Torsion points of sections of Lagrangian torus fibrations and the Chow ring of hyper-Kähler fourfolds

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Abstract

Let $\phi : X \to B$ be a Lagrangian fibration on a projective irreducible hyper-Kähler fourfold. Let $M \in \operatorname{Pic} X$ be a line bundle whose restriction to the general fiber X_b of ϕ is topologically trivial. We prove that if the fibration has maximal variation or is isotrivial, the set of points b such that the restriction $M_{|X_b}$ is torsion is dense in B. We give an application to the Chow ring of X. We prove a similar result for elliptic fibrations.

0 Introduction

Let X be an algebraic variety, $X \to B$ a projective morphism and let $M \in \operatorname{Pic} X$ be a line bundle which is topologically trivial on the fibers of ϕ. The line bundle M then determines an algebraic section ν_M of the torus fibration $\operatorname{Pic}^0(X^0/B^0) \to B^0$ (which we will call the normal function associated to M), where $X^0 \to B^0$ is the restriction of ϕ over the open set $B^0 \subset B$ of regular values of ϕ. The group of such sections is called the Mordell-Weil group of the fibration. The problem we investigate in this paper is: when do there exist points $b \in B^0$ where $\nu_M(b)$ is a torsion point? Are these points dense (for the usual or Zariski topology) in B? This question is natural only when the codimension of the locus of torsion points in $\operatorname{Pic}^0(X/B)$, that is $h^{1,0}(X_b) = \dim \operatorname{Pic}^0(X/B) - \dim B$, is not greater than $n = \dim B$, so that the section ν_M is expected to meet this locus. We will consider in this paper the case where $h^{1,0}(X_b) = \dim B$, and more precisely we will work in the following setting: Let X be a projective hyper-Kähler manifold of dimension $2n$ and let $\phi : X \to B$ be a Lagrangian fibration. According to Matsushita [17], any morphism $\phi : X \to B$ with $0 < \dim B < \dim X$ and connected fibers is a Lagrangian fibration. The smooth fibers X_b are then abelian varieties of dimension n, which are in fact canonically polarized, by the following result of Matsushita (see [19, Lemma 2.2])

Proposition 0.1. *In the situation above, the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$ has rank 1.*

Matsushita conjectured that the period map $B \dashrightarrow \mathcal{A}_{n,\alpha}$ from B to the moduli space of abelian varieties with polarization of adequate type is either generically of maximal rank n or constant (the isotrivial case). This conjecture was proved in [11] assuming that the Mumford-Tate group of the Hodge structure on the transcendental cohomology $H^2(X, \mathbb{Q})_{tr} \subset H^2(X, \mathbb{Q})$ is the full special orthogonal group of $H^2(X, \mathbb{Q})_{tr}$ equipped with the Beauville-Bogomolov intersection form, and that $b_2(X)_{tr} \geq 5$. In particular, it is satisfied by general deformations of X with fixed Picard lattice, assuming $b_2(X)_{tr} \geq 5$.

Our main result in this paper is the following:

Theorem 0.2. *Let X, ϕ be as above, and let $M \in \operatorname{Pic} X$ restrict to a topologically trivial line bundle on the smooth fibers X_b of ϕ. Assume that $\dim X \leq 4$ and the Lagrangian fibration on X satisfies Matsushita's conjecture. Then the set of points $b \in B^0$ such that $\nu_M(b) = M_{|X_b}$ is a torsion point in $\operatorname{Pic} X_b$ is dense in B^0 for the usual topology.*

Remark 0.3. In the locally isotrivial case, we will not need the condition on the dimension, so the result holds under the slightly more general assumptions that either $\dim X \leq 4$ and the period map is generically of maximal rank, or the fibration is locally isotrivial.

Applying the main theorem of [11], we conclude:

Corollary 0.4. *Let X, ϕ, M be as in Theorem 0.2. Assume $b_2(X) \geq 8$.Then the very general deformation X_t of X preserving L, M, and an ample line bundle on X (so that X_t is projective and has a deformed Lagrangian fibration ϕ_t, see [19], [11]), satisfies the conclusion of Theorem 0.2.*

The restriction to dimension 4 is certainly not essential here and we believe that the result is true in any dimension although the analysis seems to be very complicated in higher dimension. The restriction to dimension 4 appears in the analysis of the constraints on the infinitesimal variation of Hodge structures imposed by contradicting the conclusion of Theorem 0.2 (see the proof of Proposition 3.2 and its local version Proposition 3.8). These constraints, that we describe in general in Section 3.3, are very strong. In dimension 4, three different cases appear, and the number of cases to classify increases quickly with the dimension.

In the general case of an elliptic fibration over a curve, (and without any hyper-Kähler condition) some of the ingredients used in the proof of Theorem 0.2 give the following result.

Theorem 0.5. *Let $\phi : X \to B$ be an elliptic fibration, where B is a smooth projective curve and X is smooth projective, and let $B^0 \subset B$ be the open set of regular values of ϕ. Let M be a line bundle on X which is of degree 0 on the fibers of ϕ. Then either the set of points $b \in B^0$ such that $\sigma_M(b) = M_{|X_b}$ is a torsion point in $\mathrm{Pic}\, X_b$ is dense in B^0 for the usual topology, or the restriction map $H^1(X, \mathbb{Q}) \to H^1(X_b, \mathbb{Q})$ is surjective. In the second case, the fibration ϕ is locally isotrivial and the associated Jacobian fibration is rationally isogenous over B to the product $J(X_b) \times B$.*

In both cases, we will make first an infinitesimal analysis of the situation, which will be completed by a monodromy argument: the local analysis in the case where the basis and fiber dimensions are equal to 1 is extremely easy, which is why we do not need the hyper-Kähler assumption. In the higher dimensional case, the main part of the proof will be the local analysis.

One may wonder if one can prove a more general result. Let us give some examples explaining the main difficulties encountered: Consider more generally any complex torus fibration $\phi_A : A \to B$ with a section ν_A. The torus fibration is canonically isomorphic, as a real torus fibration, to the locally constant fibration $H_{1,\mathbb{R},\phi_A}/H_{1,\mathbb{Z},\phi_A}$, where $H_{1,\mathbb{R},\phi_A} := (R^1\phi_{A*}\mathbb{R})^*$. A section ν_A thus admits local liftings $\tilde{\nu}_A$ which are $\mathcal{C}^\infty$ (in fact real analytic) sections of the flat vector bundle associated with the local system $H_{1,\mathbb{R},\phi_A}$.

Example 0.6. *Assume that the section ν_A is locally the image of a constant section $\tilde{\nu}_A \in \Gamma(H_{1,\mathbb{R},\phi_A})$. Then if $\tilde{\nu}_A$ is not rational, that is, does not belong to $\Gamma(H_{1,\mathbb{Q},\phi_A})$, ν_A has no torsion point.*

This case can be in general excluded by a monodromy argument. The following example is slightly more subtle:

Example 0.7. *Assume that the torus fibration $\phi_A : A \to B$ is a fibered product*

$$A = A' \times_B A'',$$

where $\phi' : A' \to B$, $\phi'' : A'' \to B$ are torus fibrations. Choose a section $\nu'' \in A''$ which is as in Example 0.6 the image of a constant nonrational section of $(R^1\phi''_*\mathbb{R})^*$. Then for any section $\nu_{A'} : B \to A'$ of ϕ', $\nu_A = (\nu_{A'}, \nu'')$ does not have any torsion point.

In Example 0.7, there is a geometric explanation for the lack of transversality of the section ν_A with respect to the torsion sections. An abstract version of Example 0.7 involving variations of Hodge structures of weight 1 is now the following:

Example 0.8. *Assume the real variation of Hodge structure with underlying local system $H_{1,\mathbb{R},\phi_A} = (R^1\phi_{A*}\mathbb{R})^*$ splits as the direct sum*

$$H_{1,\mathbb{R}} = L' \oplus L''. \tag{1}$$

Assume that the local liftings $\tilde{\nu}_A$ can be written as $\tilde{\nu}_A = \tilde{\nu}' + \tilde{\nu}''$, where $\tilde{\nu}'$ is a locally constant section of L' (and $\tilde{\nu}''$ is a $\mathcal{C}^\infty$ section of L''). Then ν_A is not transverse to the torsion sections (or more generally to any section of the torus fibration which locally lifts to a constant section of $H_{1,\mathbb{R},\phi_A}$).

It turns out that in the situation of a Lagrangian polarized fibration, we can prove by an infinitesimal analysis, assuming furthermore that the dimension is ≤ 4, that a section of $\mathrm{Pic}^0(X^0/B^0)$ with a non dense set of torsion points is necessary locally as in Example 0.8, except for one "exotic" case (see Proposition 3.2). We can then conclude by global monodromy and transcendence arguments.

Using the local structure results for Lagrangian torus fibrations that we recall in Section 3.2, the infinitesimal analysis amounts to analyzing the $2n$-uples (a_i) appearing in the equation (see (41))

$$df = \sum_i a_i df_i, \tag{2}$$

where $f_1, \ldots, f_{2n}$ are $2n$ holomorphic functions on an open set U of $\mathbb{C}^n$, such that the $\mathrm{Re}\, f_i$ give local real coordinates, f is holomorphic, the a_i's are real analytic functions and satisfying the degeneracy condition that the map $a_. : U \to \mathbb{R}^{2n}$ is nowhere of maximal rank. This is a sort of complexified degenerate Monge-Ampère equation [14] (in fact, the real part of (2) is a degenerate Monge-Ampère equation in the coordinates $x_i = \mathrm{Re}\, f_i$) and the analysis is done in Section 3.3, the solutions being classified only when $n = 2$.

Let us explain one consequence of Theorem 0.2 which was our motivation for this work. Following the discovery made in [3] that a projective $K3$ surface S has a canonical 0-cycle o_S with the property that for any two divisors D, D' on S, the intersection $D \cdot D'$ is a multiple of o_S in $\mathrm{CH}_0(S)$, Beauville conjectured the following:

Conjecture 0.9. *(See [2]). Let X be a projective hyper-Kähler manifold. Then the cohomological cycle class restricted to the subalgebra of $\mathrm{CH}(X)_\mathbb{Q}$ generated by divisor classes is injective.*

This conjecture has been proved in [22] for varieties of the form $S^{[n]}$, where S is a $K3$ surface and $n \leq 2b_2(S)_{tr} + 4$, and in [8] for generalized Kummer varieties. It is also proved in [22] for Fano varieties of lines of cubic fourfolds, which are well-known to be irreducible hyper-Kähler fourfolds of $K3^{[2]}$ deformation type (see [1]). Finally, Conjecture 0.9 is proved by Riess [16] in the case of irreducible hyper-Kähler varieties of $K3^{[n]}$ or generalized Kummer deformation type admitting a Lagrangian fibration. Here the condition that X is a deformation of a punctual Hilbert scheme of a $K3$ surface guarantees, according to [20], that X satisfies the conjecture that any nef line bundle L on X which is isotropic for the Beauville-Bogomolov quadratic form is the pull-back of a $\mathbb{Q}$-line bundle on the basis B of a Lagrangian fibration on X. Note that, according to [4], the set of polynomial relations between degree 2 cohomology classes on an irreducible hyper-Kähler $2n$-fold X is generated as an ideal of $\mathrm{Sym}\, H^2(X,\mathbb{R})$ by the relations

$$\alpha^{n+1} = 0 \text{ in } H^{2n+2}(X,\mathbb{R}) \text{ if } q(\alpha) = 0.$$

This remains in fact true with $\mathbb{Q}$-coefficients if $b_2(X) \geq 5$ since then the quadric $q = 0$ has Zariski dense rational points. For the same reason, the cohomological relations between divisor classes $d \in \mathrm{NS}(X)_\mathbb{R} \subset H^2(X,\mathbb{R})$ are generated by the relations

$$d^{n+1} = 0 \text{ in } H^{2n+2}(X,\mathbb{R}) \quad \text{if } q(d) = 0, \tag{3}$$

and if $\rho(X) \geq 5$, the same statement holds with $\mathbb{Q}$-coefficients. If furthermore X admits a Lagrangian fibration $\phi : X \to B$, there is a nonzero class $l \in \mathrm{NS}(X)$ which satisfies $q(l) = 0$, namely $l = c_1(L)$, where L generate $\phi^*\mathrm{Pic}\, B$, so the quadric Q defined by q in $\mathrm{NS}(X)$ has a Zariski dense set of rational points and then the condition $\rho(X) \geq 5$ is not needed anymore to conclude that the cohomological relations between divisor classes $d \in \mathrm{NS}(X) \subset H^2(X, \mathbb{Q})$ are generated by the relations (3).

