On the Lefschetz standard conjecture for Lagrangian covered hyper-Kähler varieties

Claire Voisin

Abstract

We investigate the Lefschetz standard conjecture for degree 2 cohomology of hyper-Kähler manifolds admitting a covering by Lagrangian subvarieties. In the case of a Lagrangian fibration, we show that the Lefschetz standard conjecture is implied by the SYZ conjecture characterizing classes of divisors associated with Lagrangian fibration. In dimension 4, we consider the more general case of a Lagrangian covered fourfold X, and prove the Lefschetz standard conjecture in degree 2, assuming $\rho(X) = 1$ and X is general in moduli. Finally we discuss various links between Lefschetz cycles and the study of the rational equivalence of points and Bloch-Beilinson type filtrations, giving a general interpretation of a recent intriguing result of Marian and Zhao.

0 Introduction

The Lefschetz standard conjecture [18] for degree k cohomology of a smooth projective variety of dimension $N \geq k$ equipped with an ample line bundle H_X asks whether there exists a codimension k-cycle $\mathcal{Z}_{\rm lef} \in {\rm CH}^k(X \times X)$ such that, denoting $[\mathcal{Z}_{\rm lef}] \in H^{2N}(X \times X, \mathbb{Q})$ the cohomology class of $\mathcal{Z}_{\rm lef}$,

$$[\mathcal{Z}_{\rm lef}]^* : H^{2N-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is the inverse of the Lefschetz isomorphism

$$h_X^{N-k} \cup : H^k(X, \mathbb{Q}) \to H^{2N-k}(X, \mathbb{Q}),$$

where $h_X = c_1(H_X)$.

Remark 0.1. When $k = 2$, it suffices to show that $[\mathcal{Z}_{\rm lef}]^* : H^{N,N-2}(X) \to H^{2,0}(X)$ is the inverse of the Lefschetz isomorphism $h_X^{N-2} \cup : H^{2,0}(X) \to H^{N,N-2}(X, \mathbb{Q})$. Indeed, the equality of morphisms of Hodge structures $[\mathcal{Z}_{\rm lef}]^* \circ h_X^{N-2} \cup = Id$ is true on $H^2(X, \mathbb{Q})_{tr}$ once it is true on $H^{2,0}(X)$, and furthermore we have $H^2(X, \mathbb{Q}) = H^2(X, \mathbb{Q})_{tr} \oplus {\rm NS}(X)_{\mathbb{Q}}$ where the direct sum is orthogonal with respect to the nondegenerate Lefschetz intersection pairing $(\alpha, \beta)_{h_X} = \langle \alpha, h_X^{N-2}\beta \rangle_X$. Once we have a cycle $\mathcal{Z}$ satisfying $[\mathcal{Z}]^* \circ h_X^{N-2} \cup = Id$ on $H^2(X, \mathbb{Q})_{tr}$, it is easy to construct $\mathcal{Z}_{\rm lef}$ by adding to $\mathcal{Z}$ a decomposable cycle in ${\rm NS}(X) \otimes {\rm NS}(X)$, so that $[\mathcal{Z}_{\rm lef}]^* \circ h_X^{N-2} \cup = Id$ holds on the whole $H^2(X, \mathbb{Q})$.

This conjecture is essential for the theory of motives as it allows to realize motivically the Lefschetz decomposition, which has for consequences semisimplicity results for cohomological motives. The Lefschetz standard conjecture also implies the variational Hodge conjecture (see [38]). Except for hyper-Kähler manifolds that we will discuss in this paper, the main class of varieties for which Lefschetz standard conjecture is known is the class of abelian varieties by work of Lieberman [21]. In degree 2, one can show that the conjecture does not depend on the choice of polarization h_X and we will describe several equivalent formulations in Section 2.1.

If X is a projective hyper-Kähler manifold of dimension $N = 2n$, $H^{2,0}(X)$ is 1-dimensional, generated by a 2-form σ_X. For such an X, the Lefschetz standard conjecture for degree 2 is equivalent to the existence of a codimension 2-cycle $Z \in \mathrm{CH}^2(X \times X)$ whose cohomology class $[Z] \in H^4(X \times X, \mathbb{Q})$ gives the inverse $q^{-1} \in \mathrm{Sym}^2 H^2(X, \mathbb{Q})$ of the Beauville-Bogomolov form q on $H^2(X, \mathbb{Q})$. This follows from the fact that the Beauville-Bogomolov form, restricted to the transcendental cohomology, is a multiple of the Lefschetz intersection pairing $q_{\mathrm{lef}}(\alpha, \beta) = \int_X h_X^{2n-2}\alpha\beta$, as follows from the Beauville-Fujiki relations. The existence of such a cycle has been proved by Markman [23] when X is of $K3^{[n]}$-deformation type. The lefschetz standard conjecture has been proved in all degrees by Charles and Markman [9], following [23] and a programme presented in [8], for projective hyper-Kähler manifolds of $K3^{[n]}$-deformation type. Note also that the Lefschetz standard conjecture for degree 2 is satisfied by most of the explicitly known hyper-Kähler manifolds. For example, it is true for the Fano variety of lines of a cubic fourfold [3], the LLSvS 8-fold constructed from the family of degree 3 rational curves in a smooth cubic fourfold [20], the LSV 10-fold constructed in [19] as a compactification of the intermediate Jacobian fibration, and more generally the infinitely many families of hyper-Kähler manifolds constructed in [1]. This follows, using Proposition 1.1, from the existence of a natural correspondence between these varieties and the corresponding cubic fourfold, whose motive is a direct summand in the motive of a surface. Of course, except in the case of the LSV variety, one can also apply the Charles-Markman theorem, since they are all of $K3^{[n]}$ deformation type. Similarly, the double EPW sextics constructed by O'Grady [28] satisfy the conjecture, either by using the correspondence to a Fano fourfold constructed by Iliev and Manivel [16], or by applying [9].

Our goal in this paper is to study the Lefschetz standard conjecture for degree 2 cohomology of hyper-Kähler manifolds which admit a Lagrangian fibration or covering. Our first result concerns the fibered case. A central conjecture in the theory of hyper-Kähler manifolds is the following precise version of the SYZ conjecture.

Conjecture 0.2. *Let X be a projective hyper-Kähler manifold and $l \in \mathrm{NS}(X)$ satisfying the conditions*

(1) $q(l) = 0$,

(ii) l belongs to the boundary of the birational Kähler cone.

Then there exist a hyper-Kähler manifold $\psi : X \dashrightarrow X'$ birational to X and a Lagrangian fibration $X' \to B$ on X' such that $\psi^* l' = l$, where l' is the Lagrangian class of π'.

Our first result is the following.

Theorem 0.3. *Let X be a projective hyper-Kähler manifold. Assume that X admits a Lagrangian fibration $\pi : X \to B$ and that $\rho(X) = 2$. Then if one of the following conditions*

1. *X satisfies the SYZ conjecture 0.2;*
2. *There is an effective divisor Θ on X which satisfies $q(\theta) < 0$;*

holds, X satisfies the Lefschetz standard conjecture for degree 2 cohomology.

Here $\theta = [\Theta]$ and q is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{Q})$. We now turn to the more general case of a Lagrangian covering, with the following definition.

Definition 0.4. *A covering of a hyper-Kähler manifold X by Lagrangian varieties is a diagram of morphisms*

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & X \\ \pi \downarrow & & \\ B & & \end{array}$$

of smooth projective varieties, such that Φ is dominant and, for a general point $b \in B$, $\Phi_{|L_b}$ is generically finite to its image which is a (singular) Lagrangian subvariety of X.

In practice, we can assume that the map $\Phi_{|L_b}$ is birational to its image.

Example 0.5. If X contains a smooth Lagrangian subvariety $L_0 \subset X$ whose normal bundle $N_{L_0/X}$ is generated by global sections, then X admits a covering by Lagrangian varieties. This follows from the unobstructedness result of [39] for deformations of Lagrangian submanifolds of hyper-Kähler manifolds.

It is not known if such a covering exists for any projective hyper-Kähler manifold but it exists on general members of some locally complete families of hyper-Kähler manifolds with Picard number 1 as the following example shows.

Example 0.6. Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold and $X = F_1(Y)$ be its Fano variety of lines. This is a hyper-Kähler fourfold by [3]. For any hyperplane $H \subset \mathbb{P}^5$, the hyperplane section $Y_H = Y \cap H$ is a cubic threefold, whose variety of lines is, if H is general, or for any H if Y is general, a surface $\Sigma_H \subset X$ which is Lagrangian, as is observed in [39]. The family of these surfaces gives a Lagrangian covering of X.

O'Grady conjectured it should exist in general. In contrast, the Lagrangian fibered hyper-Kähler manifolds have $\rho \geq 2$, so a priori Theorem 0.7 applies to a broader setting in dimension 4.

Theorem 0.7. *Let X be a projective hyper-Kähler fourfold with $\rho(X) = 1$ admitting a covering by Lagrangian surfaces. Assume X is very general in moduli. Then X satisfies the Lefschetz standard conjecture for degree 2 cohomology.*

Remark 0.8. The Lefschetz cycle $\mathcal{Z}^2_{\rm lef}$ that exists at the fiber over a very general point can be specialized to the fiber over any point in the moduli space of smooth deformations, proving in turn the Lefschetz standard conjecture in degree 2 for any smooth member X_t of the moduli space of polarized deformations of X.

Corollary 0.9. *Let X be a projective hyper-Kähler fourfold with $\rho(X) = 1$ containing a smooth Lagrangian surface L_0 whose normal bundle is globally generated. Then X satisfies the Lefschetz standard conjecture for degree 2 cohomology.*

Proof. By using the deformations of L_0 in X as explained in Example 0.5, X admits a covering by Lagrangian surfaces. As $\rho(X) = 1$, we know by [39] that the existence of L_0 is satisfied in a Zariski open set of the moduli space of polarized deformations of X. Furthermore, as the normal bundle of a Lagrangian submanifold is isomorphic to its cotangent bundle, the dimension of its space of global sections remains constant under deformations and thus, the normal bundle remains globally generated for the deformed pair $L_{0,t} \subset X_t$, at least for t in a Zariski dense open set of the moduli space. For a very general deformation X_t of X, Theorem 0.7 thus applies. We conclude the proof for the original X using Remark 0.8. □

Remark 0.10. The assumptions of Corollary 0.9 are satisfied in Example 0.6.

In Section 1.2, we will also establish Propositions 1.7 and 1.9 which provide two steps towards proving the Lefschetz standard conjecture in degree 2 for Lagrangian covered hyper-Kähler manifolds of any dimension, assuming the base B of the Lagrangian covering has $H^{2,0}(B) = 0$ (an assumption that we do not need in dimension 4).

