient is a line bundle $\mathcal{M}$ on P, which fits into an exact sequence

$$0 \to p^*\mathcal{L}'^{\otimes l} \to q^*(E \otimes \mathcal{L}^{-a}) \to \mathcal{M} \to 0. \tag{1.6}$$

Computing determinants, we conclude that

$$\mathcal{M} = q^*(\det(E \otimes \mathcal{L}^{-a})) \otimes p^*\mathcal{L}'^{\otimes -l}.$$

Furthermore (1.6) also gives

$$q^*c_2(E \otimes \mathcal{L}^{-a}) = p^*c_1(\mathcal{L}'^{\otimes l}) \cdot (q^*c_1(E \otimes \mathcal{L}^{-a}) - p^*c_1(\mathcal{L}'^{\otimes l})) \text{ in } H^4(P, \mathbb{Z}). \tag{1.7}$$

Restricting (1.7) to the fibers of q, which are projective spaces of dimension > 1, we conclude that $l = 0$, so that

$$\mathcal{M} = q^*(\det(E \otimes \mathcal{L}^{-a})),$$

and (1.6) writes

$$0 \to \mathcal{O}_P \to q^*(E \otimes \mathcal{L}^{-a}) \to q^*(\det(E \otimes \mathcal{L}^{-a})) \to 0. \tag{1.8}$$

Next, applying R^0q_* to the exact sequence (1.8), we get that E is an extension of two line bundles on $G(k,n)$, hence is decomposable.

2) **Case where $a = b$.** In this case, the bundle $E \otimes \mathcal{L}^{-a}$ is trivial on the $\mathbb{P}^k_W$'s and in particular we have

$$q^*(E \otimes \mathcal{L}^{-a}) \cong p^*F, \tag{1.9}$$

for some vector bundle F of rank 2 on $G(k+1,n)$. The Grassmannian $G(k+1,n)$ is itself swept out by projective spaces $\mathbb{P}^{k+1}_{W''}$ of dimension $k+1$, associated to vector subspaces $W'' \subset \mathbb{C}^n$ of dimension $k+2$, and Conjecture 1.1 implies that the restriction $F_{|\mathbb{P}^{k+1}_{W''}}$ is decomposable, that is

$$F_{|\mathbb{P}^{k+1}_{W''}} = \mathcal{O}_{\mathbb{P}^{k+1}_{W''}}(a') \oplus \mathcal{O}_{\mathbb{P}^{k+1}_{W''}}(b'). \tag{1.10}$$

For any pair (W''', W'') of vector subspaces of $\mathbb{C}^n$ of respective dimensions k and $k+2$, there is a line $\Delta_{W''',W''} \subset G(k+1,n)$ parameterizing the vector subspaces V_t of dimension $k+1$ of $\mathbb{C}^n$ containing W''' and contained in W''. With the same notation as in (1.5), this line $\Delta_{W''',W''}$ admits a canonical lift $\widetilde{\Delta}_{W''',W''} \subset P$ given by the section of p

$$[V_t] \mapsto ([W'''], [V_t]) \in P,$$

which is defined on $\Delta_{W''',W''}$. Restricting equality (1.9) to the line $\widetilde{\Delta}_{W''',W''} \subset P$ which is contained in a a fiber of q, we get that $F_{|\Delta_{W''',W''}}$ is trivial, and by applying (1.10), we conclude that $F_{|\mathbb{P}^{k+1}_{W''}}$ is trivial, since $\Delta_{W''',W''}$ is contained in $\mathbb{P}^{k+1}_{W''}$. In other words, considering the incidence correspondence

$$\begin{array}{ccc} P' & \xrightarrow{q'} & G(k+1,n) \\ {\scriptstyle p'}\downarrow & & \\ G(k+2,n) & & \end{array}$$

for the family of $\mathbb{P}^{k+1}_{W''}$ covering $G(k+1,n)$, we proved that

$$q'^*F = p'^*G,$$

for some rank 2 bundle G on $G(k+2,n)$. The triviality of $E(-a)$, which is implied by the triviality of F, is thus implied by that of G. Iterating the above reasoning and using the fact that $G(n-1,n) = \mathbb{P}^{n-1}$, we finally conclude that $E(-a)$ is trivial, so Proposition 1.2 is also proved in this case.

