

A HYPER-KÄHLER COMPACTIFICATION OF THE INTERMEDIATE JACOBIAN FIBRATION ASSOCIATED TO A CUBIC FOURFOLD

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ABSTRACT. Let X be a general cubic fourfold. It was observed by Donagi and Markman that the relative intermediate Jacobian fibration \mathcal{J}/U (with $U = (\mathbb{P}^5)^\vee \setminus X^\vee$) associated to the smooth hyperplane sections of X carries a natural holomorphic symplectic form making the fibration Lagrangian. In this paper, we obtain a smooth projective compactification $\overline{\mathcal{J}}$ of \mathcal{J} with the property that the holomorphic symplectic form on \mathcal{J} extends to a holomorphic symplectic form on $\overline{\mathcal{J}}$. In particular, $\overline{\mathcal{J}}$ is a 10-dimensional compact hyper-Kähler manifold, which we show to be deformation equivalent to the exceptional example of O'Grady. This proves a conjecture by Markushevich.

INTRODUCTION

It is a problem of significant interest to construct and classify compact hyper-Kähler (HK) manifolds. In dimension 2, the HK manifolds are $K3$ surfaces. All known higher dimensional examples are obtained from $K3$ s or abelian varieties, by a moduli construction and a deformation. Specifically, Beauville [Bea83] has given two series of examples: the Hilbert scheme of points on $K3$ s, and respectively generalized Kummer varieties. Mukai [Muk84] has given a more general construction, namely he has shown that the moduli space of semi-stable sheaves on $K3$ s carries a symplectic form, and thus is a HK manifold if it is smooth. Unfortunately, these examples are deformation equivalent to those of Beauville [Yos06]. Starting from a singular moduli space of sheaves on $K3$ s, O'Grady [O'G99, O'G03] has produced two genuinely new examples: a 10-dimensional and a 6-dimensional one, that we call $OG10$ and $OG6$ respectively. It was subsequently verified ([KLS06]) that these are the only two new examples that can be obtained by this method.

It is natural to expect that hyper-Kähler manifolds can be constructed from lower dimensional objects of similar nature. Specifically, we recall that the moduli space of polarized HK manifolds in a fixed deformation class is birational via the period map to a locally symmetric variety \mathcal{D}/Γ , where \mathcal{D} is a Type IV domain, and Γ an arithmetic group. Such a locally symmetric variety contains divisors \mathcal{D}'/Γ' (known as Noether–Lefschetz or Heegner divisors) which are of the same type. It is natural to expect that some of these NL divisors are associated (at least motivically) to moduli spaces of lower dimensional HK manifolds. For instance, the Beauville–Mukai construction will give such an example (i.e. $\text{Hilb}^n(K3)$ form a NL divisor in the corresponding moduli space of $2n$ -dimensional HK manifolds). Another NL divisor in the moduli space, that we will call a Lagrangian NL divisor, is obtained by considering the HK manifolds that admit a Lagrangian fibration. Again, one can hope that they can be constructed from lower dimensional geometric objects. Note also that moduli spaces of HK manifolds are one of the rare instances when the period map has trivial Griffiths transversality relations, and thus the building blocks should be very special (essentially again HK).

The 10-dimensional examples by O'Grady have $b_2 = 24$, which means that polarized deformations of them have 21 moduli, and complete families of such varieties with Picard number 2 have 20 moduli. The construction by O'Grady in [O'G99] provides (infinitely many) 19-parameters families of such examples with Picard number 3, parameterized by any moduli space of polarized $K3$ surfaces. Cubic fourfolds have 20 moduli and they are well-known to have associated 20-parameters families of hyper-Kähler manifolds (see [BD85], [LLSvS13]), that are all deformation equivalent to Hilbert schemes of $K3$ surfaces (see [AL14]). This paper provides an algebrogeometric realization of the moduli space of cubic fourfolds as a 20-dimensional moduli space of deformations of O'Grady's 10-dimensional examples, and more precisely, as a Lagrangian NL divisor in the larger 21-dimensional deformation space (our varieties are canonically polarized). Note first that this embedding is a priori given by lattice considerations. Indeed, it is well known that the moduli

space of cubic fourfolds is birational to a 20-dimensional locally symmetric variety \mathcal{D}'/Γ' , which is associated to the lattice (up to a sign, which can be safely ignored here) $A_2 \oplus E_8^2 \oplus U^2$. On the other hand, by work of Rapagnetta [Rap08], it is known that the second cohomology of $OG10$ together with the Beauville–Bogomolov form is isometric to the lattice $A_2 \oplus E_8^2 \oplus U^3$. This shows that from an arithmetic point of view, the situation is similar to that of elliptic $K3$ s with a section (lattice $E_8^2 \oplus U^2$), aka unigonal $K3$ s, versus general $K3$ s (lattice $E_8^2 \oplus U^3$). Furthermore, it is not hard to abstractly embed the moduli of cubic fourfolds as a NL divisor into a 21-dimensional period domain of *polarized* $OG10$ manifolds (again completely analogous to the situation for unigonal $K3$ s).

Our contribution in this paper is to realize geometrically this abstract embedding by actually constructing a family of $OG10$ polarized manifolds parametrized by the moduli space of cubic 4-folds. This is done by realizing the following program that has been started by Donagi–Markman and developed by Markushevich, with further evidence provided by work of Kuznetsov and O’Grady–Rapagnetta: Starting from a general cubic fourfold X , one has the universal family \mathcal{Y}/B of cubic threefolds ($B := (\mathbb{P}^5)^\vee$) obtained as hyperplane sections, and then the associated relative Intermediate Jacobian fibration \mathcal{J}/U , where $U = B \setminus X^\vee \subset B$ is the locus of smooth hyperplane sections. In 1993, Donagi and Markman [DM96a, DM96b] had the insight that this fibration, which they showed to be algebraic, carries a holomorphic symplectic form. The question naturally was raised, as to whether or not it admitted a holomorphic symplectic compactification. If such a holomorphic symplectic compactification $\overline{\mathcal{J}}/B$ existed, then it would have Picard number at least 2 and transcendental second cohomology containing the transcendental cohomology of the cubic X . Thus, $b_2(\overline{\mathcal{J}}) \geq 24$, showing in particular that $\overline{\mathcal{J}}$ can not be deformation equivalent to $K3^{[5]}$ type, but potentially equivalent to $OG10$. Markushevich and Kuznetsov [KM09] (with further supporting evidence by O’Grady and others) conjectured that indeed a good compactification $\overline{\mathcal{J}}$ exists, and that it is of $OG10$ type. In this paper, we verify this conjecture. Specifically, we prove

Main Theorem. *(Cf. Theorem 5.6 and Corollary 6.2.) Let $X \subset \mathbb{P}^5$ be a general cubic fourfold. Let $B = (\mathbb{P}^5)^\vee$, $U = (B \setminus X^\vee)$, and \mathcal{J}/U the associated Intermediate Jacobian fibration. Then there exists a smooth projective compactification $\overline{\mathcal{J}}/B$, which carries a holomorphic symplectic form, with respect to which the fibration is Lagrangian. Furthermore, $\overline{\mathcal{J}}$ is deformation equivalent to $OG10$.*

The key issue for the theorem, and the main new content of the paper, is the construction of a smooth projective compactification of the Intermediate Jacobian fibration that has a nondegenerate holomorphic 2-form. In order to do so, it is important to understand degenerations of intermediate Jacobians as the hyperplane section of X becomes singular. The study of intermediate Jacobians from a different perspective was done in [CML09, CMGHL15], and the main tool used there is Mumford’s construction of the intermediate Jacobian as a Prym variety. This is our approach also here.

A key point that allows us to construct the compactification $\overline{\mathcal{J}}/B$ is the observation that much of the Prym construction carries through for mildly singular cubic threefolds. Namely, we recall for a smooth cubic threefold Y , the projection from a generic line l , realizes Y as a conic bundle over \mathbb{P}^2 with discriminant locus a smooth quintic C . Then, Mumford showed that $J(Y) \cong \text{Prym}(\tilde{C}/C)$, where \tilde{C} is an étale cover of C . In [CML09], we have noted that for mildly singular cubics Y there still exists *good lines* (see Def. 2.2) l such that the associated quintic C has a 1-to-1 correspondence with the singularities of Y (including the type), and that the covering \tilde{C}/C is still étale. This reduces questions about degenerations of cubic threefolds to degenerations of curves. In [CMGHL15], we have studied the degeneration of the Prym variety $\text{Prym}(\tilde{C}/C)$ from the perspective of stable abelian varieties (in the sense of Alexeev). Here we study this from the perspective of compactified relative Jacobians (or rather compactified Pryms) instead.

Finally, we should remark that this construction also provides the first example of hyperkähler manifolds admitting a Lagrangian fibration in principally polarized abelian varieties that are not Jacobians of curves. Indeed, the known constructions of compact holomorphic symplectic varieties that are fibered in principally polarized abelian varieties are either the relative compactified Jacobian of a linear system on a $K3$ surface (Beauville–Mukai system), or the relative compactified Prym associated to a linear system on a $K3$ with an anti-symplectic involution ([MT07], [ASF15]). In these last examples, either the Prym varieties are isomorphic to Jacobian of curves, or the total space of the family has singularities that admit no symplectic resolution.

Let us briefly describe the main steps of the paper.

The hyper-Kähler structure. In Section 1, we review the construction of the relative Intermediate Jacobian fibration \mathcal{J}/U and the existence of holomorphic symplectic 2-form. We then show, without much difficulty, that this symplectic form extends over the locus of cubics with a single ordinary node \mathcal{J}_1/U_1 , providing a nondegenerate holomorphic 2-form on \mathcal{J}_1/U_1 . These results are well known, and are essentially completely contained in [DM96a], [DM96b, §8.5.2], but the cycle theoretic method used here has the advantage of providing an extension to a holomorphic closed 2-form on any smooth compactification $\overline{\mathcal{J}}$ of \mathcal{J}_1 . If furthermore, $\mathcal{J}_1 \subset \overline{\mathcal{J}}$ has codimension larger than 2, then the extended form is everywhere non-degenerate. This allows us to concentrate on the problem of constructing a smooth compactification of \mathcal{J}_1/U_1 which is flat over B . The fact that the variety we construct is irreducible holomorphic symplectic (or hyper-Kähler) makes use of [LLSvS13]. Indeed, the intermediate Jacobian fibration contains a divisor which is birationally a \mathbb{P}^1 -bundle on the hyper-Kähler 8-fold recently constructed in [LLSvS13].

Local compactification. In Section 2, we briefly recall the Prym construction, and discuss the existence of a good line for all hyperplane sections of a general cubic fourfolds. Roughly speaking the existence of a good line guarantees that the moduli space of cubics can be identified locally, up to a smooth factor, with the moduli of quintic curves (see [CMGHL15, Sect. 3]). And thus, studying the degenerations of intermediate Jacobians reduces to studying degenerations of abelian varieties associated to curves (more precisely Prym varieties). Some of the results and ideas in this section occur previously in [CML09, CMGHL15]. Here, we obtain a slight strengthening applicable to our context: for Y any hyperplane section of a general cubic fourfold, there exists a very good line l , that is, a line such that the associated cover \tilde{C}/C is étale, and both curves are irreducible (with singularities in 1-to-1 correspondence with those of the cubic threefold Y).

For curves, one has a very good understanding of the degeneration of Jacobians. Namely, from the perspective of moduli spaces the correct statement is the theorem of Mumford–Namikawa that there exists an extended period map from $\overline{\mathcal{M}}_g$ to $\overline{\mathcal{A}}_g^{Vor}$ (and thus to a family of DM stable curves one can associate a family of Alexeev stable abelian varieties). If one insists instead to associating to an arbitrary family a curves \mathcal{C}/B a family of degenerate abelian varieties, without modifying the base, one gets into the theory of compactified relative Jacobians (to pass from the relative compactified Jacobian to the family of stable abelian varieties, one needs to perform a simultaneous semistable reduction, see [CML13] and the references within). The situation for Prym varieties is more complicated, but still well understood (see [FS86], [ABH02], [CMGHL14]) when the point of view is that of Beauville’s admissible covers compactification; the relative compactified Prym is less studied but a few cases, namely when the families of curves come from linear systems on surfaces, were studied in [MT07], [ASF15], and [Sac13a]. In our situation, due to the existence of very good lines, we can make an important simplifying assumption: namely, we consider only étale double covers of singular curves, and furthermore we can assume that both the cover and the base are irreducible. With these assumptions, it is easy to see that there exists a (canonical) relative compactified Prym. Our main result here is Theorem 4.31 which says that this compactification has the property that the total space is smooth, provided that the family of (base) curves gives a simultaneous versal deformation of the singularities (an analogue of the well known result for Jacobians). Moreover, this relative compactified Prym is equidimensional, in particular flat over the base. These results are discussed in Section 4. Finally, the fact for a general cubic fourfold X the versality statements needed to establish the smoothness of the compactified Prym hold is discussed in Section 3.

Descent. Let \mathcal{F}/B be the relative Fano surface, and \mathcal{F}^0/B the open subset of good lines. The results of the previous three sections give that $\mathcal{F}^0 \rightarrow B$ is a surjective smooth map (with 2 dimensional fibers), and than over \mathcal{F}^0 there is a relative compactified Prym $\overline{\mathcal{P}} \rightarrow \mathcal{F}^0$ which is proper, and has the property that $\overline{\mathcal{P}}$ is smooth. Our compactification $\overline{\mathcal{J}}$ is a descent of $\overline{\mathcal{P}}/\mathcal{F}^0/B$ to B . More precisely, in Section 5, using the relative theta divisor and a relative Proj construction, we conclude that $\overline{\mathcal{P}}$ descends to B giving a smooth compactification $\overline{\mathcal{J}}/B$ to \mathcal{J}/U and \mathcal{J}_1/U_1 , equidimensional over B . The arguments of Section 1 now allow us to conclude that $\overline{\mathcal{J}}$ is a 10-dimensional compact HK manifold, and that $\overline{\mathcal{J}}/B$ is a Lagrangian fibration (whose general fiber is an Intermediate Jacobian). This concludes (see Theorem 5.6) the proof of the first half of the Main Theorem.

Relationship with O’Grady’s varieties. In the final section, we establish that the constructed object is in fact deformation equivalent to the $OG10$ example [O’G99] (see Corollary 6.2).

Assuming the existence of a smooth compactification to \mathcal{J}/U , partial results in this direction had been established by O’Grady-Rapagnetta. Their idea was to use the degeneration of the cubic fourfold to the chordal cubic (the secant variety of the Veronese surface), for which one can show that the limiting family of intermediate Jacobians is birational to the O’Grady moduli space for an adequate $K3$ surface. A similar construction had been done by Hwang and Nagai [HN08] in the case of a singular cubic fourfold, for which the $K3$ surface is the surface of lines through the singular point. Unfortunately, in both cases, we could not deduce from the existence of these birational maps the fact that our compactified Jacobian fibration is deformation equivalent to the O’Grady moduli space, because we have no control of the singularities of the compactified Jacobian fibration at these points. We study instead, as suggested also by Markushevich and Kuznetsov [KM09], the Intermediate Jacobian fibration in the case of a Pfaffian cubic fourfold. By Beauville-Donagi [BD85], such a cubic X is Hodge theoretic equivalent to a degree 14 $K3$ surface S . Using [IM00], [MT01], and [Kuz04] we show that the fibration \mathcal{J}/U (or the compact version) is birational to the O’Grady moduli space of sheaves on S . On the other hand, using versality statements established in Section 3, we are able to prove that the compactified Jacobian fibration is smooth also for a general Pfaffian cubic fourfold, so that Huybrechts’ fundamental theorem [Huy99] applies, allowing us to conclude the equivalence to the OG10 example.

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We learned about this conjecture from a talk by D. Markushevich at the Hyper-Kähler Geometry Workshop at the Simons Center for Geometry and Physics in Fall 2012. It was he who initiated the project and subsequently and independently realized many steps towards its completion. The main result of the paper should thus partly be attributed to him. K. O’Grady and A. Rapagnetta also significantly contributed to the general project. We also acknowledge helpful discussions with E. Arbarello, S. Casalaina-Martin, K. Hulek, A. Kuznetsov, E. Macrì, L. Migliorini, K. O’Grady, and A. Rapagnetta.

Notation and Conventions. Unless otherwise specified, $X \subset \mathbb{P}^5$ denotes a (Hodge) general cubic fourfold. We let $B = (\mathbb{P}^5)^\vee$, $U = B \setminus X^\vee$, and $U_1 = B \setminus (X^\vee)^{sing}$ parameterizing hyperplane sections with at most one single ordinary node. Thus, $U \subset U_1 \subset B$ and U_1 has boundary of codimension 2 in B . Let \mathcal{Y}/B be the universal family of cubic threefolds obtained as hyperplane sections of X , and \mathcal{Y}_U and \mathcal{Y}_{U_1} the restrictions to U and U_1 respectively. Typically, Y or Y_t will denote hyperplane sections of X or fibers of \mathcal{Y} . \mathcal{J}/U or \mathcal{J}_U will denote the intermediate Jacobian fibration associated to \mathcal{Y}_U . Similarly for \mathcal{J}_1/U_1 or \mathcal{J}_{U_1} . \mathcal{J} and \mathcal{J}_1 are smooth quasi-projective varieties together with a holomorphic symplectic form, and the restriction of \mathcal{J}_1 to U is \mathcal{J} (cf. [DM96a, DM96b] and Section 1). $\overline{\mathcal{J}}/B$ will denote a proper algebraic extension of \mathcal{J}/U (and \mathcal{J}_1/U_1) over B . Of course, such $\overline{\mathcal{J}}$ always exists; the goal of the paper is to establish the existence of a smooth $\overline{\mathcal{J}}$ which is flat over B .

Let Y be a cubic threefold, and $l \subset Y$ a line (both Y and l need to satisfy some mild assumptions, to be specified in the text). The projection from l realizes Y as a quadric bundle over \mathbb{P}^2 with discriminant a plane quintic C . The lines incident with l on Y are parameterized by a curve \tilde{C} , which is a double cover (for general l , étale) of C . \mathcal{F}/B will denote the relative Fano surface, and (\tilde{C}, C) the relative family of double covers over (a suitable open of) \mathcal{F} .

We will say that two hypersurface singularities $(V(f), 0) \subset \mathbb{C}^n$ and $(V(g), 0) \subset \mathbb{C}^{n+k}$ have the same type if they differ by a suspension, i.e. in suitable analytic coordinates $g(x_1, \dots, x_{n+k}) = u \cdot (f(x_1, \dots, x_n) + x_{n+1}^2 + \dots + x_{n+k}^2)$. The deformation spaces for singularities of the same type (and also the local monodromies if $k \equiv 0 \pmod{2}$) are naturally identified.

1. HOLOMORPHIC 2-FORMS ON JACOBIAN FIBRATIONS

In [DM96b, DM96a], Donagi and Markman have performed the infinitesimal study of algebraically integrable systems, also called Lagrangian fibrations, which consist of a holomorphic family of complex tori, equipped with a nondegenerate $(2, 0)$ -form for which the fibers are Lagrangian. In this section, we provide an alternative way to construct a structure of Lagrangian or rather isotropic fibration on certain families of intermediate Jacobians. That is, we give a cycle and Hodge theoretic argument to construct a closed holomorphic 2-form vanishing on fibers for such families. The non-degeneracy of the holomorphic 2-form needs to be checked at hand. However, a strong point of our construction is that it easily implies that the $(2, 0)$ -form extends to any algebraic smooth compactification of the family of Jacobians.

1.1. The general case. Let X be a smooth projective variety of dimension $2k$, and let L be a line bundle on X . Assume that the smooth members Y of the linear system $|L|$ have the following property:

$$(1.1) \quad H^{p,q}(Y) = 0, \quad p + q = 2k - 1, \quad (p, q) \notin \{(k, k - 1), (k - 1, k)\}.$$

Note that by [Gri69] (see also [Voi07, I,12.1,12.2]) (1.1) is implied by the following property:

$$(\star) \quad \text{The Abel-Jacobi map } \Phi_Y : \text{CH}^k(Y)_{\text{hom}} \rightarrow J^{2k-1}(Y) \text{ is surjective.}$$

Conversely, (1.1) should imply (\star) , according to the Hodge conjecture (see [Voi14, 2.2.5]). Under the assumption (1.1), the intermediate Jacobians $J(Y)$ are abelian varieties, and letting $U \subset B := |L|$ be the Zariski open set parameterizing smooth members, one even has a quasiprojective variety \mathcal{J}_U with a smooth projective morphism

$$\pi_U : \mathcal{J}_U \rightarrow U$$

with fiber $J(Y_t)$ over the point $t \in U$. Here $\mathcal{Y} \subset B \times X$ is the universal family.

Let now $\eta \in H^{k+1, k-1}(X)$ and assume $\eta|_Y = 0$ in $H^{k+1, k-1}(Y)$ for any smooth member Y of $|L|$. (In our main application, where X is the cubic fourfold and $k = 2$, this assumption will be automatic, since $H^{k+1, k-1}(Y_t)$ will be 0.)

Theorem 1.2. *There exists a holomorphic 2-form*

$$\sigma_U \in H^0(\mathcal{J}_U, \Omega_{\mathcal{J}_U}^2)$$

satisfying the following properties:

- (i) *The fibers of the fibration $\pi_U : \mathcal{J}_U \rightarrow U$ are isotropic for σ_U .*
- (ii) *At any point $t \in U$, the map $\lrcorner\sigma_t : T_{U,t} \rightarrow H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t}) = H^{k-1}(Y_t, \Omega_{Y_t}^k)$ induced by σ_U using (i) is constructed as follows: As $\eta|_{Y_t} = 0$ in $H^{k-1}(\Omega_{Y_t}^{k+1})$ and $H^{k-2}(Y_t, \Omega_{Y_t}^{k+1}) = 0$ by (1.1), η determines a class $\tilde{\eta}_t \in H^{k-1}(Y_t, \Omega_{Y_t}^k(-L))$ using the exact sequence*

$$0 \rightarrow \Omega_{Y_t}^k(-L) \rightarrow \Omega_{X|Y_t}^{k+1} \rightarrow \Omega_{Y_t}^{k+1} \rightarrow 0.$$

Then $\tilde{\eta}_t$ gives by multiplication a map $T_{U,t} = H^0(Y_t, L|_{Y_t}) \rightarrow H^{k-1}(\Omega_{Y_t}^k)$ which is nothing but $\lrcorner\sigma_t$.

- (iii) *For any smooth algebraic variety $\overline{\mathcal{J}}$ containing \mathcal{J}_U as a Zariski open set, the 2-form σ_U extends to a holomorphic 2-form on $\overline{\mathcal{J}}$.*
- (iv) *The 2-form σ_U is closed.*

We will first prove the theorem assuming that the fibers satisfy property (\star) and will explain at the end how to adapt the proof when they only satisfy property (1.1).

Construction of the 2-form σ_U . We denote by $u : \mathcal{Y}_U \rightarrow U$ the first projection, where $\mathcal{Y}_U \subset U \times X$ is the universal hypersurface. We have:

Lemma 1.3. *Assuming (\star) , there exists a codimension k cycle*

$$\mathcal{Z} \in \text{CH}^k(\mathcal{J}_U \times_U \mathcal{Y}_U)_{\mathbb{Q}}$$

such that the cohomology class $\alpha := [\mathcal{Z}] \in H^{2k}(\mathcal{J}_U \times_U \mathcal{Y}_U, \mathbb{Q})$, or rather its image α_0 in $H^0(U, R^{2k}(\pi, u)_ \mathbb{Q})$, satisfies the condition that*

$$\alpha_0^* : R^{2k-1}u_* \mathbb{Q} \rightarrow R^1\pi_* \mathbb{Q}$$

is the natural isomorphism.

Proof. By assumption (\star) , for each fiber Y_t , $t \in U$, there exist a smooth and projective variety W_t and a family of codimension k cycles $\mathcal{T}_t \in \text{CH}^k(W_t \times Y_t)$, with the property that the Abel-Jacobi map

$$\Phi_{\mathcal{T}_t} : W_t \rightarrow J(Y_t), \quad w \in W_t \mapsto \Phi_{Y_t}(\mathcal{T}_{t,w})$$

is surjective. It follows that there exists a codimension k cycle $\mathcal{Z}'_t \in \text{CH}^k(J(Y_t) \times Y_t)$ such that the Abel-Jacobi map

$$\Phi_{\mathcal{Z}'_t} : J(Y_t) \rightarrow J(Y_t), \quad w \in J(Y_t) \mapsto \Phi_{Y_t}(\mathcal{T}_{t,w})$$

is N times the identity of $J(Y_t)$ for some integer $N > 0$. Indeed, we may assume that the surjective morphism $\Phi_{\mathcal{T}_t} : W_t \rightarrow J(Y_t)$ is generically finite of degree N , by replacing W_t by a linear section if necessary. Then we set

$$\mathcal{Z}'_t := (\Phi_{\mathcal{T}_t}, \text{Id}_{Y_t})_* \mathcal{T}_t.$$

The cycles \mathcal{Z}'_t have been defined fiberwise, but standard argument show that for an adequate choice of N they can be constructed in family over a smooth generically finite cover V of U by spreading the original cycles $W_{t,s}$. This provides a codimension k cycle $\mathcal{Z}'' \in \text{CH}^k(\mathcal{J}_V \times_V \mathcal{Y}_V)$ such that the class $\alpha'' := [\mathcal{Z}'']$ satisfies

$$\alpha''^* : R^{2k-1}u'_*\mathbb{Q} \rightarrow R^1\pi'_*\mathbb{Q}$$

is N times the natural isomorphism, where $u' : \mathcal{Y}_V \rightarrow V$, $\pi' : \mathcal{J}_V \rightarrow V$ are the natural maps. We can choose a partial smooth completion \bar{V} of V such that the morphism $V \rightarrow U$ extends to a proper morphism $r : \bar{V} \rightarrow U$. We next extend the cycle \mathcal{Z}'' to a cycle $\bar{\mathcal{Z}}'' \in \text{CH}^k(\mathcal{J}_{\bar{V}} \times_{\bar{V}} \mathcal{Y}_{\bar{V}})$. If $M = \deg r$, the cycle $\frac{1}{MN}\tilde{r}_*\bar{\mathcal{Z}}'' \in \text{CH}^k(\mathcal{J}_V \times_V \mathcal{Y}_V)$ satisfies the desired property, where $\tilde{r} : \mathcal{J}_{\bar{V}} \times_{\bar{V}} \mathcal{Y}_{\bar{V}} \rightarrow \mathcal{J}_V \times_V \mathcal{Y}_V$ is the natural degree M induced map. \square

Having the lemma, we now observe that there is a natural proper morphism $q' : \mathcal{J}_U \times_U \mathcal{Y}_U \rightarrow \mathcal{J}_U \times X$ given by the inclusion $Y_t \hookrightarrow X$ over any point of \mathcal{J}_t , for any t of U , and we thus get a codimension $k+1$ cycle $\mathcal{Z}_q := q'_*\mathcal{Z} \in \text{CH}^{k+1}(\mathcal{J}_U \times X)_{\mathbb{Q}}$ with Betti cohomology class $[\mathcal{Z}_q] \in H^{2k+2}(\mathcal{J}_U \times X, \mathbb{Q})$ and Dolbeault cohomology class $[\mathcal{Z}_q] \in H^{k+1}(\mathcal{J}_U \times X, \Omega_{\mathcal{J}_U \times X}^{k+1})$. For any $\eta \in H^{k-1}(X, \Omega_X^{k+1})$, the corresponding class $\sigma_U \in H^0(\mathcal{J}_U, \Omega_{\mathcal{J}_U}^2)$ is defined by the formula

$$(1.4) \quad \sigma_U = ([\mathcal{Z}_q]^{k+1, k+1})^*(\eta).$$

This completes the construction of the form σ_U . \square

Proof of properties (i), (iii) and (iv) of Theorem 1.2. With the notation $p_1 : \mathcal{J}_U \times_U \mathcal{Y}_U \rightarrow \mathcal{J}_U$ for the first projection, $p_2 : \mathcal{J}_U \times_U \mathcal{Y}_U \rightarrow \mathcal{Y}_U$ for the second projection, and $q : \mathcal{Y}_U \rightarrow X$ for the natural map, formula (1.4) gives as well

$$(1.5) \quad \sigma_U = ([\mathcal{Z}]^{k+1, k+1})^*(q^*\eta),$$

where $([\mathcal{Z}]^{k+1, k+1})^* : H^{k-1}(\mathcal{Y}_U, \Omega_{\mathcal{Y}_U}^{k+1}) \rightarrow H^0(\mathcal{J}_U, \Omega_{\mathcal{J}_U}^2)$ is defined by

$$([\mathcal{Z}]^{k+1, k+1})^*(\omega) = p_{1*}([\mathcal{Z}]^{k+1, k+1} \cup p_2^*\omega).$$

(i) follows immediately from (1.5) which gives for $t \in U$

$$\sigma_U|_{\mathcal{J}_t} = ([\mathcal{Z}]_{\mathcal{J}_t \times Y_t}^{k+1, k+1})^*(\eta|_{Y_t}),$$

and from the fact that $\eta|_{Y_t} = 0$ in $H^{k+1, k-1}(Y_t)$ by assumption. (Here we identify the fiber Y_t of the universal family and its image in X .)