The second theme of this paper, developed in Section 2, is the relation between the Lefschetz standard conjecture in degree 2 and rational equivalence of points on smooth projective varieties whose algebra of holomorphic forms is generated in degree ≤ 2. In the section 2.1 devoted to general facts about Chow groups of 0-cycles and Lefschetz conjecture, we will define geometrically the level $F^3_{dR}\mathrm{CH}_0(X) \subset F^2\mathrm{CH}_0(X)$ for any smooth complex projective varieties, where $F^2\mathrm{CH}_0(X)$ is the group of zero-cycles of degree 0 and Albanese equivalent to 0, and we will establish in Proposition 2.7 a precise relationships between the Lefschetz conjecture for degree 2 cohomology and $F^3_{dR}\mathrm{CH}_0(X)$. Whether this subgroup coincides or not with other definitions $F^3_{BB}\mathrm{CH}_0(X)$ proposed in [36] for the F^3 level of the Bloch-Beilinson filtration precisely depends on the Lefschetz conjecture in degree 2 and the Bloch conjecture for 0-cycles on a surface. We propose and discuss in Section 2.2 a conjecture (see Conjecture 2.11) on the rational equivalence of points of a smooth projective hyper-Kähler manifold. It states that two points x, y have the same class in $\mathrm{CH}_0(X)$ if and only they have the same class in $\mathrm{CH}_0(X)/F^3_{BB}\mathrm{CH}_0(X)$, where we refer to Section 2.1 for the definition of F^3_{BB}. This conjecture is motivated by the recent work [22] of A. Marian and X. Zhao, which is an interesting evidence for it. We discuss further evidence and relate it to a conjectural polynomial formula expressing partially the diagonal of a hyper-Kähler manifold as a polynomial in the Lefschetz cycle. As will be discussed there, the conjecture should hold more generally for smooth projective varieties whose algebra of holomorphic forms is generated in degree ≤ 2, but in the hyper-Kähler case, especially the Lagrangian fibered case, it seems more accessible, thanks to polynomial relations in the Chow ring that are expected to hold (see [5], [33]) and are known to hold in some cases, (see [37]). We also describe some classes of varieties for which the conjecture holds. For example, using Beauville's formulas, we prove in Proposition 2.17 that two points x, y of a Kummer variety $K = A/\pm Id$ of an abelian variety are rationally equivalent if and only if $x = y$ modulo $F^3_{BB}\mathrm{CH}_0(K)$, if and only if $x_{\leq 2} = y_{\leq 2} \in \mathrm{CH}_0(K)_{\leq 2}$, see [4] for the definition of the Beauville decomposition which is used to define $\mathrm{CH}_0(K)_{\leq 2}$.

We use the notation $\mathrm{CH}^i(X)$ for the Chow groups with $\mathbb{Q}$-coefficients.

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1 The Lefschetz standard conjecture for hyper-Kähler manifolds swept-out by Lagrangian varieties

This section is devoted to the proof of Theorems 0.3 and 0.7. We will use the following result proved in [8].

Proposition 1.1. *Let X be a smooth projective variety. Then the Lefschetz standard conjecture holds for degree 2 cohomology of X if and only if there exist a smooth projective surface Σ and a codimension 2 cycle $Z_\Sigma \in \mathrm{CH}^2(X \times \Sigma)$, such that*

$$[Z_\Sigma]^* : H^{2,0}(\Sigma) \to H^{2,0}(X)$$

is surjective.

If the Lefschetz standard conjecture holds for a very ample line bundle H_X, one can indeed take for Σ a smooth surface which is a complete intersection of hypersurfaces in $|H_X|$ and for Z_Σ the cycle $\mathcal{Z}_{\mathrm{lef}}$ restricted to $X \times \Sigma$. The other direction is more tricky.

1.1 Hyper-Kähler manifolds admitting a Lagrangian fibration

Let X be a smooth projective complex manifold of dimension $2n$ with a closed holomorphic 2-form σ_X, and $\pi : X \to B$ be a fibration whose fibers are isotropic with respect to σ_X. In this section, we will be interested in the case of a Lagrangian fibration where $\dim B = n$ and the 2-form is nondegenerate. There is a family of $n-1$-cycles on X parameterized by X and defined as follows: let H be an effective ample divisor on X with restriction $H_b \subset X_b$ on the general fiber X_b of π. Let $S \subset X$ be a multisection of π, of degree D over B. Then for $x \in X_b$, translation by $\mathrm{alb}_{X_b}(Dx - S_b)$ acts on X_b, hence on $\mathrm{Pic}\, X_b$. We denote by $H_{b,x} \subset X_b$ the divisor obtained by applying this action to H_b. This is a $n-1$-cycle in X. The collection of these $n-1$-cycles provides a self-correspondence $\Gamma_\pi \in \mathrm{CH}^{n+1}(X \times X)$ with action

$$[\Gamma_\pi]^*(\alpha) := \mathrm{pr}_{1*}([\Gamma_\pi] \cup \mathrm{pr}_2^*\alpha)$$

on the cohomology of X. Let $h := [H] \in H^2(X, \mathbb{Z})$, and let $\widetilde{B}$ be a desingularization of B. If X is hyper-Kähler, let l be the Lagrangian class associated with the fibration π, that is, l comes from the positive generator of $\mathrm{NS}(B)$. We prove the following proposition.

Proposition 1.2. *Assuming that* $H^0(\widetilde{B},\Omega^2_{\widetilde{B}})=0$*, one has*

$$[\Gamma_\pi]^*(h^{n-1}\sigma_X)=c\sigma_X \tag{1}$$

for some nonzero rational constant c. Furthermore, if X is hyper-Kähler,

$$[\Gamma_\pi]^*(h^{2n-2}\sigma_X)=c_1l^{n-1}\sigma_X \tag{2}$$

for some nonzero rational constant c_1*.*

Proof. Let us first show that (1) implies (2). The correspondence Γ_π is a relative correspondence, which means that it is contained in $X\times_B X$. It then follows from the projection formula that

$$[\Gamma_\pi]^*(l^k\alpha)=l^k[\Gamma_\pi]^*(\alpha) \tag{3}$$

for any $\alpha\in H^*(X)$. We can write

$$h=al+bl', \tag{4}$$

where $q(l')=0$, by choosing a standard hyperbolic basis of $\langle l,h\rangle\subset H^2(X,\mathbb{Q})$ equipped with the Beauville-Bogomolov form. We then have

$$h^{2n-2}\sigma_X=c_2l^{n-1}l'^{n-1}\sigma_X \text{ in } H^{4n-2}(X,\mathbb{C}) \tag{5}$$

This follows indeed from the Beauville-Fujiki relation

$$\alpha^{n+1}=0 \text{ in } H^{2n+2}(X,\mathbb{C})$$

when $q(\alpha)=0$, which by differentiation gives

$$\alpha^n\sigma=0 \text{ in } H^{2n+2}(X,\mathbb{C}) \text{ when } q(\alpha)=0,\ q(\sigma)=0,\ q(\alpha,\sigma)=0.$$

Applying this last equality to $\alpha=l$ or l' and $\sigma=\sigma_X$, we get that $l^n\sigma_X=0$ and $(l')^n\sigma_X=0$, which, using (4), implies (5) with $c_2=\binom{2n-2}{n-1}a^{n-1}b^{n-1}$. Combining (5) and (3), (1) implies (2).

We turn to the proof of (1). Let $U\subset B$ be the open set where B is smooth and over which π is smooth, and $X_U:=\pi^{-1}(U)$. As the restriction map $H^2(X,\mathbb{C})\to H^2(X_U,\mathbb{C})$ is injective for coniveau reasons on the space $H^{2,0}(X)$, it suffices to prove (1) after restriction to X_U. It also suffices to prove the equality in $H^0(X_U,\Omega^2_{X_U})$ because the restriction map $H^{2,0}(X)\to H^2(X_U)$ factors through the restriction map

$$H^0(X,\Omega^2_X)=H^0(X,\Omega^2_X)^{\rm closed}\to H^0(X_U,\Omega^2_{X_U})^{\rm closed}.$$

We now use the fact that the fibration π is Lagrangian, which means that, on X_U,

$$\sigma_X\in\Gamma(\pi^*\Omega_U\wedge\Omega_{X_U}), \tag{6}$$

We next observe that, using the condition $H^0(\widetilde{B},\Omega^2_{\widetilde{B}})=0$, in order to prove the equality (1) of $(2,0)$-forms on X_U, it suffices to prove the corresponding equality in the quotient $\Omega_U\otimes R^0\pi_*\Omega_{X_U/U}$ of $R^0\pi_*(\pi^*\Omega_U\wedge\Omega_{X_U})$. The computation is now local and fiberwise. Indeed, project σ_X to a section $\tilde{\sigma}_X$ of $\pi^*\Omega_U\otimes\Omega_{X_U/U}$, and h to a section $\tilde{h}$ of $R^1\pi_*\Omega_{X_U/U}$. Then the relative correspondence $\Gamma_{\pi,rel}\in{\rm CH}^1(X_U\times_U X_U)$ induces a morphism

$$\Gamma^*_{\pi,rel}\circ\tilde{h}^{n-1}:R^0\pi_*\Omega_{X_U/U}\stackrel{\tilde{h}^{n-1}}{\to}R^{n-1}\pi_*\Omega^n_{X_U/U}\stackrel{\Gamma^*_{rel}}{\to}R^0\pi_*\Omega_{X_U/U}, \tag{7}$$

which tensored by Ω_U will induce the sheaf-theoretic version

$$([\Gamma_\pi]^*\circ h^{n-1})_{\rm loc}:\Omega_U\otimes R^0\pi_*\Omega_{X_U/U}\to\Omega_U\otimes R^0\pi_*\Omega_{X_U/U} \tag{8}$$

of the morphism $[\Gamma_\pi]^*\circ h^{n-1}$. It thus only remains to prove that the morphism $\Gamma^*_{\pi,rel}\circ\tilde{h}^{n-1}$ of (7) is a nonzero rational multiple of the identity. This is a fiberwise statement which is a standard fact about duality in abelian varieties. □

Proof of Theorem 0.3. Let X be a projective hyper-Kähler manifold with $\rho(X) = 2$ admitting a Lagrangian fibration $\pi : X \to B$ with Lagrangian class l. As $\rho(X) = 2$, X has two isotropic classes in $\mathrm{NS}(X)$ (up to a scalar), which we call l and l'. As a consequence of fundamental results of Huybrechts in [12], Riess proved the following:

Theorem 1.3. *[29, theorem 3.3] Let l' be an isotropic class on a projective hyper-Kähler manifold X of dimension $2n$, with $q(l', \omega) > 0$ for any Kähler form on X. Then there exists a cycle $R \in \mathrm{CH}^{2n}(X \times X)$ with inverse $R^{-1} \in \mathrm{CH}^{2n}(X \times X)$ such that R^* acts as an automorphism of $\mathrm{CH}(X)$ preserving the intersection product, the action of R^* on $H^2(X, \mathbb{Q})$ preserves the Beauville-Bogomolov form q_X, and $R^*(l')$ belongs to the boundary of the birational Kähler cone of X.*

We are now faced to two possibilities: either

(a) $R^*(l')$ is proportional to l', hence l' belongs to the boundary of the birational Kähler cone of X, or

(b) $R^*(l')$ is proportional to l.