If X is any projective hyper-Kähler manifold, let $l \in H^2(X, \mathbb{R})$ be such that $q(l) = 0$. Observe now that by differentiation of (3) along the tangent space to Q at l, namely $l^{\perp q}$, one gets the relations

$$l^n h = 0 \text{ in } H^{2n+2}(X, \mathbb{R}) \text{ if } q(l) = 0 \text{ and } q(l, h) = 0. \tag{4}$$

One immediate consequence of Theorem 0.2 is the following result, proving that if l comes from a Lagrangian fibration and the dimension is 4, then (4) already holds in $\mathrm{CH}(X)_{\mathbb{Q}}$. Let $\phi : X \to B$ be a Lagrangian fibration on a projective irreducible hyper-Kähler fourfold and let L generate $\phi^*\mathrm{Pic}\, B$.

Theorem 0.10. *With the notation as above, let furthermore $D \in \mathrm{Pic}\, X$ satisfy the property that $L^2 \cdot D$ is cohomologous to 0 on X. Assume that a very general projective deformation of X preserving the Lagrangian fibration and the line bundle D satisfies Matsushita conjecture. (This is satisfied for instance if $b_2(X) \geq 8$ by [11].) Then $L^2 \cdot D = 0$ in $\mathrm{CH}^3(X)_{\mathbb{Q}}$.*

We refer to [12] for further applications of Theorem 0.10 to Conjecture 0.9. The paper is organized as follows. In the very short section 1, we will show how Theorem 0.10 follows from Theorem 0.2. We will prove Theorem 0.5 in Section 2. The proof of Theorem 0.2 will be given in Section 3 in the non-isotrivial case and in Section 4 in the isotrivial case.

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1 Proof of Theorem 0.10

We prove in this section Theorem 0.10 assuming Theorem 0.2. Let $\phi : X \to B$ be as in the introduction, with $\dim X = 4$ and let L generate $\phi^*\mathrm{Pic}\, B$ (see [17]). We want to prove that for any line bundle D on X such that $D \cdot L^2$ is cohomologous to 0 on X, then $D \cdot L^2$ is rationally equivalent to 0 on X modulo torsion. Let us introduce a universal family $\mathcal{X} \to T$ of deformations of X preserving the line bundles L, D and an ample line bundle on X. The very general fiber X_t of this family is projective and admits the deformed line bundles D_t and L_t. It also admits a Lagrangian fibration associated to L_t. The condition that $D_t \cdot L_t^2$ is rationally equivalent to 0 on X_t modulo torsion is satisfied on a countable union of closed algebraic subsets of T, so if we prove it is satisfied at the very general point $t \in T$, it will be also satisfied for any $t \in T$, hence for X. By assumption, the very general deformation (X_t, ϕ_t, L_t, D_t) satisfies Matsushita's conjecture, so we can apply Theorem 0.2. It thus suffices to prove the result when (X, ϕ, D) satisfies the conclusion of Theorem 0.2, which we assume from now on.

In the Chow ring $\mathrm{CH}(X)_{\mathbb{Q}}$, the fibers X_b, for $b \in B^0$ are all rationally equivalent (if B is smooth, then $B \cong \mathbb{P}^n$ by [10], so this is obvious; if B is not smooth, we refer to [12] for a proof of this statement). It follows that we have

$$L^2 = \mu X_b \text{ in } \mathrm{CH}^2(X)_{\mathbb{Q}}, \tag{5}$$

for some nonzero $\mu \in \mathbb{Z}$. Let $b \in B$ be a general point. The kernel of the map

$$[X_b]\cup = \frac{1}{\mu} c_1(L)^2 \cup : H^2(X, \mathbb{Q}) \to H^6(X, \mathbb{Q})$$

is equal to the kernel of the restriction map $H^2(X,\mathbb{Q}) \to H^2(X_b,\mathbb{Q})$. This is a general fact proved in [23]:

Lemma 1.1. *(See [23]) Let $j : Y \to W$ be a generically finite morphism, where Y and W are smooth projective varieties (in particular connected), and let $\dim W - \dim Y = k$, $\dim W = m$. Let $\alpha := j_*[Y]_{fund} \in H^{2k}(W,\mathbb{Q})$. Then the two $\mathbb{Q}$-vector subspaces*

$$K_1 := \mathrm{Ker}\,(j^* : H^2(W,\mathbb{Q}) \to H^2(Y,\mathbb{Q})),\;\; K_2 := \mathrm{Ker}\,(\alpha\cup : H^2(W,\mathbb{Q}) \to H^{2k+2}(W,\mathbb{Q}))$$

of $H^2(W,\mathbb{Q})$ are equal.

In our case, this also follows directly from Proposition 0.1. Indeed, the later is clearly included in the former, and in the other direction, if $[X_b] \cup D$ is cohomologous to 0 in X, $D_{|X_b}$ cannot be ample, hence it must have trivial first Chern class by Proposition 0.1. Let now $D \in \mathrm{Pic}\, X$ such that $D \cdot L^2$ is cohomologous to 0. Then $c_1(D) \cup [X_b] = 0$ by (5), and thus $c_1(D)_{|X_b} = 0$ in $H^2(X_b,\mathbb{Q})$, hence also in $H^2(X_b,\mathbb{Z})$. As we assumed that (X,ϕ,D) satisfies the conclusion of Theorem 0.2, there exist points in B^0 such that $D_{|X_b}$ is a torsion point in $\mathrm{Pic}^0(X_b)$, so that $D \cdot X_b$ is a torsion cycle in $\mathrm{CH}^3(X)$. This implies by (5) that $H \cdot L^2$ vanishes in $\mathrm{CH}^3(X)_{\mathbb{Q}}$.

2 Proof of Theorem 0.5

We will use the same notation $\phi : X^0 \to B$ for the restriction to B^0 of the elliptic fibration $\phi : X \to B$. The associated Jacobian fibration $J \to B^0$ is a complex manifold which is a fibration into complex tori, and its sheaf of holomorphic sections $\mathcal{J}$ is described in complex analytic terms as the quotient

$$\mathcal{J} = R^1\phi_*\mathcal{O}_{X^0}/R^1\phi_*\mathbb{Z} = R^1\phi_*\mathbb{C} \otimes \mathcal{O}_B/(\mathcal{H}^{1,0} \oplus R^1\phi_*\mathbb{Z}), \tag{6}$$

where $\mathcal{H}^{1,0} = R^0\phi_*\Omega_{X^0/B^0}$. Formula (6) describes J as a holomorphic torus fibration but as a $\mathcal{C}^\omega$ real torus fibration, the right formula is

$$\mathcal{J}_\mathbb{R} = \mathcal{H}^1_\mathbb{R}/R^1\phi_*\mathbb{Z},\;\; \mathcal{H}^1_\mathbb{R} := R^1\phi_*\mathbb{R} \otimes \mathcal{C}^\omega_{B^0,\mathbb{R}} \tag{7}$$

which uses the natural fiberwise isomorphisms

$$H^1(X_b,\mathbb{R}) \cong H^1(X_b,\mathbb{C})/H^{1,0}(X_b) \cong H^1(X_b,\mathcal{O}_{X_b}) \tag{8}$$

globalizing into the $\mathcal{C}^\omega_{B^0,\mathbb{R}}$ sheaf isomorphism

$$\mathcal{H}^1_\mathbb{R} \cong R^1\phi_*\mathcal{O}_{X^0} \otimes_{\mathcal{O}_B} \mathcal{C}^\omega_{B^0,\mathbb{R}} \cong \mathcal{H}^{0,1} \otimes_{\mathcal{O}_B} \mathcal{C}^\omega_{B^0,\mathbb{R}}, \tag{9}$$

where

$$\mathcal{H}^{0,1} := R^1\phi_*\mathcal{O}_{X^0} = (R^1\phi_*\mathbb{C} \otimes_\mathbb{C} \mathcal{O}_{B^0})/\mathcal{H}^{1,0}.$$

Trivializing the locally constant sheaves $R^1\phi_*\mathbb{Z}$, $R^1\phi_*\mathbb{R}$ on simply connected open sets $U \subset B$, (7) is the counterpart at the level of sheaves of sections of a local real analytic trivialization

$$J_U = U \times (H^1(X_0,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}), \tag{10}$$

where 0 is a given point of U. Let now ν be a holomorphic section of J over U. Using the trivializations (10), the section ν gives a differentiable (in fact real analytic) map

$$f_\nu : U \to H^1(X_0,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} = J(X_0),$$

and we will use the following easy criterion:

Proposition 2.1. *(i) For a section ν of a torus fibration with local associated map f_ν as above, the points x of U where $\nu(x)$ is of torsion are dense in U for the usual topology if the differential df_ν is surjective at some point $b \in U$.*

(ii) In the situation of an elliptic fibration over a 1-dimensional base, the differential df_ν is surjective if and only if it is nonzero.

Proof. (i) Observe first that f_ν being real analytic, if its differential is surjective at some point $b \in U$, it is surjective on a dense set of points of U, hence it suffices to prove density near a point b where the differential $df_{\nu,b}$ is surjective. If the differential $df_{\nu,b}$ is surjective, f_ν is an open map in a neighborhood U_b of b. As the torsion points are dense in $J(X_{b_0})$, their preimages under f_ν are then dense in U_b.

(ii) This is a consequence of the following fact:

Lemma 2.2. *The kernel of the differential $df_{\nu,b} : T_{U,b} \to T_{J(X_b)}$ is a complex vector subspace of $T_{U,b}$.*

Proof. Let us prove more generally that the fibers of f_ν are analytic subschemes of U. Let $\alpha \in J(X_0) = H^1(X_0, \mathbb{R})/H^1(X_0, \mathbb{Z})$ and let $\tilde{\alpha} \in H^1(X_0, \mathbb{R})$ be a lifting of α (such a lift is defined up to the addition of an element β of $H^1(X_0, \mathbb{R})$). The class $\tilde{\alpha}$ extends as a constant section also denoted $\tilde{\alpha}$ of the sheaf $R^1\phi_*\mathbb{R}$ on U, which induces a holomorphic section $\tilde{\alpha}^{0,1}$ of the sheaf $\mathcal{H}^{0,1} = R^1\phi^0_*\mathcal{O}_{X^0}$ on U. On the other hand, the holomorphic section ν of J lifts to a holomorphic section η of $\mathcal{H}^{0,1}$ and it is clear that

$$f_\nu^{-1}(\alpha) = \{b \in U,\ \tilde{\alpha}_b^{0,1} = \eta_b \text{ in } \mathcal{H}^{0,1}(X_b)/H^1(X_b, \mathbb{Z})\}.$$

It follows that $f_\nu^{-1}(\alpha)$ is the countable locally finite union of the closed analytic subsets defined as zero sets of the holomorphic sections

$$\tilde{\alpha}^{0,1} - \eta - \beta^{0,1} \in \Gamma(U, \mathcal{H}^{0,1})$$

of the bundle $\mathcal{H}^{0,1}$, over all sections β of $H^1_{\mathbb{Z}}$ on U. □

In the situation of (ii), the base U and the fiber $J(X_0)$ are both of real dimension 2 and Lemma 2.2 implies that the kernel of $df_{\nu,b}$ is of real dimension 0 or 2. So either $df_{\nu,b} = 0$ or $df_{\nu,b}$ is surjective. □

Proof of Theorem 0.5. Let $\phi : X^0 \to B^0$ be our elliptic fibration and ν a section of $J(X^0/B)$. Assume that the points of B where ν is a torsion point are not dense for the usual topology of B. Then by Proposition 2.1, it follows that df_ν vanishes everywhere on any open set U of B^0 where it is defined. Coming back to the sheaf theoretic language, this means equivalently that ν, seen as a section of $\mathcal{H}^1_{\mathbb{R}}/H^1_{\mathbb{Z}}$ via the isomorphism (7), is locally constant, or that our section $\nu \in \Gamma(B^0, \mathcal{H}^{0,1}/H^1_{\mathbb{Z}})$ comes from a locally constant section $\tilde{\nu} \in \Gamma(B^0, H^1_{\mathbb{R}}/H^1_{\mathbb{Z}})$. Note that, by assumption, $\tilde{\nu}$ is not of torsion and thus, fixing a base point $0 \in B$, corresponds to an element $\alpha_0 \in H^1(X_0, \mathbb{R})$ which is not rational but has the property that for any $\gamma \in \pi_1(B^0, 0)$,

$$\rho(\gamma)(\alpha_0) - \alpha_0 \in H^1(X_0, \mathbb{Z}), \tag{11}$$

where $\rho : \pi_1(B^0, 0) \to \operatorname{Aut} H^1(X_0, \mathbb{Z})$ denotes the monodromy representation of the smooth fibration $\phi : X^0 \to B^0$. We use the following easy lemma.