1.3. **The unstable variant.** An interesting and very plausible variant of Hartshorne's Conjecture 1.1 is the following statement.

Conjecture 1.6. *There exists $n'_0 \geq 5$ such that any unstable rank 2 vector bundle E on $\mathbb{P}^n$, with $n \geq n'_0$, is decomposable.*

In this statement, “unstable" means “not slope stable", and this condition says that there exists a nonzero section of $E(-a)$ for some a such that $\det E(-a) \leq 0$. Conjecture 1.6 is important and, as explained in [Sch80], it would have remarkable consequences, such as the existence of topological complex vector bundles (of rank 2) on projective space that do not have an algebraic structure. This conjecture is studied in [GS77] and [Siu15] but the proofs there have serious gaps, hence it is still open.

Let $G(k,n)$ be a Grassmannian as in the previous sections. As $\mathrm{Pic}(G(k,n)) = \mathbb{Z}\mathcal{L}$, we can speak as above of unstable rank 2 vector bundles F on $G(k,n)$. We make the following observation:

Lemma 1.7. *Let F be a unstable rank 2 vector bundle on $G(k,n)$, with $k, n-k \geq 1$. Then for any $k+1$-dimensional vector subspace $W \subset \mathbb{C}^n$, and any $k-1$-dimensional vector subspace $W' \subset \mathbb{C}^n$, the restricted vector bundles*

$$F_{|\mathbb{P}^k_W},\ F_{|\mathbb{P}^{n-k}_{W'}}$$

are unstable.

Proof. We use the notation $F(-a) := F \otimes \mathcal{L}^{\otimes -a}$. By assumption, there exists a twist $F(-a)$ of F which admits a nonzero section s, and is such that $\det F(-a) \leq 0$. If W and W' are general, the restrictions of s to $\mathbb{P}^k_W$ and $\mathbb{P}^{n-k}_{W'}$ are nonzero, implying that the restricted vector bundles $F_{|\mathbb{P}^k_W}$, $F_{|\mathbb{P}^{n-k}_{W'}}$ are unstable. By upper semi-continuity, the fact that $F_{|\mathbb{P}^k_W}(-a)$ has a nonzero section for general W implies the same property for any W, and similarly for the W''s. Hence we conclude that $F_{|\mathbb{P}^k_W}$, $F_{|\mathbb{P}^{n-k}_{W'}}$ are unstable for any W, W'. ☐

Using Lemma 1.7, the same arguments as in the previous section then give us the following unstable variant of Proposition 1.2.

Corollary 1.8. *If Conjecture 1.6 is true, then any unstable vector bundle on $G(k,n)$, with $k, n-k \geq n'_0$, is decomposable, hence its second Chern class is a multiple of $c_1(\mathcal{L})^2$.*

1.4. **Extension to higher rank bundles.** We finally prove using similar arguments an analogous but weaker statement for bundles of arbitrary rank $r \geq 2$ on $G(k,n)$. The next question, in the spirit of Hartshorne's conjectures 0.2 and 1.1, was not stated as a conjecture by Hartshorne for lack of sufficient evidence (see the end of [Har74, §6]).

Question 1.9. *Fix $r \geq 2$. Does there exist an integer $n_0(r) \geq 5$ such that any rank r vector bundle E on $\mathbb{P}^n$, with $n \geq n_0(r)$, is decomposable?*

Proposition 1.10. *If Question 1.9 has a positive answer for $r \geq 2$, then for any rank r vector bundle E on $G(k,n)$ with $n-k \geq k \geq n_0(r)$, the Chern classes $c_i(E)$ for $i = 1, \ldots, r$ satisfy*

$$\int_{G(k,n)} c_i(E) l^{n-k-i}(l^e \gamma_k - \gamma_{n-k}) = 0, \tag{1.11}$$

where $e = n-2k$, $l = c_1(\mathcal{L})$ and

$$\gamma_k \in H^{2N-2k}(G(k,n),\mathbb{Z}),\ \text{resp. } \gamma_{n-k} \in H^{2N-2n+2k}(G(k,n),\mathbb{Z}),\ N := \dim G(k,n)$$

is the class of any $\mathbb{P}^k_W$, resp. $\mathbb{P}^{n-k}_{W'}$.