(iii) We observe that if $\bar{\mathcal{J}} \supseteq \mathcal{J}_U$ is a smooth algebraic partial compactification of \mathcal{J}_U , the cycle \mathcal{Z}_q extends to a cycle $\bar{\mathcal{Z}}_q \in \text{CH}^{k+1}(\bar{\mathcal{J}} \times X)$, so that its cohomology class $[\mathcal{Z}_q]^{k+1, k+1}$ extends to a class

$$[\bar{\mathcal{Z}}_q]^{k+1, k+1} \in H^{k+1}(\bar{\mathcal{J}} \times X, \Omega_{\bar{\mathcal{J}} \times X}^{k+1}).$$

It thus follows that the form σ_U extends to a 2-form $\sigma \in H^0(\bar{\mathcal{J}}, \Omega_{\bar{\mathcal{J}}}^2)$ given by the formula

$$(1.6) \quad \sigma = ([\bar{\mathcal{Z}}_q]^{k+1, k+1})^*(\eta).$$

(iv) This is an immediate consequence of (iii). Indeed, choosing a smooth projective compactification $\bar{\mathcal{J}} \supseteq \mathcal{J}_U$ of \mathcal{J}_U , the 2-form σ_U extends by (iii) to a holomorphic 2-form σ on $\bar{\mathcal{J}}$. The 2-form σ is closed, hence the original form σ_U is closed. \square

Proof of Theorem 1.2(ii). Consider the class $q^*\eta \in H^{k-1}(\mathcal{Y}_U, \Omega_{\mathcal{Y}_U}^{k+1})$. As it vanishes on fibers Y_t , $t \in U$, of u , it provides for any $t \in U$ a morphism

$$(1.7) \quad \text{int}(\cdot)q^*\eta : T_{U,t} \rightarrow H^{k-1}(Y_t, \Omega_{Y_t}^k)$$

constructed as follows: the cotangent bundle sequence of u

$$0 \rightarrow \Omega_{U,t} \otimes \mathcal{O}_{Y_t} \rightarrow \Omega_{\mathcal{Y}_U|Y_t} \rightarrow \Omega_{Y_t} \rightarrow 0$$

induces an exact sequence

$$(1.8) \quad 0 \rightarrow \Omega_{U,t} \otimes \Omega_{Y_t}^k \rightarrow \Omega_{\mathcal{Y}_U|Y_t}^{k+1}/L^2 \rightarrow \Omega_{Y_t}^{k+1} \rightarrow 0,$$

where $L^2 \subset \Omega_{\mathcal{Y}_U|Y_t}^{k+1}$ is the subbundle $u^*\Omega_{U,t}^2 \wedge \Omega_{\mathcal{Y}_U|Y_t}^{k-1}$. From (2.18), using the fact that $H^{k-2}(Y_t, \Omega_{Y_t}^{k+1}) = 0$, we deduce that the class $q^*\eta|_{Y_t} \in H^{k-1}(Y_t, \Omega_{\mathcal{Y}_U|Y_t}^{k+1})$ lifts to a class $\widetilde{q^*\eta}_t$ in $\Omega_{U,t} \otimes H^{k-1}(Y_t, \Omega_{Y_t}^k) = \text{Hom}(T_{U,t}, H^{k-1}(Y_t, \Omega_{Y_t}^k))$, giving the desired morphism $\text{int}(\cdot)q^*\eta$ of (1.7). We use now formula (1.5) which makes obvious that for any $t \in U$, $v \in T_{U,t}$,

$$(1.9) \quad v \lrcorner \sigma_U = ([\mathcal{Z}]_{|\mathcal{J}_t \times Y_t}^{k,k})^*(\text{int}(v)q^*\eta) \text{ in } H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t}).$$

In the right hand side, we recall that, by construction, the morphism

$$([\mathcal{Z}]_{|\mathcal{J}_t \times Y_t}^{k,k})^* : H^k(Y_t, \Omega_{Y_t}^{k-1}) \rightarrow H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t})$$

is the natural isomorphism. It thus only remains to analyze the morphism $\text{int}(\cdot)q^*\eta$. We observe now that the cotangent bundle sequence (2.18) is compatible with the conormal bundle sequence of Y_t in X , since via the differential q^* of the morphism $q : \mathcal{Y} \rightarrow X$, we get the following commutative diagram :

$$(1.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{Y_t}(-L) & \longrightarrow & \Omega_{X|Y_t} & \longrightarrow & \Omega_{Y_t} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{U,t} & \longrightarrow & \Omega_{\mathcal{Y}_U|Y_t} & \longrightarrow & \Omega_{Y_t} \longrightarrow 0 \end{array}$$

The first vertical map is the natural inclusion dual to the evaluation map $T_{U,t} = H^0(Y_t, L|_{Y_t}) \rightarrow L|_{Y_t}$. Taking $k+1$ -th exterior powers, we get the following commutative diagram:

$$(1.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{Y_t}^k(-L) & \longrightarrow & \Omega_{X|Y_t}^{k+1} & \longrightarrow & \Omega_{Y_t}^{k+1} \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{U,t} \otimes \Omega_{Y_t}^k & \longrightarrow & \Omega_{\mathcal{Y}_U|Y_t}^{k+1}/L^2 & \longrightarrow & \Omega_{Y_t}^{k+1} \longrightarrow 0 \end{array}$$

It clearly follows from the commutativity of diagram (1.11) that

$$(1.12) \quad f(\widetilde{\eta}_t) = \widetilde{q^*\eta}_t \text{ in } \Omega_{U,t} \otimes H^{k-1}(Y_t, \Omega_{Y_t}^k).$$

The proof of (ii) is now a consequence of (1.9) and (1.12). Indeed, for any $v \in T_{U,t}$ we have

$$v \lrcorner \sigma_U = ([\mathcal{Z}]_{|\mathcal{J}_t \times Y_t}^{k,k})^*(\text{int}(v)q^*\eta) = ([\mathcal{Z}]_{|\mathcal{J}_t \times Y_t}^{k,k})^*(\widetilde{q^*\eta}_t(v)) \text{ in } H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t}),$$

where the first equality is (1.9) and the second equality is by definition of $\widetilde{q^*\eta}_t$. The equality (1.12) then tells that

$$\widetilde{q^*\eta}_t(v) = v\widetilde{\eta}_t,$$

where on the right, v is seen as an element of $H^0(\mathcal{O}_{Y_t}(-L))$ and $v\widetilde{\eta}_t$ is the product of v and $\widetilde{\eta}_t$. The proof is thus finished since $([\mathcal{Z}]_{|\mathcal{J}_t \times Y_t}^{k,k})^*$ is the natural identification between $H^{k-1}(Y_t, \Omega_{Y_t}^k)$ and $H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t})$. \square

The proof of Theorem 1.2 assuming property (\star) is now finished. We conclude this section explaining how to modify the arguments only assuming (1.1). If we examine the proofs given above, we see that the key tool is the algebraic cycle $\mathcal{Z} \in \text{CH}^k(\mathcal{J}_U \times_U \mathcal{Y}_U)_{\mathbb{Q}}$ and its image $\mathcal{Z}_q \in \text{CH}^{k+1}(\mathcal{J}_U \times X)_{\mathbb{Q}}$. These cycles appear only through their Dolbeault classes $[\mathcal{Z}]^{k,k}$, $[\mathcal{Z}_q]^{k+1,k+1}$, which are better seen, after extensions to smooth projective varieties, as Hodge classes. In the absence of the cycle \mathcal{Z} that we constructed using

the assumption (\star) we still have the desired Hodge classes, as follows from the following lemma. Below, a Hodge class on a smooth quasi-projective variety Y is by definition the restriction of a Hodge class on a smooth projective compactification \bar{Y} of Y . The set of Hodge classes on Y does not depend on the compactification \bar{Y} . In fact, according to Deligne [Del71], Hodge classes in $H^{2k}(Y, \mathbb{Q})$ are identified with $\text{Hdg}^{2k}(W_{2k}H^{2k}(Y, \mathbb{Q}))$, where $W_{2k}H^{2k}(Y, \mathbb{Q})$ is the smallest weight part of $H^{2k}(Y, \mathbb{Q})$, which is also the image of the restriction map $H^{2k}(\bar{Y}, \mathbb{Q}) \rightarrow H^{2k}(Y, \mathbb{Q})$ for any smooth projective compactification \bar{Y} of Y . Hodge classes $\alpha \in \text{Hdg}^{2k}(H^{2k}(Y, \mathbb{Q}))$ have a Dolbeault counterpart $\alpha^{k,k} \in H^k(Y, \Omega_Y^k)$ (which usually does not determine α in the non projective situation).

Lemma 1.13. *Let X, k, L be as above, satisfying condition (1.1). Then there exists a Hodge class $\alpha \in \text{Hdg}^{2k}(\mathcal{J}_U \times_U \mathcal{Y}_U, \mathbb{Q})$ with the property that the class $\alpha_0 \in H^0(U, R^{2k}(\pi_U, u)_*\mathbb{Q})$ induces the natural isomorphism $H^{2k-1}(Y_t, \mathbb{Q}) \cong H^1(\mathcal{J}_t, \mathbb{Q})$ at any point $t \in U$.*

Proof. The only observation to make is that the canonical isomorphism $H^{2k-1}(Y_t, \mathbb{Q}) \cong H^1(\mathcal{J}_t, \mathbb{Q})$ is an isomorphism of Hodge structures of bidegree $(-k+1, -k+1)$, by the vanishing condition (1.1). Such an isomorphism of Hodge structures provides a degree $2k$ Hodge class α_t on the product $\mathcal{J}_t \times Y_t$. We thus have a section of the local system $R^{2k}(\pi_U, u)_*\mathbb{Q}$ which is a Hodge class at any point of t of U . Deligne's global invariant cycle theorem [Del71] then says that for any smooth projective compactification M of $\mathcal{J}_U \times_U \mathcal{Y}_U$, there exists a cohomology class $\beta \in H^{2k}(M, \mathbb{Q})$ such that $\beta|_{\mathcal{J}_t \times Y_t} = \alpha_t$ for any $t \in U$. Using the facts that α_t is a Hodge class and the restriction morphism $H^{2k}(M, \mathbb{Q}) \rightarrow H^{2k}(\mathcal{J}_t \times Y_t, \mathbb{Q})$ is a morphism of polarized Hodge structures, the semisimplicity of the category of polarized rational Hodge structures allows us to conclude that the class β can be chosen to be Hodge on M (see [Voi14, 2.2.1]). The restriction of β to $\mathcal{J}_U \times_U \mathcal{Y}_U$ is then the desired Hodge class α on $\mathcal{J}_U \times_U \mathcal{Y}_U$. \square

This concludes the proof of Theorem 1.2 assuming only (1.1). We conclude this section observing that, except for (ii), we did not use the condition that \mathcal{Y} is the universal family of smooth divisors in X . Any smooth projective family mapping to X , or even only having a correspondence with X , with fibers satisfying condition (1.1) will do. In practice, (ii) gives a way of deciding whether the constructed 2-form is degenerate or not. So our arguments prove more generally the following variant of Theorem 1.2:

Theorem 1.14. *Let X be a smooth projective variety of dimension n and let $f : \mathcal{Y} \rightarrow U$ be a smooth projective morphism between smooth quasi-projective varieties. Let l, k be integers and let $Z \in \text{CH}^{n-l+k-1}(\mathcal{Y} \times X)_{\mathbb{Q}}$ be a codimension $n-l+k-1$ cycle. Assume that the fibers Y_t of f satisfy condition (1.1) for the given integer k . Let $\pi_U : \mathcal{J}_U \rightarrow U$ be the family of intermediate Jacobians $J^{2k-1}(Y_t)$, $t \in U$. Then*

- (i) *For any class $\eta \in H^{l+2,1}(X)$ such that $Z^*\eta|_{Y_t} = 0$ in $H^{k+1, k-1}(Y_t)$ for any t in U , there is a closed $(2, 0)$ -form $\sigma_U \in H^0(\mathcal{J}_U, \Omega_{\mathcal{J}_U}^2)$ for which the fibers of π_U are isotropic.*
- (ii) *For any smooth algebraic partial compactification $\mathcal{J}_U \subset \bar{\mathcal{J}}$, the $(2, 0)$ -form σ_U extends to a $(2, 0)$ -form σ on $\bar{\mathcal{J}}$.*

Example 1.15. In [IM08], Iliev and Manivel construct a Lagrangian fibration structure on the family of intermediate Jacobians of a smooth cubic fivefolds containing a given cubic fourfold X . We recover the $(2, 0)$ -form as an application of Theorem 1.14: The family \mathcal{Y}_U in this case is the universal family of these cubic fivefolds and the integer k is 3. The cycle $Z \subset X \times \mathcal{Y}_U \cong \mathcal{Y}_U \times X$ is isomorphic to $X \times U$ and will be given by the embedding of X in Y_t for any $t \in U$, hence we have $l = 1$, $n = 4$, $n-l+k-1 = 5$ in this case. Of course, some more work as in Theorem 1.2(ii) is needed to show that the $(2, 0)$ -form is nondegenerate, but our approach shows that this forms extends to any smooth projective compactification.

1.2. The case of the cubic fourfold. The paper will be devoted to the case where $X \subset \mathbb{P}^5$ is a cubic fourfold, $L = \mathcal{O}_X(1)$ and $k = 2$. One has $\dim H^{3,1}(X) = 1$ by Griffiths' theory, and a generator η of $H^{3,1}(X)$ provides thus by Theorem 1.2 a $(2, 0)$ -form σ_U on the family of intermediate jacobians of smooth hyperplane sections of X . We have the following:

Proposition 1.16. *If X is a smooth cubic fourfold, the holomorphic 2-form σ_U is nondegenerate on \mathcal{J}_U .*

Proposition 1.16 already appears in [DM96b], [IM08], [MT03], [Mar12]. The proof given here is slightly different, being an easy application of Theorem 1.2.

Proof of Proposition 1.16. We apply Theorem 1.2. In the case of the family of hyperplane sections of a cubic fourfold, the base U and the fiber $J(Y_t)$ of the family $\mathcal{J}_U \rightarrow U$ are of dimension 5. The 2-form σ_U vanishes along the fibers of π_U and in order to prove it is nondegenerate, it suffices to show that at any point $t \in U$, the map $\lrcorner(\cdot)\sigma_U : T_{U,t} \rightarrow H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t})$ is an isomorphism. Theorem 1.2(ii) tells us that $\lrcorner(\cdot)\sigma_U$ is the following map: the generator η induces for each point $t \in U$ a class $\tilde{\eta}_t \in H^1(Y_t, \Omega_{Y_t}^2(-1))$. Then, using the identification $H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t}) \cong H^1(Y_t, \Omega_{Y_t}^2)$, $\lrcorner(\cdot)\sigma : T_{U,t} = H^0(Y_t, \mathcal{O}_{Y_t}(1)) \rightarrow H^1(Y_t, \Omega_{Y_t}^2)$ is the multiplication map by $\tilde{\eta}_t$. So the statement of Proposition 1.16 is the following:

Claim. *For any $t \in U$, the class $\tilde{\eta}_t \in H^1(Y_t, \Omega_{Y_t}^2(-1))$ induces an isomorphism*

$$(1.17) \quad H^0(Y_t, \mathcal{O}_{Y_t}(1)) \rightarrow H^1(Y_t, \Omega_{Y_t}^2).$$

The proof of the claim follows from the following lemma:

Lemma 1.18. (i) *The class $\tilde{\eta}_t \in H^1(Y_t, \Omega_{Y_t}^2(-1))$ is a nonzero multiple of the extension class e of the normal bundle sequence*

$$(1.19) \quad 0 \rightarrow T_{Y_t} \rightarrow T_{\mathbb{P}^4|Y_t} \rightarrow \mathcal{O}_{Y_t}(3) \rightarrow 0,$$

using the natural identification $\Omega_{Y_t}^2(-1) \cong T_{Y_t}(-3)$.

(ii) *The extension class e has the property that the multiplication map by $e : H^0(Y_t, \mathcal{O}_{Y_t}(1)) \rightarrow H^1(Y_t, \Omega_{Y_t}^2)$ is an isomorphism.*

Proof. (ii) is Griffiths' residue isomorphism (see [Voi07, II, 6.1.3]) and in this case, the statement immediately follows from the exact sequence (1.19) and the fact that $H^0(Y_t, T_{\mathbb{P}^4|Y_t}(-2)) = 0$, and $H^2(Y_t, T_{Y_t})(-2) = 0$. As for (i), this simply follows from the fact that the class $\tilde{\eta}_t \in H^1(Y_t, \Omega_{Y_t}^2(-1))$ is nonzero because $\eta \neq 0$, and $H^1(X, \Omega_X^3(-1)) = 0$, so that $\eta|_{Y_t} \neq 0$. On the other hand, $H^1(Y_t, \Omega_{Y_t}^2(-1)) = H^1(Y_t, T_{Y_t}(-3))$ is 1-dimensional, as follows from the normal bundle sequence (1.19). \square

The proof of Proposition 1.16 is finished. \square

1.3. Another example: Quadric sections of cubic fourfolds. Note that the cubic fourfold X has another family of smooth divisors $Y \subset X$ satisfying condition (1.1), namely the smooth complete intersections $Q \cap X$, where Q is a quadric in \mathbb{P}^5 . The corresponding family $\mathcal{J}_Q \rightarrow U_Q$ of intermediate Jacobians has a basis U_Q of dimension $20 = h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)) - 1$ and fibers of dimension 20. Theorem 1.2 shows that \mathcal{J}_Q has a closed holomorphic 2-form σ_Q which extends to any smooth algebraic compactification $\overline{\mathcal{J}_Q}$ of \mathcal{J}_Q . However the 2-form in this case is only generically nondegenerate:

Lemma 1.20. *The 2-form σ_Q is nondegenerate along a fiber $\mathcal{J}_{Q,t} = J(Y_t)$, where $Y_t = Q_t \cap X$, if and only if the quadric Q_t is nondegenerate.*

Proof. The generator η of $H^1(X, \Omega_X^3) = H^1(X, T_X(-3))$ is the extension class of the normal bundle sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^5|X} \rightarrow \mathcal{O}_X(3) \rightarrow 0.$$

Restricting to $Y_t = Q_t \cap X$, we get the exact sequence

$$0 \rightarrow T_{X|Y_t} \rightarrow T_{\mathbb{P}^5|Y_t} \rightarrow \mathcal{O}_{Y_t}(3) \rightarrow 0,$$

whose extension class must come from the extension class e_t of the normal bundle sequence

$$(1.21) \quad 0 \rightarrow T_{Y_t} \rightarrow T_{Q|Y_t} \rightarrow \mathcal{O}_{Y_t}(3) \rightarrow 0$$

of Y_t in Q_t . In other words, the class $\tilde{\eta}_t \in H^1(Y_t, \Omega_{Y_t}^2(-2)) = H^1(Y_t, T_{Y_t}(-3))$ is nothing but the extension class e_t . It follows that the multiplication map

$$\tilde{\eta}_t : H^0(Y_t, \mathcal{O}_{Y_t}(2)) \rightarrow H^1(Y_t, \Omega_{Y_t}^2)$$

identifies with the map $H^0(Y_t, \mathcal{O}_{Y_t}(2)) \rightarrow H^1(Y_t, \Omega_{Y_t}^2) \cong H^1(Y_t, T_{Y_t}(-1))$ induced by the exact sequence (1.21) twisted by $\mathcal{O}_{Y_t}(-1)$. Looking at the long exact sequence associated to (1.21), we find that this map is an isomorphism if and only if $H^0(Y_t, T_{Q_t|Y_t}(-1)) = 0$. But $H^0(Y_t, T_{Q_t|Y_t}(-1)) = 0$ if and only if Q_t is not singular. \square

1.4. Extensions to nodal fibers. Let X be a smooth cubic fourfold, and $\eta \in H^1(X, \Omega_X^3)$ be a generator of $H^1(X, \Omega_X^3)$. We use as before the notation $\mathcal{Y} \rightarrow B$, $\mathcal{Y}_U \rightarrow U$ for the universal family of hyperplane sections of X . Let $U_1 \subset |\mathcal{O}_X(1)|$ be the Zariski open set parameterizing hyperplane sections of X with at most one ordinary double point. The Jacobian fibration $\pi_U : \mathcal{J}_U \rightarrow U$ has an extension $\pi_{U_1} : \mathcal{J}_{U_1} \rightarrow U_1$ which is smooth (see Lemma 5.3). As the vanishing cycle of the degeneration is not trivial at a point $t \in U_1 \setminus U$, the fiber of π_{U_1} over t is a singular compactification of a \mathbb{C}^* -bundle over $J(\tilde{Y}_t)$, where \tilde{Y}_t is the desingularization of Y_t obtained by blowing-up the node. By Theorem 1.2(iii), the 2-form σ_U extends to a 2-form σ_{U_1} on \mathcal{J}_{U_1} , for which the fibers of π_{U_1} are isotropic by Theorem 1.2(i) (and the fact that the fibers of π_{U_1} are equidimensional). Next, the smooth locus $J(Y_t)_{reg} = \mathcal{J}_{U_1, t, reg}$ is a quasiabelian variety with cotangent space isomorphic to $H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^2(\log E_Y))$, where E_Y is the exceptional divisor of the resolution $\tilde{Y}_t \rightarrow Y_t$.

Our goal in this section is to reprove the following result which can be found in [DM96b, §8.5.2]:

Proposition 1.22. *The extended 2-form σ_{U_1} is everywhere nondegenerate on \mathcal{J}_{U_1} .*

As an immediate corollary, we get the following result:

Corollary 1.23. *Assume \mathcal{J}_{U_1} has a smooth compactification $\bar{\mathcal{J}}$ with boundary $\bar{\mathcal{J}} \setminus \mathcal{J}_{U_1}$ of codimension ≥ 2 in $\bar{\mathcal{J}}$. Then $\bar{\mathcal{J}}$ is holomorphically symplectic.*

Proof. Indeed, the extended 2-form, being nondegenerate away from a codimension 2 closed analytic subset, is everywhere nondegenerate. \square

The proof of Proposition 1.22 is based on the following Lemma 1.25. We have the inclusions

$$\tilde{Y}_t \subset \tilde{X} \subset \tilde{\mathbb{P}}^5,$$

where \tilde{X} , resp. $\tilde{\mathbb{P}}^5$, is the blow-up of X , resp. \mathbb{P}^5 at the singular point of Y_t . We denote by E the exceptional divisor of $\tilde{\mathbb{P}}^5$ and E_X the exceptional divisor of \tilde{X} , so $E_Y = E_X \cap \tilde{Y}_t$. As \tilde{X} is transverse to the exceptional divisor E of $\tilde{\mathbb{P}}^5$ and belongs to the linear system $|\mathcal{O}_{\tilde{\mathbb{P}}^5}(3)(-E)|$, we have a logarithmic tangent bundles sequence

$$(1.24) \quad 0 \rightarrow T_{\tilde{X}}(\log E_X) \rightarrow T_{\tilde{\mathbb{P}}^5}(\log E)|_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}}(3)(-E_X) \rightarrow 0.$$

Here we recall that the logarithmic tangent bundle of a variety equipped with a smooth divisor D is the dual of the logarithmic cotangent bundle determined by D and can be defined as the sheaf of vector fields tangent to D along D .

Lemma 1.25. *The induced map $\lrcorner\sigma_{U_1, t} : T_{U_1, t} \rightarrow H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^2(\log E_Y))$ is constructed as follows.*

- (i) *The extension class $e \in H^1(T_{\tilde{X}}(\log E_X)(-3)(E_X))$ of (1.24), maps naturally to an element $e' \in H^1(T_{\tilde{X}}(\log E_X)(-3)(2E_X))$, which restricted to \tilde{Y}_t comes from a uniquely defined element*

$$(1.26) \quad e_Y \in H^1(\tilde{Y}_t, T_{\tilde{Y}_t}(\log E_Y)(2E_Y)(-3)).$$

- (ii) *One has $T_{\tilde{Y}_t}(\log E_Y)(2E_Y)(-3) = \Omega_{\tilde{Y}_t}^2(\log E_Y)(-1)$, thus*

$$e_Y \in H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^2(\log E_Y)(-1)).$$

- (iii) *The interior product $\lrcorner\sigma_{U_1, t} : T_{U_1, t} \rightarrow H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^2(\log E_Y))$ is given by multiplication $H^0(\tilde{Y}_t, \mathcal{O}_{\tilde{Y}_t}(1)) \xrightarrow{e_Y} H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^2(\log E_Y))$.*

Proof. (i) We write the logarithmic normal bundle sequence for $\tilde{Y}_t \subset \tilde{X}$:

$$0 \rightarrow T_{\tilde{Y}_t}(\log E_Y) \rightarrow T_{\tilde{X}}(\log E_X)|_{\tilde{Y}_t} \rightarrow \mathcal{O}_{\tilde{Y}_t}(1)(-2E_Y) \rightarrow 0$$

which we twist by $\mathcal{O}_{\tilde{Y}_t}(2E_Y)(-3)$. The conclusion then follows from the following easy vanishing statements:

$$H^1(\tilde{Y}_t, \mathcal{O}_{\tilde{Y}_t}(-2)) = 0, \quad H^0(\tilde{Y}_t, \mathcal{O}_{\tilde{Y}_t}(-2)) = 0.$$

Remark 1.27. It is easy to check that e_Y is in fact the class of the logarithmic tangent bundles exact sequence

$$0 \rightarrow T_{\widetilde{Y}_t}(\log E_Y) \rightarrow T_{\widetilde{\mathbb{P}^4}(\log E_{\mathbb{P}^4})}|_{\widetilde{Y}_t} \rightarrow \mathcal{O}_{\widetilde{Y}_t}(3)(-2E_Y) \rightarrow 0$$

associated to the embedding of \widetilde{Y}_t in the blow-up $\widetilde{\mathbb{P}^4}$ of the hyperplane \mathbb{P}^4 containing Y_t at the singular point of Y_t .

(ii) This follows from the fact that $T_{\widetilde{Y}_t}(\log E_Y)$ is dual to $\Omega_{\widetilde{Y}_t}(\log E_Y)$ and that the latter has determinant $K_{\widetilde{Y}_t}(E_Y) = \mathcal{O}_{\widetilde{Y}_t}(-2)(2E_Y)$. Thus

$$\Omega_{\widetilde{Y}_t}^2(\log E_Y) \cong T_{\widetilde{Y}_t}(\log E_Y)(K_{\widetilde{Y}_t}(E_Y)) = T_{\widetilde{Y}_t}(\log E_Y)(-2)(2E_Y).$$

(iii) The morphism $u : \mathcal{Y}_{U_1} \rightarrow U_1$ is smooth along the smooth locus $Y_{t,reg}$ of the fiber Y_t . It follows that the arguments used in the proof of Theorem 1.2(ii) apply along $Y_{t,reg}$, so that we can conclude that the conclusion of (iii) holds true in $H^1(Y_{t,reg}, \Omega_{Y_{t,reg}}^2)$. The proof is then finished using the following lemma:

Lemma 1.28. *The restriction map $H^1(\widetilde{Y}_t, \Omega_{\widetilde{Y}_t}^2(\log E_Y)) \rightarrow H^1(Y_{t,reg}, \Omega_{Y_{t,reg}}^2)$ is an isomorphism.*

Proof. Note that $Y_{t,reg} = \widetilde{Y}_t \setminus E_Y$. Denoting by $j : Y_{t,reg} \rightarrow \widetilde{Y}_t$ the inclusion map, j is an affine map and we have $\Omega_{Y_{t,reg}}^2 = j^*(\Omega_{\widetilde{Y}_t}^2(\log E_Y))$, so that

$$(1.29) \quad \begin{aligned} H^1(Y_{t,reg}, \Omega_{Y_{t,reg}}^2) &= H^1(\widetilde{Y}_t, R^0 j_*(\Omega_{\widetilde{Y}_t}^2(\log E_Y)|_{Y_{t,reg}})) \\ &= \lim_{\rightarrow k} H^1(\widetilde{Y}_t, \Omega_{\widetilde{Y}_t}^2(\log E_Y)(kE_Y)). \end{aligned}$$

The lemma then follows from the following exact sequence:

$$(1.30) \quad 0 \rightarrow \Omega_{E_Y}^2 \rightarrow \Omega_{\widetilde{Y}_t}^2(\log E_Y)|_{E_Y} \rightarrow \Omega_{E_Y} \rightarrow 0.$$

Indeed, we recall that $E_Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ and that $\mathcal{O}_{E_Y}(E_Y) = \mathcal{O}_{E_Y}(-1, -1)$. It follows that for any $k > 0$,

$$H^1(E_Y, \Omega_{E_Y}(kE_Y)) = 0, \quad H^1(E_Y, \Omega_{E_Y}^2(kE_Y)) = 0,$$

and for any $k \geq 0$,

$$H^0(E_Y, \Omega_{E_Y}(kE_Y)) = 0, \quad H^0(E_Y, \Omega_{E_Y}^2(kE_Y)) = 0,$$

Using the exact sequence (1.30), this shows that

$$H^1(E_Y, \Omega_{\widetilde{Y}_t}^2(\log E_Y)|_{E_Y}(kE_Y)) = 0$$

for $k > 0$ and $H^0(E_Y, \Omega_{\widetilde{Y}_t}^2(\log E_Y)|_{E_Y}(kE_Y)) = 0$ for $k \geq 0$. It follows that the map

$$H^1(\widetilde{Y}_t, \Omega_{\widetilde{Y}_t}^2(\log E_Y)(kE_Y)) \rightarrow H^1(Y_{t,reg}, \Omega_{Y_{t,reg}}^2((k+1)E_Y))$$

is an isomorphism for $k \geq 0$, proving the lemma by (1.29). \square

The proof of Lemma 1.25 is now complete. \square

Proof of Proposition 1.22. We just have to prove that σ_{U_1} is nondegenerate at any point of \mathcal{J}_{U_1} over $t \in U_1 \setminus U$ which is a smooth point of the fiber $\mathcal{J}_{U_1,t}$. This is equivalent to proving that the map

$$\sigma_{U_1} : T_{U_1,t} \rightarrow H^0(\mathcal{J}_{U_1,t}, \Omega_{\mathcal{J}_{U_1,t,reg}}) = H^1(\widetilde{Y}_t, \Omega_{\widetilde{Y}_t}^2(\log E_Y)(-E_Y))$$

is an isomorphism. Using Lemma 1.25, the last statement is equivalent to the fact that the multiplication map

$$(1.31) \quad e_Y : H^0(\widetilde{Y}_t, \mathcal{O}_{\widetilde{Y}_t}(3)) \rightarrow H^1(\widetilde{Y}_t, \Omega_{\widetilde{Y}_t}^2(\log E_Y)(-E_Y))$$

is an isomorphism, where $e_Y \in H^1(\widetilde{Y}_t, \Omega_{\widetilde{Y}_t}^2(\log E_Y)(-E_Y)(-1))$. We have

$$\Omega_{\widetilde{Y}_t}^2(\log E_Y)(-E_Y)(-1) \cong T_{\widetilde{Y}_t}(\log E_Y)(-3)(E_Y)$$

and the class e_Y maps to the extension class e'_Y

$$(1.32) \quad 0 \rightarrow T_{\widetilde{Y}_t}(\log E_Y) \rightarrow T_{\widetilde{\mathbb{P}^4}(\log E)}|_{\widetilde{Y}_t} \rightarrow \mathcal{O}_{\widetilde{Y}_t}(3 - 2E_Y) \rightarrow 0$$

of the logarithmic normal bundle sequence of $\widetilde{\mathcal{Y}}_t$ in $\widetilde{\mathbb{P}}^4$, via the natural map

$$H^1(\widetilde{\mathcal{Y}}_t, T_{\widetilde{\mathcal{Y}}_t}(\log E_Y)(-3)(E_Y)) \rightarrow H^1(\widetilde{\mathcal{Y}}_t, T_{\widetilde{\mathcal{Y}}_t}(\log E_Y)(-3)(2E_Y)).$$

An element in the kernel of the multiplication map by e_Y is thus also in $\text{Ker } e'_Y : H^0(\widetilde{\mathcal{Y}}_t, \mathcal{O}_{\widetilde{\mathcal{Y}}_t}(1)) \rightarrow H^1(\widetilde{\mathcal{Y}}_t, T_{\widetilde{\mathcal{Y}}_t}(\log E_Y)(-2)(2E_Y))$ induced by (1.32), hence comes from an element of

$$H^0(\widetilde{\mathcal{Y}}_t, T_{\widetilde{\mathbb{P}}^4}(\log E)_{|\widetilde{\mathcal{Y}}_t}(-2)(2E_Y),$$

and it is easily shown that this space reduces zero. The map $\lrcorner_{\sigma_{U_1}} : T_{U_1, t} \rightarrow H^0(\mathcal{J}_{U_1, t}, \Omega_{\mathcal{J}_{U_1, t}, \text{reg}})$ is thus injective, and hence an isomorphism. \square

2. GOOD AND VERY GOOD LINES

Our main tool for studying degenerations of Intermediate Jacobians of cubic threefolds is Mumford's description of the Intermediate Jacobian $J(Y)$ as a Prym variety $\text{Prym}(\widetilde{C}/C)$. The curve C (and its étale double cover \widetilde{C}) are obtained by projecting from a generic line on the smooth cubic Y . In [CML09] (and subsequently used in [CMGHL15]), we have noted that much of the Prym construction carries on to the mildly singular case provided a careful choice of a line l on (the possibly singular) Y . This provides a powerful tool for studying the degenerations of Intermediate Jacobians. We caution the reader that context in the current paper is slightly different from that of [CML09, CMGHL15] (e.g. see Remark 2.4) forcing us to reprove (under slightly different hypotheses) and strengthen certain results. For convenience, we have tried to make the exposition below mostly self-contained.