Lemma 2.3. *Let $\rho : \Gamma \to \operatorname{Aut} V_{\mathbb{Q}}$ be a finite dimensional rational representation of a group Γ. Then if the invariant space $V_{\mathbb{Q}}^{inv}$ is trivial, the set*

$$\{v \in V_{\mathbb{R}},\ \rho(\gamma)(v) - v \in V_{\mathbb{Q}} \text{ for any } \gamma \in \Gamma,\ \}$$

is equal to $V_{\mathbb{Q}}$.

As the monodromy representation is rational, this lemma tells us that the set of classes $\alpha \in H^1(X_0, \mathbb{R})$ satisfying property (11) contains a nonrational class if and only if the set

$$H^1(X_0, \mathbb{Q})^{inv} := \{\alpha \in H^1(X_0, \mathbb{Q}),\ \rho(\gamma)(\alpha) - \alpha = 0,\ \forall \gamma \in \pi_1(B^0, 0)\}$$

of monodromy invariant elements is nontrivial. By Deligne's invariant cycles theorem [5], it then follows from the existence of α_0 that the restriction map

$$H^1(X, \mathbb{Q}) \to H^1(X_0, \mathbb{Q})$$

is nontrivial, hence surjective since this a morphism of Hodge structures of weight 1 and the right hand side has dimension 2. The conclusion that X is rationally isogenous to $J(X_0) \times B$ is then immediate since X is rationally isogenous to a projective completion of the Jacobian fibration $J(X^0/B^0)$ and the later is isogenous to $J(X_0) \times B^0$ if the restriction map $H^1(X, \mathbb{Q}) \to H^1(X_0, \mathbb{Q})$ is surjective. □

3 Proof of Theorem 0.2 in the non-isotrivial case

3.1 First reduction

Let $\phi : X \to B$ be a Lagrangian fibration, where X is a projective irreducible hyper-Kähler manifold of dimension $2n$, and let $B^0 \subset X^0$ be the Zariski open set of regular values of ϕ. Denote by σ the holomorphic 2-form on X and by ω the first Chern class of an ample line bundle H on X. Let α be the type of the polarization $H_{|X_b}$. We assume in this section that the first alternative in Matsushita's conjecture is satisfied, that is, the period map $B^0 \to \mathcal{A}_{n,\alpha}$ is generically of maximal rank $n = \dim B$ and we are going to prove Theorem 0.2 in this case.

Over B^0, the fibration $\phi : X^0 \to B^0$ is a fibration into abelian varieties or rather a torsor on the corresponding Albanese fibration $\mathrm{Alb}(X^0/B^0)$, which admits a multisection $Z \to B^0$ of degree d. Using these data, $X^0 \to B^0$ is isogenous to the associated Albanese fibration $\mathrm{Alb}(X^0/B^0)$ via the map

$$i_Z : X^0 \to \mathrm{Alb}(X^0/B^0)$$

defined by $i_Z(u) = \mathrm{alb}_{X_b}(d\{u\} - Z_b),\ b = \phi(u)$. Here the notation $\{u\}$ is used to denote u as a 0-cycle in the fiber X_b (in order to avoid the confusion between addition of cycles and addition of points), so $d\{u\} - Z_b$ should be thought as a 0-cycle of degree 0 on X_b, giving rise to an element $\mathrm{alb}_{X_b}(d\{u\} - Z_b) \in \mathrm{Alb}(X_b)$. Next, using the polarisation H, the abelian fibration $\mathrm{Alb}(X^0/B^0)$ and its dual fibration $\mathrm{Pic}^0(X^0/B^0)$ are isogenous, the isogeny

$$i_H : \mathrm{Alb}(X^0/B^0) \to \mathrm{Pic}^0(X^0/B^0) \tag{12}$$

being given by

$$i_H(b, v_b) = t_{v_b}^*(H_{|X_b}) - H_{|X_b}$$

for any $b \in B^0$, $v_b \in \mathrm{Alb}(X_b)$. Here the translation t_{v_b} acts on X_b. The isogeny (12) also gives a dual isogeny

$$i_H^* : \mathrm{Pic}^0(X^0/B^0) \to \mathrm{Alb}(X^0/B^0). \tag{13}$$

An element ν_M of the Mordell-Weil group of X over B, that is a section of $\mathrm{Pic}^0(X^0/B^0)$, thus gives a section $\nu'_M = i_H^* \circ \nu_M$ of $\mathrm{Alb}(X^0/B^0)$ and the torsion points of ν_M and ν'_M coincide. From now on we work with the section ν'_M. Note now that the $(2,0)$-form σ on X is the pull-back of a $(2,0)$-form σ_A on $\mathrm{Alb}(X^0/B^0)$ via the rational map $i_Z : X \dashrightarrow \mathrm{Alb}(X^0/B^0)$. This indeed follows easily from Mumford's theorem [13], since for any two points x, y of X^0 such that $i_Z(x) = i_Z(y)$ the difference $\mathrm{alb}_{X_b}(\{x\} - \{y\})$ is a torsion point in $\mathrm{Alb}\,(X_b)$, $b = \phi(x) = \phi(y)$, hence the cycle $\{x\} - \{y\}$ is a torsion cycle in $\mathrm{CH}_0(X_b)$ and a fortiori in $\mathrm{CH}_0(X)$ (in fact it has to vanish in $\mathrm{CH}_0(X)$ since the later group has no torsion).

By the above arguments, we reduced the situation to a polarized Lagrangian torus fibration $\psi : A \to B^0$ over B^0 with a section ν (that is, we do not have to care anymore to the fact that we have a nontrivial torsor).

Our analysis of the properties of the holomorphic section ν of the torus fibration will be infinitesimal hence local, but we want first to add one restriction which is global: Let $\phi : X \to B$ be a projective irreducible hyper-Kähler manifold with a Lagrangian fibration and let $\nu : B^0 \to A = \mathrm{Alb}(X^0/B^0)$ be the section associated as above to a section ν_M of $\mathrm{Pic}(X^0/B^0)$, where M is a line bundle on X which is topologically trivial on the fibers.

Lemma 3.1. *The section ν is Lagrangian for the holomorphic 2-form σ_A.*

Proof. Let $\widetilde{B}$ be a desingularization of B. Then $H^0(\widetilde{B}, \Omega^2_{\widetilde{B}}) = 0$ since a nonzero holomorphic 2-form on $\widetilde{B}$ would provide by pull-back via the rational map $\phi : X \dashrightarrow \widetilde{B}$ a nonzero holomorphic form on X which is not proportional to σ. So we just have to prove that $\nu^*(\sigma_A)$ (which is a priori only defined as a holomorphic 2-form on B^0) extends to a holomorphic 2-form on $\widetilde{B}$. However, we can provide another construction of $\nu^*(\sigma_A)$ which will make clear that it extends to $\widetilde{B}$: the subvariety $\Gamma := i_Z^{-1}(\mathrm{Im}\,\nu)$ is algebraic in X^0 and it is a multisection of ϕ over B^0. Its Zariski closure in $\widetilde{B} \times X$ will provide a correspondence $\overline{\Gamma} \subset \widetilde{B} \times X$. Then as $\sigma = i_Z^*\sigma_A$ on X^0, we get that

$$\overline{\Gamma}^*\sigma = N\nu^*(\sigma_A) \tag{14}$$

on B^0, where N is the degree of i_Z. This gives the desired extension of $\nu^*(\sigma_A)$. □

The local part of the proof of Theorem 0.2 is the following result:

Proposition 3.2. *Let $(\phi : A \to U,\ \sigma_A \in H^0(\Omega^2_A), \omega)$ be a polarized holomorphic Lagrangian torus fibration over a connected complex manifold U and ν be a holomorphic section of A such that $\nu^*\sigma_A = 0$. Assume the period map is generically of maximal rank on U. Then if the torsion points of ν are not dense in B^0 for the usual topology, one of the following three situations holds:*

(0') The section $\nu \in \Gamma(U, \mathcal{H}^{0,1}/H^1_{\mathbb{Z}})$ admits a locally constant complex lift $\tilde{\nu} \in H^1_{\mathbb{C}}$ (which is unique up to a section of $H^1_{\mathbb{Z}}$ under our assumptions).

(1) The image of the period map $\mathcal{P} : \widetilde{U} \to Grass(2,4)$, where $\widetilde{U}$ is the universal cover of U is semi-algebraic (that is, an open set in an algebraic subvariety) and has an explicit description starting from a monodromy invariant identification of $\widetilde{U}$ with an open set in a linear projection of the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ in $\mathbb{P}^4$.

(2) The local system

$$H_{1,\mathbb{R},A} := (R^1\phi_*\mathbb{R})^*$$

admits a nontrivial local subsystem L which underlies a real subvariation of Hodge structures. Furthermore, writing the real lift (see Section 1) of the section ν as a sum $\tilde{\nu}_L + \tilde{\nu}_{L^\perp}$, where $L^\perp$ is the orthogonal variation of Hodge structures with respect to ω, then $\tilde{\nu}_L$ is a locally constant section of L.

Note that the proof of the local version of Proposition 3.2 (Proposition 3.8) will explain better how the isomorphism in (2) is constructed using the periods. The proof of Theorem 0.2 will consist in excluding by global arguments possibilities (1), (2), (3) above, when our Lagrangian torus fibration comes from a hyper-Kähler fourfold.

3.2 Local structure results

In order to prepare the proof of Proposition 3.2, we recall in this section, following [6], the local data determining a holomorphic Lagrangian polarized torus fibration. We will also add the data of our normal function (that is, holomorphic section of our torus fibration) satisfying the condition of Lemma 3.1. The holomorphic torus fibration $\phi : A \to U$ is determined by a weight 1 (or rather -1) variation of Hodge structure, that is the data of a local system $H_{1,\mathbb{Z}} = (R^1\phi_*\mathbb{Z})^*$ and a holomorphic subbundle

$$\mathcal{H}_{1,0} \subset \mathcal{H}_1 := H_{1,\mathbb{Z}} \otimes \mathcal{O}_U$$

determining a weight 1 Hodge structure at any point of U. The sheaf $\mathcal{A}$ of holomorphic sections of A identifies to

$$\mathcal{H}_{0,1}/H_{1,\mathbb{Z}},\ \mathcal{H}_{0,1} := \mathcal{H}_1/\mathcal{H}_{1,0}. \tag{15}$$

The holomorphic 2-form σ_A on A for which the fibration $\phi : A \to U$ is a Lagrangian fibration provides an isomorphism of vector bundles

$$T_U \cong R^0\phi_*\Omega_{A/U}, \tag{16}$$

which by dualization provides an isomorphism (which is canonical, given the choice of σ_A)

$$\mathcal{H}_{0,1} \cong \Omega_U. \tag{17}$$

Using the isomorphism of (17), the surjective map of holomorphic vector bundles

$$\mathcal{H}_1 \to \mathcal{H}_{0,1}$$

is thus given, on any simply connected open set U' where the flat vector bundle $\mathcal{H}_1$ is trivialized, by the evaluation morphism of $2n$ holomorphic 1-forms α_i on U', which must have the property that their real parts $\mathrm{Re}\,\alpha_i$ form a basis of $\Omega_{U',\mathbb{R}}$ at any point. We have (see [6]):

Lemma 3.3. *The forms α_i are closed on U', hence we have, up to shrinking our local open set U' if necessary, $\alpha_i = df_i$, where the f_i's are holomorphic and defined up to a constant.*

Proof. The proof will use a different description of the forms α_i. For any locally constant section γ of $H_{1,\mathbb{C}}$, we get using the closed 2-form σ_A a closed 1-form $\phi_*(\langle\gamma, \sigma_A\rangle)$ on U' (when γ is a section of $H_{1,\mathbb{Z}}$, γ is a combination of classes γ_i of oriented circle bundles $\Gamma_i \subset A$, and $\phi_*(\langle\gamma_i, \sigma_A\rangle)$ is defined as $(\phi_{|\Gamma_i})_*(\sigma_{A|\Gamma_i})$). We conclude observing that for any locally constant section γ of $H_{1,\mathbb{C}}$ on an open set U' of U, with induced section $\gamma_{0,1}$ of $\mathcal{H}_{0,1}$, providing a holomorphic form α_γ via the isomorphism (17), we have

$$\alpha_\gamma = \phi_*(\langle\gamma, \sigma_A\rangle) \text{ in } \Gamma(U', \Omega_{U'}),$$

thus proving that the forms α_i are closed. □

If we now choose the α_i to form a basis of $H_{1,\mathbb{R}}$, the corresponding $2n$ holomorphic 1-forms on U' have the properties that at any point $b \in U'$, the forms $\alpha_{i,b}$ are independent over $\mathbb{R}$. Another way to say it is that the functions $\mathrm{Re}\,f_i$ give local real coordinates on U'.