Proof. The notation is as in §1.2. Under the assumptions

$$n-k \geq k \geq n_0(r),$$

a positive answer to Question 1.9 implies that the restrictions $E_{|\mathbb{P}^k_W}$ and $E_{|\mathbb{P}^{n-k}_{W'}}$ are decomposable vector bundles and by the same arguments as in the proof of Lemma 1.4, we conclude that the type of the decomposition is the same for $E_{|\mathbb{P}^k_W}$ and $E_{|\mathbb{P}^{n-k}_{W'}}$ (and in particular it does not depend on the choice of W and W'). Hence there exist integers $a_1, \dots, a_r$ such that

$$E_{|\mathbb{P}^k_W} = \mathcal{O}_{\mathbb{P}^k_W}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^k_W}(a_r) \ \text{ and } \ E_{|\mathbb{P}^{n-k}_{W'}} = \mathcal{O}_{\mathbb{P}^{n-k}_{W'}}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{n-k}_{W'}}(a_r). \tag{1.12}$$

It follows from (1.12) that

$$c_i(E)_{|\mathbb{P}^k_W} = \sigma_i(a_j)h^i, \ c_i(E)_{|\mathbb{P}^{n-k}_{W'}} = \sigma_i(a_j)h'^i, \tag{1.13}$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^k_W}(1)) = l_{|\mathbb{P}^k_W}$, $h' = c_1(\mathcal{O}_{\mathbb{P}^{n-k}_{W'}}(1)) = l_{|\mathbb{P}^{n-k}_{W'}}$ and σ_i is the i-th symmetric function of r variables. It follows from (1.13) that

$$\int_{G(k,n)} l^{k-i}c_i(E)\gamma_k = \int_{\mathbb{P}^k_W} l^{k-i}c_i(E) = \sigma_i(a_j) \tag{1.14}$$

$$\int_{G(k,n)} l^{n-k-i}c_i(E)\gamma_{n-k} = \int_{\mathbb{P}^{n-k}_{W'}} l^{k-i}c_i(E) = \sigma_i(a_j),$$

which implies (1.11). □

The following lemma shows that the equations (1.11) impose $r-1$ independent equations to the Chern classes of E which are thus severely restricted if Question 1.9 has a positive answer.

Lemma 1.11. *Under the hypotheses of Proposition 1.10, the morphism*

$$H^{2i}(G(k,n), \mathbb{Z}) \to H^{2N}(G(k,n), \mathbb{Z})$$

of cup-product by the class $l^{n-k-i}(l^e\gamma_k - \gamma_{n-k})$ is nonzero for $i \geq 2$. Equivalently, thanks to Poincaré duality, the class $l^{n-k-i}(l^e\gamma_k - \gamma_{n-k})$ is nonzero for $i \geq 2$.

Proof. It clearly suffices to prove that the cohomology class $l^{n-k-2}(l^e\gamma_k - \gamma_{n-k})$ is nonzero in $H^{2N-4}(G(k,n), \mathbb{Z})$. This class is the difference of the class of a plane in $\mathbb{P}^k_W$ and the class of a plane in $\mathbb{P}^{n-k}_{W'}$. We construct an inclusion

$$G(2,4) = G(2,V') \subset G(k,n)$$

by choosing a subspace $V \subset \mathbb{C}^n$ of codimension $n-k-2$ and a quotient V' of V of dimension 4. This inclusion induces a surjection $H^4(G(k,n), \mathbb{Z}) \to H^4(G(2,4), \mathbb{Z})$, as one can see by computing the pull-back to $G(2,4)$ of the second Chern class of the universal bundle on $G(k,n)$, and thus an injection $H_4(G(2,4), \mathbb{Z}) \to H_4(G(k,n), \mathbb{Z})$. The Grassmannian $G(2,4)$ is a quadric in $\mathbb{P}^5$. It contains the two types of planes considered above and their classes are independent in $H_4(G(2,4), \mathbb{Z})$; hence they remain independent in $H_4(G(k,n), \mathbb{Z})$. □