Let us examine these possibilities separately in the two cases 1 and 2 of Theorem 0.3.

Case 1: *X satisfies the SYZ conjecture 0.2.* We assume that we are in situation (a); the situation (b) is treated below. The class l', being in the boundary of the birational Kähler cone of X, is the Lagrangian class of a Lagrangian fibration on some hyper-Kähler manifold X' birational to X via a rational map $\psi : X \dashrightarrow X'$. We apply the previous construction to X' and l', which provides a correspondence $\Gamma_{\pi'} \in \mathrm{CH}^{n+1}(X' \times X')$ such that

$$[\Gamma_{\pi'}]^*((h')^{4n-2}\sigma_{X'}) = c_1' l'^{n-1}\sigma_{X'}, \tag{9}$$

where $h' = (\psi^{-1})^* h$.

By the work of Huybrechts [12], there exist cycles $R_\psi \in \mathrm{CH}^{2n}(X \times X')$, $R_\psi^{-1} \in \mathrm{CH}^{2n}(X \times X')$ which act as ψ^*, $(\psi^{-1})^*$ on degree 2 cohomology and are constructed by adding corrections to the graphs of ψ and ψ^{-1}, and whose action on cohomology is compatible with cup-product. Let

$$\Gamma_{\pi',X} := R_\psi^{-1} \circ \Gamma_{l'} \circ R_\psi \in \mathrm{CH}^{n+1}(X \times X).$$

Then (9) together with compatibility of the actions with cup-product shows that

$$[\Gamma_{\pi',X}]^*(h^{4n-2}\sigma_X) = c_1' l'^{n-1}\sigma_X, \tag{10}$$

Let now

$$\Gamma_{\mathrm{lef}} := \Gamma_{\pi',X} \circ {}^t\Gamma_l \in \mathrm{CH}^2(X \times X).$$

The proof of Lefschetz standard conjecture in this case concludes, using Remark 0.1, with the following lemma.

Lemma 1.4. *One has $[\Gamma_{\mathrm{lef}}]^*(h^{2n-2}\sigma_X) = \mu\sigma_X$ with $\mu \neq 0$.*

Proof. One has $[\Gamma_{\mathrm{lef}}]^*(h^{2n-2}\sigma_X) = [{}^t\Gamma_\pi]^*([\Gamma_{\pi',X}]^*(h^{4n-2}\sigma_X))$ and by (10), this is equal to $c_1'[{}^t\Gamma_\pi]^*(l'^{n-1}\sigma_X)$, so we have to prove that $[{}^t\Gamma_\pi]^*(l'^{n-1}\sigma_X) \neq 0$. It clearly suffices to prove that

$$\langle [{}^t\Gamma_\pi]^*(l'^{n-1}\sigma_X), h^{2n-2}\overline{\sigma_X}\rangle_X \neq 0. \tag{11}$$

But $\langle [{}^t\Gamma_\pi]^*(l'^{n-1}\sigma_X), h^{2n-2}\overline{\sigma_X}\rangle_X = \langle l'^{n-1}\sigma_X, [\Gamma_\pi]^*(h^{2n-2}\overline{\sigma_X})\rangle_X$, which by Proposition 1.2, equals $c_1\langle l'^{n-1}\sigma_X, l^{n-1}\overline{\sigma_X}\rangle_X$. Thus the result follows from $\int_X (ll')^{n-1}\sigma_X\overline{\sigma_X} \neq 0$, which follows from the second Hodge-Riemann relations, since, by (5), this is proportional to $\int_X h^{2n-2}\sigma_X\overline{\sigma_X}$. □

Case 2: *There is an effective divisor Θ on X which satisfies $q(\theta) < 0$.* In this case, we claim that l' does not belong to the boundary of the birational Kähler cone of X (see also [31], where a similar situation is studied for the LSV variety constructed in [19]). Indeed, $l' = a\theta + bl$ with

$$q(l') = 0,\ q(l', l) = aq(\theta, l) > 0,\ q(l, \theta) > 0,$$

and this implies that $a > 0$ and

$$a^2 q(\theta) + 2abq(l, \theta) = 0.$$

As $q(\theta) < 0$, we find that $b > 0$ and thus

$$q(\theta, l') = aq(\theta) + bq(\theta, l) = -bq(\theta, l) < 0.$$

As θ is the class of an effective divisor, this implies the claim. It follows from the claim that we must have $R^* l' = l$, that is, we are in situation (b).

We can now in both cases assume that we are in situation (b) and conclude the proof. We have $l' = (R^{-1})^* l$ and we know that l is a Lagrangian class on X and $R^{-1} \in \mathrm{CH}^{2n}(X \times X)$ is a correspondence whose action in cohomology preserves the cup-product and is the identity on $H^{2,0}(X)$. Let $\Gamma_\pi \in \mathrm{CH}^{n+1}(X \times X)$ be constructed as before and let

$$\Gamma_{l'} := R^{-1} \circ \Gamma_\pi \circ R.$$

Then, by Proposition 1.2, we have $[\Gamma_\pi]^*(h^{4n-2}\sigma_X) = c_1 l^{n-1}\sigma_X$, hence, if $h' = (R^{-1})^* h$, we have, as in (9)

$$[\Gamma_{l'}]^*(h'^{4n-2}\sigma_X) = c_1 l'^{n-1}\sigma_X. \tag{12}$$

We can thus apply the same argument as before, defining

$$\Gamma_{\mathrm{lef}} := \Gamma_{l'} \circ {}^t\Gamma_l \in \mathrm{CH}^2(X \times X).$$

The analog of Lemma 1.4 is proved as before, and this establishes the Lefschetz standard conjecture in this case. □

In the case of dimension 4, we have the following alternative proof of Theorem 0.3 giving a slightly different statement.

Proposition 1.5. *Let X be a hyper-Kähler fourfold admitting a Lagrangian fibration $\pi : X \to B$. Assume there exists a uniruled divisor $\Theta \subset X$ such that $q(\theta, l) \neq 0$. Then X satisfies the Lefschetz standard conjecture in degree 2.*

Proof. Let $\widetilde{\Theta}$ be a desingularization of Θ and let $p : \widetilde{\Theta} \to S$ be the maximal rationally connected fibration of $\widetilde{\Theta}$. Then S is a surface since $\widetilde{\Theta}$ has a nonzero holomorphic 2-form pulled-back from X. We claim that the condition $q(\theta, l) \neq 0$ implies that $\pi_{|\Theta} : \Theta \to B$ is surjective. Indeed, if $\pi_{|\Theta} : \Theta \to B$ is not surjective, then $l^2\theta^2 = 0$ and this implies $q(l, \theta) = 0$ by the following argument. The Beauville-Fujiki relations give, for a nonzero rational constant c_X,

$$(\alpha l + \beta\theta)^4 = c_X q(\alpha l + \beta\theta)^2 = c_X(2\alpha\beta q(l, \theta) + \beta^2 q(\theta))^2,$$

and comparing the coefficients in $\alpha^2\beta^2$ we get that $l^2\theta^2$ is proportional to $q(l, \theta)^2$, which proves the claim. We then construct as in [42] a surface decomposition for X using the "sum map upto isogeny" $\mu : X \times BX \dashrightarrow X$, for which we only need to use a surface $T \subset X$ providing a multisection of π. The surface decomposition is given by the following construction: let

$$\Gamma = \widetilde{\Theta} \times_B \widetilde{\Theta},\ \psi = \mu_{|\widetilde{\Theta} \times_B \widetilde{\Theta}} : \Gamma \to X,$$

and $\phi := (p,p)_{|\widetilde{\Theta}\times_B\widetilde{\Theta}} : \Gamma \to S \times S$. The claim implies that ψ is surjective, hence generically finite since $\dim \Gamma = 4$. It is proved in [42] that these maps satisfy

$$\psi^*\sigma_X = \phi^*(\mathrm{pr}_1^*\sigma_{S,1} + \mathrm{pr}_2^*\sigma_{S,2}) \tag{13}$$

for some holomorphic forms $\sigma_{S,i}$ on S. As $\psi_*\psi^*\sigma_X = (\deg \psi)\sigma_X$, (13) implies the Lefschetz standard conjecture, using Proposition 1.1. $\square$

Remark 1.6. If X has a second Lagrangian fibration, then it is likely that one can take for Θ the locus of singular fibers of π', but to our knowledge, it is not fully proved that this divisor is uniruled.

1.2 More general coverings by Lagrangian varieties

This section is devoted to the proof of Theorem 0.7. First of all, we have the following criterion for the Lefschetz standard conjecture to hold in degree 2.