Remark 3.4. By definition, the functions $\mathrm{Re}\,f_i$, which are defined up to a constant and depend only on the choice of σ_A and of local basis α_i of $H_{1,\mathbb{R}}$, provide a local real analytic identification

$$U \cong H^1(A_{b_0}, \mathbb{R}). \tag{18}$$

Globally, they provide an affine flat structure on U.

The last piece of information we need concerning the torus fibration $\phi : A \to U$ is the polarization ω on the fibers. It is given by a monodromy invariant skew-symmetric pairing $\langle\, ,\, \rangle$ on $H_{1,\mathbb{R}}$ (we do not need here the fact that the polarization is integral), which has to polarize the Hodge structure at any point, so that for any $b \in U$

$$\begin{gathered} \langle\, ,\, \rangle = 0 \text{ on } \mathcal{H}_{1,0,b} \subset H_{1,\mathbb{C},b} \\ i\langle\alpha, \overline{\alpha}\rangle > 0, \quad \forall 0 \neq \alpha \in\ \mathcal{H}_{1,0,b}. \end{gathered} \tag{19}$$

This is as well a nondegenerate skew-symmetric form $\omega \in \bigwedge^2 H^1(A_{b_0}, \mathbb{R})$ which produces via the diffeomorphism (18) a closed 2-form ω^* on the open set U of the trivialization introduced above. The 2-form ω^* is constant in the affine coordinates introduced above.

Lemma 3.5. *The Hodge-Riemann bilinear relations (19) satisfied by* $\langle\, ,\, \rangle$ *and the Hodge structure on* $H_1(A_b, \mathbb{R})$ *at any point* $b \in U$ *are equivalent to the fact that* ω^* *is a Kähler form on* U*.*

Proof. The statement is local. Recall that by construction the holomorphic functions f_i on U' are chosen in such a way that the evaluation map

$$\begin{aligned} ev : \mathbb{C}^{2n} \otimes \mathcal{O}_{U'} &\to \Omega_{U'} \\ e_i &\mapsto df_i \end{aligned} \tag{20}$$

identifies to the variation of Hodge structure of weight 1 of our torus fibration, determined by the quotient map

$$H_{1,\mathbb{C}} \otimes \mathcal{O}_U \to \mathcal{H}_{0,1} \to 0. \tag{21}$$

Assume our chosen basis α_i of $H_1(A_b, \mathbb{R})$ is a symplectic basis in the sense that the 2-form ω is equal to $\sum_{1 \leq i \leq n} \alpha_i^* \wedge \alpha_{n+i}^*$. The 2-form ω^* on U' is then the form $\sum_{1 \leq i \leq n} dx_i \wedge dx_{i+n}$, where $x_i := \operatorname{Re} f_i$. The df_i's, for $1 \leq i \leq n$, have to generate $\Omega_{U',b}$ for any $b \in U'$, or equivalently the α_i's, for $1 \leq i \leq n$ have to generate $\mathcal{H}_{0,1,b}$. Indeed, there exists otherwise a nonzero element $u \in \mathcal{H}_{1,0,b}$ which is a complex linear combination of the α_i for $1 \leq i \leq n$. Then $\overline{u}$ is also a complex linear combination of the α_i for $1 \leq i \leq n$, and thus $\langle u, \overline{u} \rangle = 0$, which contradicts the Hodge-Riemann bilinear relations (19). This fact allows us to write for $i > n$

$$df_{i+n} = \sum_{1 \leq j \leq n} g_{ji} df_j \text{ in } \Gamma(U', \Omega_{U'}), \tag{22}$$

where the matrix g_{ji} is holomorphic on U'. A basis of

$$\mathcal{H}_{1,0} = \operatorname{Ker}(H_{1,\mathbb{R}} \otimes \mathcal{O}_{U'} \to \mathcal{H}_{0,1}$$

is then given by the sections

$$\beta_i := \alpha_{i+n} - \sum_{1 \leq j \leq n} g_{ji} \alpha_j,\ i \leq n.$$

The first Hodge-Riemann bilinear relations say that $\langle \beta_i, \beta_l \rangle = 0$ for any $i,\ l \leq n$, which says equivalently that the matrix g_{il} is symmetric. The second Hodge-Riemann bilinear relations say similarly that the matrix $\operatorname{Im} g_{ij}$ is positive definite. Let us now compute ω^*: Recalling that $dx_i = \operatorname{Re} df_i$, we have $dx_i = \frac{1}{2}(df_i + d\overline{f}_i)$ and

$$\omega^* = \frac{1}{4} \sum_{1 \leq i \leq n} (df_i + d\overline{f}_i) \wedge (df_{i+n} + d\overline{f}_{i+n}). \tag{23}$$

Using (22), formula (23) becomes

$$\omega^* = \frac{1}{4} \sum_{1 \leq i \leq n, 1 \leq j \leq n} (df_i + d\overline{f}_i) \wedge (g_{ji} df_j + \overline{g_{ij}} d\overline{f}_j). \tag{24}$$

The symmetry of the matrix g_{ij} then shows that in the develoment of (24), the terms involving $df_i \wedge df_j$ or $d\overline{f}_i \wedge d\overline{f}_j$ are 0, which proves that the form ω^* is of type $(1,1)$. Finally, the fact that $\operatorname{Im} g_{ij}$ is positive definite translates into the condition that ω^* is a positive $(1,1)$-form on U'. □

Coming back to our Lagrangian torus fibration $A \to U$, we now add to the data described above the normal function $\nu : U \to A$, which satisfies the property that $\nu^* \sigma_A = 0$. This normal function $\nu \in \Gamma(U, \mathcal{A}) = \Gamma(U, \mathcal{H}_{0,1}/H_{1,\mathbb{Z}})$ lifts locally to a section $\tilde{\nu}$ of $\mathcal{H}_{0,1} \cong \Omega_U$, hence provides locally a holomorphic 1-form α on U.

Lemma 3.6. *The form α is closed.*

Proof. It is a general fact of the theory of integrable systems (see [6]) that the isomorphism (17) is also constructed in such a way that the pull-back $\tilde{\sigma}_A$ of the holomorphic 2-form σ_A to the total space of the bundle $\mathcal{H}_{0,1}$ identifies with the canonical 2-form σ_{can} on the cotangent bundle Ω_U, which has the property that for any holomorphic 1-form β on U, seen as a section of Ω_U, one has

$$d\beta = \beta^*(\sigma_{can}). \tag{25}$$

As the section $\tilde{\nu}$ corresponds via this isomorphism to the 1-form α, the assumption that $\nu^*(\sigma_A) = 0$ says equivalently that

$$0 = \tilde{\nu}^*(\tilde{\sigma}_A) = \alpha^*(\sigma_{can}) = d\alpha.$$

□

The closed holomorphic 1-form α thus provides us on simply connected open sets U' of U with a holomorphic function f such that $df = \alpha$.

Remark 3.7. From now on, the holomorphic 2-form on our Lagrangian fibration will disappear. It is hidden in the fact that we work with the cotangent bundle of the base, which has a canonical holomorphic 2-form.

3.3 Proof of Proposition 3.2

The proof of Proposition 3.2 will be based on the following Proposition 3.8 that we first explain. We are in the local setting where the holomorphic functions f_i, f constructed in the previous section exist. Let thus V be a complex manifold, and let $f_1, \ldots, f_{2n}$ be holomorphic functions on V such that the functions $x_i := \mathrm{Re}\, f_i$ provide real coordinates on V, giving rise to a real variation of Hodge structures

$$\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathcal{O}_V \stackrel{ev}{\to} \Omega_V \to 0, \qquad e_i \mapsto df_i \tag{26}$$

with underlying local system $\mathbb{R}^{2n} = T_{V,\mathbb{R}}$ and Hodge bundles $\mathcal{H}_{1,0} = \mathrm{Ker}\, ev$, $\mathcal{H}_{0,1} = \Omega_V$. Assume the variation of Hodge structure is polarized by the standard symplectic form ω on $\mathbb{R}^{2n}$. Let f be a holomorphic function on V; as the df_i's are independent over $\mathbb{R}$ in $\Omega_{V,b}$ at any point $b \in V$, we can write

$$df = \sum_i a_i df_i, \tag{27}$$

where the a_i's are real $\mathcal{C}^\omega$ functions on V, uniquely determined by (27).

Proposition 3.8. *In the above setting, assume furthermore that $n = 2$ and the map $T_V \to Hom\,(\mathcal{H}_{1,0}, \mathcal{H}_{0,1})$ describing the infinitesimal variation of Hodge structure is injective. Let V^0 be a connected open subset of V where the map*

$$a. := (a_1, \ldots, a_4) : V \to \mathbb{R}^4 \tag{28}$$

has constant rank r. Let $\Sigma := \mathrm{Im}\, a.$, which is a submanifold of $\mathbb{R}^4$. Then r is even, $r = 4-2k$ and if r is not maximal, that is $k > 0$, either :

(0) $k = 2$ and then df is a combination of the df_i with real constant coefficients, or;

(0') df is a combination of the df_i with complex constant coefficients, or;

(1) The image of V^0 in $\mathbb{C}^5$ via the map $(f, f_1, \ldots, f_4)$ is an open set of a complete embedding of the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, or;

(2) The following three properties hold locally on V^0, modulo the action of $\mathrm{Sp}(2n, \mathbb{R})$ *acting on coordinates x_i:*

(i) With respect to the real affine structure on V given by the coordinates $x_i = \operatorname{Re} f_i$, the fibers F_σ are open sets in translates of a fixed $\mathbb{R}^2 \subset \mathbb{R}^4$.

(ii) The constant system T given by this $\mathbb{R}^2 \subset \mathbb{R}^4$ is a subvariation of Hodge structure, and its orthogonal complement is a subvariation of Hodge structure which is constant along the fibers of $a.$. In this case, Σ has a holomorphic structure and $a. : V \to \Sigma$ is holomorphic.

(iii) In the decomposition $\mathbb{R}^4 = T \oplus T^\perp$, the differential df is a sum $df = \eta_1 + \eta_2$, where η_1 is constant, and η_2 is holomorphic and pulled-back from Σ.

Remark 3.9. In the coordinates x_i above, the subspace T of $\mathbb{R}^4$ is equal to the tangent space to the fibers of the map $a.$ at any point of V^0.

Remark 3.10. Thinking in terms of the real variation of Hodge structures (26) via the construction described in Subsection 3.2, the situation (2) can be equivalently rephrased in the following form: there is a splitting

$$H = \mathbb{R}^4 = L' \oplus L'',\ L' = T,\ L'' = T^\perp$$

which is orthogonal with respect to ω and is also a splitting as real variations of Hodge structures, such that the section $\tilde{\nu} = df$ of $\Omega_V = \mathcal{H}_{0,1}$ is a sum

$$\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$$

where ν_1 is the projection to $\mathcal{L}'_{0,1}$ of a real *constant* section of L'. This is exactly the situation described in Example 0.8.

The proof of Proposition 3.8 will use a sequence of lemmas which hold in the general case where the dimension is $2n$. We have a complex manifold V^0 of dimension n equipped with holomorphic functions $f_1, \ldots, f_{2n}$ whose differentials generate $\Omega_{V^0,u}$ over $\mathbb{R}$ at any point $u \in V^0$. Denoting $x_i := \operatorname{Re} f_i$, the x_i's thus provide locally real coordinates on V^0 and we assume that the standard symplectic form on $\mathbb{R}^{2n}$ is Kähler on V^0.

Furthermore we are given a holomorphic function f on V^0, and we write the equality

$$df = \sum_i a_i df_i \tag{29}$$

of $\mathcal{C}^\omega$ forms of type $(1,0)$ on V^0, where the a_i's are real analytic.

Note that (29) is obviously equivalent, since the a_i's are real and the considered forms are of type $(1,0)$, to the equality of real 1-forms:

$$d(\operatorname{Re} f) = \sum_i a_i dx_i. \tag{30}$$

We assume that the map $a. = (a_1, \ldots, a_{2n}) : V^0 \to \mathbb{R}^{2n}$ is of constant rank $2n - 2k$ (the rank is even by Lemma 2.2).

