REMARKS ON DEGENERATIONS OF HYPER-KÄHLER MANIFOLDS

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Abstract. Using the Minimal Model Program, any degeneration of K-trivial varieties can be arranged to be in a Kulikov type form, i.e. with trivial relative canonical divisor and mild singularities. In the hyper-Kähler setting, we can then deduce a finiteness statement for monodromy acting on H^2, once one knows that one component of the central fiber is not uniruled. Independently of this, using deep results from the geometry of hyper-Kähler manifolds, we prove that a finite monodromy projective degeneration of hyper-Kähler manifolds has a smooth filling (after base change and birational modifications). As a consequence of these two results, we prove a generalization of Huybrechts' theorem about birational versus deformation equivalence, allowing singular central fibers. As an application, we give simple proofs for the deformation type of certain geometric constructions of hyper-Kähler manifolds (e.g. Debarre–Voisin [DV10] or Laza–Saccà–Voisin [LSV16]). In a slightly different direction, we establish some basic properties (dimension and rational homology type) for the dual complex of a Kulikov type degeneration of hyper-Kähler manifolds.

Introduction

The starting point of this note was the study of deformation types of hyper-Kähler manifolds. Specifically, we recall that in [LSV16] we have given, starting with a cubic fourfold X, a construction of a 10-dimensional hyper-Kähler manifold Z compactifying the intermediate Jacobian fibration associated to the family of smooth hyperplane sections of X. We then proved, via delicate geometric arguments ([LSV16, Section 6]), that when the cubic fourfold is Pfaffian, the so-constructed hyper-Kähler manifold specializes well and is birational to an OG10 exceptional hyper-Kähler manifold. By Huybrechts' theorem [Huy99, Theorem 4.6] (see Theorem 0.2 below), we thus concluded that our compactified intermediate Jacobian fibrations are deformation equivalent to OG10. While our arguments in [LSV16] establish the desired result, they are somewhat convoluted and obscure, as Pfaffian geometry is beautiful but sophisticated. As observed by O'Grady and Rapagnetta (see [OR14]) even before we started working on [LSV16], another degeneration linking in a more direct way the intermediate Jacobian fibration to OG10 varieties consists in specializing the intermediate Jacobian fibration in the case where X degenerates to the secant variety of the Veronese surface in $\mathbb{P}^5$ (see Section 5.3). There is however a serious obstruction to realize this program: starting with a well-understood or mild degeneration of cubic fourfolds $\mathcal{X}/\Delta$, the corresponding degeneration of the associated family of hyper-Kähler manifolds $\mathcal{Z}/\Delta$ can be quite singular, and a priori hard to control. This is a common occurrence that can be already observed on the family of Fano varieties of lines of cubic fourfolds when the cubic acquires a node: mild degenerations of the cubic fourfold lead to families of associated hyper-Kähler manifolds $\mathcal{Z}/\Delta$ where both the central fiber Z_0 and the family $\mathcal{Z}$ are quite singular. So even if Z_0 is birational to a known hyper-Kähler manifold, due to the singularities, it is a priori difficult to conclude that the general fiber Z_t is deformation equivalent to the given type. In [LSV16], we avoided this issue following Beauville and Donagi [BD85] by specializing to general Pfaffian cubics (for which we proved that our construction of Z has smooth specialization), while in [DV10], where another sort of couple (Fano hypersurface, hyper-Kähler manifold) was studied, an explicit resolution of $\mathcal{Z}/\Delta$ was found.

One of the results of our paper is that in fact one does not need to resolve the singular family $\mathcal{Z}/\Delta$. Specifically, the following holds:

Theorem 0.1. *Let $\mathcal{Z} \to \Delta$ be a projective morphism with Z_t smooth hyper-Kähler for $t \neq 0$. Assume that one irreducible reduced (that is, multiplicity 1) component of the central fiber is birational to a smooth hyper-Kähler manifold Z_0'. Then the smooth fibers $\mathcal{Z}_t$ are deformation equivalent to Z_0'.*

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As explained above, this theorem significantly simplifies the deformation type arguments from [LSV16], [DV10], and other geometric examples (see Section 5). We note that the particular case where the central fiber Z_0 is smooth is equivalent to the following fundamental result due to Huybrechts:

Theorem 0.2. *(Huybrechts* [Huy99]*) Let Z and Z' be two birationally equivalent projective hyper-Kähler manifolds. Then Z and Z' are deformation equivalent. (More precisely, Z and Z' have arbitrarily small deformations that are isomorphic to each other.)*

Theorem 0.1 is thus a generalization of Huybrechts' theorem but the later is in fact very much used in the proof. The proof of Theorem 0.1 is indeed an easy application of Huybrechts' theorem using the following theorem.

Theorem 0.3. *Let $\mathcal{Z} \to \Delta$ be a projective morphism with general fiber Z_t a smooth hyper-Kähler manifold. Assume that one irreducible component of the central fiber Z_0 is not uniruled. Then after a finite base change $S \to \Delta$, the family $\mathcal{Z}'_S := \mathcal{Z}_\Delta \times_\Delta S \to S$ is bimeromorphic over S to a family $\pi' : \mathcal{Z}' \to S$ that is smooth proper over S with projective hyper-Kähler fibers.*

Remark 0.4. The assumption on the central fiber is satisfied if the desingularization of one irreducible component of Z_0 has a generically nondegenerate holomorphic 2-form and this is the main situation where we will apply the theorem. Note that with this stronger assumption, Theorem 0.3 was previously announced by Todorov [Tod90].