Proposition 1.7. *Let X be a smooth projective $2n$-fold with polarization h_X and let $\Gamma \in \mathrm{CH}^{n+1}(X \times X)$ having the property that the intersection pairing $\langle\, ,\, \rangle_X$ is nondegenerate on $[\Gamma]^*(h_X^{2n-2}H^2(X,\mathbb{Q})_{tr}) \subset H^{2n}(X,\mathbb{Q})$. Then X satisfies the Lefschetz standard conjecture for degree 2 cohomology.*

Proof. We consider the codimension 2 cycle $\Gamma' := \Gamma \circ {}^t\Gamma \in \mathrm{CH}^2(X \times X)$. We claim that

$$[\Gamma']^* : h_X^{2n-2}H^2(X,\mathbb{Q})_{tr} \to H^2(X,\mathbb{Q})_{tr}$$

is an isomorphism. We note first that $[\Gamma']^* : h_X^{2n-2}H^2(X,\mathbb{Q})_{tr} \to H^2(X,\mathbb{Q})$ has image contained in $H^2(X,\mathbb{Q})_{tr}$. This is because $[\Gamma']^*$ is a morphism of Hodge structures and by definition of transcendental cohomology, any Hodge substructure of $H^2(X,\mathbb{Q})_{tr}$ which (after tensoring with $\mathbb{C}$) contains $H^{2,0}(X)$ equals $H^2(X,\mathbb{Q})_{tr}$. TYhis implies that any morphism of Hodge structures from $H^2(X,\mathbb{Q})_{tr}$ to a trivial Hodge structure is 0. Next, in order to prove the claim, it suffices to show that the intersection pairing

$$(\alpha,\beta) := \langle [\Gamma']^*\alpha, \beta\rangle_X$$

on $h_X^{2n-2}H^2(X,\mathbb{Q})_{tr}$ is nondegenerate. But we have by definition of Γ'

$$\langle [\Gamma']^*\alpha, \beta\rangle_X = \langle [\Gamma]^*\alpha, [\Gamma]^*\beta\rangle_X$$

and the pairing defined by the right hand side is nondegenerate by assumption. The claim implies the proposition since by restricting Γ' to $X \times \Sigma$ where $\Sigma \subset X$ is a surface complete intersection of hypersurfaces of class h_X, we find that $[\Gamma]^* : H^{2,0}(\Sigma) \to H^{2,0}(X)$ is surjective, so that Proposition 1.1 applies. $\square$

Assume now that a hyper-Kähler manifold X admits a Lagrangian covering, given by a diagram as in Definition 0.4

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & X \\ \pi \downarrow & & \\ B & & \end{array} \tag{14}$$

of smooth projective varieties, such that Φ is dominant and, for a general point $b \in B$, $\Phi_{|L_b}$ is generically finite to its image, which is Lagrangian subvariety of X.

If $\dim X = 2n$, so $\dim L_b = n$ and $\dim B \geq n$, as the map $\Phi_{|L_b}$ is generically finite to its image, up to replacing B by a complete intersection of ample hypersurfaces, one can assume that $\dim L = \dim X$ and Φ is dominant generically finite of degree N. We now introduce a relative Poincaré divisor $d \in \mathrm{CH}^1(L \times_B L)$, that is, for general $b \in B$, $d_b \in \mathrm{Pic}(L_b \times L_b)$ has the property that the map $y \in L_b \mapsto d_{b,y} \in \mathrm{Pic}(L_b)$ induces an isogeny $\mathrm{Alb}(L_b) \to \mathrm{Pic}^0(L_b)$.

Using the fact that $\Phi_{|L_b}$ is generically finite on its image for a general $b \in B$, for an ample line bundle H_X on X, $\Phi^* H_{X|L_b}$ is big hence can be used to construct d has in the previous section, using $\Phi^* H_X$ for divisor H. By pushing-forward d via the inclusion $L \times_B L \hookrightarrow L \times L$, this provides a codimension $n+1$-cycle $\Gamma_L \in \mathrm{CH}^{n+1}(L \times L)$. Denote by $L^1 H^{2,0}(L) \subset H^{2,0}(L)$ the Leray level

$$L^1 H^{2,0}(L) = \mathrm{Ker}\,(H^{2,0}(L) \to H^{2,0}(L_b)).$$

The fact that $\Phi : L \to X$ is a covering by Lagrangian varieties says equivalently that $\Phi^* \sigma_X \in L^1 H^{2,0}(L)$.

Lemma 1.8. *For any $\sigma \in L^1 H^{2,0}(L)$, one has*

$$[\Gamma_L]^*((\Phi^* h_X^{n-1})\sigma) = \lambda\sigma \bmod \pi^* H^{2,0}(B) \tag{15}$$

for some nonzero rational coefficient λ.

Proof. Proposition 1.2 proves the equality (15) in $H^{2,0}(L)$ assuming $H^{2,0}(B) = 0$. However, the proof given there proves this equality in general, modulo $\pi^* H^{2,0}(B)$. □

Using the generically finite map $\Phi : L \to X$, we get a codimension $n+1$-cycle

$$\Gamma_{L,X} := (\Phi, \Phi)_*(\Gamma_L). \tag{16}$$

We have the following result

Proposition 1.9. *In the situation above, assume that $\Phi_*(\pi^* H^{2,0}(B)) = 0$. Then*

$$[\Gamma_{L,X}]^*(h_X^{n-1}\sigma_X) = \lambda' \sigma_X \tag{17}$$

for some nonzero coefficient λ'. Furthermore,

$$[{}^t\Gamma_{L,X}]^*(h_X^{2n-2}\sigma_X) \neq 0 \text{ in } H^{2n}(X, \mathbb{C}). \tag{18}$$

Proof. We apply (15) to $\Phi^* \sigma_X$ and obtain

$$[\Gamma_L]^*(\Phi^*(h_X^{n-1}\sigma_X)) = \lambda \Phi^* \sigma_X + \pi^* \beta$$

for some $\beta \in H^{2,0}(B)$. Pushing forward this equation to X via Φ, and using the facts that $\Phi_*(\pi^* H^{2,0}(B)) = 0$, and $\Phi_* \circ \Phi^* = DId$, where $D = \deg \Phi$, we get

$$[\Gamma_{L,X}]^*(h_X^{n-1}\sigma_X) = D\lambda\sigma_X,$$

which proves (17), with $\lambda' = D\lambda$.

We have

$$\langle [{}^t\Gamma_{L,X}]^*(h_X^{2n-2}\sigma_X), h_X^{n-1}\overline{\sigma_X}\rangle_X = \langle h_X^{2n-2}\sigma_X, [\Gamma_{L,X}]^* h_X^{n-1}\overline{\sigma_X}\rangle_X,$$

and this is nonzero by (17), which proves (18). □

We now prove Theorem 0.7 stated in the introduction, concerning the case of dimension 4.

Proof of Theorem 0.7. Let X be a hyper-Kähler fourfold with $\rho(X) = 1$ admitting a covering by Lagrangian varieties

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & X \\ \pi \downarrow & & \\ B & & \end{array}, \tag{19}$$

where we can assume that B is a surface, up to replacing B by a complete intersection in B. We perform the construction described above, which provides us with the cycle $\Gamma_{L,X} \in \mathrm{CH}^3(X \times X)$. Suppose first that $\Phi_*(\pi^* H^{2,0}(B)) \neq 0$ in $H^{2,0}(X)$. Then, the correspondence $\Gamma \in \mathrm{CH}^2(X \times B)$ given by the Lagrangian cover (19) where B is a surface, directly solves the Lefschetz standard conjecture for degree 2 cohomology of X, since dually $[\Gamma]^* : H^{2,0}(B) \to H^{2,0}(X)$ is nonzero, hence surjective, so Proposition 1.1 applies in this case.

We can thus assume that $\Phi_*(\pi^* H^{2,0}(B)) \neq 0 = 0$ in $H^{2,0}(B)$, and then Proposition 1.9 applies, and we conclude that the morphism of Hodge structures

$$[\Gamma_{L,X}]^* : h_X^2 H^2(X, \mathbb{Q})_{tr} \to H^4(X, \mathbb{Q}) \tag{20}$$

is not equal to 0.

Assume now that X is very general in moduli, which is part of the assumptions of Theorem 0.7. Then the Mumford-Tate group of the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ is whole orthogonal group of the Beauville-Bogomolov quadratic form. We now have

Lemma 1.10. *If the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ has maximal Mumford-Tate group, any morphism of Hodge structure $\phi : H^2(X, \mathbb{Q})_{tr} \to H^4(X, \mathbb{Q})$ is given by cup-product by an element $h \in \mathrm{NS}(X)_{\mathbb{Q}}$.*

Proof. We first start with the following easy

Lemma 1.11. *If X is hyper-Kähler of dimension 4, the Hodge substructure on the orthogonal complement $H^4(X, \mathbb{Q})^{\perp SH^2}$ of $\mathrm{Sym}^2 H^2(X, \mathbb{Q})$ in $H^4(X, \mathbb{Q})$ consists of Hodge classes.*

Proof. Indeed, the cup-product map

$$\sigma_X \cup : H^1(X, \Omega_X) \to H^1(X, \Omega_X^3) = H^{3,1}(X)$$

is an isomorphism, hence the Hodge structure on $H^4(X, \mathbb{Q})^{\perp SH^2}$ has $H^{4,0} = 0$, $H^{3,1} = 0$. ☐

We conclude that the Hodge structure on $H^4(X, \mathbb{Q})$ decomposes as

$$H^4(X, \mathbb{Q}) = \mathrm{Sym}^2 H^2(X, \mathbb{Q})_{tr} \oplus \mathrm{NS}(X)_{\mathbb{Q}} \otimes H^2(X, \mathbb{Q})_{tr} \oplus K, \tag{21}$$

where K consists of Hodge classes coming either from $\mathrm{Sym}^2 \mathrm{NS}(X)_{\mathbb{Q}}$ or from $H^4(X, \mathbb{Q})^{\perp SH^2}$. The fact that the Mumford-Tate group of the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ is the whole orthogonal group implies that the Hodge structure $\mathrm{Sym}^2 H^2(X, \mathbb{Q})_{tr}$ is the sum of the 1-dimensional vector space $\mathbb{Q} c_{BB}$ generated by the Hodge class c_{BB} giving the Beauville-Bogomolov quadratic form, and an irreducible Hodge structure. As there is no nontrivial morphism of Hodge structure from $H^2(X, \mathbb{Q})_{tr}$ to a trivial Hodge structure, It follows that the morphism ϕ takes value in $\mathrm{NS}(X)_{\mathbb{Q}} \otimes H^2(X, \mathbb{Q})_{tr}$, which, as a Hodge structure, is a sum of copies of $H^2(X, \mathbb{Q})_{tr}$. As the Mumford-Tate group of the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ is the whole orthogonal group, there are no nontrivial endomorphisms of the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$, hence ϕ is given by cup-product by an element of $\mathrm{NS}(X)_{\mathbb{Q}}$. ☐

We now conclude the proof. As $\rho(X) = 1$, the divisor class h provided by Lemma 1.10 is a nonzero multiple of h_X. Thus we have by the nonvanishing of $[\Gamma_{L,X}]^*$,

$$[\Gamma_{L,X}]^* \sigma_X = \lambda'' h_X \sigma_X,$$

for some nonzero coefficient λ''. It then follows from the second Hodge-Riemann relations that

$$\langle [\Gamma_{L,X}]^* \sigma_X, \Gamma_{L,X}]^* \overline{\sigma_X} \rangle \neq 0.$$

But then the intersection pairing on $\mathrm{Im}\,([\Gamma_{L,X}]^* : h_X^2 H^2(X, \mathbb{Q})_{tr} \to H^2(X, \mathbb{Q})_{tr})$ is nondegenerate, so that Proposition 1.7 applies. ☐

2 Zero-cycles

2.1 Zero-cycles and the Lefschetz standard conjecture for H^2

We wish to discuss in this section the link between Bloch-Beilinson type filtrations on the group $\mathrm{CH}_0(X)$ of 0-cycles and the Lefschetz standard conjecture in degree 2. Given a smooth projective variety X, we define as usual

$$F^1\mathrm{CH}_0(X) = \mathrm{CH}_0(X)_{hom},\ F^2\mathrm{CH}_0(X) := \mathrm{Ker}\,(\mathrm{alb}_X : \mathrm{CH}_0(X)_{hom} \to (\mathrm{Alb}\,X)_{\mathbb{Q}}).$$

Several proposals have been made in order to construct $F^3\mathrm{CH}_0(X) \subset F^2\mathrm{CH}_0(X)$. In [32], Sh. Saito consider a subgroup that we will denote by $F^3_{dR}\mathrm{CH}_0(X)$ and is (essentially) defined as follows.

$$F^3_{dR}\mathrm{CH}_0(X) = \langle \mathrm{Im}\,(\Gamma_* : F^2\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)) \quad (22)$$
$$\text{for all } \Gamma \text{ such that } 0 = \Gamma^* : H^{2,0}(X) \to H^{2,0}(Y)\rangle.$$

Remark 2.1. In this definition, $\Gamma \in \mathrm{CH}^{\dim X}(Y \times X)$ and all smooth projective varieties Y are considered, but one can also restrict to surfaces Y by the following lemma.