Lemma 3.11. *The fibers of $a.$ are affine in the coordinates x_i.*

Proof. As the map $a.$ has locally constant rank near $b \in V^0$, it factors through a submersion $V^0 \to \Sigma$ and an immersion $\Sigma \hookrightarrow \mathbb{R}^{2n}$, where Σ is real manifold of dimension $2n - 2k$. Equation (30) says equivalently that for fixed $\sigma = (\lambda_1, \ldots, \lambda_{2n}) \in \Sigma$, the function $\operatorname{Re} f - \sum_i \lambda_i x_i$ has vanishing differential on V^0 at any point of the fiber $V^0_\sigma = a.^{-1}(\lambda_1, \ldots, \lambda_{2n})$. The function $\operatorname{Re} f - \sum_i a_i x_i$ on V^0 is thus the pull-back $a.^*\psi$ of a function ψ on Σ, that is

$$\operatorname{Re} f - \sum_i a_i x_i - a.^*\psi = 0 \tag{31}$$

on V^0. If $u = (u_1, \ldots, u_{2n}) \in T_{\Sigma,\sigma}$, we get for any $b \in V^0_\sigma$, by differentiating (31) along any $v \in T_{V^0,b}$ such that $a._*(v) = u$ and applying (30)

$$0 = -\sum_i u_i x_i - d\psi(u),$$

which says that the function $\sum_i u_i x_i$ is constant along the fiber V^0_σ. This provides $2n-2k$ real equations affine in the x_i's vanishing along V^0_σ and they define V^0_σ in V^0 since V^0_σ is a real submanifold of V of codimension $2n-2k$. □

The following lemma was partially proved in the proof of Lemma 2.2.

Lemma 3.12. *The fiber V^0_σ is a complex submanifold of V^0, along which the functions $\sum_i u_i f_i$ are constant, for any $u. \in T_{\Sigma,\sigma}$.*

Proof. The fact that the functions $\sum_i u_i f_i$ are constant along V^0_σ for any $u. \in T_{\Sigma,\sigma}$ is proved exactly as the previous lemma, replacing (30) by (29). This implies that V^0_σ is a complex submanifold of V^0 since the real parts of these functions define V^0_σ as a real submanifold of V^0 by Lemma 3.11. □

The point $\sigma \in \Sigma$ being fixed, we now choose the affine coordinates x_i in such a way that
1) The fiber V^0_σ is defined by the equations $x_i = cst$ for $i > 2k$.
2) $dx_1, \ldots, dx_{2k}$ and $dx_{2k+1}, \ldots, dx_{2n}$ generate two orthogonal subspaces of $\Omega_{V^0,\mathbb{R}}$ for the standard symplectic form ω^*. This is possible since the form ω^*, being Kähler, remains nondegenerate after restriction to V^0_σ.

Lemma 3.13. *Along V^0_σ, one has*

$$df_j = \sum_{l>2k} b_{lj} dx_l \text{ for } j > 2k, \tag{32}$$

for C^∞ complex valued functions b_{lj}, $l > 2k$, $j > 2k$, and the rank of the evaluation map

$$ev_{2k+1,\ldots,2n} : \mathbb{C}^{2n-2k} \otimes \mathcal{O}_{V^0_\sigma} \to \Omega_{V^0|V^0_\sigma},\ e_i \mapsto df_i,\ i = 2k+1, \ldots, 2n$$

is equal to $n-k$. Similarly

$$df_j = \sum_{l\leq 2k} b_{lj} dx_l \text{ for } j \leq 2k. \tag{33}$$

for C^∞ complex valued functions b_{lj}, $l \leq 2k$, $j \leq 2k$, and the rank of the evaluation map $ev_{1,\ldots,2k} : \mathbb{C}^{2k} \otimes \mathcal{O}_{V^0_\sigma} \to \Omega_{V^0|V^0_\sigma}$ is equal to k.

Remark 3.14. Using the construction described in Section 3.2, we can restate this decomposition saying that, along the fiber V^0_σ, the real variation of Hodge structure given by the evaluation map of the df_i's splits as the orthogonal sum of two real variations of Hodge structure.

Proof of Lemma 3.13. As V^0_σ is defined by the equations $x_i = cst$ for $i > 2k$, and the f_i's for $i > 2k$ are constant along V^0_σ, the df_i's for $i > 2k$ must be combinations of the dx_i for $i > 2k$, which proves (32). The fact that the evaluation map $ev_{2k+1,\ldots,2n}$ has rank $n-k$ is obvious since its image is the conormal bundle of the complex submanifold V^0_σ. We use now the fact that ω^* is Kähler on U. As the real subspace generated by the dx_i's for $i > 2k$ is a complex subspace of $\Omega_{V^0,\mathbb{R}}$ at any point of V^0_σ, so is its orthogonal complement, which is generated by the dx_i's for $i \leq 2k$. The $\mathbb{R}$-linear map

$$\Omega^{1,0}_{V^0} \to \Omega_{V^0,\mathbb{R}},\ \eta \mapsto \operatorname{Re}\eta$$

is an isomorphism of real vector bundles which has for inverse the map $\eta \mapsto \eta - iI(\eta)$ where I is the operator of complex structure acting on the bundle $\Omega_{V^0,\mathbb{R}}$. As df_i is $\mathbb{C}$-linear with $\operatorname{Re} df_i = dx_i$, and the space generated by dx_i, $i \leq 2k$ is a complex subspace of $\Omega_{V^0,\mathbb{R}}$ at any point of V^0_σ, it is stable under I, hence we conclude that for $j \leq 2k$, $df_j = dx_j - Idx_j$ is a combination of the dx_i's for $i \leq 2k$, which establishes (33). The fact that the evaluation map $ev_{1,\ldots,2k}$ has rank k follows from the fact that the restriction map $\Omega_{V^0,\mathbb{C}|V^0_\sigma} \to \Omega_{V^0_\sigma,\mathbb{C}}$ is injective on the space generated by the dx_i's for $i \leq 2k$, and that the evaluation map $ev_{1,\ldots,2k}$ composed with the restriction map $\Omega_{V^0|V^0_\sigma} \to \Omega_{V^0_\sigma}$ is obviously of rank $k = \operatorname{rank}\Omega_{V^0_\sigma}$. □

We now assume $n = 2$ and prove Proposition 3.8.

Proof of Proposition 3.8. We can assume $k = 1$. By Lemma 3.12, the fibers F_σ of the map $a.$ are complex submanifolds of dimension 1 of V^0 which satisfy 2 independent holomorphic equations of the form

$$\sum_i u_i f_i = cst,$$

where $u. \in T_{\Sigma,\sigma}$ and the constant depends on σ. We now observe that if $\sigma = (\lambda_i) \in \mathbb{R}^4$, we have by definition $df = \sum_i \lambda_i df_i$ along F_σ hence a fortiori $f - \sum_i \lambda_i f_i$ is constant along F_σ. This means that the curve $F_\sigma \subset V \subset \mathbb{C}^5$ satisfies three independent affine equations with real coefficients, where the holomorphic embedding $V \subset \mathbb{C}^5$ is given by the functions $f,\, f_1, \ldots,\, f_4$.

Lemma 3.15. *If neither (0') nor (1) hold, that is, the surface $V^0 \subset \mathbb{C}^5$ is affinely nondegenerate and is not an open set in the complete embedding of the Veronese surface, there exists a nontrivial relation $\sum_i \lambda_i df_i = 0$ with constant coefficients λ_i holding everywhere along F_σ.*

Proof. We complexify the situation and consider the complex manifold $\Sigma_{\mathbb{C}} \subset \mathbb{C}^4$ consisting of 4-uples $(\lambda_1, \ldots,\, \lambda_4)$ such that $df = \sum_i \lambda_i df_i$ along a complex curve F (deforming a fiber F_σ) in V. This must contains the complexification of the real analytic surface $\Sigma \subset \mathbb{R}^4$. We now argue by contradiction. Assuming that there is no nontrivial relation $\sum_i \lambda_i df_i = 0$ with constant coefficients λ_i holding everywhere along F_σ, then the curve F determines the parameters λ_i and thus there is actually a 2-dimensional family of curves F. Note that by complex analytic continuation, these curves F also satisfy 2 affine equations in the f_i's, so that they satisfy 3 affine equations in $f,\, f_i$. We now have the following statement:

Lemma 3.16. *Let V^0 be a smooth connected complex surface in $\mathbb{C}^5$ which satisfies the following three properties:*

a) The Gauss map of V^0 is of maximal rank 2.

b) There is a 2-dimensional family Γ of curves F contained in V^0 whose affine envelop $\langle F \rangle$ has dimension 2.

c) V^0 is not contained in a proper affine subspace in $\mathbb{C}^5$.

Then V^0 is an open set in the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

Proof. As the surface V^0 satisfies b), there is an open subset V' of V^0 such that for any $x,\, y \in V'$, there exists a curve F in the family passing through x and y. But then, the affine line generated by x and y is contained in the affine envelop $\langle F \rangle$, hence is contained in the 4-dimensional analytic subset of $\mathbb{C}^5$ defined as $\cup_{\gamma \in \Gamma} \langle F \rangle$. Thus the surface V^0 has the property that its secant variety has dimension ≤ 4. If V^0 satisfies a), its Gauss map has maximal rank, hence by [9, (6.18)], it must be an open set of the Veronese variety $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$. $\square$

Lemma 3.16 can be applied in our case since the surface $V^0 \subset \mathbb{C}^4$ embedded by the functions $f_1, \ldots, f_4$ has Gauss map of maximal rank (using the construction of section 3.2, this is equivalent to the assumption that the variation of Hodge structure has maximal rank), hence so does a fortiori the surface $V^0 \subset \mathbb{C}^5$ embedded by the functions $f,\, f_1, \ldots,\, f_4$. Hence we conclude that either there is a nontrivial affine relation between the functions $f,\, f_i$, or $V^0 \subset \mathbb{C}^5$ satisfies a), b) and c), hence is an open set in a complete Veronese embedding $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, which is property (1). In the first case, either the relation involves f, which means that df is a combination with constant complex coefficients of the df_i, that is, we are in the situation (0') of Proposition 3.8, or it does not involve f and the affine relation involves the f_i's only. This however would provide a linear relation $\sum_i \lambda_i df_i = 0$ with constant complex coefficients between the differentials df_i. Translated in terms of variations of Hodge structure, this says that the constant section $\sum_i \lambda_i e_i$ is of type $(1,0)$, and this contradicts the fact that our period map has maximal rank, using the following lemma (or rather its variation of Hodge structure translation):

Lemma 3.17. *Let $\phi : A \to V$ be a polarized Lagrangian torus fibration, with V smooth and connected. Assume that the associated period map has generically maximal rank. Then for any open set U of V, no flat section of $H^1_{\mathbb{C}} = R^1\phi_*\mathbb{C}$ is of type $(1,0)$ on U.*

Proof. Indeed, the infinitesimal version of our assumption is that the map

$$T_{V,b} \to \mathrm{Hom}(H^{1,0}(X_b), H^{0,1}(X_b))$$

is injective at a general point of B. In our situation, the existence of a Lagrangian fibration structure provides an extra duality which allows to identify this map with the map $\overline{\nabla}_b : H^{1,0}(X_b) \to \mathrm{Hom}(T_{B,b}, H^{0,1}(X_b))$. The injectivity of this last map exactly says that no nonzero element of $H^{1,0}(X_b)$ extends at first order to a flat section of $\mathcal{H}^{1,0}$ (see [24, Section 5.3]). □

This concludes the proof of Lemma 3.15. □

Coming back to the proof of Proposition 3.8, we conclude that if none of (0), (0'), (1) holds, we have $k = 1$ and there exists a nontrivial relation $\sum_i \lambda_i df_i = 0$ with constant coefficients λ_i holding everywhere along F_σ. We are now in case in (2) and prove (i), (ii) and (iii). We now by Lemma 3.11 that the fiber F_σ is affine, thus defined by two affine equations which after a real change of variables f_i can be assumed to be $x_3 = cst$, $x_4 = cst$, $x_i = \mathrm{Re}\, f_i$, and by Lemma 3.12, the two corresponding holomorphic functions f_3, f_4 are constant along F_σ. We now apply Lemma 3.13. Choosing affine coordinates x_i on V in such a way that $\omega^* = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, we get that along F_σ, df_1, df_2 are combinations of dx_1, dx_2 and df_3, df_4 are combinations of dx_3, dx_4. It follows that we have along F_σ

$$df_1 = hdf_2,\ df_3 = kdf_4, \tag{34}$$

where h, k are holomorphic functions on F_σ and the equality holds in $\Gamma(F_\sigma, \Omega_{V|F_\sigma})$. On the other hand, we know that there exists along F_σ a linear relation with constant coefficients involving the df_i. As df_1, df_2 lie in $T^* \otimes \mathbb{C}$ and df_3, df_4 belong to $(T^\perp)^* \otimes \mathbb{C}$ this is possible only if either h is constant or k is constant along F_σ.