Notation 2.1. If X is a cubic fourfold and $Y \subset X$ is a hyperplane section, we denote by $F(Y)$, resp. $F(X)$, $F(Y) \subset F(X)$, the varieties of lines in Y , resp. X . We denote by $[l] \in F(Y)$, resp. $F(X)$ the point parameterizing $l \subset Y$, resp. $l \subset X$.

2.1. Good Lines.

Definition 2.2 (also [CML09, Def. 3.4]). Let $Y \subset \mathbb{P}^4$ be a cubic threefold not containing any plane. A line $l \subset Y$ is *good* if for any plane $P \subset \mathbb{P}^4$ containing l , $P \cap Y$ consists in three distinct lines.

The notion of a good line is obviously important from the point of view of the Prym construction of the intermediate Jacobian of a cubic threefold. Projecting Y from l , we get a conic bundle $\widetilde{Y}_l \rightarrow \mathbb{P}^2$, where \widetilde{Y}_l is the blow-up of Y along l , and the discriminant curve $C_l \subset \mathbb{P}^2$ parameterizing reducible conics has degree 5. The curve \widetilde{C}_l of lines in Y intersecting l is the double cover of C_l with fiber over the point c parameterizing a reducible conic C the set of components of C . Thus if l is good the natural involution acting on \widetilde{C}_l has no fixed point.

Proposition 2.3. *Let X be a general cubic fourfold. Then any hyperplane section Y of X has a good line.*

Remark 2.4. The existence of a good line is proved in [CML09] when Y has singularities of type A_k for $k \leq 5$ or D_4 (i.e. the singularities relevant in the GIT context). Unfortunately, we need to allow some additional ADE singularities (e.g. D_5) as these can appear as singularities of hyperplane sections of general cubic fourfolds. It is very likely that the arguments of [CML09] could be extended to cover the cases needed in this paper, but we prefer to give an alternative proof here.

Proof of Proposition 2.3. Let us say that a line l in X is *special* in X , resp. in Y , if the restriction map $J_X \rightarrow H^0(l, \mathcal{O}_l(2))$, resp. $J_Y \rightarrow H^0(l, \mathcal{O}_l(2))$, has rank ≤ 2 , where J_X , resp. J_Y denotes the degree 2 part of the Jacobian ideal of X , resp. Y . As X is general, lines which are special in X are parameterized by a smooth surface $\Sigma_{sp} \subset F(X)$ (see [Ame09]). Recall from [Voi04] that the variety $F(X)$ has a rational self-map $\phi : F(X) \dashrightarrow F(X)$. The map ϕ associates to $[l] \in F(X)$ the point $[l']$ parameterizing the line $l' \subset X$ constructed as follows: if l is not special in X , that is $[l] \notin \Sigma_{sp}$, there is a unique plane $P_l \subset \mathbb{P}^5$ such that $P_l \cap X = 2l + l'$ as a divisor of P_l , where l' is also a line in X . This defines l' . When X contains no plane, the indeterminacy locus of ϕ is exactly the surface Σ_{sp} , along which the plane P_l above is not unique. Furthermore, the indeterminacies of the map ϕ are solved after blowing-up the surface Σ_{sp} and the induced morphism $\phi : \widetilde{F(X)} \rightarrow F(X)$ is finite if $F(X)$ is general (see [Ame09]). Note that the condition for a line

$l' \subset Y$ of being good will be implied by the slightly stronger fact that l' is non-special in X and for no point $[l] \in F(Y)$, one has $\phi([l]) = [l']$.

We first have:

Lemma 2.5. (i) *If X is smooth, $F(Y)$ is a surface for any hyperplane section Y of X .*

(ii) *If X is general, $F(Y)$ is irreducible and reduced for any hyperplane section of X .*

Proof. (i) We know by [BD85] that $F(X)$ is a smooth hyper-Kähler manifold. The $(2, 0)$ -form σ on $F(X)$ is deduced from the class α generating $H^{3,1}(X)$ by the formula

$$\sigma = P^* \alpha \text{ in } H^{2,0}(F(X)),$$

where $P \subset F(X) \times X$ is the incidence correspondence, so that $p : P \rightarrow F(X)$ is a \mathbb{P}^1 -bundle over $F(X)$. We claim that $F(Y) \subset F(X)$ is Lagrangian for σ . Indeed, denoting by $P_Y \subset F(Y) \times Y$ the incidence correspondence of Y , we observe that, since Y has only isolated singularities, P_Y lifts to a correspondence $P_{\tilde{Y}} \subset F(Y) \times \tilde{Y}$, where \tilde{Y} is a desingularization of Y . If U is any open set contained in the regular locus of $F(Y)_{red}$, we then have

$$\sigma|_U = P^*(\alpha)|_U = P_{\tilde{Y}}^*(j^* \alpha) \text{ in } H^{2,0}(U),$$

where $j : \tilde{Y} \rightarrow X$ is the desingularization map. Thus the vanishing of $\sigma|_U$ follows from the vanishing of $j^* \sigma$ in $H^{3,1}(\tilde{Y})$. To get the last vanishing, observe that \tilde{Y} is smooth of dimension 3 and rationally connected, so that we have $H^{2,0}(\tilde{Y}) = 0$ hence also $H^{3,1}(\tilde{Y}) = 0$. Thus the claim is proved. As $F(Y)$ is Lagrangian in the holomorphically symplectic manifold $F(X)$, it satisfies $\dim F(Y) \leq 2$.

(ii) If X is general, the space $\text{Hdg}^4(F(X))$ of rational Hodge classes of degree 4 on $F(X)$ is of dimension 2. As observed in [Voi10], the class c_2 of $F(Y) \subset F(X)$ is not in the interior of the pseudoeffective cone of classes of codimension 2 cycles in $F(X)$, hence it must be in its boundary. It follows that the class $[F(Y)] \in \text{Hdg}^4(F(X))$ cannot be written as the sum of two nonproportional effective classes. In fact it can neither be written as the sum of two proportional nonzero effective classes. Indeed, coming back to integer coefficients, the class $[F(Y)] \in \text{Hdg}^4(F(X), \mathbb{Z})$ is primitive, that is not divisible by any nonzero integer $\neq \pm 1$, because when X contains a plane P , $F(X)$ contains the dual plane P^* and $[F(Y)] \cdot [P^*] = 1$. It follows that the class $[F(Y)] \in \text{Hdg}^4(F(X), \mathbb{Z})$ cannot be written as the sum of two nonzero classes of effective cycles in $F(X)$. Thus $F(Y)$ is irreducible and reduced. \square

Coming back to the proof of Proposition 2.3, it is clear that for any hyperplane section Y of X , there is a line contained in Y which is nonspecial in X . Indeed, the surface Σ_{sp} of lines which are special in X is irreducible and not contained in the surface of lines in Y ; thus it can intersect $F(Y)$ only along a proper subset. Next, assume to the contrary that there is no good line in Y . This then means that for any $[l] \in F(Y)$, there is a $[l'] \in F(Y)$ such that for some plane $P \subset H_Y$, $P \cap Y = 2l + l'$. (Here we denote by H_Y the hyperplane cutting Y in X .) Note that the line l' is then special for Y .

There are two possibilities:

- (a) The point $[l'] \in F(Y)$ moves in a surface contained in $F(Y)$, hence by Lemma 2.5, any line in Y is special for Y .
- (b) The point $[l']$ moves in a curve $D \subset F(Y)$ and this curve is contained Σ_{sp} . Furthermore, for any $[l'] \in D$, the 3-dimensional projective space $Q_l = \cap_{x \in l} H_{X,x}$ is contained in H_Y , where $H_{X,x}$ denotes the hyperplane tangent to X at x .

In case (a), we get a contradiction as follows: the general line $l' \subset Y$ does not pass through a singular point of Y and the fact that l' is special for Y says exactly, by taking global sections in the normal bundle sequence

$$0 \rightarrow N_{l'/Y}(-1) \rightarrow N_{l'/\mathbb{P}^4}(-1) \rightarrow \mathcal{O}_{l'}(2) \rightarrow 0,$$

that $H^0(N_{l'/Y}(-1)) \neq 0$, hence that $N_{l'/Y} \cong \mathcal{O}_{l'}(1) \oplus \mathcal{O}_{l'}(-1)$. But the fact that $N_{l'/Y}$ has this form says equivalently that the map $q : P_Y \rightarrow Y$ is not submersive at any point of the fiber $P_{Y,[l']} \subset P_Y$ of $p : P_Y \rightarrow F(Y)$ over the point $[l']$. As $q(P_Y) = Y$, this contradicts the fact that $[l']$ is general in $F(Y)$.

The case (b) is excluded by the following lemma which we will use again later.

Lemma 2.6. *Let X be a general cubic fourfold. Then any hyperplane section $Y \subset X$ contains only finitely many cubic surfaces which are singular along a line.*

Proof. Assume to the contrary that there is a curve D of such surfaces and such lines l' of singularities. We note that any line l' parameterized by a point $[l'] \in D$ has to pass through a singular point of Y . Indeed, if Y is smooth along l' , then its defining equation f_Y in H_Y has at least two independent partial derivatives along l' , and thus for any \mathbb{P}^3 contained in H_Y and containing l' , the restriction $f_Y|_{\mathbb{P}^3}$ has at least one partial derivative which does not vanish in $H^0(\mathcal{O}_{l'}(2))$, hence $Y \cap \mathbb{P}^3$ cannot be singular at all points of l' . Next, a hyperplane section Y of X has finitely many singular points, hence we can assume that in case (b), the curve D consists of lines passing through a given singular point y of Y . By Lemma 2.5, the family C_y of lines in X passing through y is a curve, and thus D must be an irreducible component of C_y . In adequate homogeneous coordinates X_0, \dots, X_4 on H_y , the point y has equations $X_i = 0$, $i = 0, \dots, 3$ and Y has equation $X_4Q(X_0, \dots, X_3) + T(X_0, \dots, X_3)$, where Q and T are homogeneous of respective degrees 2, 3. The curve C_y of lines through y (in Y or X) is defined by the equations $Q = T = 0$. Let $[l'] \in C_y$ parameterize a line l' in Y such that some hyperplane H'_y in H_Y containing l' is tangent to Y everywhere along l' . This is saying that the equation $f := X_4Q(X_0, \dots, X_3) + T(X_0, \dots, X_3)$, restricted to a hyperplane H' of \mathbb{P}^3 passing through the point $[l']$, has zero derivatives along l' . Thus the equations Q and T restricted to H' , must have 0 derivative at $[l']$. It follows that the two polynomials Q and T have nonindependent derivatives at $[l']$, so that $[l']$ is a singular point of the curve C_y . In conclusion, we found that under our assumption, the curve C_y has a nonreduced component. Hence the proof of Lemma 2.6 is concluded by the proof of Lemma 2.7 below. \square

Lemma 2.7. *If X is general, the curve C_y of lines through any point $y \in X$ is reduced.*

Remark 2.8. It is not true that C_y is irreducible for any y . Indeed, a general X contains a cubic surface which is a cone over an elliptic curve, with vertex $y \in X$. Hence the elliptic curve is an irreducible component of C_y in this case.

Proof of Lemma 2.7. For any $y \in X$, the curve C_y has degree 6, and the cone over the curve C_y , with vertex y , is a degree 6 surface contained in X . We use now the fact that if X is general, $\text{Hdg}^4(X, \mathbb{Z}) = \mathbb{Z}h^2$, where $h = c_1(\mathcal{O}_X(1))$. Hence any surface in X has degree divisible by 3. Applying this to the components of this cone, the only way the curve C_y can be nonreduced is if C_y is everywhere nonreduced with multiplicity 2. The curve $C_{y,red}$ is then a curve of degree 3 which can be either a plane cubic or a normal rational curve of degree 3. If S is the cone over a normal cubic curve in \mathbb{P}^3 , the set of cubic hypersurfaces in \mathbb{P}^5 containing S has codimension 22 while the dimension of the Hilbert scheme parameterizing such an S in \mathbb{P}^5 is $5 + 4 + 15 - 3 = 21$, so that a general cubic does not contain such a surface. In the case of the cone over an irreducible plane cubic, the curve C_y is the complete intersection of a quadric and a cubic in \mathbb{P}^3 which contains an irreducible plane cubic with multiplicity 2. The only possibility is then that the quadric itself is a double plane. However, one can easily check that for general X , there is no point $y \in X$ where the Hessian of the defining equation of X defines a nonreduced quadric in \mathbb{P}^3 . \square

The proof of Proposition 2.3 is now complete. \square

2.2. Existence of very good lines. For constructing compactified Jacobians (and similarly compactified Pryms) irreducibility assumptions are crucial. This leads us to the following strengthening of the requirement of good line.

Definition 2.9. Let Y be a cubic threefold. We will say that a line $l \subset Y$ is *very good* if l is good (cf. Def. 2.2) and the curve $\tilde{C}_l = \tilde{C}_{l,Y} = \{\text{lines in } Y \text{ meeting } l\}$ is irreducible.

Proposition 2.10. *Let X be a general cubic 4-fold. Then for any hyperplane section $Y \subset X$, there exists a line $l \subset Y$ such that the curve $\tilde{C}_{l,Y}$ is irreducible.*

Corollary 2.11. *If X and Y are as above, a general line in Y is very good.*

Proof. Proposition 2.3 shows the existence of a good line, and this is an open property on $F(Y)$. Proposition 2.10 shows the existence a line $l \subset Y$ such that the curve $\tilde{C}_{l,Y}$ is irreducible and this is also an open property on $F(Y)$. As we know by Lemma 2.5 that $F(Y)$ is irreducible, it follows that a general line is very good. \square

Proof of Proposition 2.10. The incidence variety $P_Y \subset F(Y) \times Y$ is a \mathbb{P}^1 -bundle $p : P_Y \rightarrow F(Y)$ over Y . We proved in Lemma 2.5 that X and Y being as above, $F(Y)$ is irreducible and reduced, thus P_Y satisfies the

same properties. In particular, the degree of the map $q : P_Y \rightarrow Y$ is 6 as for a smooth Y , and the degree of the map $q' = pr_2 : P_Y \times_Y P_Y \setminus \Delta_{P_Y} \rightarrow P_Y$ is 5. We have the following lemma:

Lemma 2.12. *Let X be a general cubic fourfold and let Y be any hyperplane section of X . Then if for any line $l \subset Y$, the curve \tilde{C}_l is reducible, the generically finite cover $q : P_Y \rightarrow Y$ has the property that $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ has at least two irreducible components dominating Y .*

Proof. We observe first that for any line $l \subset Y$, the curve $\tilde{C}_l \subset F(Y)$ (minus the point $[l]$ when l is special for Y) identifies naturally with $q^{-1}(l)$ away from its intersection with the vertical curve $P_{Y,[l]} = p^{-1}([l])$. Indeed, $q^{-1}(l)$ is, away from the vertical fiber $P_{Y,[l]}$, the set of pairs $([l'], x)$ such that $[l] \neq [l']$ and $x \in l \cap l'$. The curve \tilde{C}_l (away from $[l]$ when l is special) thus maps to it via the map

$$[l'] \mapsto ([l'], x), \{x\} = l \cap l'.$$

We will in fact see \tilde{C}_l (minus the point $[l]$) as contained in $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ by the map

$$(2.13) \quad [l'] \mapsto ([l'], [l], x), \{x\} = l \cap l'.$$

Now suppose that \tilde{C}_l is reducible for any l . For general l , \tilde{C}_l is reduced and the morphism $q'_l = q'_{|\tilde{C}_l} : \tilde{C}_l \rightarrow l$ has degree 5. So, for general l , we must be in one of the following situations:

- (1) \tilde{C}_l has no component of degree 1 over l and has a unique component $\tilde{C}_{l,2}$ of degree 2 over l . We then denote $\tilde{C}_{l,3}$ the Zariski closure of $\tilde{C}_l \setminus \tilde{C}_{l,2}$.
- (2) \tilde{C}_l has $1 \leq k < 4$ components $\tilde{C}_{l,1,i}$ of degree 1 over l and the rest $\tilde{C}_{l,rem} = \overline{\tilde{C}_l \setminus \cup_i \tilde{C}_{l,1,i}}$ has all its components of degree > 1 over \tilde{C}_l .
- (3) \tilde{C}_l has five components of degree 1 over l .

If case 1 or case 2 happens, then $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ has at least two irreducible components dominating P_Y , namely the two varieties $\cup_{[l] \in F(Y)} \tilde{C}_{l,2}$ and $\cup_{[l] \in F(Y)} \tilde{C}_{l,3}$ in case 1, and the two varieties $\cup_{[l] \in F(Y), 1 \leq i \leq k} \tilde{C}_{l,1,i}$ and $\cup_{[l] \in F(Y)} \tilde{C}_{l,rem}$ in case 2. It thus suffices to show that 3 cannot happen. This however follows from Proposition 2.3. Indeed we may assume that l is good, so that the involution on \tilde{C}_l has no fixed point. But if \tilde{C}_l has five irreducible components all isomorphic to \mathbb{P}^1 , one of them is fixed under the involution which then has fixed points. \square

Proposition 2.10 now follows from Lemma 2.12 and from the following:

Lemma 2.14. *If X is general, for any hyperplane section Y of X , the variety $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ is irreducible.*

Let us prove a few intermediate statements:

Lemma 2.15. *If X is general, and Y is any hyperplane section of X , any irreducible component of $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ dominates P_Y by the second projection.*

Proof. The only possibility for an irreducible component Z of $P_Y \times_Y P_Y$ not to dominate P_Y by the second projection is if there is a curve $W \subset Y$ such that for any $y \in W$, there is a curve D_y of lines in Y passing through y . However, this is impossible by the following claim.

Claim 2.16. *For general X , and for any hyperplane section Y of X , there are only finitely many points $y \in Y$ such that the set of lines in Y through y contains a curve D_y .*

Proof. Assume by contradiction that this set is a curve W . Then the union over $y \in W$ of the D_y 's would then be a surface contained in $F(Y)$, and since we know by Lemma 2.5 that $F(Y)$ is irreducible, this surface would be the whole of $F(Y)$. Thus Y has the property that any line in Y meets the curve $W \subset Y$. But the general point $[l] \in F(Y)$ is a smooth point of $F(Y)$ parameterizing a line l with normal bundle $N_{l/X} \cong \mathcal{O}_l \oplus \mathcal{O}_l$, which means that the morphism $q : P_Y \rightarrow Y$ is étale in a neighborhood of the fiber $P_{Y,[l]}$ of P_Y over $[l] \in F(Y)$, so that the general deformation of l in Y does not intersect Z . \square

Lemma 2.15 is thus proved. \square

The variety $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ has several rational involutions. We will denote by τ the involution exchanging factors and by i the involution of $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ which maps (l_1, l_2) , $l_1 \cap l_2 \neq \emptyset$ to (l_3, l_2) where l_3 is the residual line of the intersection $P_{l_1, l_2} \cap Y$, P_{l_1, l_2} being the plane generated by l_1 and l_2 . Recall from (2.13) that the fiber of the composite map

$$P_Y \times_Y P_Y \xrightarrow{pr_2} P_Y \xrightarrow{p} F(Y)$$

over $[l] \in F(Y)$ identifies with the curve \tilde{C}_l of lines in Y meeting l and that i acts on \tilde{C}_l as the Prym involution. The quotient \tilde{C}_l/i is the discriminant curve C_l of the conic bundle $\pi_l : \tilde{Y}_l \rightarrow \mathbb{P}^2$.

Lemma 2.17. *If X is general, and Y is any hyperplane section of X , the quotient*

$$(P_Y \times_Y P_Y \setminus \Delta_{P_Y})/i$$

is irreducible.

Proof. Using Lemma 2.15, it suffices to show that, X and Y being as above, for general $[l] \in F(Y)$, the curve C_l is irreducible. The curve C_l is a quintic curve, so if it is not irreducible, it must decompose either as

- (a) the union of a line and a quartic, or
- (b) the union of a smooth conic and a cubic.

Case (a) is excluded as follows. A line component in the discriminant curve provides a cubic surface $S_l = \mathbb{P}_l^3 \cap Y$ which contains a one parameter family of lines meeting l . The surface S_l is irreducible because X is general so does not contain a plane or a quadric surface. Furthermore S_l cannot be a cone over an elliptic curve, because l is general, hence by Claim 2.16, through any point y of l there are only finitely many lines in Y through y . For any plane $P \subset \mathbb{P}_l^3$ containing l , the intersection $P_l \cap S_l$ contains l and a residual reducible conic. The singular point of the conic moves in S_l , hence by Bertini, the singular locus of S_l consists of a curve \bar{Z} and thus must be a line which is a special line of X whose associated \mathbb{P}^3 is contained in H_Y . We know by Lemma 2.6 that there are finitely many such surfaces contained in Y , so the general line l in Y cannot be contained in such a surface.

Case (b) is excluded as follows : Suppose the discriminant curve C_l has a component which is a smooth conic C . As l is a good line, the double cover $r : \tilde{C}_l \rightarrow C_l$ is étale, hence split over C : $r^{-1}(C) = C_1 \cup C_2$. Let $\pi_l : \tilde{Y}_l \rightarrow \mathbb{P}^2$ be the linear projection from l , and let $T := \pi_l^{-1}(C)$. Then T is a reducible surface, $T = T_1 \cup T_2$, where T_1 is swept-out by lines in C_1 and T_2 is swept-out by lines in C_2 . On the other hand, as X is general, any surface in X has degree divisible 3, thus T_1 and T_2 must have degree 3. The surfaces T_1 and T_2 are ruled surfaces using their 1-parameter family of lines intersecting l , and none of them can be contained in a projective subspace $P_3 \subset H_Y$, since otherwise P_3 would contain l and thus would project via π_l to a line in \mathbb{P}^2 while the image $\pi_l(T_i)$ is our smooth conic.

Finally, a ruled nondegenerate degree 3 surface in \mathbb{P}^4 is a cone over a rational normal curve or a projection of a Veronese surface from one of its points. The first case corresponds to the vector bundle $\mathcal{O} \oplus \mathcal{O}(3)$ on \mathbb{P}^1 and the second case corresponds to the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(2)$ on \mathbb{P}^1 . We already explained by counting parameters that the general cubic hypersurface X in \mathbb{P}^5 does not contain a cone over a rational cubic curve in \mathbb{P}^3 . It is also true that the general cubic hypersurface X in \mathbb{P}^5 does not contain the projection of a Veronese surface from one of its point, but this does not follow from an immediate dimension count. One has to argue as follows: this surface V is smooth with $c_1(V)^2 = 8$, $c_2(V) = 4$. If $V \subset X$, the normal bundle $N_{V/X}$ fits into the exact sequence

$$(2.18) \quad 0 \rightarrow T_V \rightarrow T_{X|V} \rightarrow N_{V/X} \rightarrow 0.$$

As

$$c_2(T_X) = 6h^2, \quad c_1(T_X) = 3h,$$

one gets from (2.18) the following equalities in $H^*(V, \mathbb{Q})$:

$$(2.19) \quad \begin{aligned} c_1(N_{V/X}) &= -c_1(T_V) + 3h_V, \quad c_2(N_{V/X}) = -c_2(T_V) - c_1(N_{V/X}) \cdot c_1(T_V) + 6h_V^2 \\ &= -c_2(T_V) - (-c_1(T_V) + 3h_V) \cdot c_1(T_V) + 6h_V^2, \end{aligned}$$

where $h_V = c_1(\mathcal{O}_V(1))$ and $h_V^2 = 3$. In the ruled surface $T = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \xrightarrow{\pi} \mathbb{P}^1$, one has

$$K_V = -c_1(T_V) = -2h_V + \pi^* \mathcal{O}_{\mathbb{P}^1}(3),$$

which combined with (2.19) gives

$$c_2(N_{V/X}) = -4 - (-2h_V + \pi^* c_1(\mathcal{O}_{\mathbb{P}^1}(3)) + 3h_V) \cdot (2h_V - \pi^* c_1(\mathcal{O}_{\mathbb{P}^1}(3))) + 18 = 5.$$

This shows that the self-intersection of V in X is equal to 5 so that the class of V is not equal to h^2 . Hence such a surface does not exist for general X . \square

We get the following corollary (where again X is general and Y is any hyperplane section of X):

Corollary 2.20. (i) *The fibered product $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ has at most two irreducible components, and if it is reducible, they are exchanged by the rational involution i acting on each curve $\tilde{C}_l \times [l] \subset P_Y \times_Y P_Y$.*
(ii) *Let τ be the involution of $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ exchanging factors. Then if it is reducible, its two components $\mathcal{C}_1, \mathcal{C}_2$ are exchanged by τ .*

Proof. (i) is an immediate consequence of Lemmas 2.17 and 2.15. The proof of (ii) goes as follows: We observe that the two rational involutions σ and τ are part of an action of the symmetric group \mathfrak{S}_3 on $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ by birational maps. Indeed, $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ can also be seen as the set of labelled triangles, that is, triples (l_1, l_2, l_3) of lines in Y , such that for some plane $P \subset H_Y$, $P \cap Y = l_1 + l_2 + l_3$. The action of \mathfrak{S}_3 is simply the permutation of the labels. Note that these birational maps are well defined at each generic point of $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ by Lemma 2.15. The involution i is the involution $(l_1, l_2) \mapsto (l_3, l_2)$ while the involution τ is the involution $(l_1, l_2) \mapsto (l_2, l_1)$. In any case, these two involutions are conjugate in \mathfrak{S}_3 . We know by (i) that if there are two irreducible components, they are exchanged by i . Hence they must be also exchanged by τ . \square

Proof of Lemma 2.14. let X be a general cubic fourfold and let Y be any hyperplane section of X . Assume by contradiction that $P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ is not irreducible. By Corollary 2.20, it has then exactly two components $\mathcal{C}_1, \mathcal{C}_2$. Both components dominate Y by Lemma 2.15. Let $1 \leq k_1 < k_2$, $k_1 + k_2 = 5$ be the respective degrees of $pr_2 : \mathcal{C}_1 \rightarrow P_Y, \mathcal{C}_2 \rightarrow P_Y$. One has $(k_1, k_2) = (2, 3)$ or $(k_1, k_2) = (1, 4)$. For a general point $y \in Y$, denote by $\{l_1, \dots, l_6\} = q^{-1}(y) =: E_y \subset P_Y$. For $l_i \neq l_j$, we have $(l_i, l_j) \in P_Y \times_Y P_Y \setminus \Delta_{P_Y}$ and thus we can write $E_y \times E_y \setminus \Delta_{E_y}$ as a disjoint union

$$(2.21) \quad E_y \times E_y \setminus \Delta_{E_y} = E_{1,y} \sqcup E_{2,y},$$

where

$$E_{1,y} := (E_y \times E_y \setminus \Delta_{E_y}) \cap \mathcal{C}_1, \quad E_{2,y} := (E_y \times E_y \setminus \Delta_{E_y}) \cap \mathcal{C}_2.$$

The partition (2.21) satisfies:

(a) For any $i \in \{1, \dots, 6\}$, the set of $j \neq i$ such that $(l_i, l_j) \in E_{1,y}$ has cardinality k_1 and the set of $j \neq i$ such that $(l_i, l_j) \in E_{2,y}$ has cardinality k_2 .

(b) For any $i, j \in \{1, \dots, 6\}$ with $i \neq j$, $(l_i, l_j) \in E_{1,y} \Leftrightarrow (l_j, l_i) \in E_{2,y}$.

Indeed, (b) is exactly Corollary 2.20(ii).

The contradiction is now obvious: Indeed, (a) shows that the cardinality of $E_{1,y}$ is $6k_1$ and the cardinality of $E_{2,y}$ is $6k_2$, with $6k_1 \neq 6k_2$, while (b) implies that both sets have the same cardinality. \square

Proposition 2.10 is thus proved. \square

3. TRANSVERSALITY ARGUMENTS

This section is devoted to applying transversality arguments in order to deduce that some statements which hold for general cubic threefolds in large codimension hold for any hyperplane sections of a general cubic fourfold. In particular, we will first prove Lemma 3.8 which guarantees that the versality statement of [CML09] is actually satisfied by the family of quintic curves associated to the family of hyperplane sections of a general cubic fourfold and a local choice of good lines in them. This will be needed in Section 4. In Section 3.2, we will extend this result to the case of a general Pfaffian cubic fourfold. We will also prove by applying a similar transversality argument the existence of a very good line in any hyperplane section of a general Pfaffian cubic fourfold. This will be needed in order to make the arguments of Section 4 apply as well when the cubic fourfold is a general Pfaffian cubic (see Section 6).