Lemma 2.2. *Let Y be a smooth projective variety. Then*

$$F^2\mathrm{CH}_0(Y) = \langle \mathrm{Im}\,(\Gamma_* : F^2\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)),\ \text{where } Z \text{ is a surface}\rangle.$$

Proof. Indeed, we can assume $\dim Y \geq 2$. Any 0-cycle z of Y is supported on a surface $Z \subset Y$ which is a smooth complete intersection of ample hypersurfaces. By Lefschetz theorem on hyperplane sections, $\mathrm{Alb}\,Z = \mathrm{Alb}\,Y$, so if $\mathrm{alb}_Y(z) = 0$, $\mathrm{alb}_Z(z) = 0$. □

The definition of $F^3_{dR}\mathrm{CH}_0$ given in (22) is essentially motivated by the axioms of the Bloch-Beilinson filtration (see [6]). Indeed, suppose Y is a surface, and $\Gamma \in \mathrm{CH}^{\dim X}(Y \times X)$ is a correspondence satisfying the property that $\Gamma^* : H^{2,0}(X) \to H^{2,0}(Y)$ is zero. Then one gets by Künneth decomposition on the surface Y (see [26]) that the class $[\Gamma]$ of Γ decomposes as

$$[\Gamma] = \sum_i [\Gamma_i \times \Gamma'_i] + [R_1] + [R_2]$$

for some cycles Γ_i, resp. Γ'_i in Y, resp. X, such that $\mathrm{codim}\,\Gamma_i + \mathrm{codim}\,\Gamma'_i = 2$, where the cycle R_1 is supported on $C_1 \times X$ for a curve $C_1 \subset Y$, and the cycle R_2 is supported on $Y \times C_2$ for a curve $C_2 \subset X$. A decomposable cycle $\sum_i \Gamma_i \times \Gamma'_i$ and a cycle R_1 supported on $C_1 \times X$ act trivially on 0-cycles of degree 0; the cycle R_2 acts trivially on 0-cycles of Y which are of degree 0 and annihilated by alb_Y, that is, which belong to $F^2\mathrm{CH}_0(Y)$. Hence we get

$$\Gamma_* = (\Gamma - \sum_i \Gamma_i \times \Gamma'_i - R_1 - R_2)_* : F^2\mathrm{CH}_0(Y) \to F^2\mathrm{CH}_0(X).$$

As the cycle $\Gamma - \sum_i \Gamma_i \times \Gamma'_i - R_1 - R_2$ is cohomologous to 0, the image $\Gamma_*(F^2\mathrm{CH}_0(Y))$ must be contained in the F^3 level of the Bloch-Beilinson filtration.

We now turn to a different definition, introduced in [36], also a candidate for the third level of the Bloch-Beilinson filtration.

$$F^3_{BB}\mathrm{CH}_0(X) := \cap_{\Sigma,\Gamma}\mathrm{Ker}\,(\Gamma_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\Sigma)), \quad (23)$$

where the intersection is over all smooth projective surfaces Σ and correspondences $\Gamma \in \mathrm{CH}^2(X \times \Sigma)$. Again, this definition is dictated by the Bloch-Beinlinson axioms since Γ_* should preserve the Bloch-Beilinson filtration and for a Bloch-Beilinson filtration F, $F^3\mathrm{CH}_0(\Sigma) = 0$ for any smooth projective surface.

Remark 2.3. It is obvious from the definition that $F^3_{BB}\mathrm{CH}_0(S) = 0$ for a smooth projective surface S.

Let us recall the statement of the Bloch conjecture for 0-cycles on a surfaces (see [6]):

Conjecture 2.4. *(Bloch) Let $\Gamma \in \mathrm{CH}^2(S\times T)$ where T and S are smooth projective surfaces. Then if $\Gamma^* : H^{2,0}(T) \to H^{2,0}(S)$ is zero, $\Gamma_* : F^2\mathrm{CH}_0(S) \to \mathrm{CH}_0(T)$ is zero.*

We now have the following implications.

Proposition 2.5. *The inclusion $F^3_{dR}\mathrm{CH}_0(X) \subset F^3_{BB}\mathrm{CH}_0(X)$ holds for any X if and only if the Bloch conjecture 2.4 holds for 0-cycles on surfaces.*

Proof. Using Remark 2.1, $F^3_{dR}\mathrm{CH}_0(X)$ is generated by cycles Γ_*z, where S is a smooth projective surface, $\Gamma \in \mathrm{CH}^{\dim X}(S \times X)$ has the property that $\Gamma^* : H^{2,0}(X) \to H^{2,0}(S)$ is zero, and $z \in F^2\mathrm{CH}_0(\Sigma)$. Let $w = \Gamma_*z$ be such a cycle and let now $\Gamma' \in \mathrm{CH}^2(X \times \Sigma)$ be a correspondence, where Σ is a smooth projective surface. Then $\Gamma'_*w = (\Gamma' \circ \Gamma)_*z$ and the correspondence

$$\Gamma' \circ \Gamma \in \mathrm{CH}^2(S \times \Sigma)$$

has the property that $(\Gamma' \circ \Gamma)^* : H^{2,0}(\Sigma) \to H^{2,0}(S)$ is zero. Hence if Bloch conjecture 2.4 holds, $(\Gamma' \circ \Gamma)_*z = 0$ in $\mathrm{CH}_0(\Sigma)$. Hence $w \in F^3_{BB}\mathrm{CH}_0(X)$.

Conversely, assume the inclusion $F^3_{dR}\mathrm{CH}_0(X) \subset F^3_{BB}\mathrm{CH}_0(X)$ for any smooth projective X. Let S, T be smooth projective surfaces and let $\Gamma \in \mathrm{CH}^2(S\times T)$ such that $\Gamma^* : H^{2,0}(T) \to H^{2,0}(S)$ vanishes. Then by definition,

$$\Gamma_*(F^2\mathrm{CH}_0(S)) \subset F^3_{dR}\mathrm{CH}_0(T)$$

and $F^3_{dR}\mathrm{CH}_0(T) \subset F^3_{BB}\mathrm{CH}_0(T)$ by our assumption. As $F^3_{BB}\mathrm{CH}_0(T) = 0$ by Remark 2.3, we get $\Gamma_*(F^2\mathrm{CH}_0(S)) = 0$. □

Proposition 2.6. *The inclusion $F^3_{BB}\mathrm{CH}_0(X) \subset F^3_{dR}\mathrm{CH}_0(X)$ holds for X if X satisfies the Lefschetz standard conjecture for degree 2 cohomology.*

Proof. Assuming the Lefschetz standard conjecture for degree 2 cohomology on X, there exist a surface Σ and correspondences $\Gamma \in \mathrm{CH}^2(X \times \Sigma)$, $\Gamma' \in \mathrm{CH}^{\dim X}(\Sigma \times X)$ such that $\Gamma' \circ \Gamma \in \mathrm{CH}^{\dim X}(X \times X)$ acts as the identity on $H^{2,0}(X)$. For any $z \in F^2\mathrm{CH}_0(X)$,

$$z - (\Gamma' \circ \Gamma)_*z \in F^3_{dR}\mathrm{CH}_0(X) \tag{24}$$

since the correspondence $\Delta_X - \Gamma' \circ \Gamma$ acts trivially on $H^{2,0}(X)$. Let now $z \in F^3_{BB}\mathrm{CH}_0(X)$. Then $\Gamma_*z = 0$ in $\mathrm{CH}_0(\Sigma)$, hence $(\Gamma' \circ \Gamma)_*z = 0$ in $\mathrm{CH}_0(X)$. Hence $z \in F^3_{dR}\mathrm{CH}_0(X)$ by (24). □

Proposition 2.7. *The Lefschetz standard conjecture for degree 2 cohomology holds on X if and only if there exists a surface $\Sigma \overset{j}{\hookrightarrow} X$ such that $j_* : \mathrm{CH}_0(\Sigma) \to \mathrm{CH}_0(X)/F^3_{dR}\mathrm{CH}_0(X)$ is surjective.*

Proof. The "only if" direction follows from the arguments given in the previous proof. Indeed, we proved that the Lefschetz standard conjecture implies the equality (24) for $z \in F^2\mathrm{CH}_0(X)$, where the cycle $(\Gamma' \circ \Gamma)_*z$ is supported on a surface $S \overset{j}{\hookrightarrow} X$, namely, we can take for S the image in X of $\mathrm{Supp}\,\Gamma'$ by the second projection. This equality thus says $F^2\mathrm{CH}_0(X)/F^3_{dR}\mathrm{CH}_0(X)$ is contained in the image of $j_* : \mathrm{CH}_0(S) \to \mathrm{CH}_0(X)/F^3_{dR}\mathrm{CH}_0(X)$. If furthermore S contains an ample complete intersection surface, then the map $j_* : \mathrm{CH}_0(S) \to \mathrm{CH}_0(X)/F^2\mathrm{CH}_0(X)$ is also surjective because the Lefschetz hyperplane section theorem implies that $j_* : \mathrm{Alb}\,S \to \mathrm{Alb}\,X$ is surjective. Hence $j_* : \mathrm{CH}_0(S) \to \mathrm{CH}_0(X)/F^3_{dR}\mathrm{CH}_0(X)$ is surjective.