Lemma 3.18. *The case $df_1 = hdf_2$ with h constant along F_σ is not possible.*

Proof. If we have this extra relation, then we have an extra affine equation $f_1 - hf_2 = Cst$ along F_σ, which means that via the embedding $V \to \mathbb{C}^4$ given by $f_1, \ldots, f_4$, the fibers F_σ are mapped to lines in $\mathbb{C}^4$. The family of lines F_σ is then a 1-parameter holomorphic family $(F_t)_{t\in C}$ of (open sets in) lines in $V^0 \subset \mathbb{C}^4$. Let s denote a local holomorphic coordinate on C, and let t be a linear function on $\mathbb{C}^4$, which restricts to a coordinate on the general line F_t. The surface $V^0 \subset \mathbb{C}^4$ is then parameterized by (s,t), with a parameterization which takes the form:

$$f.(s,t) = \alpha.(s) + t\beta.(s), \tag{35}$$

with $f. = (f_1, \ldots, f_4)$ and similarly for $\alpha.$ and $\beta.$. We now have the function f which is affine linear along the fibers F_σ, hence writes

$$f = f_0(s) + tf_1(s). \tag{36}$$

We get from (35) and (36)

$$df_i = (\dot{\alpha}_i(s) + t\dot{\beta}_i(s))ds + \beta_i(s)dt,\ df = (\dot{f}_0(s) + t\dot{f}_1(s))ds + f_1(s)dt. \tag{37}$$

We write now

$$df = \sum_i \lambda_i(s) df_i \tag{38}$$

where the λ_i's are real functions and hence uniquely defined by (38), and depend only on s. We then get:

$$\dot{f}_0(s) + t\dot{f}_1(s) = \sum_i \lambda_i(s)(\dot{\alpha}_i(s) + t\dot{\beta}_i(s)), \tag{39}$$
$$f_1(s) = \sum_i \lambda_i(s)\beta_i(s),$$

which also provides

$$\dot{f}_0(s) = \sum_i \lambda_i(s)\dot{\alpha}_i(s),\ \dot{f}_1(s) = \sum_i \lambda_i(s)\dot{\beta}_i(s),$$
$$f_1(s) = \sum_i \lambda_i(s)\beta_i(s).$$

We now observe that the left hand side in each of the equations in (40) is a holomorphic function of s, as are the functions $\dot{\alpha}_i(s)$, $\dot{\beta}_i(s)$, $\beta_i(s)$. Applying the operator $\frac{\partial}{\partial \overline{s}}$ to these equations, we thus get:

$$0 = \sum_i \frac{\partial \lambda_i}{\partial \overline{s}}\dot{\alpha}_i(s),\ 0 = \sum_i \frac{\partial \lambda_i}{\partial \overline{s}}\dot{\beta}_i(s),$$
$$0 = \sum_i \frac{\partial \lambda_i}{\partial \overline{s}}\beta_i(s).$$

These equations say that the three vectors $(\beta_i(s))$, $(\dot{\beta}_i(s))$, $(\dot{\alpha}_i(s))$ satisfy the linear equation defined by $(\frac{\partial \lambda_i}{\partial \overline{s}})$. Note now that these three vectors are generically linearly independent in $\mathbb{C}^4$, as otherwise, the Gauss map of the surface V is not of maximal rank. Hence they generate a hyperplane which varies holomorphically with s, and at the asme time is defined by the equation $(\frac{\partial \lambda_i}{\partial \overline{s}})$ which is antiholomorphic in s. So either $(\frac{\partial \lambda_i}{\partial \overline{s}}) = 0$ or the hyperplane is constant. The first case says that the λ_i's are in fact constant (since they are real), so we are not in the situation (2). The second case say that the surface V^0 is contained in an affine space in $\mathbb{C}^4$, which we also excluded. □

We conclude from Lemma 3.18 that we must have $df_3 = kdf_4$ with k constant along F_σ, and $df_1 = hdf_2$ with h nonconstant holomorphic. In particular, the only affine equations satisfied by F_σ in $\mathbb{C}^4$ are (in the given coordinates) $z_3 = Cst$, $z_4 = Cst$. Note that as we assume that we are not in the situation (1), by Lemma 3.16, V^0 does not contain a two-parameter family of curves whose affine envelop is a plane, hence there is only a one-parameter holomorphic family of such curves, which must be the family of fibers F_σ. In particular, Σ has a holomorphic structure for which $a_\cdot : V \to \Sigma$ is holomorphic. Each fiber F_σ generates an affine $\mathbb{C}^2_{aff}$. The corresponding linear space $\mathbb{C}^2 \subset \mathbb{C}^4$ is defined by the equations $z_3 = z_4 = 0$ and z_3, z_4 are real combinations of the original linear coordinates on $\mathbb{C}^4$. If we have a holomorphic family of linear subspaces $\mathbb{C}^2 \subset \mathbb{C}^4$ which are all defined over $\mathbb{R}$, this family must be constant. This proves property (i), and property (ii) is proved in the equations (34).

The last statement (iii) follows easily from (i) and (ii). Indeed, by (i) we have coordinates $x_i = \operatorname{Re} f_i$ in which all the fibers F_σ are defined by equations $x_3 = cst$, $x_4 = cst$. We now write $df = \sum_i \lambda_i df_i$ with λ_i real and constant along the fibers F_σ, which means that λ_i depend only on the coordinates x_3, x_4. Let $\eta_1 = \lambda_1 df_1 + \lambda_2 df_2$, and $\eta_2 = \lambda_3 df_3 + \lambda_4 df_4$. Then using (34), we see that the decomposition $df = \eta_1 + \eta_2$ is the decomposition of df according to the orthogonal decomposition $\mathbb{R}^4 = \mathbb{R}\langle dx_1, dx_2\rangle \oplus \mathbb{R}\langle dx_3, dx_4\rangle$. The $d\lambda_i$'s involve only dx_3 and dx_4, and thus the equation $df = 0 = \sum_i d\lambda_i \wedge df_i$ clearly implies that $d\eta_1 = 0$, $d\eta_2 = 0$. In particular, these 1-forms are holomorphic, since they are of type $(1, 0)$. As the form η_2 is a section of the subbundle $a_\cdot^*\Omega_\Sigma \subset \Omega_{V^0}$, the fact that it is closed implies that η_2 is the pull-back of a holomorphic form on Σ. Finally the equation $d\eta_1 = 0$ with

$\mathrm{Re}\,\eta_1 = \lambda_1 dx_1 + \lambda_2 dx_2$, the functions λ_1, λ_2 being independent of x_1, x_2, implies that η_1 is a form with constant coefficients. The proof of Proposition 3.8 is finished. □

We now conclude the proof of Proposition 3.2.

Proof of Proposition 3.2. Let $\phi : A \to U$ be a polarized Lagrangian torus fibration equipped with its holomorphic 2-form σ_A and let ν be a holomorphic section of A such that $\nu^*\sigma_A = 0$. We assume furthermore that the corresponding infinitesimal variation of weight 1 Hodge structure $T_U \to Hom\,(\mathcal{H}_{1,0}, \mathcal{H}_{0,1})$ is generically of maximal rank and $\dim A = 4$. We use the same infinitesimal criterion as in Proposition 2.1, (i). It says that if the set of points $b \in U$ such that $\nu(b)$ is a torsion point is not dense in U, then the locally associated map $g_\nu : V \to A_{b_0}$ defined by composing ν with a real analytic local trivialization $A \cong V \times A_{b_0}$ is nowhere of maximal rank. Writing A as a quotient $\mathcal{H}_{0,1,A}/H_{1,\mathbb{Z}}$, such a real analytic trivialization is obtained using the natural isomorphism (9) between the flat real analytic vector bundle $\mathcal{H}_{1,\mathbb{R}} = \mathcal{H}_{1,\mathbb{Z}} \otimes \mathcal{C}^\omega(\mathbb{R})$ associated with the local system $H_{1,\mathbb{R}}$ and the holomorphic vector bundle $\mathcal{H}_{0,1}$, and by trivializing locally the local system $H_{1,\mathbb{Z}}$ on small open sets $V \subset U$. Let $\nu \in \Gamma(U, \mathcal{H}_{0,1}/H_{1,\mathbb{Z}})$ with local lift $\tilde{\nu} \in \Gamma(V, \mathcal{H}_{0,1})$. Then the map g_ν is defined by choosing a trivializing basis $\alpha_1, \ldots, \alpha_4$ of $H_{1,\mathbb{R}}$ on V, which allows to write uniquely

$$\tilde{\nu} = \sum_i a_i \alpha_i \tag{40}$$

for some real analytic functions a_i, the equality being in $\mathcal{H}_{1,\mathbb{R}} \cong \mathcal{H}_{0,1}$. The map g_ν is then given by

$$g_\nu = (a_1, \ldots, a_4) \in \mathbb{R}^4$$

(followed by the projection $\mathbb{R}^4 \to \mathbb{R}^4/\mathbb{Z}^4$ if the basis α_i is chosen to be integral). Next the section $\tilde{\nu}$ of $\mathcal{H}_{0,1}$ on V identifies with a section α_ν of Ω_V which is closed by Lemma 3.6, hence can be written locally as $\alpha_\nu = df_\nu$. Similarly, recall from Lemma 3.3 that the forms on B given by the holomorphic sections α_i of $\mathcal{H}_{0,1} \cong \Omega_V$ are closed, so that we can assume up to shrinking V that they are exact: $\alpha_i = df_i$. Equality (40) thus rewrites on small open sets $V \subset U$ as

$$df_\nu = \sum_i a_i df_i \text{ in } \Gamma(V, \Omega_V), \tag{41}$$

which is exactly the equation studied in Proposition 3.8. Furthermore, as we assumed that the infinitesimal variation of Hodge structure is generically of maximal rank on U, Proposition 3.8 can be applied. We know that the map $g_\nu = a_. : V \to \mathbb{R}^4$ is nowhere of maximal rank, hence by Proposition 3.8, on any open subset $V^0 \subset V$ where g_ν is of locally constant rank $4 - 2k < 4$, one of the possibilities (0') to (2) holds on V^0. Comparing with the statement of proposition 3.2, we see that we only have to explain how to pass in each of these three cases from the local statement to the global statement. There is a small subtlety here since denoting by U^0 the union of the sets V^0 above, the complement $U \setminus U^0$ is a closed real analytic subset of U, so that U^0 could be disconnected and the natural map $\pi_1(U^0) \to \pi_1(U)$ could be nonsurjective.

Case (0') of Proposition 3.8. In this case, there is a simply connected open subset $V^0 \subset U$ and a constant section ϕ of $H_{1,\mathbb{C}}$ on V^0 such that $\tilde{\nu} = \phi^{0,1}$ in $\Gamma(V^0, \mathcal{H}_{0,1})$, where $\phi^{0,1}$ is the projection of ϕ in $\mathcal{H}_{0,1} = H_{1,\mathbb{C}} \otimes \mathcal{O}_{V^0}/\mathcal{H}_{1,0}$. Let $\widetilde{U}$ be the universal cover of U. Then ϕ extends to a section that we still denote ϕ of the constant (trivial) local system $H_{1,\mathbb{C}}$ on $\widetilde{U}$ and the local lift $\tilde{\nu}$ of ν also extends to a lift $\tilde{\nu}$ of ν in $\Gamma(\widetilde{U}, \mathcal{H}_{1,0})$. The equality $\phi^{0,1} = \tilde{\nu}$ on V^0 extends by analytic continuation to an equality $\phi^{0,1} = \tilde{\nu}$ in $\Gamma(\widetilde{U}, \mathcal{H}_{0,1})$, which gives us case (0') of Proposition 3.2.

Case (1) of Proposition 3.8. The local data f_ν, f_i are globally defined on $\widetilde{U}$. Indeed, the f_i's were obtained locally, first of all by integrating the $(2,0)$-form on local sections of

$H_{1,\mathbb{Z}}$, thus getting sections of $\mathcal{H}_{0,1}$, then identifying these sections to closed holomorphic forms, and finally choosing a primitive of these closed forms. Similarly f_ν was obtained by choosing a lift of ν to a section of $\mathcal{H}_{0,1} \cong \Omega_U$, then taking a primitive of the closed holomorphic form so obtained. As the $(2,0)$-form is globally defined on A, all this can be done globally once we work on a simply connected manifold. It suffices then to observe that if the image $(f_\nu,\, f_i)(V^0)$ in $\mathbb{C}^5$ is contained in the (affine part) of a Veronese surface in $\mathbb{P}^5$ for some $V^0 \subset \widetilde{U}$, then $(f_\nu,\, f_i)(\widetilde{U})$ is also contained by analytic continuation in this Veronese surface.