2. Construction of symplectic submanifolds

2.1. **Construction of topological vector bundles.** Let X be a connected and simply connected CW-complex. The *rationalization* of X in the sense of Sullivan is a continuous map $\rho : X \to X_{\mathbb{Q}}$ of connected and simply connected CW-complexes that satisfies the following equivalent properties:

(i) for all $i \geq 1$, the abelian group $H_i(X_{\mathbb{Q}}, \mathbb{Z})$ is a $\mathbb{Q}$-vector space and the morphism $\rho_* : H_i(X, \mathbb{Q}) \to H_i(X_{\mathbb{Q}}, \mathbb{Q})$ is an isomorphism;

(ii) for all $i \geq 1$, the abelian group $\pi_i(X_{\mathbb{Q}})$ is a $\mathbb{Q}$-vector space and the morphism $\rho_* : \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_i(X_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism.

It exists and it is unique up to homotopy (see for instance [MP12, Theorem 6.1.2]).

Let M be an abelian group and fix $n \geq 2$. Let $K(M, n)$ denote an Eilenberg–Maclane CW-complex, whose only nontrivial homotopy group is M in degree n. It is a classifying space for $H^n(-, M)$ (see [Hat02, Theorem 4.57]). The characterization (ii) shows that the rationalization of $K(M, n)$ is $K(M \otimes_{\mathbb{Z}} \mathbb{Q}, n)$.

Proposition 2.1. *Let X be a finite CW-complex. Fix $r \geq 1$ and, for $1 \leq i \leq r$, a class $\alpha_i \in H^{2i}(X, \mathbb{Z})$. Then there exist $m \geq 1$ and a complex topological vector bundle E of rank r on X such that $c_i(E) = m \cdot \alpha_i$ in $H^{2i}(X, \mathbb{Z})$ for all $1 \leq i \leq r$.*

Proof. Let $\mathrm{BU}(r)$ be the classifying space for complex topological vector bundles of rank r, constructed as the increasing union of the complex Grassmannians $G(r, n)$ for all $n \geq r$. Let $\rho : \mathrm{BU}(r) \to \mathrm{BU}(r)_{\mathbb{Q}}$ be its rationalization.

The Chern classes induce a continuous map $(c_1, \ldots, c_r) : \mathrm{BU}(r) \to \prod_{i=1}^r K(\mathbb{Z}, 2i)$. On the one hand, $H^*(\mathrm{BU}(r), \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_r]$ (see [MS74, Theorem 14.5]). On the other hand, $H^*(K(\mathbb{Z}, 2i), \mathbb{Q})$ is a polynomial ring in the tautological element of $H^{2i}(K(\mathbb{Z}, 2i), \mathbb{Q})$ (see [Ser51, Proposition 4 p. 501]). It therefore follows from the Künneth formula that the rationalized map

$$(c_1, \ldots, c_r) : \mathrm{BU}(r)_{\mathbb{Q}} \to \prod_{i=1}^r K(\mathbb{Q}, 2i) \tag{2.1}$$

induces an isomorphism between integral cohomology groups, and hence between integral homology groups by the universal coefficient theorem (as these groups are $\mathbb{Q}$-vector spaces). We deduce from [May83, Theorem B] that (2.1) is a homotopy equivalence, and we denote by $\Phi : \prod_{i=1}^r K(\mathbb{Q}, 2i) \to \mathrm{BU}(r)_{\mathbb{Q}}$ a homotopy inverse.