We now prove the reverse implication. Assume that there is a surface $\Sigma \overset{j}{\hookrightarrow} X$ such that $j_* : \mathrm{CH}_0(\Sigma) \to \mathrm{CH}_0(X)/F^3_{dR}\mathrm{CH}_0(X)$ is surjective. Then for a general point $x \in X$, there exist a smooth projective surface Y, a correspondence $\Gamma \in \mathrm{CH}^{\dim X}(Y \times X)$, such that $\Gamma^* : H^{2,0}(X) \to H^{2,0}(Y)$ is zero, a 0-cycle $z_x \in F^2\mathrm{CH}_0(Y)$ and a 0-cycle $z'_x \in \mathrm{CH}_0(\Sigma)$ such that $x = \Gamma_*(z_x) + z'_x$ in $\mathrm{CH}_0(X)$.

As it follows from Bloch-Srinivas arguments [7], these data can be spread over a finite extension of the function field of X, and this produces

(1) a smooth projective variety $\mathcal{Y}$, a dominant morphism $f : \mathcal{Y} \to X$ of relative dimension 2 and a cycle $\Gamma \in \mathrm{CH}^{\dim X}(\mathcal{Y} \times X)$, such that $\Gamma^*_x : H^{2,0}(X) \to H^{2,0}(\mathcal{Y}_x)$ is 0 for a general point $x \in X$;

(2) two cycles

$$\mathcal{Z} \in \mathrm{CH}^2(\mathcal{Y}),\ \mathcal{Z}' \in \mathrm{CH}^2(X \times \Sigma),$$

such that $\mathcal{Z}_x \in F^2\mathrm{CH}_0(\mathcal{Y}_x)$, and for some integer $N \neq 0$

$$N\Delta_X = (f, \Gamma)_*(\mathcal{Z}) + (Id_X, j)_*\mathcal{Z}' + W \text{ in } \mathrm{CH}^{\dim X}(X \times X) \tag{25}$$

where $W \in \mathrm{CH}^{\dim X}(X \times X)$ is a cycle which is supported on $D \times X$, for some divisor $D \subset X$. We now examine how the various cycles appearing in (25) act on $H^{2,0}(X)$. One has $[W]^* = 0$ on $H^{2,0}(X)$ because W is supported on $D \times X$, for some divisor $D \subset X$. Next we claim that the cycle $A := (f, \Gamma)_*(\mathcal{Z}) \in \mathrm{CH}^{\dim X}(X \times X)$ satisfies $[A]^*(\eta) = 0$ for any $\eta \in H^{2,0}(X)$. To see this, we note that

$$[A]^*(\eta) = f_*([\mathcal{Z}] \cup \Gamma^*\eta),\ \ \forall\ \eta \in H^{2,0}(X). \tag{26}$$

Recalling that $\mathcal{Z}_x \in F^2\mathrm{CH}_0(\mathcal{Y}_x)$, we get that, at least after restriction to $\mathcal{Y}_U = f^{-1}(U)$ for some Zariski dense open set U of X, $[\mathcal{Z}]_{|\mathcal{Y}_U} \in L^2H^4(\mathcal{Y}_U, \mathbb{Q})$, where $L^\bullet$ denotes the Leray filtration associated with the map f. On the other hand, as $\Gamma^*\eta_{|\mathcal{Y}_x} = 0$, we get that $[\Gamma^*\eta] \in L^1H^2(\mathcal{Y}_U, \mathbb{C})$. It follows that

$$[\mathcal{Z}] \cup \Gamma^*\eta_{|\mathcal{Y}_U} \in L^3H^6\mathcal{Y}_U, \mathbb{C}) \subset \mathrm{Ker}\,(f_* : H^6\mathcal{Y}_U, \mathbb{C}) \to H^2(U, \mathbb{C})).$$

Thus the class $f_*([\mathcal{Z}] \cup \Gamma^*\eta)$ vanishes in $H^2(U, \mathbb{C})$, hence vanishes in $H^{2,0}(X)$ since the composite map

$$H^{2,0}(X) \to H^2(X, \mathbb{C}) \to H^2(U, \mathbb{C})$$

is injective by coniveau (see [10]). We thus conclude from the claim and (25) that for any $\eta \in H^{2,0}(X)$,

$$N\eta = [(Id_X, j)_*\mathcal{Z}']^*\eta \text{ in } H^{2,0}(X). \tag{27}$$

But $[(Id_X, j)_*\mathcal{Z}']^*\eta = [\mathcal{Z}']^*(j^*\eta)$, so (27) implies that $[\mathcal{Z}']^* : H^{2,0}(\Sigma) \to H^{2,0}(X)$ is surjective. Then X satisfies the Lefschetz standard conjecture in degree 2 by Proposition 2.7. □

We conclude this section with the following result that will be used in next section.

Lemma 2.8. *Let X, Y be smooth projective varieties, and let $\Gamma \in \mathrm{CH}^2(X \times Y)$. Then for any $z \in F^3_{BB}\mathrm{CH}_0(X)$, $\Gamma_* z = 0$ in $\mathrm{CH}^2(Y)$.*

Proof. As $z \in F^3_{BB}\mathrm{CH}_0(X)$, the cycle $w := \Gamma_* z \in \mathrm{CH}^2(Y)$ has the property that for any surface $T \subset Y$, $w_{|T} = 0$ in $\mathrm{CH}_0(T)$. One then applies the following theorem of Joshi [17].

Theorem 2.9. *Let Y be a smooth projective variety and let $w \in \mathrm{CH}^2(Y)$ be a cycle such that $w_{|T} = 0$ for any surface $T \subset Y$. Then $w = 0$.*

□

2.2 A conjecture on 0-cycles of hyper-Kähler manifolds

We conclude with the formulation of a conjecture concerning the rational equivalence of points on a smooth projective hyper-Kähler manifold X. Recall from Section 2.1 the filtration

$$F^3_{BB}\mathrm{CH}_0(X) \subset F^2\mathrm{CH}_0(X) := \mathrm{Ker}\,(\mathrm{alb}_X : \mathrm{CH}_0(X)_{hom} \to \mathrm{Alb}(X)\otimes\mathbb{Q}).$$

The rational equivalence of points on a hyper-Kähler manifold is a very intriguing subject, that started with work of Beauville and the author [5], and continued in [13], [27], [40], [41], and more recently in the paper [22] which establishes the following remarkable result:

Theorem 2.10. *Let $\mathcal{M}$ be a smooth projective moduli space of semistable sheaves on a projective $K3$ surface S. Then two points $[\mathcal{F}]$, $[\mathcal{G}] \in \mathcal{M}$ have the same rational equivalence class in $\mathcal{M}$ if and only if $c_2(\mathcal{F}) = c_2(\mathcal{G})$ in $\mathrm{CH}_0(S)$.*

Of course, the condition is necessary, since, assuming for simplicity there is a universal sheaf $\mathcal{F}_{\rm univ}$ on $\mathcal{M}\times S$, the Chern class

$$c_2(\mathcal{F}_{\rm univ}) \in \mathrm{CH}^2(\mathcal{M}\times S)$$

induces a group morphism $\mathrm{CH}_0(\mathcal{M}) \to \mathrm{CH}_0(S)$ which maps the class of $[\mathcal{F}]$ to $c_2(\mathcal{F})$. The other direction is far from obvious and is a very intriguing statement. The following conjecture is motivated by Theorem 2.10.

Conjecture 2.11. *Let X be a smooth projective manifold whose algebra of holomorphic forms is generated in degree ≤ 2 (for example, a projective hyper-Kähler manifold). Then for x, $y \in X$, $x = y$ in $\mathrm{CH}_0(X)$ if and only if $x = y$ in $\mathrm{CH}_0(X)/F^3_{BB}\mathrm{CH}_0(X)$.*

Theorem 2.10 establishes the conjecture when $X = \mathcal{M}$ is as above a smooth projective moduli space of sheaves on a $K3$ surface. Indeed, if $[\mathcal{F}] = [\mathcal{G}]$ in $\mathrm{CH}_0(\mathcal{M})/F^3_{BB}\mathrm{CH}_0(\mathcal{M})$, then $c_2(\mathcal{F}) = c_2(\mathcal{G})$ in $\mathrm{CH}_0(S)$ by the definition (23) of F^3_{BB}, and using the existence of the correspondence $c_2(\mathcal{F}_{\rm univ})$. Thus Theorem 2.10 gives $[\mathcal{F}] = [\mathcal{G}]$ in $\mathrm{CH}_0(\mathcal{M})$.

Let us observe that the Lefschetz standard conjecture in degree 2, the generalized Hodge conjecture, and the nilpotence conjecture together imply Conjecture 2.11. To explain this, let us assume, as this is the case when X is hyper-Kähler, that $\dim X = 2n$ and the algebra of holomorphic forms is generated in degree 2, with $H^{2,0}(X) = \mathbb{C}\sigma_X$. Then $H^{i,0}(X) = 0$ for i odd and $H^{2i,0}(X)$ is generated by σ_X^i. Denoting by $h_X \in \mathrm{NS}(X)_{\mathbb{Q}} \subset H^2(X,\mathbb{Q})$ a polarizing class on X, and assuming the Lefschetz standard conjecture in degree 2 holds for X, there exist a codimension 2 cycle $\mathcal{Z}_{\rm lef} \in \mathrm{CH}^2(X\times X)$ such that

$$[\mathcal{Z}_{\rm lef}]^*(h_X^{2n-2}\sigma_X) = \sigma_X. \tag{28}$$

Lemma 2.12. *One has for any $i \leq n$,*

$$[\mathcal{Z}_{\rm lef}^i]^* h_X^{2n-2i}\sigma_X^i = \lambda_i\sigma_X^i, \tag{29}$$

for some nonzero rational number λ_i.