3.1. Transversality results for general cubic fourfolds. Below, we denote by \mathcal{M}_X^0 the open subset of $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))$ parameterizing smooth cubic fourfolds, and by \mathcal{M}_Y the projective space $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$. By restriction from \mathbb{P}^5 to a given $\mathbb{P}^4 \subset \mathbb{P}^5$ we get a morphism

$$r : \mathcal{M}_X^0 \rightarrow \mathcal{M}_Y$$

which is obviously smooth, since the fibers are Zariski open subsets of $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$. With these notations, the following transversality lemma holds:

- Lemma 3.1.** (i) *Let $Z \subset \mathcal{M}_Y$ be a closed algebraic subset of codimension ≥ 6 , which is invariant under the action of $\mathrm{PGL}(5)$. Then for a general $[X] \in \mathcal{M}_X$, no hyperplane section of X is isomorphic to a cubic threefold Y parameterized by a point of Z .*
- (ii) *Let $\mathcal{M}'_X \subset \mathcal{M}_X^0$ be a hypersurface which is invariant under $\mathrm{PGL}(5)$, and let $Z \subset \mathcal{M}_Y$ be a closed algebraic subset of codimension ≥ 7 , which is invariant under the action of $\mathrm{PGL}(5)$. Then for a general $[X] \in \mathcal{M}'_X$, no hyperplane section of X is isomorphic to a cubic threefold Y parameterized by a point of Z .*

Proof. (i) Indeed, as r is smooth (actually, flat would suffice), $r^{-1}(Z) \subset \mathcal{M}_X^0$ has codimension 6 in \mathcal{M}_X^0 . The group $\mathrm{PGL}(6, 5) \subset \mathrm{PGL}(6)$ of automorphisms of \mathbb{P}^5 preserving \mathbb{P}^4 acts on \mathcal{M}_X^0 preserving $r^{-1}(Z)$ since Z is invariant under $\mathrm{PGL}(5)$. It thus follows that

$$\dim \mathrm{PGL}(6) \cdot r^{-1}(Z) \leq 5 + \dim r^{-1}(Z),$$

or equivalently that $\mathrm{codim} \mathrm{PGL}(6) \cdot r^{-1}(Z) \geq 1$. Thus $\mathrm{PGL}(6) \cdot r^{-1}(Z)$ is not open in \mathcal{M}_X^0 , which proves (i).

(ii) The same argument as in (1) shows that $\mathrm{PGL}(6) \cdot r^{-1}(Z)$ has codimension at least 2 in \mathcal{M}_X^0 , hence cannot contain the hypersurface \mathcal{M}'_X . \square

Remark 3.2. We will see in Section 3.2 an improved version of Lemma 3.1 (ii), where under a certain assumption on the hypersurface \mathcal{M}'_X , the estimate on codimension of Z will be also 6, not 7. The hypersurface of interest for us will be the locus of Pfaffian cubics.

The above lemma allows us to exclude from considerations highly singular cubic threefolds. Specifically, for our purposes a mildly singular cubic threefold means the following:

Definition 3.3. Let Y be a cubic threefold. We say Y is *allowable* (or mildly singular) if Y has at worst isolated singularities and $\tau_{tot}(Y) \leq 6$. Here τ denotes the Tjurina number of an isolated hypersurface singularity, and $\tau_{tot}(Y)$ is the sum of the associated Tjurina numbers, i.e. $\tau_{tot}(Y) = \sum_{p \in \mathrm{Sing}(Y)} \tau(Y_p)$ (where Y_p denotes the germ of Y at p).

Remark 3.4. We recall that for an isolated hypersurface singularity $(V(f), 0) \subset \mathbb{C}^n$, the Tjurina number is defined to be $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n] / \langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$, and it represents the expected codimension in moduli to encounter that singularity. In other words, by dimension count, we expect that all hyperplane sections Y of a general cubic fourfold X are allowable in the sense of Definition 3.3. The results below say that this is indeed the case.

Remark 3.5. The arguments involving good lines are closely related and inspired by those in [CML09]. However, as already noted in Remark 2.4, the results of [CML09] do not suffice here. Namely, in [CML09] the focus was on GIT stable/semistable cubic threefolds, while here we focus on hyperplane sections of general cubic fourfolds (or general Pfaffian cubics). Thus, our notion of allowable is slightly different from that of [CML09, Def. 2.2].

Proposition 3.6. *Let Y be an allowable cubic threefold (c.f. Def. 3.3). Then, the following hold:*

- (0) *Y has worst ADE (in particular planar) singularities.*
- (1) *The deformations of Y in \mathbb{P}^4 induce a simultaneous versal deformation of the singularities of Y . This means equivalently that the natural map from the first order deformation space of Y , that is $H^0(Y, \mathcal{O}_Y(3))$, to the product $\prod_{p \in \mathrm{Sing} Y} T_{Y_p}^1$, where $T_{Y_p}^1$ classifies the first order deformations of the germ of singularities of Y at p , is surjective.*

- (2) Assume additionally that there exists a good line $l \subset Y$. Let \tilde{C}_l be the curve of lines in Y meeting l . Then the singular points of Y are in bijection with the singular points of the curve $C_l = \tilde{C}_l/\iota$ (which is a plane quintic curve), the analytic types of the singularities of Y and C_l coincide and the deformation theory of the singular points of Y and C_l coincide. Furthermore, the deformations of C_l give simultaneous versal deformations of the singularities of C_l (which is compatible with the deformations of the singularities of Y).
- (3) The locus of cubic hypersurfaces $Y \subset \mathbb{P}^4$ with non-allowable singularities has codimension ≥ 7 in the space $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$ of all cubic threefolds.
- (4) Y does not admit a non-trivial vector field.

Proof. A non-ADE hypersurface singularity has Tjurina number $\tau \geq 7$, giving (0).

The simultaneous versality statement (1) is a specialization of a result of Shustin–Tyomkin [ST99, Main Theorem] to the case of cubic threefolds (in fact, $\tau_{tot}(Y) \leq 7$ suffices; see also [dPW98, Lemma 3.3(i)] which gives the stronger results that $\tau_{tot}(Y) \leq 15$ suffices for cubic threefolds; [CML09, (5) on p. 35] gives the simultaneous versality for GIT stable cubics).

The correspondence of singularities under the projection from a good line is [CML09, Proposition 3.6]. Clearly we get $\tau_{tot}(C_l) \leq 6$, which then implies that the deformations of C_l give simultaneous versal deformations of its singularities (e.g. [Shu87] says that $\tau_{tot}(C_l) < 4(d-1) = 16$ suffices). Finally, the compatibility between the global-to-local deformations of Y and C_l is discussed in [CMGHL15, §3.3].

The expected codimension for the equisingular deformations of a singular cubic Y is $\tau_{tot}(Y)$. Thus, the locus of cubics with $\tau_{tot}(Y) \geq 7$ is expected to have codimension 7 in $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$. The simultaneous versality statements cited above (esp. [dPW98, Lemma 3.3(i)]) guarantee that the expected codimension is the actual codimension for cubics with $\tau_{tot}(Y) \leq 15$. It remains to check that the more degenerate cases (cubics with $\tau_{tot}(Y) > 15$ or non-isolated singularities) have still codimension ≥ 7 . This is an easy case by case analysis that we omit (N.B. the main tool here is to study a singular cubic via the associated (2, 3) complete intersection in \mathbb{P}^3 , see [CML09, §3.1]).

The cases when Y is GIT semistable (in particular, if Y has at worst A_1, \dots, A_5 or D_4 singularities) and has positive dimensional stabilizer are classified by Allcock (e.g. [All03]). It follows that either $\tau_{tot}(Y) \geq 10$ or Y has non-isolated singularities (in fact Y is the chordal cubic). Assuming that Y is not GIT semistable and that $\tau_{tot}(Y) \leq 6$ leads to a small number of cases that can be excluded by a case by case analysis. Namely, Y is stabilized either by G_m or G_a . The unipotent case can be seen not to occur using the classification of [dPW10]. Finally, if Y is stabilized by G_m it has at least two singularities. Under our assumptions (in particular, GIT unstable), the only possibility is Y has a D_5 singularity and an A_1 singularity, which can be then excluded by studying the associated (2, 3) curve (obtained by projecting from the A_1 singular point). \square

An immediate consequence of Propositions 2.3 and 3.6(3) and Lemma 3.1 is:

Corollary 3.7. *If X is a general cubic fourfold, any hyperplane section Y of X is allowable, hence satisfies properties (0)-(4) of Proposition 3.6. Moreover, it has a very good line.*

Note however that since we are restricting to the universal family \mathcal{Y}/B of hyperplane sections of a fixed cubic fourfold X , the simultaneous versal statement of Proposition 3.6(1) does not suffice for our purposes. What is needed instead is the following lemma which follows from Proposition 3.6 and a transversality argument.

Lemma 3.8. *Let $X \subset \mathbb{P}^5$ be a general cubic fourfold, and let Y be any hyperplane section of X . Then the natural morphism $H^0(Y, \mathcal{O}_Y(1)) \rightarrow \prod_{p \in \text{Sing } Y} T_{Y,p}^1$ is surjective. (In other words, the family of deformations of Y in X induces a versal deformation of the singularities of Y .)*

Corollary 3.9. *In the situation of Lemma 3.8, denote by \mathcal{F}_{good} the universal family of good lines in hyperplane sections of X :*

$$\mathcal{F} = \{([l], t) \in G(2, 6) \times (\mathbb{P}^5)^*, l \subset Y_t\},$$

Then if X is general, $Y_0 \subset X$ is any hyperplane section and $l \subset Y_0$ is a general good line, the natural map $T_{\mathcal{F}_{good}, ([l], 0)} \rightarrow \bigoplus_{p \in \text{Sing } C_{l, Y_0}} T_{C_{l,p}}^1$ is surjective. Furthermore, for a local analytic or étale section $B \subset \mathcal{F}_{good}$ of the second projection defined near 0, the natural map $T_{B, ([l], 0)} \rightarrow \bigoplus_{p \in \text{Sing } C_{l, Y_0}} T_{C_{l,p}}^1$ is surjective.

The corollary is indeed an immediate consequence of the lemma and of Proposition 3.6(2) above.

Proof of Lemma 3.8. We first note that for a general cubic fourfold X , and for any hyperplane section $Y \subset X$, Y has no nonzero vector field, or equivalently, the isotropy group $\{g \in \mathrm{GL}(5), g^* f_Y = f_Y\}$ is finite, where $f_Y \in H^0(\mathcal{O}_{\mathbb{P}^4}(3))$ is the defining equation of Y . This follows from Lemma 3.1 and Proposition 3.6(4). From now on, we will work with the Zariski open subset $\mathcal{M}_X^{00} \subset \mathcal{M}_X^0$ of cubic fourfolds with no nontrivial automorphisms and whose hyperplane sections have no nonzero infinitesimal automorphisms. Next, using Proposition 3.6(3) and Lemma 3.1(i), we get that for a general cubic fourfold $X \subset \mathbb{P}^5$, any hyperplane section Y of X satisfies property (1) of Proposition 3.6. Let now k be a number and $z = (z_1, \dots, z_k)$ be the data of k analytic isomorphism classes of germs of allowable hypersurfaces singularities. Let $N := \sum_i \dim T_{z_i}^1$. Let $\mathcal{M}_{Y,z} \subset \mathcal{M}_Y$ be the set of cubic threefolds admitting exactly k singular points with local germs z_i . Using property (1), we conclude that $\mathcal{M}_{Y,z}$ is smooth locally closed of codimension N in \mathcal{M}_Y . It follows that its inverse image $r^{-1}(\mathcal{M}_{Y,z})$ is smooth of codimension N in \mathcal{M}_X^{00} . The group $\mathrm{PGL}(6, 5)$ of automorphisms of \mathbb{P}^5 preserving \mathbb{P}^4 acts now on \mathcal{M}_X^{00} preserving $r^{-1}(\mathcal{M}_{Y,z})$ and using the definition of \mathcal{M}_X^{00} , we find that the fiber L_X over a general point $[X] \in \mathcal{M}_X^{00}/\mathrm{PGL}(6)$ of the quotient map

$$\mathcal{M}_X^{00}/\mathrm{PGL}(6, 5) \rightarrow \mathcal{M}_X^{00}/\mathrm{PGL}(6)$$

is smooth, isomorphic to $\mathbb{P}(H^0(\mathcal{O}_X(1)))$. (This statement is in fact not completely correct due to the presence of hyperplane sections of X which have finite automorphisms, but it is true at the infinitesimal level.) Sard's theorem then tells us that for general X , the locus $L_X \cap r^{-1}(\mathcal{M}_{Y,z})$ is smooth of codimension N in L_X , which exactly means that for the given type z , and for any $Y \subset X$ having z as singularities, the map $H^0(Y, \mathcal{O}_Y(1)) \rightarrow \prod_{p \in \mathrm{Sing} Y} T_{Y_p}^1$ is surjective. The conclusion then follows from the fact that there are finitely many analytic isomorphism classes of allowable singularities (i.e. all are ADE with $\tau \leq 6$). \square

3.2. Transversality results in the Pfaffian case. Recall that a Pfaffian cubic hypersurface is a linear section of the Pfaffian cubic in $\mathbb{P}^{14} = \mathbb{P}(\bigwedge^2 W_6)$ defined by the equation $\omega^3 = 0$ in $\bigwedge^6 W_6$. Pfaffian cubic fourfolds are parameterized by a hypersurface \mathcal{P} in the moduli space of all cubic fourfolds (see [BD85]). This hypersurface is smooth away from the locus where the cubic has two different Pfaffian structures. In general, it is a divisor with normal crossings, with one branch for each Pfaffian structure. This follows from the fact that the period map for cubic fourfolds is étale. As we want to apply the results of Section 4 also to the case of a general Pfaffian cubic fourfold, we have to prove that a general Pfaffian cubic fourfold satisfies the needed assumptions, namely Lemmas 3.13 and 3.14, that will be obtained as easy consequences of the following lemma.

Lemma 3.10. *Let X be a general Pfaffian cubic fourfold. Then for any hyperplane section $Y \subset X$ with equation $f_Y \in H^0(X, \mathcal{O}_X(1))$, the subspace $f_Y H^0(X, \mathcal{O}_X(2)) \subset H^0(X, \mathcal{O}_X(3))$ is not contained in the tangent space to the Pfaffian hypersurface at the point $[X]$.*

Proof. The Pfaffian cubic fourfolds are characterized by the fact that they contain quintic del Pezzo surfaces: if X contains a quintic del Pezzo surface Σ , the Pfaffian rank 2 vector bundle \mathcal{E} with $c_2 = 2$ and $c_1 = 0$ on X is deduced from Σ by the Serre construction. Conversely, if X is Pfaffian with Pfaffian rank 2 vector bundle \mathcal{E} , there is a five dimensional family of quintic del Pezzo surfaces Σ in X , obtained as zero-sets of sections of \mathcal{E} (see [Bea00, Proposition 9.2]). Let $\Sigma \subset X$ be such a pair, and let $\sigma := [\Sigma] \in H^2(X, \Omega_X^2)$ be the cohomology class of Σ . The cup-product with σ induces a composite morphism

$$(3.11) \quad \sigma \cup : H^0(X, \mathcal{O}_X(3)) \xrightarrow{\rho} H^1(X, T_X) \rightarrow H^{1,3}(X),$$

where the first map ρ is the Kodaira-Spencer map, and the general theory of variations of Hodge structures tells us that the tangent space to the Pfaffian hypersurface \mathcal{P} at $[X]$ identifies to $\mathrm{Ker} \sigma \cup$. Note that ρ identifies to the quotient map

$$H^0(X, \mathcal{O}_X(3)) \rightarrow R_f^3 := H^0(X, \mathcal{O}_X(3))/J_f^3,$$

where f is the defining equation for X and J_f^3 is the degree 3 piece of the Jacobian ideal of f . Griffiths' residue theory (see [Voi07, II,6.2]) provides isomorphisms

$$H^{2,2}(X)_{\mathrm{prim}} \cong R_f^3, \quad H^{1,3}(X) \cong R_f^6$$

such that the second map in (3.11) identifies to multiplication by $\tilde{\sigma} : R_f^3 \rightarrow R_f^6$, where $\tilde{\sigma} \in R_f^3$ is the representative of σ , or rather of its projection in $H^{2,2}(X)_{\text{prim}} = H^{2,2}(X)/\langle h^2 \rangle$ (here, $h = c_1(\mathcal{O}_X(1))$). Lemma 3.10 can thus be rephrased as follows: For a general Pfaffian cubic fourfold with equation f , the Pfaffian class $\tilde{\sigma} \in R_f^3$ is not annihilated by yR_f^2 , for any $0 \neq y \in H^0(X, \mathcal{O}_X(1))$. Note that by Macaulay's theorem [Voi07, II,6.2.2], to say that $y\tilde{\sigma}R_f^2 = 0$ in R_f^6 is equivalent to saying that $y\tilde{\sigma} = 0$ in R_f^4 . So what we have to prove is the following:

Claim. *For a general Pfaffian cubic fourfold with equation f and Pfaffian class $\tilde{\sigma} \in R_f^3$, and for any $0 \neq y \in H^0(X, \mathcal{O}_X(1))$, $y\tilde{\sigma} \neq 0$ in R_f^4 .*

This statement can be proved as follows: we observe that a smooth linearly normal quintic del Pezzo surface is a hypersurface of bidegree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ Segre embedded in \mathbb{P}^5 . We now choose a reducible $\Sigma \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 2)|$, consisting of the union of $x \times \mathbb{P}^2$ (a plane P_1 in \mathbb{P}^5) and two hypersurfaces in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0, 1)|$, which will provide quadrics Q_2, Q_3 in \mathbb{P}^5 . Of course, the configuration P_1, Q_2, Q_3 is not arbitrary: Q_1 and Q_2 have to meet along a line Δ in \mathbb{P}^5 and P_1 meets Q_2 , resp. Q_3 , along a line Δ_2 resp. Δ_3 in \mathbb{P}^5 . Let X be a general smooth cubic fourfold containing P_1, Q_2 and Q_3 , and let P_2 , resp. P_3 , be the planes residual to Q_2 , resp. Q_3 in X . Then P_2 and P_3 do not meet and they each intersect P_1 in one point. Note that we have an equality of primitive cohomology classes

$$\sigma = p_1 - p_2 - p_3,$$

where p_i is the class of P_i in $H^{2,2}(X)_{\text{prim}}$ for $i = 1, 2, 3$. We claim that conversely, if a smooth cubic fourfold contains three planes P_1, P_2, P_3 such that P_2 and P_3 do not intersect but each of them intersect P_1 in respective points x_2, x_3 , then X is Pfaffian with primitive Pfaffian class $\sigma = p_1 - p_2 - p_3$. To see this, we choose the residual quadrics Q_2 and Q_3 in such a way that they each intersect P_1 along a line. We need also that Q_2 and Q_3 intersect along a line, and for this it suffices to choose a line Δ in X passing through a point x of P_1 and intersecting both P_2 and P_3 . Such a line exists and in general the point x is not on the line $\langle x_2, x_3 \rangle$. Let \mathbb{P}_2^3 be generated by P_2 and Δ and \mathbb{P}_3^3 be generated by P_3 and Δ . Let Q_2 , resp. Q_3 be the residual quadric of P_2 , resp. P_3 , in $X \cap \mathbb{P}_2^3$, resp. $X \cap \mathbb{P}_3^3$. Then $Q_2 \cap P_1$ is the line $\langle x_2, x \rangle$, $Q_3 \cap P_1$ is the line $\langle x_3, x \rangle$ and $Q_2 \cap Q_3$ is the line Δ . We thus constructed the (degenerate) quintic del Pezzo surface $\Sigma = P_1 \cup Q_2 \cup Q_3$.

Remark 3.12. The claim could also be proved by observing that if $P_1, P_2, P_3 \subset X$ are three planes as above, then the integral cohomology class $\sigma = [P_1] - [P_2] - [P_3] + 2h^2 \in H^4(X, \mathbb{Z})$ has the same numerical properties as the class of a quintic del Pezzo surface, namely $\sigma^2 = 13$ and $\sigma \cdot h^2 = 5$.

Coming back to the proof of the statement, it suffices now to prove that for a general cubic fourfold containing three planes P_1, P_2, P_3 such that P_2 and P_3 do not intersect but each of them intersects P_1 in one point, then for any $0 \neq y \in H^0(X, \mathcal{O}_X(1))$, the class $p_1 - p_2 - p_3 \in H^{2,2}(X)_{\text{prim}} = R_f^3$ satisfies $y(p_1 - p_2 - p_3) \neq 0$ in R_f^4 . This computation can be made explicitly on the Fermat cubic, where such configurations of planes are easy to exhibit and where the computations in the Jacobian ring are easy to perform. \square

We have the following applications.

Lemma 3.13. *Let X be a general Pfaffian cubic fourfold. Then for any hyperplane section Y of X , the natural map $H^0(Y, \mathcal{O}_Y(1)) \rightarrow \bigoplus_{p \in \text{Sing } Y} T_Y^1$ is surjective.*

Proof. Let \mathcal{P}^{00} be the Zariski open subset of the Pfaffian hypersurface which is defined as the intersection of \mathcal{P} with the Zariski open set \mathcal{M}_X^{00} . Note that \mathcal{P}^{00} is non-empty by Lemma 3.1(ii), using the fact that the set of $[Y] \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(3)))$ admitting a nontrivial vector field has codimension ≥ 7 (see Proposition 3.6(4)). We now consider the natural map $r_{pf} : \mathcal{P}^{00} \rightarrow \mathcal{M}_Y$ defined as the restriction to $\mathcal{P}^{00} \subset \mathcal{M}_X^{00}$ of $r : \mathcal{M}_X^{00} \rightarrow \mathcal{M}_Y$. The fiber of r_{pf} over $[Y] \in \mathcal{M}_Y$ consists in those Pfaffian cubic fourfolds which intersect \mathbb{P}^4 along Y . Let X be a general Pfaffian cubic fourfold. Then r_{pf} is smooth at any $[X']$ parameterizing a cubic isomorphic to X . Indeed, the map r is smooth, and r_{pf} is the restriction of r to \mathcal{P}^{00} . Thus, if r_{pf} was not smooth at a point $[X']$ with $[Y] = r_{pf}([X'])$, then the fiber of r would be tangent to \mathcal{P}^{00} at $[X']$ which exactly means that $f_Y H^0(X', \mathcal{O}_{X'}(2)) \subset H^0(X', \mathcal{O}_{X'}(3))$ is contained in the tangent space to the Pfaffian hypersurface at

the point $[X']$. As X is general, Lemma 3.13 tells us that this does not happen at any X' isomorphic to X . The end of the proof is now identical to the proof of Lemma 3.8. Indeed, by Lemma 3.14 below and Proposition 3.6, property (1) of Proposition 3.6 is satisfied by any hyperplane section of a general Pfaffian cubic, replacing $r^{-1}(\mathcal{M}_{Y,z})$ by its Pfaffian analogue $r_{pf}^{-1}(\mathcal{M}_{Y,z})$ which we know to be smooth. \square

Recall from Definition 2.9 that a very good line in a cubic threefold containing no plane is a line which is good and such that the curve \tilde{C}_l of lines in Y meeting l is irreducible.

Lemma 3.14. *Let X be a general Pfaffian cubic fourfold. Then any hyperplane section Y of X is allowable and admits a very good line.*

Proof. We know by Propositions 2.3 and 2.10 and Corollary 3.7 that if X is a general cubic fourfold, then any hyperplane section of X allowable and contains a very good line. Let us say that Y is bad if it does not admit a very good line or has non-allowable singularities and X is bad if it has a hyperplane section Y which is bad. The locus of bad cubic fourfolds is a proper closed algebraic subset $\mathcal{M}_X^{00,bad}$ of \mathcal{M}_X^{00} and its irreducible components are constructed as follows: for each irreducible component $Z \subset \mathcal{M}_Y$ of the locus of bad cubic threefolds, let $Z_X = r^{-1}(Z) \subset \mathcal{M}_X$. Then Z_X is the locus of bad cubic fourfolds such that the cubic threefold $X \cap \mathbb{P}^4$ is parameterized by a point of Z , and thus $\mathrm{PGL}(6) \cdot Z_X$ is the set of cubic fourfolds X such that some hyperplane section of X is isomorphic to a cubic threefold parameterized by a point of Z . It is thus clear that we get any irreducible component of $\mathcal{M}_X^{00,bad}$ as $\mathrm{PGL}(6) \cdot Z_X$, with Z as above. We conclude from this that any irreducible component Z' of $\mathcal{M}_X^{00,bad}$ has the property that, for any $[X] \in Z'$, there exists a hyperplane section $Y \subset X$ such that all cubic fourfolds containing Y as a hyperplane section are parameterized by points of Z' . In particular, if $f_y \in H^0(X, \mathcal{O}_X(1))$ is the equation of Y in X , $f_Y H^0(X, \mathcal{O}_X(2))$ must be contained in the Zariski tangent space of Z' . If any Pfaffian cubic fourfold was bad, then the Pfaffian hypersurface \mathcal{P}^{00} would be an irreducible component Z' of $\mathcal{M}_X^{00,bad}$ and we would get a contradiction with Lemma 3.10. \square

4. RELATIVE COMPACTIFIED PRYM VARIETIES

As previously mentioned, our main tool for compactifying the intermediate Jacobian fibration \mathcal{J}/U is the Prym construction that identifies the Intermediate Jacobian $J(Y)$ with a Prym variety $\mathrm{Prym}(\tilde{C}/C)$ (where (\tilde{C}, C) are obtained from Y via the projection from a general line). The Prym construction works well in a relative setting over the smooth locus U (and for the open set of good lines \mathcal{F}^{good}/U), reducing (at least locally) the problem of understanding degenerations of Intermediate Jacobians to that of understanding degenerations of Prym varieties. This is of course a well studied problem: Beauville [Bea77] gave a compactification for the moduli of pairs (\tilde{C}, C) , and many people studied degenerations of Pryms as abelian varieties (e.g. [FS86], [ABH02], [CMGHL14]). Here we need to understand a compactification, and its local structure over given base B . A few instances of this have already been studied in [MT07], [ASF15], and [Sac13a]). Below, we define and prove a number of results for the relative compactified Prym for families of étale double covers of irreducible curves. Specifically, the results of Sections 2 and 3 say that for a general cubic fourfold X , we can replace (locally on B) the family \mathcal{Y}/B of hyperplane sections by a family of double covers (\tilde{C}, C) such that each fiber (\tilde{C}_t, C_t) is an étale double cover with both curves irreducible (cf. Cor. 2.11). Furthermore, the singularities of \tilde{C}_t and C_t are planar, and we can assume (cf. Cor. 3.9) that the family \mathcal{C} gives a simultaneous versal deformation of the singularities of any fiber C_t .

Notation 4.1. In this section below, B will stand for an arbitrary base, not necessarily $(\mathbb{P}^5)^\vee$ as elsewhere in the paper.

We proceed as follows: as in [ASF15] we can define, for any family $\tilde{C}_B \rightarrow \mathcal{C}_B$ of étale double covers of irreducible locally planar curves, parametrized by a base B , a relative compactified Prym variety $\mathrm{Prym}(\tilde{C}_B/\mathcal{C}_B) \rightarrow B$ whose fibers over the locus parametrizing smooth curves are usual Prym varieties. The relative Prym variety is defined as (one component) of the fixed locus of an involution on the relative compactified Jacobian $\overline{\mathrm{Jac}}(\tilde{C}_B)$ of the family $\tilde{C}_B \rightarrow B$. From this definition, it follows immediately that if $\overline{\mathrm{Jac}}(\tilde{C}_B)$ is smooth then so is $\mathrm{Prym}(\tilde{C}_B/\mathcal{C}_B)$. Unfortunately, in general $\overline{\mathrm{Jac}}(\tilde{C}_B)$ is not smooth. However, one can sometimes think of $\tilde{C}_B \rightarrow B$ as the restriction of a larger family $\tilde{C}_{\tilde{B}} \rightarrow \tilde{B}$, $B \subset \tilde{B}$, with the property

that a) there exists two compatible involutions on $\tilde{\mathcal{C}}_{\tilde{B}}$ and on \tilde{B} , such that the first is an extension of the given involution on $\tilde{\mathcal{C}}_B$ and second has the property that the fixed locus on \tilde{B} is equal to B ; b) the relative compactified Jacobian $\overline{\text{Jac}}(\tilde{\mathcal{C}}_{\tilde{B}})$ is smooth. Under these assumptions the relative Prym variety $\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$ is smooth. An instance of this already appeared in [ASF15]. As discussed below, the versality statements valid in our setup allows us to conclude the the relative compactified Prym is indeed smooth in our situation.

The results in this section build on a well known result for compactified relative Jacobians, namely the Fantechi–Göttsche–van Straten [FGvS99] criterion.

4.1. Relative Compactified Pryms (the étale case). Let $f : \tilde{C} \rightarrow C$ be an étale double cover of smooth projective curves, and let $\iota : \tilde{C} \rightarrow \tilde{C}$ be the corresponding involution on \tilde{C} . We denote by g be the genus of C , and by h the genus of \tilde{C} , so that $h = 2g - 1$. Recall that the Prym variety of \tilde{C} over C , which we will denote by $\text{Prym}(\tilde{C}/C)$, is the identity component of the fixed locus of the involution

$$\tau := -\iota^* : \text{Pic}^0(\tilde{C}) \rightarrow \text{Pic}^0(\tilde{C}).$$

The Prym variety $\text{Prym}(\tilde{C}/C)$ is a principally polarized abelian variety [Mum74] of dimension $g - 1$. Equivalently [Mum74], the Prym can be defined as the identity component of the Norm map

$$\text{Nm} : \text{Pic}^0(\mathcal{C}) \rightarrow \text{Pic}^0(C), \quad \mathcal{O}_{\tilde{C}}(\sum p_i) \mapsto \mathcal{O}_C(\sum f(p_i)),$$

or as the image of

$$(1 - \iota^*) : \text{Pic}^0(\tilde{C}) \rightarrow \text{Pic}^0(\tilde{C}).$$

Now suppose that $f : \tilde{C} \rightarrow C$ is an étale double cover of singular, but irreducible, curves, and let $\tilde{n} : \tilde{D} \rightarrow \tilde{C}$ and $n : D \rightarrow C$ be the normalizations of the two curves. The involution ι on \tilde{C} lifts to a compatible involution

$$(4.2) \quad \varepsilon : \tilde{D} \rightarrow \tilde{D}$$

so that the natural morphism $\tilde{D} \rightarrow D$ is an étale double cover with associated involution ε . Let $\{x_1, \dots, x_k\}$ be the singular points of C and let

$$\{p_1, \dots, p_k, q_1, \dots, q_k\}$$

be the singular points of \tilde{C} , with $f^{-1}(x_i) = \{p_i, q_i\}$. The identity component of the Picard group, or generalized Jacobian, of \tilde{C} fits into the natural short exact sequence of groups

$$(4.3) \quad 1 \rightarrow A \times A \rightarrow \text{Pic}^0(\tilde{C}) \xrightarrow{\tilde{n}^*} \text{Pic}^0(\tilde{D}) \rightarrow 1,$$

where $A := H^0(\tilde{C}, \oplus_{i=1}^k (n_* \mathcal{O}_{\tilde{D}} / \mathcal{O}_{\tilde{C}})_{p_i}) \cong H^0(\tilde{C}, \oplus_{i=1}^k (n_* \mathcal{O}_{\tilde{D}} / \mathcal{O}_{\tilde{C}})_{q_i})$ is a commutative affine group. The involution $-\iota^*$ still acts on $\text{Pic}^0(\tilde{C})$ and we can define, in analogy with the generalized Jacobian, the generalized Prym variety of \tilde{C} over C to be the identity component of the fixed locus of $-\iota^*$

$$\text{Prym}(\tilde{C}/C) := \text{Fix}(-\iota^*)_{\text{id}}.$$

The involution $-\iota^*$ is compatible, via n^* , with $-\varepsilon^*$ on $\text{Pic}^0(\tilde{D})$ and it acts on $A \times A$ via $(a, b) \mapsto (b^{-1}, a^{-1})$ (N.B. We use multiplicative notation for these groups, even though A can be a product of both additive and multiplicative groups.). We therefore get a short exact sequence

$$1 \rightarrow A \rightarrow \text{Prym}(\tilde{C}/C) \xrightarrow{\tilde{n}^*} \text{Prym}(\tilde{D}/D) \rightarrow 1,$$

where the inclusion $A \hookrightarrow A \times A = \ker \tilde{n}^*$ is given by $a \mapsto (a, a^{-1})$. Set $\delta := \dim A$, so that $g(D) = g - \delta$, and $g(\tilde{D}) = h - 2\delta$. Since $\dim \text{Prym}(\tilde{D}/D) = g - \delta - 1$, we see that

$$\dim \text{Prym}(\tilde{C}/C) = g - 1.$$

For example, if C is a nodal irreducible curve, then $\text{Prym}(\tilde{C}/C)$ is a semi-abelian variety.