Define $Y := \prod_{i=1}^r K(\mathbb{Z}, 2i)$. By cellular approximation, we may assume that the continuous maps $\mu_m : Y \to Y$ induced by multiplication by m on the coefficients $\mathbb{Z}$ preserve the d-skeleton Y_d of Y for all $d \geq 0$.

Lemma 2.2. *For all $d \geq 0$, there exist a continuous map $\Psi_d : Y_d \to \mathrm{BU}(r)$ and an integer $m_d \geq 1$ such that the following diagram commutes up to homotopy:*

$$
\begin{tikzcd}
Y_d \arrow[d, hook] \arrow[ddr, dotted, "\Psi_d"] & \\
Y \arrow[d, "\mu_{m_d}"'] & \\
Y \arrow[d] & \mathrm{BU}(r) \arrow[d, "\rho"'] \\
\prod_{i=1}^r K(\mathbb{Q}, 2i) \arrow[r, "\Phi", "\sim"'] & \mathrm{BU}(r)_{\mathbb{Q}}.
\end{tikzcd}
\tag{2.2}
$$

Proof of Lemma 2.2. We argue by induction on d. The base case $d = 0$ is obvious as both $\mathrm{BU}(r)$ and $\mathrm{BU}(r)_{\mathbb{Q}}$ are connected. Assume from now on the existence of a lift $\Psi_d : Y_d \to \mathrm{BU}(r)$ of the composition $Y \xrightarrow{\mu_{m_d}} Y \to \prod_{i=1}^r K(\mathbb{Q}, 2i) \xrightarrow{\Phi} \mathrm{BU}(r)_{\mathbb{Q}}$ to $\mathrm{BU}(r)$, in restriction to Y_d. After possibly modifying it, our goal is to extend this lift to Y_{d+1}, at the expense of precomposing by a well-chosen multiplication map $\mu_m : Y \to Y$ (we will then set $m_{d+1} := m \cdot m_d$).

By obstruction theory [Ste51, Theorem 34.2], the obstruction to the existence of such a lift lives in $H^{d+1}(Y, \pi_d(F))$, where F is the homotopy fiber of ρ. To conclude, it suffices to show that this obstruction class is killed by pull-back by $\mu_m : Y \to Y$ for some $m \geq 1$.

As $\rho : \mathrm{BU}(r) \to \mathrm{BU}(r)_{\mathbb{Q}}$ is the rationalization of $\mathrm{BU}(r)$, and as the homology groups of $\mathrm{BU}(r)$, hence also its homotopy groups, are finitely generated [Ser51, Proposition 1 p. 491], the exact sequence

$$\pi_{d+1}(\mathrm{BU}(r)) \to \pi_{d+1}(\mathrm{BU}(r)_{\mathbb{Q}}) \to \pi_d(F) \to \pi_d(\mathrm{BU}(r)) \to \pi_d(\mathrm{BU}(r)_{\mathbb{Q}})$$

shows that $\pi_d(F)$ is the direct sum of a finite abelian group and of finitely many copies of $\mathbb{Q}/\mathbb{Z}$. In addition, since the homology groups of Y are finitely generated [Ser51, Corollaire 1 p. 500], one can verify that $H^{d+1}(Y, \mathbb{Q}/\mathbb{Z}) = \mathrm{colim}_{l \geq 1} H^{d+1}(Y, \mathbb{Z}/l)$ by applying the universal coefficient theorem. Consequently, we only need to show that, for all $l \geq 1$, any class in $H^{d+1}(Y, \mathbb{Z}/l)$ is killed by pull-back by $\mu_m : Y \to Y$ for some well-chosen $m \geq 1$.