Proof. We write

$$[\mathcal{Z}_{\rm lef}] = [\mathcal{Z}_{\rm lef}]^{2,2} + [\mathcal{Z}_{\rm lef}]^{4,0} + [\mathcal{Z}_{\rm lef}]^{0,4}, \tag{30}$$

where $[\mathcal{Z}_{\rm lef}]^{i,j} \in H^i(X,\mathbb{Q})\otimes H^j(X,\mathbb{Q})$ denotes the (i,j) Künneth component of $[\mathcal{Z}_{\rm lef}]$. Note that the Künneth components are algebraic in this case, since $[\mathcal{Z}_{\rm lef}]^{4,0}$, resp. $[\mathcal{Z}_{\rm lef}]^{0,4}$, identifies to the restriction to $X\times x$, resp. $x\times X$, of $[\mathcal{Z}_{\rm lef}]$, where x is any point of X. Furthermore, developing the i-th power of (30), one sees immediately that

$$[\mathcal{Z}_{\rm lef}^i]^*(h_X^{2n-2i}\sigma_X^i) = (([\mathcal{Z}_{\rm lef}]^{2,2})^i)^*(h_X^{2n-2i}\sigma_X^i).$$

In other words, we can assume that $[\mathcal{Z}_{\rm lef}]=[\mathcal{Z}_{\rm lef}]^{2,2}$. Over $\mathbb{C}$, we have the Hodge decomposition

$$H^2(X,\mathbb{C})=\mathbb{C}\sigma_X\oplus H^{1,1}(X)\oplus\overline{\mathbb{C}\sigma_X},$$

and the equation (28) says that

$$[\mathcal{Z}_{\rm lef}]=\lambda({\rm pr}_1^*\sigma_X\cup{\rm pr}_2^*\overline{\sigma_X}+{\rm pr}_1^*\overline{\sigma_X}\cup{\rm pr}_2^*\sigma_X)+R, \tag{31}$$

where the residual term R belongs to the term $H^{1,1}(X)\otimes H^{1,1}(X)$ of the Hodge-Künneth decomposition of $H^4(X\times X,\mathbb{C})$. When we take the i-th power of (31), only the term $\lambda^i({\rm pr}_1^*\sigma^i\cup{\rm pr}_2^*\overline{\sigma}^i)$ acts nontrivially on $h_X^{2n-2i}\sigma_X^i$ and it acts on $h_X^{2n-2i}\sigma_X^i$ by multiplication by the coefficient $\lambda^i\int_X h_X^{2n-2i}\sigma_X^i\overline{\sigma_X}^i$, which proves the result. □

Corollary 2.13. *There exists a polynomial*

$$P=\sum_{i=0}^n\mu_i\mathcal{Z}_{\rm lef}^i\cdot{\rm pr}_2^*h_x^{2n-2i}\in{\rm CH}^{2n}(X\times X),$$

such that

$$[P]^*\sigma_X^i=\sigma_X^i \tag{32}$$

for $i=0,\ldots,n$.

Proof. Let $\Gamma_i:=\mathcal{Z}_{\rm lef}^i\cdot{\rm pr}_2^*h_x^{2n-2i}$. We have

$$[\Gamma_i]^*\sigma_X^j=0 \text{ for } j>i,$$

$$[\Gamma_i]^*\sigma_X^j=\lambda_{ij}\sigma_X^j \text{ for } j\leq i,$$

with $\lambda_j\neq 0$ when $j=i$ by Lemma 2.12. The matrix $M:=(\lambda_{ji})$ is thus invertible, and our condition on the column vector ${}^t(\mu_1,\ldots,\mu_n)$ is

$$M\begin{pmatrix}\mu_1\\ \vdots\\ \mu_n\end{pmatrix}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

□

As the cycle $P=\sum_{i=0}^n\mathcal{Z}_{\rm lef}^i\cdot{\rm pr}_2^*h_X^{2n-2i}\in{\rm CH}^{2n}(X\times X)$ acts as the identity on $H^{*,0}(X)$, the cycle $\Delta_X-P\in{\rm CH}^{2n}(X\times X)$ acts as 0 on $H^{*,0}(X)$, so that its image

$${\rm Im}\,[\Delta_X-P]^*:H^*(X,\mathbb{Q})\to H^*(X,\mathbb{Q})$$

has Hodge coniveau ≥ 1. Assuming the generalized Hodge conjecture, there should exist a divisor $D\subset X$ and a cycle $\mathcal{Z}'\in{\rm CH}^{2n}(X\times X)$ supported on $D\times X$ such that

$$[\Delta_X-P]=[\mathcal{Z}'] \text{ in } H^{4n}(X\times X,\mathbb{Q}).$$

Equivalently, the cycle $\Delta_X-P-\mathcal{Z}'$ is cohomologous to 0, and the nilpotence conjecture [35] predicts that a power $(\Delta_X-P-\mathcal{Z}')^{\circ N}$ vanishes in ${\rm CH}^{2n}(X\times X)$, so in particular the endomorphism

$$\phi:=(\Delta_X-P-\mathcal{Z}')_*:{\rm CH}_0(X)\to{\rm CH}_0(X)$$

is nilpotent. We observe that, as $\mathcal{Z}'$ is supported on $D\times X$, $\mathcal{Z}'_*=0$ on ${\rm CH}_0(X)$, so

$$\phi=Id_{{\rm CH}_0(X)}-P_*.$$

Now assume $x,y\in X$ and $x=y$ modulo $F^3_{BB}{\rm CH}_0(X)$. Then as already noticed,

$$\mathcal{Z}_{{\rm lef},x}=\mathcal{Z}_{{\rm lef},y} \text{ in } {\rm CH}^2(X),$$

hence $\mathcal{Z}^i_{{\rm lef},x}=\mathcal{Z}^i_{{\rm lef},y}$ in ${\rm CH}^{2i}(X)$ for any i, so that $P_*x=P_*y$ in ${\rm CH}_0(X)$. It thus follows that $\phi(x-y)=x-y$. But ϕ is nilpotent, so the last condition implies $x-y=0$, as we wanted.

Remark 2.14. One may wonder why this proof does not work as well to show (assuming the general conjectures) that for any 0-cycles $z=\sum_i n_i x_i$ of degree 0, if $z=0$ modulo $F^3_{BB}\mathrm{CH}_0(X)$, then $z=0$, a statement that is clearly not expected to be true since $\mathrm{CH}_0(X)/F^3_{BB}\mathrm{CH}_0(X)$ is expected to be a direct summand in the CH_0 of a surface. The reason is that for a cycle $x-y$

$$P_*(x-y)=\sum_i \lambda_i h_X^{2n-2i}(\mathcal{Z}^i_{\mathrm{lef},x}-\mathcal{Z}^i_{lef,y})$$

is divisible by $\mathcal{Z}_{lef,x}-\mathcal{Z}_{lef,y}$, while for general 0-cycles of degree 0, $P_*(z)$ is not a priori divisible by $(\mathcal{Z}_{\mathrm{lef}})_*(z)$.

The same argument as above gives the following statement linking the Lefschetz standard conjecture for degree 2 cohomology and Conjecture 2.11.

Proposition 2.15. *Let X be a hyper-Kähler manifold with a codimension 2 cycle $\mathcal{Z}_{\mathrm{lef}}\in \mathrm{CH}^2(X\times X)$ such that there exists a polynomial $P(y)=\sum_{i=0}^n \gamma_i y^i$ with coefficients in $\gamma_i\in\mathrm{CH}^{2n-2i}(X)$, such that for any $x\in X$,*

$$x=P(\mathcal{Z}_x) \text{ in } \mathrm{CH}_0(X). \tag{33}$$

Then X satisfies Conjecture 2.11.

Proof. By Lemma 2.8, a 0-cycle $z\in F^3_{BB}\mathrm{CH}_0(X)$ satisfies $\mathcal{Z}_*(z)=0$ in $\mathrm{CH}^2(X)$. If $x=y$ modulo $F^3_{BB}\mathrm{CH}_0(X)$, we thus get $\mathcal{Z}_x=\mathcal{Z}_y$ in $\mathrm{CH}^2(X)$ and thus (33) implies that $x=y$ in $\mathrm{CH}_0(X)$. □

Proposition 2.15 applies when $X=F_1(Y)$ is the Fano variety of lines of a smooth cubic fourfold Y (see [3]). Indeed in this case the following quadratic formula, which is a particular case of (33)

$$[l]=\alpha S_l^2+\Gamma.S_l+\gamma' \text{ in } \mathrm{CH}_0(X), \tag{34}$$

is proved in [37], for any line $l\subset Y$, where $S_l\subset F_1(Y)$ is the surface of lines meeting l.

This suggests that, in the hyper-Kähler case, one can formulate a conjecture even stronger than Conjecture 2.11 and we refer to [33] for a more precise formulation, namely that the cycle $\Delta_X-P-\mathcal{Z}'$ that appears in the argument above is in fact 0, and not only nilpotent.

Conjecture 2.16. *Let X be a projective hyper-Kähler manifold with polarization h_X. There any Lefschetz cycle $\mathcal{Z}_{\mathrm{lef}}\in\mathrm{CH}^2(X\times X)$ for h_X satisfies a polynomial equation*

$$\Delta_X=P(\mathcal{Z}_{\mathrm{lef}})+W \text{ in } \mathrm{CH}^{2n}(X\times X), \tag{35}$$

where $P(\mathcal{Z}_{\mathrm{lef}})=\sum_i \mathcal{Z}^i_{\mathrm{lef}}\mathrm{pr}_2^*\gamma_i$ for some cycle $\gamma_i\in\mathrm{CH}^{2n-2i}(X)$, and W is supported on $D\times X$ for some divisor $D\subset X$.

Turning to non hyper-Kähler projective manifolds, an easy example of a variety satisfying Conjecture 2.11 is a product of curves and surfaces. Indeed, if $X=\prod_{i=1}^m S_i$, where S_i is either a curve or a surface, and $x=(x_1,\ldots,x_m)\in X$, $y=(y_1,\ldots,y_m)\in X$, then $x=y$ in $\mathrm{CH}_0(X)/F^3_{BB}\mathrm{CH}_0(X)$ implies, using Definition (23), that $x_i=y_i$ in $\mathrm{CH}_0(S_i)$, so that $x=y$ in $\mathrm{CH}_0(X)$. Another class of varieties satisfying the conjecture appears in the following proposition.

Proposition 2.17. *Let X be a desingularization of the quotient $K=A/\pm Id$ of an abelian variety by the -1-involution. Then X satisfies Conjecture 2.11.*

Proof. We first observe that $\mathrm{CH}_0(X)$ is isomorphic by pull-back to the $-Id$-invariant part of $\mathrm{CH}_0(A)$, so we only have to show that if $x,y\in A$ and $\{x\}+\{-x\}=\{y\}+\{-y\}$ in $\mathrm{CH}_0(A)/F^3_{BB}\mathrm{CH}_0(A)$, then $\{x\}+\{-x\}=\{y\}+\{-y\}$ in $\mathrm{CH}_0(A)$. Here we denote as usual by $\{x\}$ the 0-cycle corresponding to the point x, so as to distinguish the addition in A and the addition in $\mathrm{CH}_0(A)$. Let Θ be an ample divisor on A, determining an isogeny $A \to \widehat{A}$

$$x \mapsto D_x = \Theta_x - \Theta.$$

Recalling the Pontryagin product $*$ defined on 0-cycles of A by

$$z * z' = \mu_*(z \times z'),$$

where $\mu : A \times A \to A$ is the sum map, we have Beauville's formulas in [2, Proposition 6] which give in particular the following equalities:

$$\frac{\theta^{g-k}}{(g-k)!} D_x^k = \frac{\theta^g}{g!} * \gamma(x)^{*k}, \tag{36}$$

where

$$\gamma(x) := \{0_A\} - \{x\} + \frac{1}{2}(\{0_A\} - \{x\})^{*2} + \cdots + \frac{1}{g}(\{0_A\} - \{x\})^{*g} = -\log(\{x\}) \text{ in } \mathrm{CH}_0(A). \tag{37}$$

Here the logarithm is taken with respect to the Pontryagin product $*$ and the expansion is finite because 0-cycles of degree 0 are nilpotent for the Pontryagin product. Note that $\{x\} * \{-x\} = \{0_A\}$, that is, $\{-x\}$ is the inverse of $\{x\}$ for the Pontryagin product, so that $\gamma(x) = -\gamma(-x)$.