Case (2) of Proposition 3.8. There exists now a simply connected open set $V^0 \subset U$ where the real analytic map $g_\nu = a.$ has constant rank 2, and properties (i), (ii), (iii) holds on V^0, namely the fibers F_σ of g_ν are affine and parallel in the coordinates $\mathrm{Re}\, f_i$, and furthermore, denoting by $T \subset \mathbb{R}^4$ the common tangent space of the F_σ's, the orthogonal decomposition

$$\mathbb{R}^4 = T \oplus_{\perp\omega} T^{\perp\omega}$$

is a decomposition of the real variation of Hodge structures on $H_{1,\mathbb{R}}$ as a sum of two variations of Hodge structures $H_{1,\mathbb{R}} = L \oplus L'$, and L' has trivial variation of Hodge structure along the fibers F_σ. Finally, in the induced decomposition of $\tilde{\nu}$ as $\tilde{\nu} = \tilde{\nu}_L + \tilde{\nu}_{L'}$, the term $\tilde{\nu}_L$ comes from a real constant section of L while $\tilde{\nu}_{L'}$ is only constant along F_σ. Let us see V^0 as an open subset of $\widetilde{U}$. On $\widetilde{U}$, the subspace $T \subset H_{1,\mathbb{R}}$ is globally defined and the fact that it is a real subvariation of Hodge structure L, as is its orthogonal complement $T^\perp$ giving rise to the variation of Hodge structure L', is a closed analytic property, which thus must hold everywhere on $\widetilde{U}$ by analytic continuation. As the real affine coordinates x_i are globally defined on $\widetilde{U}$, T also determines a foliation on $\widetilde{U}$ whose fibers intersect V^0 along the curves F_σ. The fact that the variation of Hodge structure on L' is constant along the fibers of this foliation is then a real (in fact complex) analytic property on $\widetilde{U}$, hence satisfied everywhere. Finally the fact that $\tilde{\nu}_L$ is constant is also real analytic, hence true everywhere. In order to complete the proof in this case, we just have to explain why the vector space T determines a local subsystem of $H_{1,\mathbb{R}}$ on U, or equivalently, why it is invariant by monodromy. This is however clear by uniqueness of the foliation determined by T and satisfying all the properties above. The proof of Proposition 3.2 is now complete. □

3.4 Proof of Theorem 0.2 in the nonisotrivial case

We explain in this section how Theorem 0.2 in the nonisotrivial case follows from Proposition 3.2 and its local version Proposition 3.8.

Proof of Theorem 0.2. The proof is by contradiction. Under the global assumptions of Theorem 0.2, assume that the fibration is not locally isotrivial (hence is generically of maximal rank), but that the torsion points of the section ν_M are not dense in B. Then over B^0, all the assumptions of Proposition 3.2 are satisfied by Lemma 3.1. Proposition 3.2 thus applies and we are in one of the three situations (0'), (1), (2) of Proposition 3.2, that we have to exclude.

Case (0'). This case is easy to exclude. Indeed, (1) says that the section $\nu_M \in \Gamma(B^0, \mathcal{J})$ is a section of $H^1_{\mathbb{C}}/H^1_{\mathbb{Z}} \subset \mathcal{J}$. The inclusion here is due to Lemma 3.17 which gives under our asumptions the injectivity of the natural map of sheaves $H^1_{\mathbb{C}} \to \mathcal{H}_{0,1}$.

Denote by $H^2(X,\mathbb{Z})_{X_b}$ the kernel of the restriction map $H^2(X,\mathbb{Z}) \to H^2(X_b,\mathbb{Z})$ and similarly for X^0. Recall that the image of $c_1(M)$ in $H^1(B^0, R^1\phi_*\mathbb{Z})$ by the composition of the restriction map $H^2(X,\mathbb{Z})_{X_b} \to H^2(X^0,\mathbb{Z})_{X_b}$ (whose kernel consists of divisor classes supported over the discriminant locus) and the natural Leray map $H^2(X^0,\mathbb{Z})_{X_b} \to H^1(B^0, R^1\phi_*\mathbb{Z})$, can be constructed as the image of $\nu_M \in \Gamma(B^0, \mathcal{J})$ by the connecting map of the exact sequence

$$0 \to H^1_{\mathbb{Z}} \to \mathcal{H}^{0,1} \to \mathcal{J} \to 0,$$

(see [24, 8.2.2]). It then follows that the cohomology class $c_1(M)$ restricted to X^0 vanishes in $H^1(B^0, R^1\phi_*\mathbb{C})$, so that a multiple $Nc_1(M)$ vanishes in $H^1(B^0, R^1\phi_*\mathbb{Z})$. Thus, looking at the Leray spectral sequence, $Nc_1(M)_{|X^0}$ is the pull-back $\phi^*\alpha$ of a degree 2 cohomology class on B^0. It is then immediate that one can choose α to be a divisor class on B^0, so that finally $Nc_1(M)$ can be written as the sum of a divisor class pulled-back from B and a divisor class supported on $X \setminus X^0$. This implies the same result in $\operatorname{Pic} X$ since $\operatorname{Pic}^0 X = 0$. It follows that $N\nu_M$ vanishes, or ν_M is of torsion, a contradiction.

Case (1). We have to exclude by monodromy considerations the possibility that the image of the locally defined holomorphic map $(f, f_1, \dots, f_4) : V \to \mathbb{C}^5$ is an open set in (the affine part of) a Veronese surface in $\mathbb{P}^5$. Assuming we are in this situation, it follows in particular that the image of the period map $V \to \mathcal{D} \subset G(2,4)_{lag}$ is an open set in an algebraic subvariety of $G(2,4)_{lag}$ (in the terminology of [7], it is semi-algebraic). Here $G(2,4)_{lag}$ is the Lagrangian Grassmannian parameterizing locally the polarized deformations of the abelian surface. It follows then from [7] or [21] that this image, being 2-dimensional, must be a Noether-Lefschetz divisor, that is, there is (maybe after taking a finite cover) a second global section $\omega' \in \bigwedge^2 H^*_{1,\mathbb{Q}}$ such that ω' is of type $(1,1)$ on any fiber X_b, or equivalently, the period point $H_{1,0,b} \subset H_1(X_b, \mathbb{C})$ is also Lagrangian for the form ω'. In our situation (and forgetting about rational coefficients), this is saying, using the translation described in section 3.2, that our surface $V \subset \mathbb{C}^4$ is Lagrangian for two 2-forms with constant coefficients on $\mathbb{C}^4$.

Remark 3.19. This says in particular that our abelian surfaces X_b have $\rho(X_b) \geq 2$. Unfortunately we cannot conclude simply by applying Matsushita's result (Proposition 0.1), because a nontrivial finite monodromy can act on the Néron-Severi group of the geometric generic fiber (or very general fiber), while Matsushita's result concerns the Néron-Severi group of the generic fiber.

We use now the following easy result:

Lemma 3.20. *Let $V \subset \mathbb{C}^4$ be (an open subset of) the affine part of the projection of a Veronese surface in $\mathbb{P}^5$. Then, if there are two constant coefficients 2-forms vanishing on V, the curve at infinity of V is a double line. Hence V has a canonical affine structure.*

Proof. This is an explicit computation. The affine part V of a Veronese surface in $\mathbb{C}^5$ is either the complement of a smooth conic, resp. the complement of a nodal conic, or the complement of a double line. In the first two cases, one easily shows that there is no nonzero constant coefficients 2-form on $\mathbb{C}^5$ vanishing on V, resp. there is only one nonzero constant coefficients 2-form on $\mathbb{C}^5$ vanishing on V (which is in fact of rank 2). So in both cases, an affine projection of V is not Lagrangian with respect to two constant coefficients 2-forms on $\mathbb{C}^4$, and thus only the last case is possible. □

Lemma 3.20 can be rephrased saying that our surface $V \subset \mathbb{C}^4$ must be an affine Veronese surface embedded in $\mathbb{C}^4$ via (inhomogeneous) polynomials of degree ≤ 2 in two variables. We next have the following:

Lemma 3.21. *If an affine Veronese surface in $\mathbb{C}^4$ is Lagrangian for at least one nondegenerate constant coefficients 2-forms on $\mathbb{C}^4$, the embedding is given by two homogeneous quadratic polynomials and two linear polynomials.*

Proof. Indeed, let s, t be affine coordinates on $\mathbb{C}^2$. Consider 4 quadratic polynomials (modulo constants) $f_1, \dots, f_4$, three of which having independent quadratic homogeneous terms, giving a morphism $\phi : \mathbb{C}^2 \to \mathbb{C}^4$. Then for a given choice of affine coordinates s, t, we have $f_1 = s^2 + a_1$, $f_2 = t^2 + a_2$, $f_3 = st + a_3$, $f_4 = a_4$, where the a_i's are affine in s and t. Then we have

$$\phi^*(dz_1 \wedge dz_2) = (4st + l_{12})ds \wedge dt,$$

$$\phi^*(dz_1 \wedge dz_3) = (2s^2 + l_{23})ds \wedge dt,$$

$$\phi^*(dz_2 \wedge dz_3) = (2t^2 + l_{23}) ds \wedge dt,$$

where the l_{ij} are affine linear functions, and the other pull-backs $\phi^* dz_i \wedge dz_j$ are 2-forms with affine coefficients on $\mathbb{C}^2$. It follows that a 2-form ω with constant coefficients vanishing under ϕ^* is a combination of $dz_1 \wedge dz_4$, $dz_2 \wedge dz_4$, $dz_3 \wedge dz_4$, hence is degenerate. This computation shows that the quadratic parts of our polynomials generate only a 2-dimensional space, hence that there are at least two affine functions $f_1 = s$, $f_2 = t$ in our system. Removing the relevant affine functions of s, t from the two remaining functions f_3, f_4, we may assume they are quadratic homogeneous. □

We now look at the monodromy acting on the space $\mathbb{C}^4 = H_1(X_b, \mathbb{C})$, which we can also see as a space of functions $f_1, \ldots, f_4$ on V, defined modulo constants. Note first of all that since our family of abelian surfaces is a two-parameter family of abelian surfaces with Picard number 2, there cannot be any line in $\mathbb{C}^4 = H_1(X_b, \mathbb{C})$ (or point in $\mathbb{P}(H_1(X_b, \mathbb{C}))$ left invariant by a finite index subgroup of the monodromy group. On the other hand, by Lemma 3.21, the monodromy acting on our space of functions preserves a rank 2 subspace L consisting of affine linear functions. Furthermore, the rank 2 system of quadratic homogeneous part found in Lemma 3.21 cannot have a base point, since this would distinguish a point in $\mathbb{P}(L)$. It follows that for an adequate choice of affine coordinates s, t on $\mathbb{C}^2$, our functions can be written as $\phi = (s^2, t^2, s, t)$, where the subspace L is generated by s and t. But then there are two natural subspaces in $\mathbb{C}^4$, namely the one generated by s^2, s and the one generated by t^2, t. These subspaces are related to the two locally constant 2-forms ω and ω' as follows: the only 2-forms vanishing on $\mathbb{C}^2$ via ϕ^* are $dz_1 \wedge dz_3$ and $dz_2 \wedge dz_4$. Thus we must have $\langle \omega, \omega' \rangle = \langle dz_1 \wedge dz_3, dz_2 \wedge dz_4 \rangle$, and the two linear subspaces above are the kernels of the two degenerate forms in the pencil $\langle \omega, \omega' \rangle$. These two subspaces can be exchanged under monodromy but there is a finite index subgroup fixing each of them. This finite index subgroup then fixes each of the lines generated by e_3 or e_4. This gives a contradiction and excludes case (1).

Case (2). We now have a nontrivial local subsystem $L \subset H_{1,\mathbb{R}}$ which is a subvariation of Hodge structures. Let us recall the following lemma from [11].