The strategy to compactify the generalized Prym variety is to extend the involution to the compactified Jacobian of \tilde{C} .

By definition, the degree d compactified Jacobian of an integral projective curve Γ is the moduli space of rank 1 torsion free sheaves on Γ of degree d . If Γ has planar singularities, i.e. if locally around every singular point Γ is isomorphic to a plane curve, then every component is irreducible [Reg80] of dimension equal to the arithmetic genus of Γ . The degree zero component $\overline{\text{Jac}}(\Gamma)$ contains $\text{Pic}^0(\Gamma)$ as an open dense subset.

For the proof of Propositions 4.14 and 4.31 we will need further properties of the compactified Jacobian, in particular, we will need a description of the complement $\overline{\text{Jac}}(\Gamma) \setminus \text{Pic}^0(\Gamma)$ (cf. [Reg80] and [Coo98]; see Proposition 4.15 below), and a smoothness criterion due to Fantechi–Göttsche–van Straten [FGvS99] (recalled below in Proposition 4.28). If Γ has locally planar singularities, then any coherent torsion free sheaf F on Γ is reflexive, in the sense that

$$(F^\vee)^\vee = F, \quad \text{where } F^\vee := \mathcal{H}\text{om}_{\mathcal{O}_\Gamma}(F, \mathcal{O}_\Gamma).$$

Lemma 4.4. *Assume that C , and therefore \tilde{C} , has planar singularities. Then assignment*

$$(4.5) \quad \begin{aligned} \tau : \overline{\text{Jac}}(\tilde{C}) &\longrightarrow \overline{\text{Jac}}(\tilde{C}) \\ F &\longmapsto \iota^* F^\vee \end{aligned}$$

defines a regular involution which extends $-\iota^$ over the complement of $\text{Pic}^0(\tilde{C})$.*

Proof. By [AK79, Thm 3.4], there is a universal sheaf \mathcal{F} on $\overline{\text{Jac}}(\tilde{C}) \times \tilde{C}$. Hence the family $\mathcal{H}\text{om}(\iota^* \mathcal{F}, \mathcal{O}_{\tilde{C}})$ of rank one torsion free sheaves determines a morphism $\overline{\text{Jac}}(\tilde{C}) \rightarrow \overline{\text{Jac}}(\tilde{C})$ which sends any sheaf F to $\iota^* F^\vee$ and is an involution since the sheaves are reflexive. \square

Definition 4.6. The compactified Prym variety $\overline{\text{Prym}}(\tilde{C}/C)$ of an étale double cover $\tilde{C} \rightarrow C$ of integral curves with planar singularities is the irreducible component containing the identity of the fixed locus $\text{Fix}(\tau) \subset \overline{\text{Jac}}(\tilde{C})$.

Remark 4.7. Notice that $\text{Prym}(\tilde{C}/C) \subset \overline{\text{Prym}}(\tilde{C}/C)$ is a dense open subset, so that, in particular, $\dim \overline{\text{Prym}}(\tilde{C}/C) = g - 1$. For example, if C is irreducible with one node, then $\overline{\text{Prym}}(\tilde{C}/C)$ is a rank one degeneration of an abelian variety. For a other explicit description of Prym varieties of singular curves see [Sac13a].

Remark 4.8. One could also drop the assumption of integrality, and define the relative compactified Prym variety for étale double covers of arbitrary curves with locally planar singularities. In this situation, however, a choice has to be made, namely that of a polarization on the family of double covers and the relative compactified Prym depends on this choice (for the case of family of curves lying on smooth projective surfaces see [ASF15] and [Sac13a]). Since the curves we will deal with in our situation turn out to be integral by Proposition 2.10, we will restrict ourselves to the case of integral curves.

Now let B be an irreducible base scheme and let

$$(4.9) \quad \begin{array}{ccc} \tilde{\mathcal{C}}_B & \xrightarrow[\substack{f \\ 2:1}]{} & \mathcal{C}_B \\ & \searrow & \swarrow \\ & B & \end{array}$$

be a family, parametrized by B , of étale double covers of reduced and irreducible curves with planar singularities. Let us denote by g the genus of the curves in the family $\mathcal{C}_B \rightarrow B$, and by $h = 2g - 1$ the genus of their double covers $\tilde{\mathcal{C}}_B \rightarrow B$. We let

$$\iota : \tilde{\mathcal{C}}_B \rightarrow \tilde{\mathcal{C}}_B,$$

be the involution associated to the covering, and define a 2-torsion line bundle η_B on $\tilde{\mathcal{C}}_B$ by setting

$$f_* \mathcal{O}_{\tilde{\mathcal{C}}_B} = \mathcal{O}_{\mathcal{C}_B} \oplus \eta_B.$$

For any $b \in B$, we denote by

$$f_b : \tilde{\mathcal{C}}_b \rightarrow \mathcal{C}_b, \quad \iota_b : \tilde{\mathcal{C}}_b \rightarrow \tilde{\mathcal{C}}_b$$

and $\eta_b \in \text{Pic}^0(\mathcal{C}_b)$ the restrictions of f , ι and of η to the fiber over b . Consider the degree 0 relative compactified Jacobians

$$\xi : \overline{\text{Jac}}(\tilde{\mathcal{C}}_B) \rightarrow B, \quad \pi : \overline{\text{Jac}}(\mathcal{C}_B) \rightarrow B.$$

The fiber of ξ (respectively π) over a point $b \in B$ is the compactified Jacobian of the curve $\tilde{\mathcal{C}}_b$ (respectively \mathcal{C}_b) and is a reduced and irreducible l.c.i. of dimension h (respectively g). The smooth locus of ξ (respectively π) is the relative generalized Jacobian $\text{Pic}_{\tilde{\mathcal{C}}_B}^0$ (resp. $\text{Pic}_{\mathcal{C}_B}^0$) which parametrizes line bundles. Since they are

a group schemes, both admit a zero section, and the morphisms $-\iota^*$ and $1 - \iota^*$ are well defined group homomorphisms of $\text{Pic}_{\tilde{\mathcal{C}}_B}^0$. Also, $\text{Pic}_{\mathcal{C}_B}^0$ has a section determined by η_B .

Lemma 4.10. *The “ -1 ” morphism on the group scheme $\text{Pic}_{\tilde{\mathcal{C}}_B}^0$ extends to a regular morphism on the whole $\overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$. In particular, there is a regular involution*

$$\begin{aligned} \tau : \overline{\text{Jac}}(\tilde{\mathcal{C}}_B) &\longrightarrow \overline{\text{Jac}}(\tilde{\mathcal{C}}_B) \\ F &\longmapsto \iota^* F^\vee \end{aligned}$$

Proof. Again, this follows from [AK79, Thm 3.4]. Locally on B we can find a section of the smooth locus of $\tilde{\mathcal{C}}_B \rightarrow B$, hence by loc. cit. there is a universal sheaf on $\overline{\text{Jac}}(\tilde{\mathcal{C}}_B) \times \tilde{\mathcal{C}}_B$, which in turn induces an involution like in Lemma 4.4. Since the assignment is independent of the choice of a section, these local morphisms glue together to a global involution on $\overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$. \square

We can now define the relative compactified Prym.

Definition 4.11. Let $f : \tilde{\mathcal{C}}_B \rightarrow \mathcal{C}_B$ be as in (4.9) an étale double cover between two families of reduced and irreducible curves with planar singularities. The relative compactified Prym variety of $\tilde{\mathcal{C}}_B$ over \mathcal{C}_B , denoted $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$ is defined to be the irreducible component of the fixed locus $\text{Fix}(\tau) \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$ that contains the zero section:

$$\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) := \text{Fix}(\tau)_{\text{id}} \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$$

Lemma 4.12. *The fixed locus $\text{Fix}(-\iota^*) \subset \text{Pic}_{\tilde{\mathcal{C}}_B}^0$ has four isomorphic connected components.*

Proof. In [Mum74], it is proved that for any $b \in B$ such that \mathcal{C}_b is smooth, $\text{Nm}^{-1}(\mathcal{O}_{\mathcal{C}_b}) \subset \text{Pic}_{\tilde{\mathcal{C}}_b}^0$ has two connected components. Hence, so has $\text{Nm}^{-1}(\eta_b)$. Moreover,

$$f^*(\text{Nm}(\mathcal{O}_{\tilde{\mathcal{C}}_b}(\sum c_i))) = \mathcal{O}_{\tilde{\mathcal{C}}_b}(\sum c_i + \iota c_i),$$

and $\ker(f^*) = \langle \mathcal{O}_{\tilde{\mathcal{C}}_b}, \eta_b \rangle$, so that $\text{Fix}(-\iota_b^*) = \text{Nm}^{-1}(\eta_b) \cup \text{Nm}^{-1}(\mathcal{O}_{\mathcal{C}_b})$ and hence the fixed locus of $-\iota_b^*$ in $\text{Pic}^0(\tilde{\mathcal{C}}_b)$ has four connected components. Over B , since f is étale and $f_*\mathcal{O}_{\tilde{\mathcal{C}}_B} = \mathcal{O}_{\mathcal{C}_B} \oplus \eta_B$ is locally free, we can consider the relative norm map [Gro61, §6.5] $\text{Nm}_B : \text{Pic}_{\tilde{\mathcal{C}}_B}^0 \rightarrow \text{Pic}_{\mathcal{C}_B}^0$ and hence we may consider the inverse images of the zero section of $\text{Pic}_{\mathcal{C}_B}^0 \rightarrow B$ and of the section determined by η_B . By [Gro61, Prop. 6.5.8] on each fiber the norm map is compatible with the norm map associated to the double cover $\tilde{D}_b \rightarrow D_b$ between the normalizations of the two curves (cf. (4.2)). Hence, it restricts to a norm map $m : A \times A \rightarrow A$ (notation as in (4.3)), which is nothing but the multiplication map $(a, b) \rightarrow ab$. Hence, on every fiber the kernel of the norm map is an extension of the kernel of the norm map of an étale double cover of smooth curves (namely, the normalizations) by $\ker m \cong A$. It follows that the inverse image under Nm_B of the zero section has two connected components: the one containing the zero section of $\text{Pic}_{\tilde{\mathcal{C}}_B}^0$ and the remaining one. Hence, also the inverse image of the section determined by η_B has two connected components and the lemma is proved. \square

Let

$$(4.13) \quad \mathcal{P}_1, \dots, \mathcal{P}_4$$

be the four connected components of $\text{Fix}(-\iota^*) \subset \text{Pic}_{\tilde{\mathcal{C}}_B}^0$, with \mathcal{P}_1 the component

$$\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) := \text{Fix}(-\iota^*)_{\text{id}} = \text{Im}(1 - \iota^*) \subset \text{Pic}_{\tilde{\mathcal{C}}_B}^0$$

containing the zero section. We will call this component, which is a group scheme of dimension $g - 1$ over T , the relative generalized Prym variety of $\tilde{\mathcal{C}}_B$ over \mathcal{C}_B . It is dense in the relative compactified Prym, so that its closure $\overline{\mathcal{P}}_1$ satisfies

$$\overline{\mathcal{P}}_1 = \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) = \overline{\text{Im}(1 - \iota^*)}.$$

By restricting π to the relative Prym we get a morphism

$$\nu_B : \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \rightarrow B,$$

whose fiber over a point $b \in B$ corresponding to a double cover between smooth curves $\tilde{\mathcal{C}}_b \rightarrow \mathcal{C}_b$ is isomorphic to the usual Prym variety $\text{Prym}(\tilde{\mathcal{C}}_b/\mathcal{C}_b)$. Notice that

$$\dim \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) = \dim B + g - 1.$$

We now need to show that this definition is well posed, in the sense that it is compatible with base change. For example, we would like to verify that the fiber over an arbitrary point $b \in B$ of the relative compactified Prym variety is the compactified Prym variety of $\tilde{\mathcal{C}}_b$ over \mathcal{C}_b .

Proposition 4.14. (1) *For every $b \in B$ we have $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)|_b = \overline{\text{Prym}}(\tilde{\mathcal{C}}_b/\mathcal{C}_b)$.*
(2) *The base change property holds for arbitrary base change, i.e., for any morphism $B' \rightarrow B$, setting $\tilde{\mathcal{C}}_{B'} = \tilde{\mathcal{C}}_B \times_B B'$ and $\mathcal{C}_{B'} = \mathcal{C}_B \times_B B'$, we have*

$$\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \times_B B' = \overline{\text{Prym}}(\tilde{\mathcal{C}}_{B'}/\mathcal{C}_{B'}).$$

(3) *The morphism $\nu_B : \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \rightarrow B$ is equidimensional.*

We claim that it is sufficient to prove (1), since it implies the other two points. Let us see why. To show that (1) \Rightarrow (2) we may assume without loss of generality that B' is irreducible. Hence $\overline{\text{Prym}}(\tilde{\mathcal{C}}_{B'}/\mathcal{C}_{B'})$ is irreducible. Since it is contained in $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \times_B B'$, we only need to show that this is also irreducible of the same dimension. By (1) and Remark 4.7, the fiber $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)|_b$ over any $b \in B$ is irreducible of dimension $g - 1$. Hence, the morphism $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \times_B B' \rightarrow B'$ is equidimensional and $\dim \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \times_B B' = \dim B' + g - 1$.

To prove (1) we first need a few lemmas, as well as the first statement of Proposition 4.31 below. Before stating and proving these lemmas, we need to recall a few properties of compactified Jacobians (cf. [Reg80], [Coo98]).

Let Γ be an integral projective curve, with normalization $n : \bar{\Gamma} \rightarrow \Gamma$. Given a rank 1 torsion free sheaf F on Γ , there exists [Coo98] a partial normalization

$$n' : \Gamma' \rightarrow \Gamma,$$

with the property that $\mathcal{E} \text{nd}(F) \cong n'_* \mathcal{O}_{\Gamma'}$ and that the rank 1 torsion free sheaf $F' := n'^* F / \text{tors}$ on Γ' , satisfies

$$F = n'_* F'.$$

For later use, we highlight that, given F , we can define the curve Γ' by setting

$$\Gamma' := \text{Spec}_{\mathcal{O}_{\Gamma}} \mathcal{E} \text{nd}(F)$$

We define a *local type* [Coo98] of rank one torsion free sheaf to be a collection $\{M_p\}_{p \in \text{Sing}(\Gamma)}$ of isomorphism classes of rank one torsion free $\mathcal{O}_{\Gamma, p}$ -modules, where p runs in the set of singular point of Γ .

Two rank one torsion free sheaves F and G on Γ are said to be of the same local type if for any $p \in \Gamma$ the localizations F_p and G_p are isomorphism $\mathcal{O}_{\Gamma, p}$ -modules.

Proposition 4.15 ([Coo98], [Reg80]). *The relative compactified Jacobian $\overline{\text{Jac}}(\Gamma)$ is stratified based on the local type: for every local type $\{M_p\}_{p \in \text{Sing}(\Gamma)}$ there exist a rank one torsion free sheaf F of degree zero with $F_p \cong M_p$ for every p . Furthermore, $\text{Pic}(\Gamma)$ acts transitively on the set of rank one torsion free sheaves of a fixed type $\{M_p\}$, with stabilizer $\ker[\text{Pic}^0(\Gamma) \rightarrow \text{Pic}^0(\Gamma')]$, with $\Gamma' := \text{Spec}_{\mathcal{O}_{\Gamma}} \mathcal{E} \text{nd}(F)$ as above. If Γ has planar singularities, then $\overline{\text{Jac}}(\Gamma)$ contains the generalized Jacobian as a dense open subset.*

Let us now return to our situation. Our aim is to understand the fixed locus of τ in $\overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$. We start by viewing the group homomorphism $(1 - \iota^*) : \text{Pic}_{\tilde{\mathcal{C}}_B}^0 \rightarrow \text{Pic}_{\tilde{\mathcal{C}}_B}^0$, $L \mapsto L \otimes \iota^* L^\vee$ as a rational map

$$(4.16) \quad (1 - \iota^*) : \overline{\text{Jac}}(\tilde{\mathcal{C}}_B) \dashrightarrow \overline{\text{Jac}}(\tilde{\mathcal{C}}_B).$$

Let us focus on a neighborhood of the fiber $\overline{\text{Jac}}(\tilde{\mathcal{C}}_{b_0})$ over a point $b_0 \in B$. Let $\{x_1, \dots, x_k\}$ be the singular points of \mathcal{C}_{b_0} and let $\{p_1, \dots, p_k, q_1, \dots, q_k\}$ be the singular points of $\tilde{\mathcal{C}}_{b_0}$, with $f^{-1}(x_i) = \{p_i, q_i\}$. For every subset $I \subset \{1, \dots, k\}$, we can consider the open subset

$$V_I \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$$

of sheaves that are locally free in a neighborhood of $\{p_i\}_{i \in I}$ and of $\{q_j\}_{j \notin I}$, so that

$$V = \cup_I V_I,$$

is the open set of sheaves that for every i are locally free at least on one of the two points p_i or q_i .

Lemma 4.17. *The rational map (4.16) is defined in an open neighborhood of $\overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$ containing V .*

Proof. It is enough to show that if $F \in V_I$ then $F \otimes \iota^* F^\vee$ is torsion free. We only need to check this condition at the singular points $\{p_1, \dots, p_k, q_1, \dots, q_k\}$ and, by symmetry, it is enough to check in p_i , for every i . We have $(F \otimes \iota^* F^\vee)_{p_i} = F_{p_i} \otimes F_{q_i}^\vee$. Since by construction at least one between F_{p_i} and $F_{q_i}^\vee$ is locally free, while the other is torsion free, their tensor product is torsion free. \square

Lemma 4.18. *Given $F \in \text{Fix}(\tau) \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$, supported on $\tilde{\mathcal{C}}_{b_0}$ there exist a $G' \in V \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$ such that the rank 1, τ -invariant, torsion free sheaf*

$$G := (1 - \iota^*)G' \in \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$$

is of the same local type as F .

Proof. Since $\tau(F) = F$, we have

$$F_{p_i} = (\iota^* F^\vee)_{p_i} = F_{q_i}^\vee$$

so that the local type of F is determined by the localizations F_{p_1}, \dots, F_{p_k} at only half of the singular points. Consider the local type $\{F_{p_1}, \dots, F_{p_k}, \mathcal{O}_{q_1}, \dots, \mathcal{O}_{q_k}\}$, where $\mathcal{O}_{q_i} := \mathcal{O}_{\tilde{\mathcal{C}}_{b_0, q_i}}$, and let G' be a sheaf in $\overline{\text{Jac}}(\tilde{\mathcal{C}}_{b_0})$ with this local type, which exists by Proposition 4.15. With this notation it is clear that

$$G := G' \otimes \iota^* G'^\vee$$

is of the same local type as F . Since $G \in \overline{\text{Im}}(1 - \iota^*) = \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$, the Lemma is proved. \square

Set

$$\Gamma' := \text{Spec}_{\mathcal{O}_F} \mathcal{E} \text{nd}(F),$$

and let $n' : \Gamma' \rightarrow \Gamma$ be the natural partial normalization morphism. Since G and F are of the same local type, by Proposition 4.15 we know that there exists an $L \in \text{Pic}_{\tilde{\mathcal{C}}_{t_0}}^0$, well defined up to an element of $\ker[n'^* : \text{Pic}^0(\Gamma) \rightarrow \text{Pic}^0(\Gamma')]$, such that

$$G = F \otimes L.$$

Lemma 4.19. *Up to changing L by an element of $\ker[\text{Pic}^0(\Gamma) \rightarrow \text{Pic}^0(\Gamma')]$, we can assume that $\tau(L) = L$.*

Proof. Since $\tau(F) = F$ and $\tau(G) = G$, we have that $M := L \otimes \tau(L)^\vee$ lies in $\ker[\text{Pic}^0(\Gamma) \rightarrow \text{Pic}^0(\Gamma')]$ and satisfies $M^\vee \cong \tau(M)$. This last equality implies that $M = \iota^*(M)$, and it is not hard to see (cf. for example [Sac13b, Lem. 2.8]) that this implies the existence of an $M' \in \ker[\text{Pic}^0(\Gamma) \rightarrow \text{Pic}^0(\Gamma')]$ such that $M = M' \otimes \iota^* M'$. From

$$L = \iota^* L^\vee \otimes M = \iota^* L^\vee \otimes M' \otimes \iota^* M',$$

we deduce that

$$L' := L \otimes M'^\vee$$

satisfies $\iota^* L'^\vee = L'$ and since $M' \in \ker[\text{Pic}^0(\Gamma) \rightarrow \text{Pic}^0(\Gamma')]$ we still have

$$G = F \otimes L'.$$

\square

Let us now go back to the components $\mathcal{P}_1, \dots, \mathcal{P}_4$, defined in (4.13). The morphism

$$\begin{aligned} \mathcal{P}_1 \otimes \mathcal{P}_i &\longrightarrow \mathcal{P}_i \\ (L, M) &\longmapsto L \otimes M \end{aligned}$$

extends to a morphism

$$(4.20) \quad \begin{aligned} \overline{\mathcal{P}}_1 \otimes \mathcal{P}_i &\longrightarrow \overline{\mathcal{P}}_i \\ (G, M) &\longmapsto G \otimes M \end{aligned}$$

where for every i , $\overline{\mathcal{P}}_i$ denotes the closure of \mathcal{P}_i and hence is an irreducible component of $\text{Fix}(\tau)$. Recall that we defined \mathcal{P}_1 so that $\overline{\mathcal{P}}_1 = \overline{\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)}$.

The last element we need is the following Lemma

Lemma 4.21. *For $i \neq j$ the closures $\overline{\mathcal{P}}_i$ and $\overline{\mathcal{P}}_j$ do not intersect.*

Proof. This is Corollary 4.32 below, which is based only on the smoothness of the relative compactified Jacobian over the versal family of an integral, locally planar curve. \square

Corollary 4.22. *Every irreducible component of $\text{Fix}(\tau) \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_B)$ is of the form $\overline{\mathcal{P}}_i$ for some $i = 1, \dots, 4$. In particular, in Definition 4.11 we can replace “irreducible” component with “connected” component.*

Proof. Consider an $F \in \text{Fix}(\tau)$ and let G and L be as in Lemmas 4.18 and 4.19 respectively. Since $\tau(L) = L$ and the fixed locus of τ on $\text{Pic}_{\tilde{\mathcal{C}}_B}^0$ is equal to $\coprod \mathcal{P}_i$, $L \in \mathcal{P}_i$ for some i . Since $G \in \overline{\mathcal{P}}_1$, it follows by (4.20) that $F \in \overline{\mathcal{P}}_i$. \square

We finally get to the proof of (1) of Proposition 4.14:

Corollary 4.23. $\overline{\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)}|_{b_0} = \overline{\text{Prym}(\tilde{\mathcal{C}}_{b_0}/\mathcal{C}_{b_0})}$

Proof. We only need to prove that $\overline{\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)}|_{b_0} \subset \overline{\text{Prym}(\tilde{\mathcal{C}}_{b_0}/\mathcal{C}_{b_0})}$ since the reverse inclusion is clear. Consider an $F \in \overline{\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)}|_{b_0}$. As in Lemmas 4.18 and 4.19, we can find a τ -invariant L and a $G \in \overline{\mathcal{P}}_1$ such that $G = F \otimes L$. By Lemma 4.21, and the fact that $F \in \overline{\mathcal{P}}_1$, we necessarily have $L \in \mathcal{P}_1$. Since $\mathcal{P}_1 = (1 - \iota^*)\text{Pic}_{\tilde{\mathcal{C}}_B}^0$, we can find $L'' \in \text{Pic}_{\tilde{\mathcal{C}}_{t_0}}^0$ such that $L = (1 - \iota^*)L''$. By construction, $G = (1 - \iota^*)F'$ and hence

$$F = (1 - \iota_{t_0}^*)(F' \otimes L''^\vee),$$

from which we see that

$$F \in \overline{\text{Im}(1 - \iota_{t_0}^*)} = \overline{\text{Prym}(\tilde{\mathcal{C}}_{t_0}/\mathcal{C}_{t_0})}.$$

\square

4.2. Smoothness results for the relative compactified Prym. The next step is to study the local structure of the relative compactified Prym variety. This will allow us to formulate a criterion that has to be satisfied by a family $\tilde{\mathcal{C}}_B \rightarrow \mathcal{C}_B$ of étale double covers of irreducible, locally planar curves in order for the relative compactified Prym to be smooth. Since this criterion will be deduced by an analogue criterion for the smoothness of the relative compactified Jacobian, we start by reviewing rapidly, following closely [FGvS99], the results we need on this topic.

Let D be a reduced projective curve, with planar singularities. We denote by $\text{Def}(D)$ be the deformation functor of the curve D and, for any $p \in D$, we let $\text{Def}(D_p)$ be the deformation functor of the local ring $\mathcal{O}_{D,p}$. For more precise definitions see [Ser06, §2.4.1]. Let $\text{Sing}(D) \subset D$ denote the singular locus of D , set

$$\text{Def}^{loc}(D) := \prod_{p \in \text{Sing}(D)} \text{Def}(D_p)$$

and consider the natural transformation of functors

$$\Phi : \text{Def}(D) \rightarrow \text{Def}^{loc}(D)$$

which to a deformation of the global curve assigns the induced deformation of local rings at the singular points. Since D is reduced, $\text{Def}(D)$ and $\text{Def}(D_p)$ are unobstructed [Ser06, Example 2.4.9] and hence they admit smooth semi-universal deformation spaces, i.e., there exists a smooth affine scheme S , a point $s \in S$, and a transformation of functors

$$\Psi : (S, s) \rightarrow \text{Def}(D),$$

(here, we denote by (S, s) the deformation functor induced by the germ of the complex space) that is smooth and an isomorphism at the level of tangent spaces, and analogously for the $\text{Def}(D_p)$. We say that the semi-universal space is centered at $s \in S$. The tangent spaces to these deformation functors fit into the local to

global exact sequence

$$0 \rightarrow H^1(T_D) \rightarrow \underbrace{\text{Ext}^1(\Omega_D^1, \mathcal{O}_D)}_{T \text{Def}(D)} \rightarrow \bigoplus_{p \in \text{Sing}(D)} \underbrace{H^0(\mathcal{E} \text{xt}_{\mathcal{O}_{D,p}}^1(\Omega_{D,p}^1, \mathcal{O}_{D,p}))}_{T \text{Def}(D_p)} \rightarrow 0,$$

where $T_D := \mathcal{H} \text{om}(\Omega_D^1, \mathcal{O}_D)$, and where $H^1(T_D)$ is the tangent space to the subfunctor $\text{Def}(D)'$ of the deformations of D that are locally trivial. Let $\mathcal{D} \rightarrow S$ be the semi-universal family for D , centered at s , and let $\overline{\text{Jac}}(\mathcal{D}) \rightarrow S$ be the relative compactified Jacobian. For any sheaf $F \in \overline{\text{Jac}}(\mathcal{D})$ we can consider the deformation functor of the pair $\text{Def}(F, D)$ and, for any $p \in \text{Sing}(D)$, also of the pair $\text{Def}(F_p, D_p)$. By [FGvS99, Prop. A3], the $\text{Def}(F_p, D_p)$ are smooth functors. As above, there are natural transformations $\Psi' : (\overline{\text{Jac}}(\mathcal{D}), F) \rightarrow \text{Def}(F, D)$ and $\Phi' : \text{Def}(F, D) \rightarrow \prod \text{Def}(F_p, D_p)$. There is a commutative diagram of functors

$$(4.24) \quad \begin{array}{ccc} (\overline{\text{Jac}}(\mathcal{D}), F) & \xrightarrow{\Phi' \circ \Psi'} & \prod_{p \in \text{Sing}(D)} \text{Def}(F_p, D_p) \\ \downarrow & & \downarrow \\ (S, s) & \xrightarrow{\Phi \circ \Psi} & \prod_{p \in \text{Sing}(D)} \text{Def}(D_p) \end{array}$$

This diagram is not necessarily cartesian, but the horizontal maps are smooth maps of functors by what was said above and by [FGvS99, Prop. A1]. In particular, $\overline{\text{Jac}}(\mathcal{D})$ is smooth along $\overline{\text{Jac}}(\mathcal{D}) = \overline{\text{Jac}}(\mathcal{D}_s)$. Now let $\mathcal{D}_B \rightarrow B$ be a family of integral locally planar curves, with $\mathcal{D}_0 = D$ for some $0 \in B$ and with B smooth. There is a morphism $B \rightarrow S$, with $0 \mapsto s$, inducing a diagram

$$\begin{array}{ccc} \overline{\text{Jac}}(\mathcal{D}_B) & \longrightarrow & \overline{\text{Jac}}(\mathcal{D}) \\ \downarrow & & \downarrow \\ B & \longrightarrow & S \end{array}$$

Since the diagram is cartesian and we are assuming that B is smooth, $\overline{\text{Jac}}(\mathcal{D}_B)$ is smooth at a point $F \in \overline{\text{Jac}}(\mathcal{D}_0)$ if and only if the image of the tangent space $T_0 B$ in $T_s S$ is transversal to the image of $T_F \overline{\text{Jac}}(\mathcal{D})$ in $T_s S$. Hence, in order to be able to check whether $\overline{\text{Jac}}(\mathcal{D}_B)$ is smooth at a point F , we need to understand the image of $T_F \overline{\text{Jac}}(\mathcal{D})$ in $T_s S$. This is done by analyzing what happens in diagram (4.24) at the level of tangent spaces.