We can assume that $\frac{\theta^g}{g!} = d\{0_A\}$ for some nonzero integer d, so that the formula becomes

$$\frac{\theta^{g-k}}{(g-k)!} D_x^k = d\gamma(x)^{*k} \tag{38}$$

for any k. We now add-up formulas (38) for x and $-x$. For odd k we get 0, using

$$D_x = -D_{-x}, \ \ \gamma(x) = -\gamma(-x),$$

and for even $k = 2r$ we get

$$2\frac{\theta^{g-2r}}{(g-2r)!} D_x^{2r} = d\gamma(x)^{*2r} \text{ in } \mathrm{CH}_0(A). \tag{39}$$

We now observe that if $\{x\} + \{-x\} = \{y\} + \{-y\}$ in $\mathrm{CH}_0(A)/F^3_{BB}\mathrm{CH}_0(A)$, then, using Lemma 2.8, $D_x^2 = D_y^2$ in $\mathrm{CH}^2(A)$, since clearly $x \mapsto D_x^2$ is both -1-invariant and induced by a codimension 2 self-correspondence of A. So we conclude from (39) that, for any r

$$\gamma(x)^{*2r} = \gamma(y)^{*2r} \text{ in } \mathrm{CH}_0(A). \tag{40}$$

Finally, we have (using again nilpotence so that the formal series reduce in fact to finite sums).

$$\{x\} = \exp(\gamma(x)), \ \{-x\} = \exp(-\gamma(x)) \text{ in } \mathrm{CH}_0(A)$$

and similarly for y, so that

$$\{x\} + \{-x\} = \exp(\gamma(x)) + \exp(-\gamma(x)) = 2\sum_{r\geq 0} \frac{\gamma(x)^{*2r}}{(2r)!} \text{ in } \mathrm{CH}_0(A), \tag{41}$$

and similarly for y. Equations (41) and (40) imply $\{x\}+\{-x\} = \{y\}+\{-y\}$ in $\mathrm{CH}_0(A)$. □

We conclude with the following analogue of Theorem 2.10 in the case of punctual Hilbert schemes.

Theorem 2.18. *Let Σ be a smooth surface. Then for any n, and any $z, z' \in \Sigma^{[n]}$, z is rationally equivalent to z' in $\Sigma^{[n]}$ if and only if the corresponding 0-cycles Z, Z' of Σ are rationally equivalent in Σ. A fortiori $\Sigma^{[n]}$ satisfies Conjecture 2.11 since a 0-cycle in $F^3_{BB}\mathrm{CH}_0(\Sigma^{[n]})$ is annihilated by I_*, where $I \subset \Sigma^{[n]}) \times \Sigma$ is the incidence correspondence.*

Proof. Note that $\mathrm{CH}_0(\Sigma^{[k]}) = \mathrm{CH}_0(\Sigma^{(k)})$, so we will work with $\Sigma^{(n)}$. It is a standard fact that the quotient singularities of $\Sigma^{(n)}$ allow to do intersection theory at least with $\mathbb{Q}$-coefficients. The computations we make should be thought as computations in the invariant part under $\mathfrak{S}_n$ of the Chow groups of Σ^n). We start with the following lemma.

Lemma 2.19. *For any integer $k \geq 0$ and any $z \in \Sigma^{(k)}$, the addition*

$$\mu_z : \Sigma^{(n)} \to \Sigma^{(k+n)},$$

$$w \mapsto w + z,$$

induces an injective map $\mu_{z*} : \mathrm{CH}_0(\Sigma^{(n)}) \to \mathrm{CH}_0(\Sigma^{(n+k)})$.

Proof. Reasoning by induction on k, it clearly suffices to prove the result when $k = 1$, so z consists of one point $z \in \Sigma$. The incidence correspondence

$$\Gamma \in \mathrm{CH}^{2n}(\Sigma^{(n+1)} \times \Sigma^{(n)})$$

given by the nested symmetric product (one could work with the Hilbert schemes at this point) induces a morphism

$$\Gamma_{n+1,*} : \mathrm{CH}_0(\Sigma^{(n+1)}) \to \mathrm{CH}_0(\Sigma^{(n)}).$$

One clearly has

$$\Gamma_{n+1,*} \circ \mu_{z*} = Id_{\mathrm{CH}_0(\Sigma^{(n)})} + \mu_{z*} \circ \Gamma_{n*}. \tag{42}$$

A cycle in $\mathrm{Ker}\, \mu_{z*}$ is thus in the image of μ_{z*}. Using the equations (42) for $n-1, n-2, \ldots 1$, one proves similarly that it is in the image of $\mu_{(2z),*}$, $\mu_{(3z),*}$ and finally that it is zero. □

Let now $z, z' \in \Sigma^{[n]}$ be two points such that the corresponding 0-cycles Z, Z' are rationally equivalent in Σ. Then there exist an effective 0-cycle $W \in \Sigma^{(k)}$ and a rational curve in $\Sigma^{(n+k)}$ passing through the two points $\mu_w(z)$ and $\mu_w(z')$. Thus $\mu_{w*}(z) = \mu_{w*}(z')$ in $\mathrm{CH}_0(\Sigma^{(n+k)})$. By Lemma 2.19, $z = z'$ in $\mathrm{CH}_0(\Sigma^{(n)})$. □

We can also prove the result by establishing a polynomial formula of the form (35) in this case. Concretely, for the symmetric product $X = \Sigma^{(n)}$, consider the codimension 2 cycle $\mathcal{Z}_{\mathrm{lef}} \in \mathrm{CH}^2(X \times X)$ defined as the incidence correspondence:

$$\mathcal{Z}_{\mathrm{lef}} = \{(z_1, z_2) \in \Sigma^{(n)} \times \Sigma^{(n)},\ Z_1 \cap Z_2 \neq \emptyset\}.$$

Proposition 2.20. *There exist cycles $\gamma_i \in \mathrm{CH}^{2n-2i}(X)$ such that the cycle $P = \sum_i \mathcal{Z}^i_{\mathrm{lef}} \mathrm{pr}_2^* \gamma_i$ satisfies, for any $z \in X$,*

$$z = P(\mathcal{Z}_{\mathrm{lef},z}) \text{ in } \mathrm{CH}_0(X). \tag{43}$$

Proof. It suffices to check equality (43) after pull-back to Σ^n. Let $z = x_1 + \ldots + x_n \in \Sigma^{(n)}$. Then $\mathcal{Z}_{\mathrm{lef},z} = \sum_i x_i + \Sigma^{(n-1)}$. In order to compute the self-intersection of this cycle, we pull-back to Σ^n, and denote the resulting cycle by $\mathcal{Z}'_{\mathrm{lef},z}$. We get by definition, assuming the points x_i are distinct

$$\mathcal{Z}'_{\mathrm{lef},z} = \sum_{i,j} \mathrm{pr}_j^* x_i.$$

Using the fact that cycles $Z_{ij} := \mathrm{pr}_j^* x_i$ satisfy $Z_{ij}Z_{kj} = 0$ for any j, k, we get

$$(\mathcal{Z}'_{\mathrm{lef},z})^n = \sum_f Z_f,$$

where f runs through the set of applications from $\{1, \ldots, n\}$ to itself and

$$Z_f = \prod_{i=1}^n Z_{f(i)i} = (x_{f(1)}, \ldots, x_{f(n)}) \in \mathrm{CH}_0(\Sigma^n). \tag{44}$$

We now stratify the set of maps f according to their combinatorics. Up to the action of the symmetric group $\mathfrak{S}_n$, such a map is characterized by the image of f and the partition of $\{1, \ldots, n\}$ given by the non-empty preimages $f^{-1}(s) \subset \{1, \ldots, n\}$. The partition $(1, \ldots, 1)$ where all sets have cardinality 1 produce the term $\sum_{\sigma \in \mathfrak{S}_n} (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ which is the pull-back $\tilde{z} \in \mathrm{CH}_0(\Sigma^n)$ of the point $z = x_1 + \ldots + x_n \in \Sigma^{(n)}$. The next case is the case of a partition $(2, 1, \ldots, 1)$ for which exactly two points have the same image, the map f being injective on the remaining set. The contribution of these partitions to (44) is the sum

$$\sum_{i \neq j, \sigma \in \mathfrak{S}_n} \sigma(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, \widehat{x_{j+1}}, \ldots, x_n), \tag{45}$$

where in each n-uple, exactly one point appears twice and another point is missing. We observe that (45) appears in the development of

$$\mathcal{Z}'_{\mathrm{lef},z})^{n-1} \cdot \Delta_{(2,1,\ldots,1)}, \tag{46}$$

where the codimension 2 cycle $\Delta_{(2,1,\ldots,1)}$ is the sum of the partial diagonals Δ_{ij} where $y_i = y_j$. Developing (46), we get that

$$\tilde{z} = (\mathcal{Z}'_{\mathrm{lef},z})^n - \mathcal{Z}'_{\mathrm{lef},z})^{n-1} \cdot \Delta_{(2,1,\ldots,1)} + R_z \text{ in } \mathrm{CH}_0(\Sigma^n),$$

where the cycle R_z consists in a sum of terms $(x_{f(1)}, \ldots, x_{f(n)})$ for which the image of the map f has cardinality $\leq n-2$. More generally, one checks by induction on the cardinality of f that Proposition 2.20 holds, where one can take for the cycles γ_i a combination with rational coefficients of the diagonals Δ_I where $|I| = n - i$. Here the cardinality of $I = \{I_1, \ldots, I_k\}$ with $\{1, \ldots, n\} = I_1 \sqcup \ldots \sqcup I_k$ is the number k and the diagonal $\Delta_I \subset \Sigma^n$ is defined by the equations $y_j = y_k$ when i, $k \in I_l$ for some l. □

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Collège de France 3 rue d'Ulm, 75005 Paris, France
claire.voisin@imj-prg.fr