As $\mathbb{Z}/l$ is an injective $\mathbb{Z}/l$-module (see [Wei94, Exercise 2.3.1]), the functor $\mathrm{Hom}(-, \mathbb{Z}/l)$ is exact on the abelian category of $\mathbb{Z}/l$-modules. It follows that

$$H^{d+1}(Y, \mathbb{Z}/l) = \mathrm{Hom}(H_{d+1}(Y, \mathbb{Z}/l), \mathbb{Z}/l).$$

Using that $H_{d+1}(Y, \mathbb{Z}/l)$ is finitely generated, we see that it remains to verify that any class in $H_{d+1}(Y, \mathbb{Z}/l)$ is killed by push-forward by $\mu_m : Y \to Y$ for some well-chosen $m \geq 1$. Equivalently, we must prove that any class in $H_{d+1}(Y, \mathbb{Z}/l)$ vanishes in the homology of the homotopy colimit (constructed as a mapping telescope, see [Hat02, p. 312]) of the diagram

$$Y \xrightarrow{\mu_2} Y \xrightarrow{\mu_3} Y \xrightarrow{\mu_4} Y \xrightarrow{\mu_5} \cdots . \tag{2.3}$$

This homotopy colimit identifies with the rationalization of Y (to see it, compute its homotopy groups and use characterization (ii) above). Its integral homology groups are therefore $\mathbb{Q}$-vector spaces, and we deduce that its homology groups with $\mathbb{Z}/l$ coefficients vanish. This concludes the proof. ∎

Let us resume the proof of Proposition 2.1. Let $f : X \to Y = \prod_{i=1}^r K(\mathbb{Z}, 2i)$ be a continuous map classifying the classes $\alpha_1, \ldots, \alpha_r$. As X is a finite CW-complex, we may assume by cellular approximation that $f(X) \subset Y_d$ for some well-chosen $d \geq 0$. Let $\Psi_d : Y_d \to \mathrm{BU}(r)$ and $m_d \geq 1$ be as in Lemma 2.2. Let $m' \geq 1$ be an integer killing the torsion subgroup of $H^*(X, \mathbb{Z})$. Set $m := m' \cdot m_d$.

Let E' and E be the complex topological vector bundles of rank r on X classified by $\Psi_d \circ f : X \to \mathrm{BU}(r)$ and $\Psi_d \circ \mu_{m'} \circ f : X \to \mathrm{BU}(r)$ respectively. One has $c_i(E') = m_d \cdot \alpha_i$ in $H^{2i}(X, \mathbb{Q})$ for $1 \leq i \leq r$ by commutativity of (2.2). Our choice of m' implies that $m' \cdot c_i(E') = m \cdot \alpha_i$ in $H^{2i}(X, \mathbb{Z})$ for $1 \leq i \leq r$. As $c_i(E) = m' \cdot c_i(E')$ in $H^{2i}(X, \mathbb{Z})$ for $1 \leq i \leq r$, the proof of the proposition is complete. □

Remark 2.3. One can obtain many variants of Proposition 2.1 using the same argument. For instance, it is possible to construct a complex topological vector bundle E of rank r on X with the property that $c_i(E) = m^i \cdot \alpha_i$ for $1 \leq i \leq r$, by replacing everywhere the map $\mu_m : Y \to Y$ by the map $\lambda_m : Y \to Y$ induced by multiplication by m^i on the i-th factor of $Y = \prod_{i=1}^r K(\mathbb{Z}, 2i)$.

2.2. **The Auroux–Donaldson theorem.** The next theorem is Auroux's extension to vector bundles [Aur97, Corollary 1] of Donaldon's construction of symplectic hypersurfaces [Don96] in a symplectic manifold (M, ω) (see also [Sik98]).

Theorem 2.4 (Auroux–Donaldson)**.** *Let (M, ω) be a compact symplectic $\mathcal{C}^\infty$ manifold. Then there exists a complex line bundle L on M such that the following holds. For all complex vector bundles E on M, and for all $k \gg 0$, there exists a $\mathcal{C}^\infty$ section of $E \otimes L^{\otimes k}$ which is transversal to 0 and whose zero locus is a symplectic $\mathcal{C}^\infty$ submanifold of M.*

To be precise, the Auroux–Donaldson formulation assumes that the de Rham cohomology class $\frac{[\omega]}{2\pi}$ is is in the image of the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. This restriction is well-known to be immaterial, as indicated in [Don96, Corollary 6] and [Sik98, Remarque p. 233]. For the convenience of the reader, we give a few more details on this argument.