Set $P := \mathbb{C}[[x, y]]$. Since D has locally planar singularities, for any $p_i \in \text{Sing}(D)$ there exist an $f_i \in P$ such that the completion of the local ring \mathcal{O}_{D,p_i} is isomorphic to $R_i = P/f_i$. With this notation, we have

$$T_{D_{p_i}}^1 (= T \text{Def}(D, p_i)) = P/(f_i, \partial_x f_i, \partial_y f_i).$$

For any i , let $\overline{R}_i \supset R_i$ be the normalization of R_i . We denote by $I_i \subset R_i$ the conductor ideal, i.e., $I_i := \text{Hom}(\overline{R}_i, R_i)$, and we let

$$(4.25) \quad V(D_{p_i}) \subset T_{D_{p_i}}^1$$

be the image in the Jacobian ring $P/(f_i, \partial_x f_i, \partial_y f_i)$ of the conductor ideal. It is a codimension δ_i subspace, with $\delta_i := \dim \overline{R}_i/R_i = \dim \overline{R}_i/I_i$. Let

$$(4.26) \quad V(D) \subset T_s S$$

be the inverse image of $\prod V(D_{p_i})$ under the tangent map $T_s S \rightarrow \prod T_{D_{p_i}}^1$. It is known that $V(D)$ is the support of the tangent cone to the deformations of D that keep the geometric genus constant, and has codimension in $T_s S$ equal to the cogenus $\delta = \sum \delta_i$, i.e. the difference between the arithmetic and the geometric genera of D .

Let M be rank one torsion free R_i -module. Recall that M admits a length one free resolution

$$0 \rightarrow P^n \xrightarrow{\varphi} P^n \rightarrow M \rightarrow 0$$

and that the j -th Fitting ideal $\mathcal{F}_j(M) \subset R_i$ of M is the ideal of R_i generated by the $(n-i)$ -minors of the matrix φ . It is independent of the choice of the resolution.

For example, the 0–th Fitting ideal is the ideal generated by the local equation f_i of the curve at p_i , i.e., $(\det \varphi) = (f_i) \subset R_i$. As for the first Fitting ideal, by [FGvS99, Prop. C2] it is the image in R_i of the valuation map

$$M \times \text{Hom}(M, R_i) \rightarrow R_i.$$

For later use, we highlight the following remark

Remark 4.27. $\mathcal{F}_1(M) = \mathcal{F}_1(M^\vee) \subset R_i$.

We can now formulate the following key Proposition of [FGvS99].

Proposition 4.28. [FGvS99]

- a) For any $F \in \overline{\text{Jac}}(D)$, the image of the tangent space $T_F \overline{\text{Jac}}(D)$ in $T_s S$ contains the space $V(D)$ defined in (4.26).
- b) There exists an F in $\overline{\text{Jac}}(D)$ such that the image is exactly $V(D)$.
- c) $\overline{\text{Jac}}(\mathcal{D}_B)$ is smooth along $\overline{\text{Jac}}(D)$ if and only if the image of $T_0 B$ in $T_s S$ is transversal to $V(D)$.

Proof. By [FGvS99, Prop. C1] the image of the tangent space $T \text{Def}(F_{p_i}, D_{p_i})$ in $T \text{Def}(p_i)$ equals to the image

$$(4.29) \quad W(F_{p_i}) \subset R_i / (\partial_x f_i, \partial_y f_i) = T \text{Def}(p_i)$$

of the first Fitting ideal $\mathcal{F}_1(F_i) \subset R_i$. By [FGvS99, Cor. C3], $\mathcal{F}_1(F_i) \supset I_i$ and hence the first statement follows from the definition of $V(D)$ and the fact that the tangent map $T_F \overline{\text{Jac}}(D) \rightarrow \prod T \text{Def}(F_{p_i}, D_{p_i})$ is surjective. The third statement follows from the fact that Remark C4 of [FGvS99] observes that $\mathcal{F}_1(\overline{R}_i) = I_i$, and from the fact that by Proposition 4.15 above there exist an $F \in \overline{\text{Jac}}(D)$ with local type $\{\overline{R}_i\}$. Statement c) is clear, once we notice that we are assuming that B is smooth and that $\overline{\text{Jac}}(D_B) = \overline{\text{Jac}}(D) \times_S B$. \square

Let us now get back to our situation and consider

$$(4.30) \quad f : \tilde{C} \rightarrow C, \quad \text{and} \quad \iota : \tilde{C} \rightarrow \tilde{C},$$

an étale double cover of reduced and irreducible curves with planar singularities and the corresponding involution on \tilde{C} .

We denote by $\text{Def}(\tilde{C}, C)$ the deformation functor of the map $\tilde{C} \rightarrow C$ [Ser06, Def. 3.4.1] whose tangent space can be identified by [Ran89] with $T_C^1 = \text{Ext}^1(\Omega_C^1, \mathcal{O}_C) (= T \text{Def}(C))$, viewed as the ι -invariant part of $T_{\tilde{C}}^1 = \text{Ext}^1(\Omega_{\tilde{C}}^1, \mathcal{O}_{\tilde{C}})$. A semi-universal family for this functor can be described as follows. Let $\mathcal{C} \rightarrow S$ be a semi-universal family for C , centered at a point $s \in S$. As in [CMGHL15], we can consider the finite group scheme $\text{Pic}^0(\mathcal{C})[2] \subset \text{Pic}^0(\mathcal{C})$ over S which parametrizes 2-torsion line bundles on the curves in $\mathcal{C} \rightarrow S$. Since we are in characteristic zero, the morphism $\text{Pic}^0(\mathcal{C})[2] \rightarrow S$ is étale and therefore the natural transformation

$$(\text{Pic}^0(\mathcal{C})[2], \eta) \rightarrow \text{Def}(\tilde{C}, C),$$

which one can easily check to be smooth, is an isomorphism at the level of tangent spaces. Hence, $\text{Pic}^0(\mathcal{C})[2]$ is the base of a semi-universal family for $\text{Def}(\tilde{C}, C)$. Since $\text{Pic}^0(\mathcal{C})[2] \rightarrow S$ is étale, and an isomorphism on tangent spaces, we can replace S by $\text{Pic}^0(\mathcal{C})[2]$ so that (S, s) is a semi-universal space for both $\text{Def}(C)$ and $\text{Def}(\tilde{C}, C)$. In particular, we have a family

$$f : \tilde{\mathcal{C}}_S \rightarrow \mathcal{C}_S := C, \quad \iota : \tilde{\mathcal{C}}_S \rightarrow \tilde{\mathcal{C}}_S$$

of étale double covers of integral curves with planar singularities.

We can finally state and prove the smoothness criterion for the relative Prym variety.

Theorem 4.31. *Let the notation be as above.*

- (1) *The relative compactified Prym variety $\overline{\text{Prym}}(\tilde{\mathcal{C}}_S/\mathcal{C}_S)$ over the semi-universal deformation space is smooth.*

- (2) For any smooth base B and any family of double covers as in (4.9), the relative compactified Prym variety $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$ is smooth along $\text{Prym}(\tilde{\mathcal{C}}_b/\mathcal{C}_b) = \nu_B^{-1}(b)$ if and only if the image of the tangent map $T_b B \rightarrow T\text{Def}(\mathcal{C}_b)$ of the classifying morphism is transversal to the space $V(\mathcal{C}_b) \subset T\text{Def}(\mathcal{C}_b)$ defined in (4.26);
- (3) $\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$ is smooth along $\text{Prym}(\tilde{\mathcal{C}}_b/\mathcal{C}_b) = \nu^{-1}(b)$ if and only if $\overline{\text{Jac}}(\mathcal{C}_B)$ is smooth along $\overline{\text{Jac}}(\mathcal{C}_b) = \pi^{-1}(b)$;

Proof. Consider the curve \tilde{C} and the involution $\iota : \tilde{C} \rightarrow \tilde{C}$. A result of Rim (Corollary in [Rim80]) ensures that we may consider a ι -equivariant semi-universal family $\tilde{\mathcal{C}}_S \rightarrow \tilde{S}$ for \tilde{C} . By definition, this is a semi-universal family for $\text{Def}(\tilde{C})$ that has the additional property of admitting compatible actions of ι on \tilde{C} and on \tilde{S} . Let us then consider such a family. We set $T =: \text{Fix}(\iota) \subset \tilde{S}$ and we denote by $\tilde{\mathcal{C}}_T \rightarrow T$ the restriction of the semi-universal family to T . Then

$$\mathcal{C}_T := \tilde{\mathcal{C}}_T/\iota \rightarrow T$$

is a family of integral curves with locally planar singularities. If \tilde{S} is centered at $s \in \tilde{S}$, then $\mathcal{C}_s = C$, and the tangent space of T at s is the ι -invariant subspace $(T_s \tilde{S})^\iota = T_C^1 = T\text{Def}(\tilde{C}, C)$. This shows that T is a semi-universal space for $\text{Def}(\tilde{C}, C)$. Hence, to prove (1) it is enough to prove that $\text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T)$ is smooth. But this is clear, since $\text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T)$ is the just the component of the fixed locus of τ that contains the zero section of $\overline{\text{Jac}}(\tilde{\mathcal{C}}_S)$. Since $\overline{\text{Jac}}(\tilde{\mathcal{C}}_S)$ is smooth, so is every component of fixed locus of any involution acting on it.

For item (2), we can reason as in the proof of Proposition 4.28, provided we understand for, any $F \in \text{Prym}(\tilde{\mathcal{C}}_b/\mathcal{C}_b)$, the image in $T\text{Def}(\mathcal{C}_b, \mathcal{C}_b)$ of the tangent space of the relative compactified Prym variety over a semi-universal deformation space T for $\text{Def}(\tilde{\mathcal{C}}_b, \mathcal{C}_b)$. Indeed, since by Proposition 4.14 $\text{Prym}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) = \text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T) \times_T B$, it is sufficient to prove that for any $F \in \text{Prym}(\tilde{\mathcal{C}}_b/\mathcal{C}_b)$, the image of $T_F \text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T)$ in $T\text{Def}(\tilde{\mathcal{C}}_b, \mathcal{C}_b) = T_{\mathcal{C}_b}^1$ contains the support $V(\mathcal{C}_b)$ of the tangent cone to the equigeneric locus. To see this, we argue as follows. Set $C = \mathcal{C}_b$ and $\tilde{C} = \mathcal{C}_b$ and let $\{p_1, \dots, p_k, q_1, \dots, q_k\}$ be the singular points of \tilde{C} , with $f^{-1}(x_i) = \{p_i, q_i\}$. Consider an $F \in \text{Prym}(\tilde{C}/C)$ and let \tilde{S} and T be as above. The tangent map

$$\Xi : T_F \overline{\text{Jac}}(\tilde{\mathcal{C}}_S) \longrightarrow \prod_{i=1}^k T_{\tilde{\mathcal{C}}_{p_i}}^1 \times T_{\tilde{\mathcal{C}}_{q_i}}^1,$$

is equivariant with respect to the two involutions τ , which acts on $T_F \overline{\text{Jac}}(\tilde{\mathcal{C}}_S)$ with fixed locus $T_F \text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T)$, and ι , which acts on $\prod_{i=1}^k T_{\tilde{\mathcal{C}}_{p_i}}^1 \times T_{\tilde{\mathcal{C}}_{q_i}}^1$ by interchanging $T_{\tilde{\mathcal{C}}_{p_i}}^1$ with $T_{\tilde{\mathcal{C}}_{q_i}}^1$ (which are isomorphic since $\iota(p_i) = q_i$). By item (a) in Proposition 4.28, we know that the image of Ξ is

$$\prod_i W(F_{p_i}) \times W(F_{q_i}).$$

Here as in (4.29) $W(F_{p_i})$ denotes the image of the first Fitting ideal of F_{p_i} in $T_{\tilde{\mathcal{C}}_{p_i}}^1$. Since $\tau(F) = F$, $F_{q_i} = F_{p_i}^\vee$ and hence, by Remark 4.27, $W(F_{p_i}) \cong W(F_{q_i})$. It follows that the image of $T_F \text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T)$ in $\prod_i T\text{Def}(\tilde{C}, p_i) \times T\text{Def}(\tilde{C}, q_i)$, which is nothing but the ι -invariant subspace of $\text{Im } \Xi$, is equal to the product of diagonals $\prod \Delta_{W(F_{p_i})}$. Under the identification $T_C^1 = (T_s \tilde{S})^\iota$, the subspace $V(C) \subset T_C^1$ corresponds to the preimage in $T_s \tilde{S}$ of the product $\prod \Delta_{V(\tilde{C}, p_i)} \subset \prod T_{\tilde{\mathcal{C}}_{p_i}}^1 \times T_{\tilde{\mathcal{C}}_{q_i}}^1$. Since we know that $W(F_{p_i}) \supset V(\tilde{C}, p_i)$, it follows that $\prod \Delta_{W(F_{p_i})} \supset \prod \Delta_{V(\tilde{C}, p_i)}$ and hence that the image of $T_F \text{Prym}(\tilde{\mathcal{C}}_T/\mathcal{C}_T)$ in T_C^1 contains $V(C)$.

The only thing we are left to prove is that there exist an $F \in \text{Prym}(\tilde{C}/C)$ such that this image is exactly $V(C)$. This is done, like in Lemma 4.18, by considering a sheaf $F' \in \overline{\text{Jac}}(\tilde{C})$ of local type $\{\overline{\mathcal{O}}_{\tilde{C}, p_1}, \dots, \overline{\mathcal{O}}_{\tilde{C}, p_k}, \mathcal{O}_{\tilde{C}, p_1}, \dots, \mathcal{O}_{\tilde{C}, p_k}\}$, where $\overline{\mathcal{O}}_{\tilde{C}, p_i}$ is the normalization of $\mathcal{O}_{\tilde{C}, p_i}$, and setting

$$F = F' \otimes \iota * F'^\vee.$$

As for statement (3), it follows from (2) and statement (c) of Proposition 4.28. \square

Corollary 4.32 (of the proof of (1) of Proposition 4.31). *For any base T , the closure $\overline{\mathcal{P}}_i$ of the connected components of $\text{Fix}(-\iota^*) \subset \text{Pic}_{\tilde{\mathcal{C}}_B}^0$ do not intersect.*

Proof. Since $\text{Fix}(\tau) \subset \overline{\text{Jac}}(\tilde{\mathcal{C}}_{\tilde{\mathcal{S}}})$ is smooth, its irreducible components are smooth and disjoint. In particular, the closure in $\overline{\text{Jac}}(\tilde{\mathcal{C}}_{\tilde{\mathcal{S}}})$ of the components of $\text{Fix}(-\iota^*) \subset \text{Pic}_{\tilde{\mathcal{C}}}^0$, which are irreducible components of $\text{Fix}(\tau)$ are smooth and disjoint. If this is true over the semi-universal family, it is *a fortiori* true that the closures of the \mathcal{P}_i are disjoint over an arbitrary base. \square

Corollary 4.33. *If $\overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B)$ and B are smooth, then $\nu_B : \overline{\text{Prym}}(\tilde{\mathcal{C}}_B/\mathcal{C}_B) \rightarrow B$ is flat.*

5. DESCENT – FROM THE RELATIVE PRYM TO THE RELATIVE INTERMEDIATE JACOBIAN

In the previous section, we developed a method for associating to any family of double covers of irreducible locally planar curves a relative compactified Prym variety. We now apply these results to the double cover of curves that come up in our situation and get a relative compactified Prym $\overline{\mathcal{P}}$. The transversality arguments of Section 3 guarantee the smoothness of $\overline{\mathcal{P}}$. Unfortunately, this flat family $\overline{\mathcal{P}}$ (of relative dimension 5) lives over the relative Fano variety \mathcal{F} (or more precisely an open subset of it) and not over the base $B = (\mathbb{P}^5)^\vee$ as would be needed in order to compactify the intermediate Jacobian fibration \mathcal{J}/U . It is therefore necessary to descend $\overline{\mathcal{P}}$ to a family $\overline{\mathcal{J}}$ over B that will give the desired compactification of \mathcal{J}/U . This descent argument is the content of this section.

Let X be a general cubic fourfold (or a general Pfaffian cubic). Let $p : \mathcal{F} \rightarrow B$ be the relative Fano surface, let \mathcal{F}^0/B the non empty open subset of very good lines. Recall that, by Proposition 2.10, $\mathcal{F}^0 \rightarrow B$ is surjective and let

$$\begin{array}{ccc} \tilde{\mathcal{C}}_{\mathcal{F}^0} & \xrightarrow{\quad} & \mathcal{C}_{\mathcal{F}^0} \\ & \searrow \quad \swarrow & \\ & \mathcal{F}^0 & \end{array}$$

be the associated family of plane quintic curves with their étale double covers. By Proposition 2.10, the curves in the two families are reduced and irreducible. We may therefore apply the results of Section 4 and construct the relative compactified Prym variety

$$\nu : \overline{\mathcal{P}}_{\mathcal{F}^0} := \overline{\text{Prym}}(\tilde{\mathcal{C}}_{\mathcal{F}^0}/\mathcal{C}_{\mathcal{F}^0}) \rightarrow \mathcal{F}^0.$$

Proposition 5.1. *The relative compactified Prym $\overline{\mathcal{P}}_{\mathcal{F}^0}$ is smooth and the morphism $\nu : \overline{\mathcal{P}}_{\mathcal{F}^0} \rightarrow \mathcal{F}^0$ is flat of relative dimension 5.*

Proof. This follows immediately from (2) of Proposition 4.31 and from Corollary 3.9. \square

As usual, we let U (respectively U_1) be the open subset of B parametrizing hyperplane sections that are smooth (respectively the hyperplane sections that have a single ordinary node). We set $\mathcal{F}_U^0 = \mathcal{F}^0 \times_U B$ and let $\tilde{\mathcal{C}}_{\mathcal{F}_U^0}$ be the restriction of the family of curves. Analogously for U_1 . For any $t \in B$, and any $\ell \in \mathcal{F}_t^0 = F(\mathcal{Y}_t)$, the curve $\tilde{\mathcal{C}}_{(\ell,t)}$ is the curve of lines in \mathcal{Y}_t that meet the line $\ell \subset \mathcal{Y}_t$. We let $\mathcal{L}_U \subset \tilde{\mathcal{C}}_{\mathcal{F}_U^0} \times_U \mathcal{Y}_U$ be the corresponding universal family of lines of the smooth hyperplane sections. For any $x \in \tilde{\mathcal{C}}_{(\ell,t)}$ we let \mathcal{L}_x be the corresponding line in \mathcal{Y}_t . There is a relative Abel–Jacobi map

$$\Phi_{\mathcal{L}_U} : \tilde{\mathcal{C}}_{\mathcal{F}_U^0} \rightarrow \mathcal{J}_U, \quad \tilde{\mathcal{C}}_{(\ell,t)} \ni x \mapsto \Phi_{\mathcal{Y}_t}(\mathcal{L}_x - \ell) \in \mathcal{J}_t$$

inducing a morphism

$$\text{Jac}(\tilde{\mathcal{C}}_{\mathcal{F}_U^0}) \rightarrow \mathcal{J}_{\mathcal{F}_U^0} = \mathcal{J}_U \times_U \mathcal{F}_U^0.$$

Since for every $x \in \tilde{\mathcal{C}}_{(\ell,t)}$, the rational equivalence class of the cycle $\mathcal{L}_x + \iota\mathcal{L}_x$ in \mathcal{Y}_t is constant, the morphism Ψ factors via $(1 - \iota)\text{Jac}(\tilde{\mathcal{C}}_{\mathcal{F}_U^0}) = \text{Prym}(\tilde{\mathcal{C}}_{\mathcal{F}_U^0}/\mathcal{C}_{\mathcal{F}_U^0})$ inducing a morphism from the generalized relative Prym variety

$$(5.2) \quad \text{Prym}(\tilde{\mathcal{C}}_{\mathcal{F}_U^0}/\mathcal{C}_{\mathcal{F}_U^0}) \rightarrow \mathcal{J}_{\mathcal{F}_U^0},$$

which is an isomorphism by a result of Mumford [Tju72]. In particular, over the smooth locus the relative Prym variety descends to the intermediate Jacobian fibration. The following lemma shows that it is the case also over the locus U_1 parametrizing one-nodal hyperplane sections.

Lemma 5.3. *The isomorphism (5.2) extends to an isomorphism $\text{Prym}(\tilde{\mathcal{C}}_{\mathcal{F}_{U_1}^0}/\mathcal{C}_{\mathcal{F}_{U_1}^0}) \rightarrow \mathcal{J}_{\mathcal{F}_{U_1}^0} = \mathcal{J}_1 \times_{U_1} \mathcal{F}_{U_1}^0$.*

Proof. The extensions of the families $\text{Prym}(\tilde{\mathcal{C}}_{\mathcal{F}_U^0}/\mathcal{C}_{\mathcal{F}_U^0})$ and $\mathcal{J}_{\mathcal{F}_U^0}$ as families of semi-abelian varieties over the boundary $\mathcal{F}_{U_1}^0 \setminus \mathcal{F}_U^0$, are determined by the monodromy of the local systems over \mathcal{F}_U^0 : indeed, if $\mathcal{H}_{\text{Prym}}^{0,1}$ and $\mathcal{H}_{\mathcal{J}}^{0,1}$ are the Hodge bundles of the families $\nu : \text{Prym}(\tilde{\mathcal{C}}_{\mathcal{F}_U^0}/\mathcal{C}_{\mathcal{F}_U^0}) \rightarrow \mathcal{F}_U^0$ and $\rho : \mathcal{J}_{\mathcal{F}_U^0} \rightarrow \mathcal{F}_U^0$, respectively, then the two families of semi-abelian varieties $\text{Prym}(\tilde{\mathcal{C}}_{\mathcal{F}_{U_1}^0}/\mathcal{C}_{\mathcal{F}_{U_1}^0})$ and $\mathcal{J}_{\mathcal{F}_{U_1}^0}$ are given, respectively, by

$$\overline{\mathcal{H}}_{\text{Prym}}^{0,1}/j_*R^1\nu_*\mathbb{Z}, \quad \text{and} \quad \overline{\mathcal{H}}_{\mathcal{J}}^{0,1}/j_*R^1\rho_*\mathbb{Z}.$$

Here $\overline{\mathcal{H}}_{\text{Prym}}^{0,1}$ and $\overline{\mathcal{H}}_{\mathcal{J}}^{0,1}$ are the canonical extensions of the Hodge bundle across $\mathcal{F}_{U_1}^0 \setminus \mathcal{F}_U^0$, which is smooth, and $j : \mathcal{F}_U^0 \rightarrow \mathcal{F}_{U_1}^0$ is the inclusion. Since the two families are isomorphic over \mathcal{F}_U^0 by (5.2), so are the corresponding local systems and Hodge bundles, and hence so are the canonical extensions. \square

Notice that Lemma 5.3 also shows that \mathcal{J}_{U_1} is smooth. This lemma says that the relative Prym fibration $\overline{P}_{\mathcal{F}^0}$, restricted to $\mathcal{F}_{U_1}^0$ is the pull-back of the intermediate Jacobian fibration over U_1 . The aim of this section is to show a similar result over the whole of B , that is, to construct a projective compactification $\overline{\mathcal{J}}$ of \mathcal{J}_1 that is flat over B , whose pull-back to \mathcal{F}^0 will be isomorphic to $\overline{P}_{\mathcal{F}^0}$. Then $\overline{\mathcal{J}}$ will be clearly smooth.

The morphism $\pi_1 : \mathcal{J}_1 \rightarrow U_1$ is projective. In fact, there is a canonical Theta divisor $\Theta_1 \subset \mathcal{J}_1$ defined as the Zariski closure in \mathcal{J}_1 of the divisor $\Theta \subset \mathcal{J}$ defined as follows: For any $t \in U$, consider the family $\mathcal{C}_t \rightarrow \mathcal{H}_t$ of rational cubic curves in Y_t . We have the Abel-Jacobi map

$$\Phi_{\mathcal{C}_t} : \mathcal{H}_t \rightarrow J(Y_t), \quad s \mapsto \Phi_{Y_t}(\mathcal{C}_{t,s} - h_t^2),$$

where $h_t = c_1(\mathcal{O}_{Y_t}(1)) \in \text{CH}^2(Y_t)$, so that $\mathcal{C}_{t,s} - h_t^2$ is homologous to 0 in Y_t . The image of this map is a (-1) -invariant divisor in $J(Y_t)$ by the following observation ([HRS05]):

Lemma 5.4. *Let $C \subset Y$ be a general cubic rational normal curve in a smooth cubic threefold. Then C is rationally equivalent in Y to the difference of two lines. Conversely, the difference of two general lines in Y_t is rationally equivalent in Y_t to a smooth rational cubic curve.*

Proof. We admit that, when correctly defined, the family \mathcal{H}_t of such curves is irreducible. The curve C generates a \mathbb{P}^3 which intersects Y in a cubic surface $S \subset Y$ which is smooth because C is generic. The linear system $|\mathcal{O}_S(C)|$ is a \mathbb{P}^2 which provides a birational map $\phi : S \rightarrow \mathbb{P}^2$, contracting 6 lines L_i in S to points p_i . The curve C belongs to the linear system $|\phi^*\mathcal{O}_{\mathbb{P}^2}(1)|$. Choose a line L contracted to a point p , and consider the proper transform Γ in S of a conic passing through all points p_i except p . Then Γ is a line in S which belongs to the linear system $|\phi^*\mathcal{O}_{\mathbb{P}^2}(2)(-\sum_{p_i \neq p} L_i)|$ and thus $\Gamma - L$ is rationally equivalent in S to $\phi^*\mathcal{O}_{\mathbb{P}^2}(2)(-\sum_i L_i)$. But $K_S = -h|_S = \phi^*\mathcal{O}_{\mathbb{P}^2}(-3)(\sum_i L_i)$, hence we get

$$-h|_S + \phi^*\mathcal{O}_{\mathbb{P}^2}(1) = L - M$$

in $\text{CH}^1(S)$, and thus $C = h^2 + L - \Gamma$ in $\text{CH}^2(Y)$. Conversely, if L, Γ are two lines in Y_t , the $\mathbb{P}^3 = P_3$ generated by L and Γ intersects Y_t along a smooth cubic surface, and for a given point $x \in L$, the plane $\langle \Gamma, x \rangle \subset P_3$ intersects Y_t along the union of Γ and a conic Γ' meeting L at x . The curve $L \cup \Gamma'$ is a reducible rational cubic curve in S (which deforms to a smooth rational cubic curve). \square

It follows that the divisor $\text{Im } \Phi_{\mathcal{C}_t}$ (or rather 6 times this divisor) is also equal to the image in $J(Y)$ of the difference map $F(Y) \times F(Y) \rightarrow J(Y), (l_1, l_2) \mapsto l_1 - l_2$. Thus it is invariant under the involution (-1) of $J(Y_t)$. In fact, by [CG72, Section 13], this is a Theta divisor in $J(Y_t)$ (see also [HRS05]). We now have

Lemma 5.5. *For any dominant morphism $f : M \rightarrow U_1$ with M irreducible, the (-1) -invariant part of $\text{Pic } \mathcal{J}_M/M$, where $\mathcal{J}_M := \mathcal{J} \times_{U_1} M$, is generated modulo torsion by $f^*\Theta$.*

Proof. The relative Picard group $\text{Pic } \mathcal{J}_M/M := \text{Pic } \mathcal{J}_M/\pi_M^* \text{Pic } M$ (where $\pi_M : \mathcal{J}_M \rightarrow M$ is the natural map,) injects in the Picard group of the fiber $J(Y_t)$, where $t \in M$ is very general, because the fibers of the map π_M (which are also the fibers of the map $\pi_1 : \mathcal{J}_1 \rightarrow U_1$) are all reduced and irreducible. In particular, it injects into $\text{Pic } \mathcal{J}_{M_U}/M_U$, where $M_U := f^{-1}(U) \subset M$. We are thus reduced to proving that the (-1) -invariant part of $\text{Pic } J(Y_t)$ is modulo torsion generated by Θ_{Y_t} for t very general in U_1 . However, modulo torsion, the (-1) -invariant part of $\text{Pic } J(Y_t)$ is isomorphic to the Néron-Severi group of $J(Y_t)$. Finally, recall that we have a canonical isomorphism $H^3(Y_t, \mathbb{Q}) \cong H^1(J(Y_t), \mathbb{Q})$ which provides more generally an isomorphism of local systems over U :

$$R^3 u_* \mathbb{Q} \cong R^1 \pi_* \mathbb{Q}.$$

For the local system on the left, the corresponding monodromy group

$$\text{Im}(\rho : \pi_1(U, t) \rightarrow \text{Aut } H^3(Y_t, \mathbb{Z}))$$

is the full symplectic group of the intersection pairing by Picard-Lefschetz theory [Bea86]. Hence the same is true for the local system on the right. As M is irreducible and the morphism $f : M \rightarrow B$ is dominating, the image of the morphism

$$f_{U*} : \pi_1(M_U, m_t) \rightarrow \pi_1(U, t),$$

where $f_U : M_U \rightarrow U$ is the restriction of f and $f_U(m_t) = t$, is a subgroup of finite index in $\pi_1(U, t)$. Thus the monodromy group of the family $\mathcal{J}_{M_U} \rightarrow M_U$ acts via a subgroup of finite index of $Sp(H^1(\mathcal{J}_{m_t}, \mathbb{Z}))$ on the cohomology of \mathcal{J}_{m_t} . On the other hand, it is a general fact that the monodromy acts on $H^2(J(Y_{m_t}), \mathbb{Q}) = \bigwedge^2 H^1(J(Y_{m_t}), \mathbb{Q})$ by preserving (for very general t) the Néron-Severi group of $J(Y_t)$ and with finite orbits on $\text{NS}(J(Y_t))_{\mathbb{Q}}$. The only elements of $\bigwedge^2 H^1(J(Y_{m_t}), \mathbb{Z})$ which have finite orbit under $Sp(H^1(\mathcal{J}_{m_t}, \mathbb{Z}))$ are the multiples of the class Θ_t and we conclude that $\text{NS}(J(Y_t))_{\mathbb{Q}} = \mathbb{Q}\Theta_{Y_t}$. \square