Lemma 3.22. *Let $\phi : X \to B$ be a Lagrangian fibration on a hyper-Kähler manifold. Then there is no proper nontrivial real subvariation of Hodge structure of $H_{1,\mathbb{R}}$ on B^0.*

Proof. According to Matsushita's Proposition 0.1, the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$ has rank 1. By Deligne's invariant cycle theorem, this implies that the local system $R^2\phi_*\mathbb{R}$ on B^0 has only one global section (provided by the polarization ω and its real multiples). If the local system $(R^1\phi_*\mathbb{R})^*$ contains a nontrivial local subsystem L which carries a real variation of Hodge structures, the restriction of ω to L is nondegenerate and the orthogonal complement $L^{\perp\omega}$ with respect to ω is also a local subsystem of $(R^1\phi_*\mathbb{R})^*$ on which ω is nondegenerate. Thus we have a decomposition

$$(R^1\phi_*\mathbb{R})^* = L \oplus L^\perp \tag{42}$$

and if both L and $L^\perp$ were nontrivial, we would get two independent sections of $R^2\phi_*\mathbb{R} = \bigwedge^2 (R^1\phi_*\mathbb{R})^{**}$, namely $\pi_L^* \omega_{|L}$ and $\pi_{L^\perp}^*(\omega_{|L^\perp})$, where π_L and $\pi_{L^\perp}$ are the two projections associated with the decomposition (42). This is a contradiction, hence a proper nontrivial such L does not exist. □

Thus Case (2) is also excluded. □

4 Proof of Theorem 0.2 when the fibration is isotrivial

Theorem 0.2 when the fibration is isotrivial works in any dimension. It will be obtained as a consequence of the following analogue of Proposition 3.2 in the isotrivial case.

Proposition 4.1. *Let $\phi : A \to B$ be a locally isotrivial Lagrangian torus fibration, where A is projective and has a generically nondegenerate holomorphic form σ_A. Let $\nu : B \to A$ be a section such that $\nu^*\sigma_A = 0$. Then if the torsion points of ν are not dense in B^0 for the usual topology, the local system*

$$H_{1,\mathbb{R},A} := (R^1\phi_*\mathbb{R})^*$$

admits a nontrivial local subsystem L which underlies a real subvariation of Hodge structures. Furthermore, writing the real lift (see Section 1) of the section ν as a sum $\tilde{\nu}_L + \tilde{\nu}_{L^\perp}$, where $L^\perp$ is the orthogonal variation of Hodge structures with respect to ω, then $\tilde{\nu}_L$ is a locally constant section of L.

Proof. The local analysis made in Subsection 3.2 works similarly, but we are going to prove directly that the fibers V^0_σ of the map ϕ_ν (with local form $a.$) are parallel, so that ϕ_ν factors through an affine projection (in the local affine coordinates described in 3.2). The local data described in Section 3.2 take the following form: there are locally $2n$ holomorphic functions $f_1, \ldots, f_{2n}$ on simply connected open subsets $U \subset B$, which are independent over $\mathbb{R}$ and provide real coordinates $x_1, \ldots, x_{2n}$, $x_i = \mathrm{Re}\, f_i$. By isotriviality, modulo the constants, the f_i's provide only n independent holomorphic functions independent over $\mathbb{C}$. In particular the open set B^0 over which ϕ is smooth and σ_A is nondegenerate is endowed with a flat holomorphic structure.

Lemma 4.2. *This flat holomorphic structure, after passing to a generically finite cover B' of B^0, is induced by the choice of n holomorphic 1-forms on a smooth projective completion $\overline{B'}$ of B'.*

Proof. Indeed, the fibration $\phi : A \to B$ is trivialized over a finite cover B' of a Zariski open set of B^0 as

$$A' := A \times_B B' \cong B' \times J_0,$$

where $0 \in B'$ is a given point, and J_0 is the fiber of $\phi' : A' \to B'$ over it. The local flat holomorphic coordinates on B^0, pulled back to B', come from the holomorphic 2-form on $B' \times J_0$ which is of the form

$$\sum_i pr^*_{B'}\alpha_i \otimes pr^*_{J_0}\beta_i,$$

for a basis β_i of $H^0(\Omega_{J_0})$. The flat structure on B^0 is given by the holomorphic forms α_i. We now claim that the forms α_i extend to holomorphic 1-forms on $\overline{B'}$. This is seen as follows: More generally, the transcendental cohomology $H^2(A,\mathbb{Q})_{tr}$, pulled-back to $B' \times J_0$ falls in the $(1,1)$ part $H^1(B',\mathbb{Q}) \otimes H^1(J_0,\mathbb{Q})$ of the Künneth decomposition of $H^2(B',\mathbb{Q})$, and this implies that it is contained in the image of $H^1(\overline{B'},\mathbb{Q}) \otimes H^1(J_0,\mathbb{Q})$ in $H^1(B',\mathbb{Q}) \otimes H^1(J_0,\mathbb{Q})$ by Deligne's strictness theorem for morphisms of mixed Hodge structures [5]): indeed, we get from the above a morphism of mixed Hodge structures

$$\alpha : H^2(X,\mathbb{Q})_{tr} \otimes H^1(J_0,\mathbb{Q})^* \to H^1(B',\mathbb{Q})$$

with image $K \subset H^1(B',\mathbb{Q})$ such that $H^2(X,\mathbb{Q})_{tr}$, pulled-back to $B' \times J_0$ is contained in $K \otimes H^1(J_0,\mathbb{Q})$. By Deligne strictness theorem [5], this morphism α has its image contained in the pure (smallest weight) part of $H^1(B',\mathbb{Q})$, namely $H^1(\overline{B'},\mathbb{Q})$. This finishes the proof. □

As a consequence, we get the following:

Corollary 4.3. *Let $(F_t)_{t\in\Sigma}$ be a continuous family of complex submanifolds of an open set V^0 of B^0. Assume that each F_t is flat for the flat holomorphic structure described above, and is algebraic, that is a connected component of the intersection of an algebraic subvariety C_t of B with V^0. Then the F_t's are parallel, that is, F_t being affine in the flat coordinates, the linear part of their defining equations is constant.*

Proof. The linear part of the defining equations for F_t is determined by Lemma 4.2 by the vector space of holomorphic forms on $\overline{B'}$ vanishing on F_t, or equivalently on C_t. On the other hand, by desingularization and up to shrinking our family, we can assume that the C_t form a family of smooth projective varieties mapping to $\overline{B'}$. It is then clear that the space of holomorphic forms on $\overline{B'}$ vanishing on C_t does not depend on t. □

The proof of Proposition 4.1 will finally use the following Lemma 4.4. We add to the data of our locally isotrivial fibration $\phi : A \to B$ the section ν, which satisfies the condition $\tilde{\nu}^*\sigma_A = 0$. This section lifts locally on B^0 to a section $\tilde{\nu}_M$ of the sheaf $H_{1,\mathbb{R}} \otimes \mathcal{C}^\infty_{\mathbb{R}}$ where $H_{1,\mathbb{R}}$ is the the real local system $(R^1\phi_*\mathbb{R})^*$. This section is well-defined up to translation by a constant (integral) section of $H_{1,\mathbb{R}}$, and by Proposition 2.1, if the torsion points of ν_M are not dense in B^0 for the usual topology, then the map (defined on simply connected open sets $V \subset B^0$)

$$\tilde{\nu}_M : V \to H_1(J_0, \mathbb{R})$$

via a trivialization of $H_{1,\mathbb{R}}$ has positive dimensional fibers F_t.

Lemma 4.4. *In the isotrivial case, the fibers F_t are algebraic.*

Proof. Indeed, after passing to the finite cover B' already introduced above, the normal function ν_M gives a morphism of algebraic varieties

$$\nu'_M : B' \to \mathrm{Pic}^0(J_0) \cong_{isog} J_0,$$

which to $b' \in B'$ associates $M_{|X'_{b'}}$ with $X'_{b'} \cong J_0$ canonically. It is imediate that the fibers of $\tilde{\nu}_M$ coincide with the fibers of ν'_M, via the morphism $B' \to B^0$ and after restricting to the adequate open sets. □

We can now apply Corollary 4.3 to the fibers $V^0_\sigma \subset V^0$ of the local map $a.$ restricted to the open subset $V^0 \subset V$ where $a.$ has constant rank. These fibers are both complex and affine by Lemmas 3.11 and 3.12. We thus conclude that they vary in a parallel way. The properties (i), (ii), (iii) stated in (2) of Proposition 3.8 are then proved as in the general case, which proves the proposition. □

Proof of Theorem 0.2 in the isotrivial case. Let $\phi : X \to B$ be a Lagrangian isotrivial fibration, X projective hyper-Kähler, and let ν_M be the normal function associated to a line bundle M on X which is topologically trivial on the smooth fibers of ϕ. Proposition 4.1 applies and shows that if the set of points $b \in B$ where $\nu_M(b)$ is a torsion point is not dense in B, there is a decomposition of the local system $R^1\phi_*\mathbb{R}$ on B^0 as a direct sum of two subvariations of Hodge structures L and L', with $L \neq 0$, such that in the corresponding decomposition $\tilde{\nu}_M = \tilde{\nu}_{M,L} + \tilde{\nu}_{M,L'}$, $\tilde{\nu}_{M,L}$ is locally constant with real coefficients.

Lemma 3.22 says that L must be equal to $R^1\phi_*\mathbb{R}$ and thus $\tilde{\nu}$ is locally constant, so that ν_M is a section of $R^1\phi_*\mathbb{R}/R^1\phi_*\mathbb{Z}$. But then Lemma 2.3 says that a nontorsion section $R^1\phi_*\mathbb{R}/R^1\phi_*\mathbb{Z}$ can exist only if the invariant part of $H^1(X_b, \mathbb{R})$ is not trivial, which provides a contradiction since $H^1(X, \mathbb{R}) = 0$. □

References

[1] A. Beauville, R. Donagi. La variété des droites d'une hypersurface cubique de dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 703–706.

[2] A. Beauville. On the splitting of the Bloch-Beilinson filtration, in *Algebraic cycles and motives* (vol. 2), London Math. Soc. Lecture Notes 344, 38-53; Cambridge University Press (2007).

[3] A. Beauville, C. Voisin. On the Chow ring of a $K3$ surface, J. Algebraic Geom. 13 (2004), no. 3, 417–426.

[4] F. Bogomolov. On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). Geom. Funct. Anal. 6 (1996), no. 4, 612-618.

[5] P. Deligne. Théorie de Hodge II, Inst. Hautes Études Sci. Publ. Math. No. 40 , 5-57 (1971).

[6] R. Donagi, E. Markman. Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles, in *Integrable systems and quantum groups* (Montecatini Terme, 1993), 1-119, Lecture Notes in Math., 1620, Springer, Berlin, 1996.

[7] R. Friedman, R. Laza. Semialgebraic horizontal subvarieties of Calabi-Yau type. Duke Math. J. 162 (2013), no. 12, 2077-2148.

[8] L. Fu. Beauville-Voisin conjecture for generalized Kummer varieties, Int. Math. Res. Not. IMRN 2015, no. 12, 3878-3898.

[9] Ph. Griffiths, J. Harris. Algebraic geometry and local differential geometry. Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 3, 355-452.

[10] J.-M. Hwang. Base manifolds for fibrations of projective irreducible symplectic manifolds. Invent. math. 174, 625-644 (2008).

[11] B. van Geemen, C. Voisin. On a conjecture of Matsushita, to appear in IMRN.

[12] H.-Y. Lin. Lagrangian constant cycle subvarieties in Lagrangian fibrations, arXiv:1510.01437.

[13] D. Mumford. Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9 (1968) 195-204.

[14] J. Rauch, B. Taylor. The Dirichlet problem for the multidimensional Monge-Ampère equation. Rocky Mountain J. Math. 7 (1977), no. 2, 345-364.

[15] U. Riess. On the Chow ring of birational irreducible symplectic varieties, Manuscripta Math. 145 (2014), no. 3-4, 473-501.

[16] U. Riess. On the Beauville conjecture, arXiv:1409.3484, to appear in IMRN.

[17] D. Matsushita. On fibre space structures of a projective irreducible symplectic manifold, Topology 38 (1999), no. 1, 79-83.

[18] D. Matsushita. On base manifolds of Lagrangian fibrations. Sci. China Math. 58 (2015), no. 3, 531-542.

[19] D. Matsushita. On deformations of Lagrangian fibrations, arXiv:0903.2098.

[20] D. Matsushita. On isotropic divisors on irreducible symplectic manifolds, arXiv:1310.0896.

[21] E. Ullmo, A. Yafaev. A characterization of special subvarieties. Mathematika 57 (2011), no. 2, 263-273.

[22] C. Voisin. On the Chow ring of certain algebraic hyper-Kähler manifolds, Pure and Applied Mathematics Quarterly, Volume 4, Number 3, (2008).

[23] C. Voisin. Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes, in *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, 294-303, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[24] C. Voisin. *Hodge theory and complex algebraic geometry* II. Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, (2003).

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