Proof of Theorem 2.4. Fix a Riemannian metric g on M. Let $(\omega_i)_{1 \leq i \leq N}$ be a collection of closed $\mathcal{C}^\infty$ 2-forms on M whose de Rham cohomology classes form a basis of $H^2(M, \mathbb{R})$. For $\underline{t} := (t_1, \ldots, t_N) \in \mathbb{R}^N$ small enough, the 2-form $\omega_{\underline{t}} := \omega + \sum_{i=1}^N t_i \omega_i$ is still symplectic. For such values of $\underline{t}$, the almost complex structure $J_{\underline{t}} := J_{g, \omega_{\underline{t}}}$ (in the notation of [MS17, Proposition 2.5.6]) depends continuously on $\underline{t}$, and is $\omega_{\underline{t}}$-compatible in the sense that the pairing $(v, w) \mapsto \omega_{\underline{t}}(v, J_{\underline{t}}(w))$ defines a Riemannian metric on M.

For $\underline{t}$ small enough, one has $\omega(v, J_{\underline{t}}(v)) > 0$ for all nonzero tangent vectors v to M. As the set of $\underline{t} \in \mathbb{R}^N$ such that $\frac{[\omega_{\underline{t}}]}{2\pi} \in H^2(M, \mathbb{Q}) \subset H^2(M, \mathbb{R})$ is dense in $\mathbb{R}^N$, we may fix from now on a $\underline{t}$ satisfying both conditions. Set $\omega' := \omega_{\underline{t}}$ and $J' := J_{\underline{t}}$.

As the class $\frac{[\omega']}{2\pi}$ is rational, some integral multiple of it is the first Chern class of a complex line bundle L on M. By [Aur97, Corollary 1], there exist $\mathcal{C}^\infty$ sections of $E \otimes L^{\otimes k}$ for $k \gg 0$ which are asymptotically holomorphic (with respect to J') and transverse to 0

in the sense of [Aur97, Definitions 1 and 2]. Their zero loci $N_k \subset M$ are therefore asymptotically J'-holomorphic (see [Aur97, Proposition 1]) in the sense that the subbundles TN_k and $J'(TN_k)$ of $TM|_{N_k}$ are very close when $k \gg 0$ (with respect to a fixed metric on the Grassmannian bundle associated with TM). The positivity of $\omega(v, J'(v))$ for all nonzero tangent vectors v to N_k now implies that $\omega|_{N_k}$ is nondegenerate, so the $\mathcal{C}^\infty$ submanifold $N_k \subset M$ is symplectic. □

Theorem 2.5. *Let (M, ω) be a compact symplectic $\mathcal{C}^\infty$ manifold. For all $c \geq 0$, the $\mathbb{Q}$-vector space $H^{2c}(M, \mathbb{Q})$ is generated by fundamental classes of codimension $2c$ symplectic $\mathcal{C}^\infty$ submanifolds of M.*

Proof. We may assume that $c \geq 1$. Choose $\alpha \in H^{2c}(M, \mathbb{Q})$. By Proposition 2.1, after possibly replacing α by a positive integral multiple, there exists a complex vector bundle E of rank c on M such that $c_i(E) = 0$ in $H^{2i}(M, \mathbb{Q})$ for $1 \leq i \leq c-1$, and $c_c(E) = \alpha$ in $H^{2c}(M, \mathbb{Q})$. Let L be as in Theorem 2.4. For all $k \gg 0$, one can find a $\mathcal{C}^\infty$ section of $E \otimes L^{\otimes k}$ which is transversal to 0 and whose zero locus $N_k \subset M$ is symplectic. One computes that $[N_k] = c_c(E \otimes L^{\otimes k}) = \alpha + k^c c_1(L)^c$, so

$$\alpha = \frac{(k+1)^c}{(k+1)^c - k^c}[N_k] - \frac{k^c}{(k+1)^c - k^c}[N_{k+1}]. \qquad \square$$

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