Using the above results and the local compactification of Section 4, we now conclude:

Theorem 5.6. *Let $j_1 : U_1 \rightarrow B$ be the inclusion. Then*

(i) *For any sufficiently large integer $d > 0$, the sheaf of algebras*

$$\mathcal{E}^* := R^0 j_{1*}(\oplus_{k \geq 0} R^0 \pi_{1*} \mathcal{O}_{\mathcal{J}_1}(kd\Theta))$$

is a sheaf of algebras of finite type, and each summand $R^0 j_{1}(R^0 \pi_{1*} \mathcal{O}_{\mathcal{J}_1}(kd\Theta))$ is a locally free coherent sheaf on B .*

(ii) *The variety $\overline{\mathcal{J}} := \text{Proj}(\mathcal{E}^*) \rightarrow B$ is a smooth projective compactification of \mathcal{J}_1 .*

(iii) *The variety $\overline{\mathcal{J}}$ is irreducible hyper-Kähler.*

Proof. (i) and (ii) It suffices to prove the existence of d locally in the Zariski topology. For any $b \in B$, there exists by Corollary 2.11 a very good line $l_b \in Y_b$. As the family $\mathcal{F} \rightarrow B$ of lines in the fibers of u is smooth at a point parameterizing a very good line, we may assume up to an étale base change $c : M \rightarrow B$, $m \rightarrow b$, that there is a section $M \rightarrow \mathcal{F}_M$, $m \mapsto l_m$. Furthermore, the conclusion of Corollary 3.9 holds, so that the corresponding family of plane quintic curves C_{l_m} induces a versal deformations of $\text{Sing}(C_{l_m})$. We can then apply the results of Section 4 and especially Theorem 4.31 which provides a smooth projective flat compactification $\pi_M : \mathcal{J}_M \rightarrow M$ of $\mathcal{J}_{M_1} := \mathcal{J}_1 \times_{U_1} M_1$, where $M_1 = c^{-1}(U_1) \subset M$. If \mathcal{L} is a π_M -relatively ample line bundle on \mathcal{J}_M , for some l_0 large enough, $R^i \pi_{M*} \mathcal{L}^{\otimes n} = 0$ for $n \geq l_0$, $i > 0$, and thus we conclude by flatness of π_M that $R^0 \pi_{M*} \mathcal{L}^{\otimes n}$ is locally free for $n \geq d_0$. Next we observe that the involution (-1) acting on \mathcal{J}_{M_1} extends to an involution that we will denote by $(-1)_M$ acting on \mathcal{J}_M over M : this involution can be defined either as i^* where i is the relative involution of the family of curves D_{l_m} associated with the section $m \mapsto l_m \in F(Y_m)$, or as the involution $\mathcal{F} \mapsto \mathcal{F}^*$ acting on reflexive sheaves on D_{l_m} . It follows that starting from any π_M -relatively ample line bundle \mathcal{L} on \mathcal{J}_M , we can construct a π_M -relatively ample line bundle \mathcal{L}' which is relatively ample and $(-1)_M$ -invariant, namely

$$\mathcal{L}' = \mathcal{L} \otimes (-1)_M^* \mathcal{L}.$$

We now apply Lemma 5.5 to the dominant morphism $c_{U_1} : M_1 \rightarrow U_1$ and the (-1) -invariant line bundle $\mathcal{L}'|_{\mathcal{J}_{M_1}}$. It says that up to replacing \mathcal{L}' by a multiple, we have

$$(5.7) \quad \mathcal{L}'|_{\mathcal{J}_{M_1}} = \tilde{c}_{U_1}^*(d\Theta) \otimes \pi_{M_1}^* \mathcal{N}_1$$

for some integer d , where \mathcal{N}_1 is a line bundle on M_1 , $\pi_{M_1} : \mathcal{J}_{M_1} \rightarrow M_1$ is the restriction of π_M over M_1 and $\tilde{c}_{U_1} : \mathcal{J}_{M_1} \rightarrow \mathcal{J}_{U_1}$ is the natural morphism above $c_{U_1} : M_1 \rightarrow U_1$. We observe now that $\text{Pic } M_1 = \text{Pic } M$ because M is smooth and $\text{codim}(M \setminus M_1 \subset M) \geq 2$. Thus the line bundle \mathcal{N}_1 extends to a line bundle \mathcal{N} on M . Let $j_{M_1} : M_1 \rightarrow M$ be the inclusion map. We have by (5.7)

$$(5.8) \quad \begin{aligned} R^0 j_{M_1*} (R^0 \pi_{M_1*} (\mathcal{L}'^{\otimes k}_{|\mathcal{J}_{M_1}})) &= R^0 j_{M_1*} (R^0 \pi_{M_1*} (\tilde{c}_{U_1}^* \mathcal{O}_{\mathcal{J}_1}(kd\Theta) \otimes \pi_{M_1}^* \mathcal{N}_1^{\otimes k})) \\ &= \mathcal{N}^{\otimes k} \otimes R^0 j_{M_1*} (R^0 \pi_{M_1*} (\tilde{c}_{U_1}^* \mathcal{O}_{\mathcal{J}_1}(kd\Theta))). \end{aligned}$$

As the morphism $M \rightarrow B$ is étale, proving that $R^0 j_{1*} (R^0 \pi_{1*} \mathcal{O}_{\mathcal{J}_1}(kd\Theta))$ is locally free on B and that the sheaf of algebras $\oplus_k R^0 j_{1*} (R^0 \pi_{1*} \mathcal{O}_{\mathcal{J}_1}(kd\Theta))$ is of finite type is equivalent to proving that $R^0 j_{M_1*} (R^0 \pi_{M_1*} \tilde{c}_{U_1}^* (\mathcal{O}_{\mathcal{J}_1}(kd\Theta)))$ is locally free on M and that the sheaf of algebras $\oplus_k R^0 j_{M_1*} (R^0 \pi_{M_1*} (\tilde{c}_{U_1}^* \mathcal{O}_{\mathcal{J}_1}(kd\Theta)))$ is of finite type. Observe that we have

$$(5.9) \quad R^0 j_{M_1*} (R^0 \pi_{M_1*} (\mathcal{L}'^{\otimes k}_{|\mathcal{J}_{M_1}})) = R^0 \pi_{M*} (\mathcal{L}'^{\otimes k}_{|\mathcal{J}_{M_1}})$$

because \mathcal{J}_M and M are smooth and, by flatness of \mathcal{J}_M over M , $\mathcal{J}_{M_1} \subset \mathcal{J}_M$ is the complement of a closed algebraic subset of codimension ≥ 2 in \mathcal{J}_M , as is $M_1 \subset M$. Finally, by (5.8) and (5.9), the sheaf $R^0 j_{M_1*} (R^0 \pi_{M_1*} \tilde{c}_{U_1}^* (kd\Theta))$ is isomorphic to

$$\mathcal{N}^{\otimes -k} \otimes R^0 \pi_{M*} (\mathcal{L}'^{\otimes k}),$$

which is locally free by vanishing and base change for the flat family $\pi_M : \mathcal{J}_M \rightarrow M$, for $k \geq l_0$. The relative ampleness of \mathcal{L}' implies that

$$\oplus_k R^0 \pi_{M*} (\mathcal{L}'^{\otimes k}_{|\mathcal{J}_{M_1}})$$

is a sheaf of algebras of finite type over M , and so is $\oplus_k R^0 j_{M_1*} (R^0 \pi_{M_1*} (\tilde{c}_{U_1}^* \mathcal{O}_{\mathcal{J}_1}(kd\Theta)))$ by the isomorphism above. Finally, we have over M

$$\mathcal{J}_M = \text{Proj}(\oplus_k R^0 \pi_{M*} \mathcal{L}'^{\otimes k}).$$

and thus by (5.8), the variety $\overline{\mathcal{J}} := \text{Proj } \mathcal{E}^* \rightarrow B$ satisfies $\overline{\mathcal{J}}_M := \overline{\mathcal{J}} \times_B M \cong \mathcal{J}_M$. It follows that $\overline{\mathcal{J}}$ is smooth because \mathcal{J}_M is smooth and $M \rightarrow B$ is étale.

(iii) We know by Proposition 1.22 that \mathcal{J}_1 has a nondegenerate holomorphic 2-form which by Theorem 1.2 (iii) extends to a nondegenerate holomorphic 2-form on $\overline{\mathcal{J}}$, since $\text{codim}(\overline{\mathcal{J}} \setminus \mathcal{J}_1 \subset \overline{\mathcal{J}}) \geq 2$. What remains to be done is to prove that $\overline{\mathcal{J}}$ is irreducible hyperkähler. We have the following lemma:

Lemma 5.10. *The holomorphic 2-forms on any finite étale cover $\tilde{\mathcal{J}}$ of $\overline{\mathcal{J}}$ are multiples of the form σ coming from X .*

Proof. The variety $\overline{\mathcal{J}}$ contains the Zariski closure of the Theta divisor $\Theta \subset \mathcal{J}$ which is birational to a \mathbb{P}^1 -bundle over the Lehn-Lehn-Sorger-van Straten variety $F_3(X)$. Indeed, recall from [LLSvS13] that $F_3(X)$ parameterizes birationally pencils $|D|$ of rational cubic curves on cubic surfaces $S \subset X$. Consider the \mathbb{P}^1 -bundle $\mathbb{P} \rightarrow F_3(X)$ with fiber over (S, D) the \mathbb{P}^1 of hyperplanes in \mathbb{P}^5 containing S . Then \mathbb{P} admits a morphism to $(\mathbb{P}^5)^*$ whose fiber over H_Y parameterizes the pencils of cubic rational curves in Y . We already mentioned that via the map

$$|D| \mapsto \Phi_X(D - h^2) \in J(Y),$$

the set of such pencils dominates (in fact, is birational to) the Theta divisor of $J(Y)$. This construction in family over U provides the rational map $\mathbb{P} \dashrightarrow \Theta \subset \mathcal{J} \subset Z$.

The proof of Lemma 5.10 is now immediate: The variety $F_3(X)$ is simply connected (it is a deformation of $S^{[4]}$ for some K3 surface S , see [AL14]), hence \mathbb{P} is simply connected, so the rational map $\mathbb{P} \dashrightarrow \overline{\mathcal{J}}$ constructed above lifts to a rational map $\mathbb{P} \dashrightarrow \tilde{\mathcal{J}}$ for any finite étale cover $\tilde{\mathcal{J}}$ of $\overline{\mathcal{J}}$. As $F_3(X)$ is an irreducible hyperkähler manifold and $\mathbb{P} \rightarrow F_3(X)$ is a \mathbb{P}^1 -bundle, the holomorphic 2-forms on \mathbb{P} are all multiples of the restriction of $\sigma_{\overline{\mathcal{J}}}$. It thus only suffices to show that if α is a holomorphic 2-form on $\tilde{\mathcal{J}}$ which vanishes on the image Θ of \mathbb{P} , $\alpha = 0$. That follows however immediately from the fact that, fiberwise, as $\Theta_t \subset \mathcal{J}_t$ is an ample divisor, the restriction map

$$H^0(\mathcal{J}_t, \Omega_{\mathcal{J}_t}^i) \rightarrow H^0(\Theta_t^0, \Omega_{\Theta_t^0}^i)$$

is injective for $t \in U$, and $0 \leq i \leq 2$, where Θ_t^0 is the smooth locus of Θ_t . \square

By the Beauville-Bogomolov decomposition theorem [Bea83], Lemma 5.10 implies that $\overline{\mathcal{J}}$ is irreducible hyper-Kähler. \square

6. CONSTRUCTION OF A BIRATIONAL MAP IN THE PFAFFIAN CASE

Let X be a Pfaffian cubic fourfold. This means that there is a 6-dimensional vector space V_6 and a 6-dimensional vector space $W_6 \subset \bigwedge^2 V_6^*$ such that $X \subset \mathbb{P}(W_6)$ is the Pfaffian cubic hypersurface defined by the Pfaffian equation $\omega^3 = 0$. Following [BD85], let $\Sigma \subset G(2, V_6)$ be the surface defined as follows:

$$\Sigma = \{[l] \in G(2, V_6), \omega|_{V_l} = 0 \forall \omega \in W_6\},$$

where we denote by $V_l \subset V_6$ the rank 2 vector subspace corresponding to $[l]$ (l will denote the corresponding projective line in $\mathbb{P}(V_6)$). Being defined (for general X as above) as the complete intersection of 6 linear Plücker forms on $G(2, 6)$, Σ is a smooth degree 14 K3 surface.

Our goal in this section is to prove the following result:

Theorem 6.1. *The intermediate Jacobian fibration \mathcal{J} of X is birational to the O'Grady moduli space $\mathcal{M}_{2,0,4}(\Sigma)$ parameterizing rank 2 semistable sheaves on Σ with $c_1 = 0$, $c_2 = 4$.*

Corollary 6.2. *The compactified Jacobian fibration $\overline{\mathcal{J}}$ of Theorem 5.6 is a deformation of O'Grady's 10-dimensional variety.*

Proof. Indeed, Theorem 5.6 is true when X is a general Pfaffian cubic fourfold, thanks to the results of Section 3.2 which guarantee that the assumptions needed to make the local construction of Section 4 work are satisfied in the general Pfaffian case. Hence we can rephrase Theorem 6.1 saying that our family of hyper-Kähler compactified Jacobians $\overline{\mathcal{J}}_X$ parameterized by an open set of the moduli space of cubic fourfolds has a smooth member which is birational to O'Grady's 10-dimensional variety $OG10$, which is also a smooth hyper-Kähler manifold. We then apply [Huy99] to conclude that the varieties $\overline{\mathcal{J}}_X$ are deformation equivalent to $OG10$. \square

We will heavily use the results of [IM00] (based on [MT01]) and their generalization in [Kuz04]. The proof of Theorem 6.1 will be completed in subsection 6.3. As the intermediate Jacobian fibration is a fibration over $\mathbb{P}(W_6^*)$ into intermediate Jacobians of cubic 3-folds $Y_3 = H \cap X$ which come equipped with a Pfaffian representation induced from the one of X , we will devote sections 6.1 and 6.2 to the Pfaffian 3-fold geometry, which is the subject of the papers [IM00], [Kuz04]. The cubic threefold Y_3 is represented as the Pfaffian cubic in $\mathbb{P}(W_5)$ for some 5-dimensional vector space $W_5 \subset \bigwedge^2 V_6^*$. Associated to this data, we get the index 1, degree 14 Fano 3-fold $V_{14} \subset G(2, V_6)$ defined as

$$(6.3) \quad V_{14} = \{[l] \in G(2, V_6), \omega|_{V_l} = 0 \forall \omega \in W_5\}.$$

The variety V_{14} comes equipped with the tautological rank 2 dual vector bundle, that we will denote by \mathcal{E}_{14} . The Pfaffian cubic Y_3 comes equipped with the natural rank 2 vector bundle \mathcal{E}_3 with fiber $(\text{Ker } \omega)^*$ over a 2-form $\omega \in \mathbb{P}(\bigwedge^2 V_6^*)$ of rank 4. As $V_6^* = H^0(Y_3, \mathcal{E}_3) = H^0(V_{14}, \mathcal{E}_{14})$, we have natural maps

$$\phi : \mathbb{P}(\mathcal{E}_{14}) \rightarrow \mathbb{P}(V_6), \quad \psi : \mathbb{P}(\mathcal{E}_3) \rightarrow \mathbb{P}(V_6)$$

which are easily seen to have the same image $Q \subset \mathbb{P}(V_6)$. According to [Kuz04], Q is a quartic hypersurface which is singular along a curve and ϕ, ψ are two small resolutions of Q . In particular, $\mathbb{P}(\mathcal{E}_{14})$ and $\mathbb{P}(\mathcal{E}_3)$ are birational, and even related by a flop $\theta = \psi^{-1} \circ \phi$.

We will need the following lemma: Denote by Δ_{14} , resp. Δ_3 , the general fiber of the projective bundle $\mathbb{P}(\mathcal{E}_{14}) \rightarrow V_{14}$, resp. $\mathbb{P}(\mathcal{E}_3) \rightarrow Y_3$. Via the birational map θ , these curves (which do not meet the indeterminacy locus of θ or θ^{-1}) can be seen as curves either in $\mathbb{P}(\mathcal{E}_{14})$ or in $\mathbb{P}(\mathcal{E}_3)$, and we will denote by $\deg_{V_{14}}(\cdot)$, resp. $\deg_{Y_3}(\cdot)$ the degree of their projection in V_{14} , resp. Y_3 with respect to the canonical polarizations. We will also denote by $\deg_Q(\cdot)$ the degree of their projections in $Q \subset \mathbb{P}^5$ via ϕ , resp. ψ .

Lemma 6.4. *One has $\deg_Q \Delta_3 = \deg_Q \Delta_{14} = 1$ and*

$$(6.5) \quad \begin{aligned} \deg_{V_{14}} \Delta_{14} &= 0, \quad \deg_{Y_3} \Delta_{14} = 4, \\ \deg_{Y_3} \Delta_3 &= 0, \quad \deg_{V_{14}} \Delta_3 = 4. \end{aligned}$$

Proof. Let $\omega \in Y_3$ be a general point. The fiber Δ_3 over ω is thus by definition the projective line $L_\omega = \mathbb{P}(\text{Ker } \omega)$. Its image C_ω in V_{14} is the set of points $[l]$ in V_{14} such that the corresponding projective line l meets L_ω . As $L_\omega = \mathbb{P}(\text{Ker } \omega)$, one has $\omega|_{V_l} = 0$ for any line l meeting L_ω . Thus C_ω is the vanishing locus of the 4-dimensional space $W_5/\langle \omega \rangle$ of Plücker equations on the set of lines in V_6 meeting L_ω . It is easily proved that this locus has degree 4 and this proves the last statement. Next let $[l] \in V_{14}$ be a general point. Let C_l be the image in Y_3 of the fiber Δ_{14} over $[l]$. Then C_l is the set of $\alpha \in W_5$ whose kernel intersects V_l nontrivially, and the map $l \mapsto C_l$ associates to $x \in l$ the unique form $\omega \in \mathbb{P}(W_5)$ such that $x \in \mathbb{P}(\text{Ker } \omega)$. As all forms in W_5 vanish on l , the natural morphism

$$V_6 \otimes \mathcal{O}_l(-1) \rightarrow W_5^* \otimes \mathcal{O}_l$$

of vector bundles over $l = \mathbb{P}^1$ factors through $(V_6/V_l) \otimes \mathcal{O}_l(-1)$, and its cokernel has thus degree 4, which proves the second statement in (6.4). The other statements are immediate. \square

6.1. The cubic 3-fold side. We know by [MT01], [IM00] that the moduli space of stable vector bundles of rank 2 on Y_3 with $c_1 = 0$ and $\deg_{Y_3} c_2 = 2$ is birationally isomorphic to $J(Y_3)$ via the Abel-Jacobi map. Let E be such a vector bundle.

Lemma 6.6. *If E is general, then:*

(i) $\dim H^0(Y_3, E^* \otimes \mathcal{E}_3) = 4$.

(ii) Furthermore, if (Y_3, E) is general, for a general section $\sigma \in H^0(Y_3, E^* \otimes \mathcal{E}_3)$, we have an exact sequence

$$(6.7) \quad 0 \rightarrow E \xrightarrow{\sigma} \mathcal{E}_3 \rightarrow \mathcal{O}_S(C) \rightarrow 0,$$

where $S \in |\mathcal{O}_{Y_3}(2)|$ is a smooth K3 surface and $C \subset S$ is a degree 9, genus 5 curve.

Proof. (i) Note that $E \cong E^*$, so (i) is equivalent to $\dim H^0(Y_3, E \otimes \mathcal{E}_3) = 4$. We now specialize to the case where E is not locally free, namely $E = \mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}$, where l_1, l_2 are two general lines in Y_3 . Then $h^0(Y_3, \mathcal{E}_3 \otimes \mathcal{I}_{l_i}) = 2$ for $i = 1, 2$ and $H^p(Y_3, \mathcal{E}_3 \otimes \mathcal{I}_{l_i}) = 0$ for $i = 1, 2$ and $p > 0$, which implies the result for general E by standard deformation arguments.

(ii) As E is stable, for any $0 \neq \sigma \in H^0(Y_3, E^* \otimes \mathcal{E}_3)$, the generic rank of $\sigma : E \rightarrow \mathcal{E}_3$ must be 2, so that we have the exact sequence (6.7). We claim that the surface S defined by the vanishing of the determinant of σ is smooth so that, in particular, the rank of σ is 1 along S and $\text{Coker } \sigma$ is a line bundle on S . We clearly have $S \in |\mathcal{O}_{Y_3}(2)|$ so that assuming the claim, S is a smooth K3 surface. Finally the exact sequence (6.7) and the fact that $\deg_{Y_3} c_2(E) = 2$, $\deg_{Y_3} c_2(\mathcal{E}_3) = 5$ immediately imply that the degree of C is 9. To conclude, we observe that $H^0(Y_3, E) = 0$, $H^1(Y_3, E) = 0$ and the exact sequence (6.7) give $H^0(S, \mathcal{O}_S(C)) = 6$, that is, $g(C) = 5$.

We now prove the claim. As the smoothness of the degeneracy surface S is an open property (on the moduli space of 4-uples (Y_3, E_3, E, σ) with $\dim \text{Hom}(E, E_3) = 4$, which is known to be irreducible by (i) and [MT01], [IM00]), it suffices to show that the conclusion holds for at least one 4-uple satisfying the property that $\dim \text{Hom}(E, E_3) = 4$. It turns out that the result is true generically even in the degenerate case where $E = \mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}$. This is proved by working more generally on the universal Pfaffian cubic Y_{Pf} in \mathbb{P}^{14} , of which Y_3 is a generic linear section. One easily checks that given two general lines l_1, l_2 contained in Y_{Pf} , and two sections σ , resp. τ of $\mathcal{E}_{Pf} \otimes \mathcal{I}_{l_1}$, resp. $\mathcal{E}_{Pf} \otimes \mathcal{I}_{l_2}$, the quadratic equation $\sigma \wedge \tau \in H^0(Y_{Pf}, \det \mathcal{E}_{Pf}) = H^0(Y_{Pf}, \mathcal{O}_{Y_{Pf}}(2))$ is of the form $\omega \mapsto \omega|_{W_4}^2$ for some 4-dimensional subspace $W_4 \subset V_6$, hence defines a rank 6 quadric Q on \mathbb{P}^{14} . Its intersection with Y_{Pf} is thus smooth away from the set of ω whose kernel is contained in the codimension 2 linear subspace $W_4 \subset W_6$, which has codimension 4 in Y_{Pf} . The generic linear section $Y_3 \subset Y_{Pf}$ thus intersects $Q \cap Y_{Pf}$ along a smooth surface. \square

Note that if we restrict (6.7) to S , and then to C , we conclude, using $\mathcal{O}_S(C)|_C = K_C$, that there is a surjective morphism

$$(6.8) \quad \mathcal{E}_3|_C \rightarrow K_C,$$

hence a section

$$\phi_\sigma : C \rightarrow \mathbb{P}(\mathcal{E}_3|_C)$$

such that $\phi_\sigma^* \mathcal{O}_{\mathbb{P}(\mathcal{E}_3)}(1) = K_C$. This morphism induces a surjection $H^0(\mathbb{P}(\mathcal{E}_3), \mathcal{O}_{\mathbb{P}(\mathcal{E}_3)}(1)) \rightarrow H^0(C, K_C)$ since both maps $H^0(Y_3, \mathcal{E}_3) \rightarrow H^0(S, \mathcal{O}_S(C))$ and $H^0(S, \mathcal{O}_S(C)) \rightarrow H^0(C, K_C)$ are surjective. As $H^0(Q, \mathcal{O}_Q(1)) = H^0(\mathbb{P}(\mathcal{E}_3), \mathcal{O}_{\mathbb{P}(\mathcal{E}_3)}(1))$, we get as well a surjective map $H^0(Q, \mathcal{O}_Q(1)) \rightarrow H^0(C, K_C)$. Thus the image of C in $Q \subset \mathbb{P}(V_6)$ via $\psi \circ \phi_\sigma$ is a linearly normal canonical genus 5 curve.

The following lemma shows conversely how to recover the bundle from the curve $C \subset \mathbb{P}(\mathcal{E}_3)$:

Lemma 6.9. *Let $C \subset \mathbb{P}(\mathcal{E}_3)$ be a general genus 5 curve such that the image C' of C in Y_3 has degree 9, the morphism $C \rightarrow C'$ is an isomorphism, and the image of C in $Q \subset \mathbb{P}^5$ via $\psi \circ \phi_\sigma$ is a linearly normal canonical curve. Then there exists a unique stable vector bundle E on Y_3 with $c_1 = 0$, $\deg_{Y_3} c_2 = 2$ and a unique $\sigma \in H^0(Y_3, E^* \otimes \mathcal{E}_3)$ determining the line bundle $\mathcal{O}_S(C)$ as in Lemma 6.6.*

Here “general” means that there is a component of the Hilbert scheme of such curves on which the conclusion holds generically. In fact, this component is the one containing the curves $\phi_\sigma(C)$ appearing in Lemma 6.6 and its proof.

Proof. The curve $C' \subset Y_3$ has degree 9 and genus 5. Assuming the Abel-Jacobi class of C' is general in $J(Y_3)$, then (1) C' is not contained in a hyperplane section of Y_3 and (2) C' is contained in a unique surface $S \subset Y_3$, with $S \in |\mathcal{O}_{Y_3}(2)|$. Indeed, curves contained in a hyperplane section of Y_3 cannot have a general Abel-Jacobi class, by Lemma 6.10 below. This proves the first statement. We have $h^0(Y_3, \mathcal{O}_{Y_3}(2)) = 15$ and $h^0(C', \mathcal{O}_{Y_3(2)|_{C'}}) = 14$, thus C' is contained in at least one quadric section of X . If C' is contained in two surfaces S, S' as above, then as S and S' have no common component by the first statement, C' is a component of the complete intersection $S \cap S'$ which has degree 12. Thus C' is residual to a degree 3 curve, and its Abel-Jacobi point is, up to a sign and a constant, the Abel-Jacobi point of a degree 3 curve which again by Lemma 6.10 cannot be general in $J(Y_3)$. This proves the second statement. Note that, according to Lemma 6.6 (ii), the surface S is smooth for general C in the considered component of the Hilbert scheme of $\mathbb{P}(\mathcal{E}_3)$. We now observe that the restriction map $H^0(S, \mathcal{E}_{3|S}^*(C')) \rightarrow H^0(C', \mathcal{E}_{3|C'}^*(K_{C'}))$ is surjective. This indeed follows from the fact that $H^1(S, \mathcal{E}_{3|S}^*) = 0$ (see [Kuz04]). Let $\sigma' \in H^0(S, \mathcal{E}_{3|S}^*(C'))$ be a lift of the natural section $\sigma \in H^0(C', \mathcal{E}_{3|C'}^*(K_{C'}))$ giving the embedding of C' in $\mathbb{P}(\mathcal{E}_3)$ with image C . We have $\dim H^0(Y_3, \mathcal{E}_3) = 6 = \dim H^0(S, \mathcal{O}_S(C))$ and thus the property that the map

$$\sigma' : H^0(Y_3, \mathcal{E}_3) \rightarrow H^0(S, \mathcal{O}_S(C))$$

is an isomorphism is an open property. Furthermore, the line bundle $\mathcal{O}_S(C)$ is generically globally generated, and we thus conclude that for generic C , we get a surjective morphism

$$\sigma' : \mathcal{E}_{3|S} \rightarrow \mathcal{O}_S(C'),$$

hence as well a surjective morphism

$$\mathcal{E}_3 \rightarrow \mathcal{O}_S(C')$$

of sheaves on Y_3 . Its kernel provides the desired bundle E . That E is stable follows from $H^0(Y_3, E) = 0$, which is a consequence of the surjectivity, hence injectivity, of the map $H^0(Y_3, \mathcal{E}_3) \rightarrow H^0(S, \mathcal{O}_S(C))$ which implies that $H^0(Y_3, E) = 0$. \square

We used above the following lemma:

Lemma 6.10. *Let M be a smooth variety and $Z \subset M \times Y_3$ be a codimension 2 subvariety. Assume that for general $m \in M$, the curve $Z_m \subset Y_3$ is contained in a hyperplane section of Y_3 . Then the Abel-Jacobi map $\Phi_Z : M \rightarrow J(Y_3)$ is not dominating.*

Proof. This immediately follows from the fact that the transpose of the differential of the Abel-Jacobi map

$$d\Phi_Z^* : \Omega_{J(Y_3),0} \rightarrow \Omega_{M,m}$$

factors through the restriction map

$$H^0(Y_3, \mathcal{O}_{Y_3}(1)) \rightarrow H^0(Z_m, \mathcal{O}_{Z_m}(1)),$$

where one uses the natural identification (see section 1.2)

$$\Omega_{J(Y_3),0} \cong H^0(Y_3, \mathcal{O}_{Y_3}(1)).$$

Our assumptions thus say that $d\Phi_Z^*$ is not injective so Φ_Z is nowhere a submersion on M . \square

The two lemmas 6.6 and 6.9 together show that a component of the family of genus 5 curves in $\mathbb{P}(\mathcal{E}_3)$, of Y_3 -degree 9 and Q -degree 8 is birationally a \mathbb{P}^5 -bundle over a \mathbb{P}^3 -bundle over a Zariski open set of $J(Y_3)$, which is itself birational to a moduli space of rank 2 vector bundles on Y_3 .

Remark 6.11. We believe that the \mathbb{P}^3 -bundle is not Zariski locally trivial, that is, is not the projectivization of a vector bundle, over any Zariski open set of $J(Y_3)$.

6.2. The V_{14} side. Let V_{14} be a general 3-dimensional linear section of $G(2, 6)$ and let $C \subset V_{14}$ be a genus 5, degree 13 general curve such that $h^0(\mathcal{E}_{14|C}) = 6$. Then by Riemann–Roch, there is a morphism (which is generically a surjection)

$$(6.12) \quad \sigma_C : \mathcal{E}_{14|C} \rightarrow K_C \rightarrow 0,$$

or equivalently a section $\phi_\sigma : C \rightarrow \mathbb{P}(\mathcal{E}_{14})$ such that $\phi_\sigma^*(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{14})}(1)) = K_C$. We now have:

Lemma 6.13. (i) *Let \mathcal{L} be the Plücker line bundle. Then $\dim H^0(C, \mathcal{L}|_C) = 9$ and the restriction map $H^0(V_{14}, \mathcal{L}) \rightarrow H^0(C, \mathcal{L}|_C)$ is surjective. Hence C is contained in exactly one K3 surface $S \in |\mathcal{L}|$.*

(ii) *The surface S is smooth.*

Proof. For this we observe that the moduli space of genus 5 curves equipped with a degree 13 line bundle \mathcal{L} and a rank 2 vector bundle \mathcal{E} with determinant \mathcal{L} and $h^0(C, \mathcal{E}) = 6$ is irreducible: In fact, such vector bundles are obtained as a generic extension $0 \rightarrow \mathcal{L} \otimes K_C^{-1} \rightarrow \mathcal{E} \rightarrow K_C \rightarrow 0$, with $\deg \mathcal{L} = 13$. Next, one has to prove that given such a generic pair (C, \mathcal{E}) , the wedge-product map $\bigwedge^2 H^0(C, \mathcal{E}) \rightarrow H^0(C, \mathcal{L})$ is surjective, and gives an embedding of C in the Grassmannian $G(2, 6) \subset \mathbb{P}^{14}$. For the last two points, we can consider the case where $\mathcal{E} = L_1 \oplus L_2$, where L_1 is a generic line bundle of degree 7 on C and L_2 is a generic line bundle of degree 6 with 3 sections on C . Finally one checks that C is generically defined by linear equations in the Grassmannian. Hence by Bertini, a general K3 linear section of the Grassmannian containing C is smooth. \square

Coming back to our general curve $C \subset V_{14}$, we now compute that $\chi(S, \mathcal{E}_{14|S}^*(C)) = 1$ and thus either $H^0(S, \mathcal{E}_{14|S}^*(C)) \neq 0$ or $H^0(S, \mathcal{E}_{14|S}(-C)) \neq 0$. As we have $H^0(S, \mathcal{L}(-C)) = 0$, and $\mathcal{L} = \det \mathcal{E}_{14}$, the second case is excluded so that we have a nonzero morphism $\sigma : \mathcal{E}_{14} \rightarrow \mathcal{O}_S(C)$, extending the morphism σ_C of (6.12). We now compute

$$c_2(\mathcal{E}_{14|S}^*(C)) = c_2(\mathcal{E}_{14|S}^*) - \deg_{V_{14}} C + C^2 = 5 - 13 + 8 = 0.$$

It thus follows that either σ vanishes nowhere on S , or σ vanishes along a curve in S which does not meet C . The second case can only occur if $\rho(S) \geq 3$ while a dimension count shows that the family of surfaces S appearing in this construction has dimension 8, so that the generically S satisfies $\rho(S) = 2$. Hence σ is everywhere surjective and we thus get a rank 2 vector bundle E on V_{14} with trivial determinant fitting in the exact sequence

$$(6.14) \quad 0 \rightarrow E \rightarrow \mathcal{E}_{14} \rightarrow \mathcal{O}_S(C) \rightarrow 0.$$

One easily computes that $\deg_{V_{14}}(c_2(E)) = 4$. The following will be useful:

Lemma 6.15. *The restriction of E to a smooth hyperplane section $\Sigma \subset V_{14}$ is a rank 2 vector bundle on Σ with trivial determinant and $c_2 = 4$. If $\rho(\Sigma) = 1$, $E|_\Sigma$ is stable.*

Proof. The first statement is obvious. The stability follows from the vanishing $H^0(\Sigma, E|_\Sigma) = 0$ which is implied by $H^0(V_{14}, E) = 0$ and $H^1(V_{14}, E \otimes \mathcal{L}^{-1}) = 0$, which are both implied by the exact sequence (6.14) and the fact that $(C - c_1(\mathcal{L}))^2 = -4$ on Σ , and $H^0(\Sigma, \mathcal{L}^{-1}(C)) = H^2(\Sigma, \mathcal{L}^{-1}(C)) = 0$, implying that $H^2(\Sigma, \mathcal{L}^{-1}(C)) = 0$. \square

Remark 6.16. Note that the vector bundle E constructed above from the data of the K3 surface S and the line bundle $\mathcal{O}_S(C)$ satisfies $\dim \text{Hom}(E, \mathcal{E}_{14}) = 4$ as easily follows from (6.14). It follows that the 13-dimensional family of genus 5, degree 13 curves on V_{14} corresponds in fact to a 5-dimensional family of vector bundles on V_{14} .

6.3. Construction of the rational map. We now make the following construction: Let (Y_3, \mathcal{E}_3) be a general cubic threefold with Pfaffian structure, and let E be a general rank 2 stable vector bundle on Y_3 with $c_1(E) = 0$ and $\deg_{Y_3} c_2(E) = 2$. By Lemma 6.6, there is an associated 8-dimensional family of genus 5 curves $C \subset \mathbb{P}(\mathcal{E}_3)$ satisfying $\deg_{Y_3}(C) = 9$, $\deg_Q(C) = 8$ such that the image $\psi(C) \subset Q$ is a linearly normal canonical curve of genus 5. We claim that the curve $\psi(C)$ is the complete intersection of 3 quadrics in \mathbb{P}^4 : For this, we have to show that C is not trigonal. However, from our construction, we see that C is contained in a general K3 surface S with Picard lattice generated by h and the class c of C , with intersection lattice

$$h^2 = 6, h \cdot c = 9, c^2 = 8.$$

the fact that C is not trigonal then follows from [GL87]. As $\psi(C) \subset Q$ is the complete intersection of three quadrics in a hyperplane section $H \cap Q$ of Q , we can write

$$(6.17) \quad q|_H = s_1 q_1 + s_2 q_2 + s_3 q_3,$$

where q is the defining equation for Q , and the q_i 's are the defining equations for $\psi(C)$. Here the s_i 's are also quadratic polynomials on H . It follows that Q contains another set of canonical curves of genus 5, namely, viewing the expression in the right hand side of (6.17) as a quadric in the 6 variables $q_i, s_i, i = 1, 2, 3$, the plane defined by the q_i 's determines one ruling of this quadric (these planes are parameterized by a \mathbb{P}^3) and the planes in the other rulings will correspond to a second \mathbb{P}^3 of linearly normal degree 8 genus 5 canonical curves in $Q \cap H$. Concretely the curve $C_1 \subset Q \cap H$ defined in H by $q_1 = q_2 = q_3 = 0$ is such a curve. The important point for us is that the original curve C is a general member of a linear system $|\mathcal{O}_S(C)|$ on a K3 surface $S \subset \mathbb{P}(\mathcal{E}_3)$, hence it does not meet the surface $\Sigma_3 \subset \mathbb{P}(\mathcal{E}_3)$ which is contracted by ψ , which means that $\psi(C)$ does not meet the singular curve of Q . The residual curve C_1 constructed above thus moves freely in Q and also avoid the singular locus of Q which is the indeterminacy locus of the rational map ϕ^{-1} . Thus it lifts to a curve $C' = \phi^{-1}(C_1) \subset \mathbb{P}(\mathcal{E}_{14})$.

Lemma 6.18. *The genus 5 curve C' satisfies $\deg_{V_{14}} C' = 13$.*

Proof. The rank 2 vector space $A^3(\mathbb{P}(\mathcal{E}_{14})) = \text{Hdg}^6(\mathbb{P}(\mathcal{E}_{14})) = A^3(\mathbb{P}(\mathcal{E}_3))$ of curve classes in either of these two spaces is generated by the classes $[\Delta_3]$ and $[\Delta_{14}]$. We can thus write in this space

$$[C] = \alpha[\Delta_3] + \beta[\Delta_{14}].$$

Next, as the curve $C_1 = \phi(C') \subset Q$ is residual to $\psi(C)$ in the complete intersection in Q of a hyperplane H and two quadrics, we get that $[C'] = 4h_Q^3 - [C]$ in $A^3(\mathbb{P}(\mathcal{E}_{14}))$, where h_Q is the pull-back to $\mathbb{P}(\mathcal{E}_{14})$ of $c_1(\mathcal{O}_Q(1))$. Note that we also have $h_Q = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{14})}(1))$. It thus follows that

$$(6.19) \quad \deg_{V_{14}} C' = 4\deg_{V_{14}} h_Q^3 - \deg_{V_{14}} C.$$

As $h_Q = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{14})}(1))$, the standard theory of Chern classes (see [Ful98]) says that $\pi_{V_{14}*}(h_Q^3) = s_2(\mathcal{E}_{14}) = c_1^2(\mathcal{E}_{14}) - c_2(\mathcal{E}_{14})$ and thus $\deg_{V_{14}}(h_Q^3) = 14 - 5 = 9$. Next we have $\deg_{Y_3}(C) = 9$, $\deg_Q(C) = 8$, which by Lemma 6.4 gives

$$\alpha + \beta = 8, 4\beta = 9.$$

We thus deduce that $4\alpha = 23$. This finally gives using (6.19):

$$\deg_{V_{14}} C' = 36 - 4\alpha = 36 - 23 = 13.$$

□

By a dimension count (or by the reversibility of the construction), we observe that for generic C , the genus 5, V_{14} -degree 13, Q -degree 8 curve C' is generic in $\mathbb{P}(\mathcal{E}_{14})$ and we can thus apply the construction of Section 6.2 to get from C' a stable rank 2 vector bundle E' on V_{14} with trivial determinant and $\deg_{V_{14}} c_2(E') = 4$.

Proof of Theorem 6.1. Let X be a general Pfaffian cubic fourfold and $u : \mathcal{Y}_U \rightarrow U$ be the universal family of smooth hyperplane sections of X . The general fiber of u is thus a general cubic threefold with pfaffian structure. For each point $t \in U$, there is a canonical morphism from the moduli space $\mathcal{M}_{2,0,2,t}$ of rank 2 vector bundles on Y_t with trivial determinant and $\deg_{Y_t} c_2 = 2$ to the intermediate Jacobian $J(Y_t)$ which maps E to $\Phi_{Y_t}(c_2(E) - c_2(\mathcal{E}_{3,t}(-1)))$. Here $\mathcal{E}_{3,t}$ denotes the restriction to Y_t of the Pfaffian vector bundle \mathcal{E} on X . This morphism is birational by [IM00]. This way we conclude that the moduli space \mathcal{M} of sheaves

on X supported on a hyperplane section and with the same numerical data as E (seen as a sheaf on X) is birational to \mathcal{J}_U , where $\mathcal{J}_U \rightarrow U$ is the family of intermediate Jacobians.

Remark 6.20. For general X , this birational isomorphism does not exist, or rather takes values in a torsor under \mathcal{J} .

On the other hand, we also have the universal family

$$v : \mathcal{V}_{14,U} \rightarrow U$$

of corresponding linear sections of the Grassmannian. For each $t \in U$ corresponding to a $W_{5,t} \subset W_6 \subset \bigwedge^2 V_6^*$, the fiber $\mathcal{V}_{14,t}$ is the complete intersection of $G(2, V_6)$ with 5 Plücker hypersurfaces defined by $W_{5,t}$. We thus have a natural inclusion $\Sigma \subset \mathcal{V}_{14,t}$ as a Plücker hypersurface since by definition $\Sigma \subset G(2, V_6)$ is the vanishing locus in $G(2, V_6)$ of the 6 Plücker equations defined by W_6 .

The construction described above done in family over U now gives us the following: There exists a smooth projective variety W which admits a morphism $g : W \rightarrow \mathcal{M} \cong_{\text{birat}} \mathcal{J}_U$ with rationally connected fibers and a rational map

$$f : W \dashrightarrow \mathcal{M}_{2,0,4}(\Sigma).$$

The general point of W parameterizes a general rank 2 vector bundle E with $c_1 = 0$ and $\deg_{Y_t} c_2 = 2$ on a fiber Y_t of \mathcal{Y}_U , the choice of a general nonzero morphism $\sigma : E \rightarrow \mathcal{E}_{3,t}$ defined up to a coefficient, a general member C of the linear system $|\mathcal{O}_S(C)|$, where $\mathcal{O}_S(C)$ is defined by the exact sequence (6.7), and a general $(2, 2, 2, 1)$ complete intersection curve C' contained in Q_t , residual in Q_t to the $(2, 2, 2, 1)$ complete intersection curve $\psi_t(C) \subset Q_t$ (we will see in fact C' as living in $\mathbb{P}(\mathcal{E}_{14,t})$). Thus the general fiber of the map g has dimension 10 and W has dimension $20 = 10 + 10$. The rational map f associates then to these data the vector bundle $E'_{|\Sigma}$, which is stable by Lemma 6.15, where the vector bundle $E' = E_{S',C'}$ on $\mathcal{V}_{14,t}$ is associated as in Section 6.2 to the curve C' . Here S' is the generically unique Plücker hyperplane section of $\mathcal{V}_{14,t}$ containing C' .

Remark 6.21. One easily checks that E' does not depend on the choice of C or the residual curve C' . The only reason to introduce these curves was the fact that they do not meet the singular locus of Q_t , which is not true for the associated K3 surfaces where they lie. It will also appear below that E' neither depends on the choice of σ .

The proof of Theorem 6.1 will be completed using the following:

Proposition 6.22. *The rational map f factors through \mathcal{J}_U and induces a birational isomorphism*

$$g : \mathcal{J}_U \dashrightarrow \mathcal{M}_{2,0,4}(\Sigma).$$

Proof. We know by Theorem 5.6 that \mathcal{J}_U is not uniruled (this is indeed implied by the fact that a smooth projective completion of \mathcal{J}_U admits a generically nondegenerate holomorphic 2-form). It follows that \mathcal{J}_U is birational to the basis of the maximal rationally connected fibration of W . We now have the following lemma:

Lemma 6.23. *(i) The rational map f is dominating.*

(ii) The general fiber of f is rationally connected.

Proof. (i) The variety W has two holomorphic (in fact algebraic) 2-forms, namely the pull-back $\widetilde{\sigma}_{\mathcal{J}}$ to W of the holomorphic 2-form $\sigma_{\mathcal{J}}$ on \mathcal{J}_U constructed in Theorem 5.6, and the form $f^* \sigma_{\mathcal{M}_{2,0,4}}$. We claim that for some $\lambda \neq 0$

$$(6.24) \quad \widetilde{\sigma}_{\mathcal{J}} = \lambda f^* \sigma_{\mathcal{M}_{2,0,4}}.$$

This equation immediately implies the surjectivity of f , since the generic rank of $\widetilde{\sigma}_{\mathcal{J}}$ is equal to $\dim \mathcal{J}_U = 10$ and the rank of $f^* \sigma_{\mathcal{M}_{2,0,4}}$ is not greater than the rank of f , so the equality (6.24) implies that the generic rank of f is $10 = \dim \mathcal{M}_{2,0,4}(\Sigma)$, implying that f is dominant. We prove now the claim: Note that W is a fibration over \mathcal{J}_U (or rather a smooth projective compactification $\overline{\mathcal{J}}$ of \mathcal{J}_U) with rationally connected general fiber, hence $H^{2,0}(W) = H^{2,0}(\overline{\mathcal{J}})$ is of dimension 1 by Theorem 5.6(iii). As $\widetilde{\sigma}_{\mathcal{J}} \neq 0$, it thus suffices

to prove that $f^*\sigma_{\mathcal{M}_{2,0,4}} \neq 0$. This can be proved by a Chow-theoretic argument using Mumford's theorem [Mum68]. Indeed, there is a natural inclusion

$$\Sigma \times U \subset \mathcal{V}_{14,U}$$

which is the restriction over U of the natural inclusion

$$j : \Sigma \times \mathbb{P}^5 \subset \mathcal{V}_{14},$$

where \mathcal{V}_{14} is the universal family of V_{14} -threefolds containing Σ . It is immediate to check that $pr_{1*} \circ j^* : \text{CH}_1(\mathcal{V}_{14})_{\text{hom}} \rightarrow \text{CH}_0(\Sigma)_{\text{hom}}$ (which is just the restriction map j_t^* on each $\text{CH}_1(\mathcal{V}_{14,t})_{\text{hom}}$, where j_t is the inclusion of Σ in $\mathcal{V}_{14,t}$), is an isomorphism. On the other hand, \mathcal{V}_{14} is birationally equivalent to the universal family \mathcal{Y} of hyperplane sections of X , which satisfies $\text{CH}_1(\mathcal{Y})_{\text{hom}} \cong \text{CH}_1(X)_{\text{hom}}$ since $\text{CH}_0(X)_{\text{hom}} = 0$ and \mathcal{Y} is a projective bundle over X . This fibered birational isomorphism induces an isomorphism between the intermediate Jacobian fibrations over U (see [IM00]) hence a fiberwise isomorphism

$$(6.25) \quad \text{CH}_1(\mathcal{V}_{14,t})_{\text{hom}} \cong \text{CH}_1(Y_t)_{\text{hom}},$$

since for rationally connected threefolds Y , the Abel-Jacobi map $\text{CH}_1(Y)_{\text{hom}} \rightarrow J(Y)$ is an isomorphism (see [BS83]). This easily implies that

$$\text{CH}_1(\mathcal{V}_{14})_{\mathbb{Q},\text{hom}} \cong \text{CH}_1(\mathcal{Y})_{\mathbb{Q},\text{hom}}$$

since $\text{CH}_0(\mathcal{V}_{14,t})_{\text{hom}} = 0$, $\text{CH}_0(Y_t)_{\text{hom}} = 0$. We now observe that each point w of fiber W_t of the variety W over U parameterizes vector bundles E'_w , resp. E_w , on fibers $\mathcal{V}_{14,t}$, resp. Y_t , and that for each $t \in U$, the two maps

$$\begin{aligned} c_{\mathcal{V}} : W_t &\rightarrow \text{CH}_1(\mathcal{V}_{14,t}), \quad c_{\mathcal{Y}} : W \rightarrow \text{CH}_1(Y_t), \\ c_{\mathcal{V}}(w) &= c_2(E'_w), \quad c_{\mathcal{Y}}(w) = c_2(E_w) \end{aligned}$$

coincide up to sign and a constant via (6.25). With the notation above, this follows from the construction for a given t of the curve C' as residual to the curve C in a $(2, 2, 1)$ complete intersection in Q_t . Combining these observations, we conclude that the map

$$\begin{aligned} \text{CH}_0(W)_{\text{hom}} &\rightarrow \text{CH}_0(\Sigma)_{\text{hom}}, \\ w &\mapsto c_2(E'_w|_{\Sigma}) \end{aligned}$$

is surjective, hence by Mumford's theorem [Mum68] that the corresponding pull-back of the holomorphic 2-form on Σ is nonzero. However, by construction of the holomorphic 2-form on $\mathcal{M}_{2,0,4}(\Sigma)$, this pull-back is nothing but $f^*\sigma_{\mathcal{M}_{2,0,4}}$.

(ii) Let E be a general stable rank 2 vector bundle on Σ with trivial determinant and $c_2 = 4$. The fiber of f over E essentially consists of vector bundles $E_{S',C',t}$ on threefolds $\mathcal{V}_{14,t}$ containing Σ such that

$$E_{S',C',t|_{\Sigma}} \cong E.$$

More precisely, for each such vector bundle, one can apply the results of section 6.2: choosing a general section $\tilde{\sigma}$ of $\text{Hom}(E_{S',C',t}, \mathcal{E}_{14,t})$ one gets a degeneracy K3 surface $S_{\tilde{\sigma}} \subset \mathcal{V}_{14,t}$ and a line bundle $\mathcal{O}_{S_{\tilde{\sigma}}}(D)$ on $S_{\tilde{\sigma}}$ which is a quotient of $\mathcal{E}_{14,t}|_{S_{\tilde{\sigma}}}$, providing a section $\phi_{\tilde{\sigma}} : S_{\tilde{\sigma}} \rightarrow \mathbb{P}(\mathcal{E}_{14,t})$. For a general curve $D_0 \in |\mathcal{O}_{S_{\tilde{\sigma}}}(D)|$, the lifted curve $\phi_{\tilde{\sigma}}(D_0)$ is of genus 5, Q -degree 8 and V_{14} -degree 13 and using the map $\phi : \mathbb{P}(\mathcal{E}_{14,t}) \rightarrow Q_t$, it provides a complete intersection curves of type $(1, 2, 2, 2)$ contained in Q_t and the residual curve D'_0 in a $(2, 2, 1)$ complete intersection of Q_t provides a genus 5 curves in $\mathbb{P}(\mathcal{E}_{3,t})$ of Q -degree 8 and Y_3 -degree 9. Applying Lemma 6.9, we then reconstructed an element of W with image E under f . Given the vector bundle $E_{S',C',t}$ on a threefold $\mathcal{V}_{14,t}$, the extra data described above, namely the choice of σ and of the curves D_0, D'_0 , are parameterized by a rationally connected variety so the proof will be finished once we know that $E_{S',C',t}$ is determined by E .

Lemma 6.26. *Let $E_{S',C',t}$ be a general rank 2 vector bundle on $\mathcal{V}_{14,t}$ constructed as in Section 6.2 and let E be its restriction to Σ . Then the restriction map*

$$\text{Hom}(E_{S',C',t}, \mathcal{E}_{14,t}) \rightarrow \text{Hom}(E, \mathcal{E}_{14|\Sigma})$$

is an isomorphism, where $\mathcal{E}_{14|\Sigma}$ denotes the Plücker rank 2 vector bundle restricted to Σ .

Proof. Indeed, the injectivity is obvious and on the other hand, both sides have dimension 4. This was already proved in Remark 6.16 for the left hand side. For the right hand side, we can specialize the general vector bundle E to the case where $E^* = \mathcal{I}_{z_1} \oplus \mathcal{I}_{z_2}$ where z_1, z_2 are two length two subschemes of Σ ; then $H^0(\Sigma, E^* \otimes \mathcal{E}_{14|\Sigma}) = H^0(\Sigma, \mathcal{E}_{14|\Sigma} \otimes \mathcal{I}_{z_1}) \oplus H^0(\Sigma, \mathcal{E}_{14|\Sigma} \otimes \mathcal{I}_{z_2})$ has dimension 4 while $H^i(\Sigma, E^* \otimes \mathcal{E}_{14|\Sigma}) = H^i(\Sigma, \mathcal{E}_{14|\Sigma} \otimes \mathcal{I}_{z_1}) \oplus H^i(\Sigma, \mathcal{E}_{14|\Sigma} \otimes \mathcal{I}_{z_2}) = 0$ for $i > 0$. The conclusion then follows from a deformation argument. \square

Let now $\sigma \in \text{Hom}(E, \mathcal{E}_{14|\Sigma})$ be a general section. Then we get a degeneracy curve $D_\sigma \in |\mathcal{L}_\Sigma|$, where \mathcal{L} is the Plücker line bundle, and an exact sequence

$$0 \rightarrow E \rightarrow \mathcal{E}_{14|\Sigma} \rightarrow \mathcal{O}_{D_\sigma}(Z) \rightarrow 0$$

where Z is a divisor of degree 13 on D_σ . This gives a section $\phi_\sigma : D_\sigma \rightarrow \mathbb{P}(\mathcal{E}_{14})$ with image \widetilde{D}_σ . For each vector bundle $E_{S', C', t}$ on some $\mathcal{V}_{14, t} \supset \Sigma$ restricting to E on Σ , the section σ extends to a section $\tilde{\sigma}$ by Lemma 6.26, and thus there is a K3 surface

$$\widetilde{S}_{\tilde{\sigma}} := \phi_{\tilde{\sigma}}(S_{\tilde{\sigma}}) \subset \mathbb{P}(\mathcal{E}_{14}),$$

which is a lift of the degeneracy surface $S_{\tilde{\sigma}} \subset \mathcal{V}_{14, t}$. The surface $\widetilde{S}_{\tilde{\sigma}}$ intersects $\mathbb{P}(\mathcal{E}_{14|\Sigma})$ along the curve $\widetilde{D}_\sigma = \phi_\sigma(D_\sigma)$. The surface $S_{\tilde{\sigma}}$ carries a line bundle $\mathcal{O}_{S_{\tilde{\sigma}}}(C_\sigma)$ which restricts to $\mathcal{O}_{D_\sigma}(Z)$ on D_σ . Note that the curve $D_\sigma \subset S_{\tilde{\sigma}}$ is a member of $|\mathcal{L}_{|S_{\tilde{\sigma}}}|$. The uniqueness of $E_{S', C', t}$ then follows from the results of Section 6.2 and from the following:

Lemma 6.27. *For a general curve $\widetilde{D}_\sigma \subset \mathbb{P}(\mathcal{E}_{14})$ as above, there exists a unique surface $\widetilde{S}_{\tilde{\sigma}} \subset \mathbb{P}(\mathcal{E}_{14})$ satisfying the conditions above, that is, lifting a K3 surface in some $\mathcal{V}_{14, t}$ containing Σ and intersecting $\mathbb{P}(\mathcal{E}_{14|\Sigma})$ along the curve \widetilde{D}_σ .*

Proof. Let $N_{\widetilde{D}_\sigma/\mathbb{P}(\mathcal{E}_{14})}$ be the normal bundle of \widetilde{D}_σ in $\mathbb{P}(\mathcal{E}_{14})$. There is an exact sequence

$$(6.28) \quad 0 \rightarrow T_{\mathbb{P}(\mathcal{E}_{14})/G(2,6)}|_{\widetilde{D}_\sigma} \rightarrow N_{\widetilde{D}_\sigma/\mathbb{P}(\mathcal{E}_{14})} \rightarrow N_{D_\sigma/G(2,6)} = \mathcal{L}_{|D_\sigma}^7 \rightarrow 0,$$

and each surface $\widetilde{S}_{\tilde{\sigma}}$ extending \widetilde{D}_σ as above provides an inclusion

$$N_{D_\sigma/S_{\tilde{\sigma}}} = \mathcal{L}_{|D_\sigma} \subset N_{\widetilde{D}_\sigma/\mathbb{P}(\mathcal{E}_{14})},$$

or equivalently a nonzero section of $N_{\widetilde{D}_\sigma/\mathbb{P}(\mathcal{E}_{14})} \otimes \mathcal{L}^{-1}$. It is not hard to see that this section determines the surface $\widetilde{S}_{\tilde{\sigma}}$, so we only have to prove that, for general \widetilde{D}_σ as above:

$$(6.29) \quad h^0(\widetilde{D}_\sigma, N_{\widetilde{D}_\sigma/\mathbb{P}(\mathcal{E}_{14})} \otimes \mathcal{L}^{-1}) = 1.$$

In order to prove (6.29), we write the normal bundle sequence twisted by \mathcal{L}^{-1} for $\widetilde{D}_\sigma \subset \widetilde{S}_{\tilde{\sigma}} \subset \mathbb{P}(\mathcal{E}_{14})$. This gives

$$0 \rightarrow \mathcal{O}_{D_\sigma} \rightarrow N_{\widetilde{D}_\sigma/\mathbb{P}(\mathcal{E}_{14})} \otimes \mathcal{L}^{-1} \rightarrow (N_{\widetilde{S}_{\tilde{\sigma}}/\mathbb{P}(\mathcal{E}_{14})})|_{\widetilde{D}_\sigma} \otimes \mathcal{L}^{-1} \rightarrow 0$$

and (6.29) will follow from $h^0(\widetilde{D}_\sigma, (N_{\widetilde{S}_{\tilde{\sigma}}/\mathbb{P}(\mathcal{E}_{14})})|_{\widetilde{D}_\sigma} \otimes \mathcal{L}^{-1}) = 0$ which itself will be a consequence of

$$(6.30) \quad \begin{aligned} h^0(\widetilde{S}_{\tilde{\sigma}}, N_{\widetilde{S}_{\tilde{\sigma}}/\mathbb{P}(\mathcal{E}_{14})} \otimes \mathcal{L}^{-1}) &= 0, \\ h^1(\widetilde{S}_{\tilde{\sigma}}, N_{\widetilde{S}_{\tilde{\sigma}}/\mathbb{P}(\mathcal{E}_{14})} \otimes \mathcal{L}^{-2}) &= 0. \end{aligned}$$

The second vanishing statement is obtained by writing the normal bundle sequence (6.28) for $\widetilde{S}_{\tilde{\sigma}}$:

$$(6.31) \quad 0 \rightarrow (T_{\mathbb{P}(\mathcal{E}_{14})/G(2,6)})|_{\widetilde{S}_{\tilde{\sigma}}} \rightarrow N_{\widetilde{S}_{\tilde{\sigma}}/\mathbb{P}(\mathcal{E}_{14})} \rightarrow N_{S_{\tilde{\sigma}}/G(2,6)} = \mathcal{L}_{|S_{\tilde{\sigma}}}^6 \rightarrow 0,$$

where the line bundle $(T_{\mathbb{P}(\mathcal{E}_{14})/G(2,6)})|_{\widetilde{S}_{\tilde{\sigma}}}$ is isomorphic to $\mathcal{L}_{|S_{\tilde{\sigma}}}^{-1}(2C_\sigma)$. One then concludes using

$$H^1(S_{\tilde{\sigma}}, \mathcal{L}_{|S_{\tilde{\sigma}}}^{-1}) = 0, \quad H^1(S_{\tilde{\sigma}}, \mathcal{L}_{|S_{\tilde{\sigma}}}^{-3}(2C_\sigma)) = 0,$$

which both follow from standard vanishing theorems on the K3 surface $S_{\tilde{\sigma}}$. It remains to prove the first vanishing statement. However, according to Section 6.2, the deformation space of $\widetilde{S}_{\tilde{\sigma}}$ in $\mathbb{P}(\mathcal{E}_{14})$ is smooth and isomorphic to a \mathbb{P}^3 -bundle over the 10-dimensional moduli space of sheaves on \mathcal{V}_{14} supported on fibers $\mathcal{V}_{14, t}$ and are locally free on $\mathcal{V}_{14, t}$ of rank 2, with trivial determinant and $\deg_{\mathcal{V}_{14, t}} c_2 = 4$. By (i) and Lemma

6.26, the restriction map $(E_t, \tilde{\sigma}) \rightarrow (E_{t|\Sigma}, \tilde{\sigma}|_\Sigma)$ has generically surjective differential, hence also injective differential. It is clear however that sections of $N_{\widetilde{S_\sigma}/\mathbb{P}(\mathcal{E}_{14})} \otimes \mathcal{L}^{-1}$, seen as sections of $N_{\widetilde{S_\sigma}/\mathbb{P}(\mathcal{E}_{14})}$ vanishing on $\widetilde{D_\sigma}$, belong to the kernel of this differential. Hence they must be trivial. \square

This concludes the proof of Lemma 6.23. \square

Lemma 6.23 implies Proposition 6.22 as follows: since f is dominating by (i) and $\mathcal{M}_{2,0,4}(\Sigma)$ is not uniruled, f must factor through the MRC fibration of W , that is, through \mathcal{J}_U . The general fiber of the induced rational map $g : \mathcal{J}_U \dashrightarrow \mathcal{M}_{2,0,4}(\Sigma)$ are then rationally connected by (ii). But as \mathcal{J}_U is not uniruled, the general fiber of g is a point, so g is birational. \square

The proof of Theorem 6.1 is now finished. \square

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