Remark 0.5. In the assumptions of Theorem 0.3, we did not ask that the considered irreducible component V be reduced. This is because after base change and normalization, we can remove multiplicities, still having a component satisfying the main assumption, but now reduced. In this process, the considered component V, when it has multiplicity > 1, is replaced by a generically finite cover of V, hence it is not in general birational to V. This is why we need the multiplicity 1 assumption in Theorem 0.1, whose statement actually involves the birational model of M.

A first important step in the proof of Theorem 0.3 is the following result:

Theorem 0.6. *Let $\mathcal{Z} \to \Delta$ be a projective morphism with general fiber Z_t a smooth hyper-Kähler manifold. Assume that one irreducible component of the central fiber Z_0 is not uniruled. Then, the monodromy action on the degree 2 cohomology of the smooth fiber Z_t is finite.*

Remark 0.7. For the geometric applications we mentioned above, checking finiteness of monodromy can be done directly, as we will explain for completeness. This is due to the fact that we are considering (a family of) badly degenerating hyper-Kähler manifolds associated to (a family of) mildly degenerating Fano hypersurfaces, for which the finiteness of monodromy is clear.

Once one has finiteness of the monodromy acting on H^2, Theorem 0.3 is a consequence of the following variant of Theorem 0.3 whose proof uses the surjectivity of the period map proved by Huybrechts and Verbitsky's Torelli theorem (see [Ver13], and also [Huy12]).

Theorem 0.8. *Let $\mathcal{Z} \to \Delta$ be a projective morphism with general fiber Z_t a smooth hyper-Kähler manifold. Assume the monodromy acting on $H^2(Z_t, \mathbb{Q})$ is finite. Then after a finite base change $S \to \Delta$, the family $\mathcal{Z}'_S := \mathcal{Z}_\Delta \times_\Delta S \to S$ is bimeromorphic over S to a family $\pi' : \mathcal{Z}' \to S$ which is smooth proper over S with projective hyper-Kähler fibers.*

Remark 0.9. Let us emphasize that this is a result specific for hyper-Kähler manifolds. There are examples of families of Calabi-Yau varieties for which the monodromy is finite, but not admitting a smooth filling after base change. The first example is due to Friedman [Fri86] who noticed that a generic degeneration to a quintic threefold with an A_2 singularity has finite order monodromy. Wang [Wan97, §4] then checked that there is no smooth filling. Another example, this time for Calabi-Yau fourfolds, is that of a Lefschetz 1-nodal degeneration of a sextic hypersurface in $\mathbb{P}^5$ which is treated in [Voi90]. For this example, Morgan [Mor83] shows that the monodromy is finite in the group of isotopy classes of diffeomorphisms of the smooth fibers X_t, so that after finite base change, the family admits a $\mathcal{C}^\infty$ filling. It is proved in [Voi90] that for no base change, the base-changed family admits a filling with a smooth Moïshezon fiber.

Theorem 0.8 tells us that under the same assumptions on $\mathcal{Z} \to \Delta$, there is, after base change, a family $\mathcal{Z}' \to \Delta$ birationally equivalent to $\mathcal{Z}$ over Δ, with smooth central fiber. The monodromy action on the whole cohomology of the fiber Z'_t is thus finite. With a little more work, we will prove in Section 3 that the monodromy action on the whole cohomology of the original fiber Z_t is also finite (see Corollary 3.2).

Theorem 0.6 rests on the application of MMP (see Section 1) to understanding the degenerations of K-trivial varieties (such as Calabi-Yau or hyper-Kähler manifolds). For a long time it was understood that the MMP plays a central role in this enterprise. Namely, the model result here is the Kulikov–Persson–Pinkham (KPP) Theorem which says that a 1-parameter degeneration of $K3$ surfaces can be arranged to be a semistable family satisfying the additional condition that the relative canonical class is trivial. As an application of this result, one obtains a control of the monodromy for the degenerations of $K3$ surfaces in terms of the central fiber and then a properness result for the period map. In higher dimensions, the analogue of the KPP theorem is that any 1-parameter degeneration of K-trivial varieties can be modified such that all the fibers have mild singularities and that the relative canonical class is trivial (this is nothing but a relative minimal model). More precisely, a higher dimensional analogue of the KPP theorem is given by Fujino [Fuj11] (building on [BCHM10]). We state a refinement of Fujino's result in Theorem 1.1, which provides some additional control on the behavior of the central fiber under the semistable reduction, followed by the minimal model program.

To complete the proof of Theorem 0.6, we use the fact that the singularities occurring in the MMP are mild from a cohomological point of view. This follows by combining the results of Kollár–Kovács [KK10] and Steenbrink [Ste81], which give a vast generalization and deeper understanding of the results of Shah [Sha79, Sha80] on degenerations of $K3$ surfaces. These arguments apply to degenerations of any K-trivial varieties, but since the cohomologically mild condition refers only to the holomorphic part of the cohomology (i.e. the $H^{k,0}$ pieces of the Hodge Structure), controlling the monodromy in terms of the central fiber is possible only for H^1 and H^2 (see Theorems 2.6 and 0.6 for the case of hyper-Kähler fibers). Since the degree 2 cohomology controls the geometry of hyper-Kähler manifolds, we obtain in Section 3 the much stronger result (that can not follow from general MMP) that certain degenerations of hyper-Kähler manifolds have smooth fillings.

As explained above, the smooth filling of finite monodromy degenerations (Theorem 0.8) is a result specific to hyper-Kähler manifolds. The proof given in Section 3 depends on deep properties of the period map. In Section 4, we give a completely different proof of Theorems 0.3 and 0.8, which again depends on specific results in the geometry of hyper-Kähler manifolds. Specifically, starting with a degeneration of hyper-Kähler manifolds $\mathcal{Z}/\Delta$ with a component of the central fiber not uniruled, by applying the MMP results of Section 1 and ideas similar to those in Section 2 (i.e. mild MMP singularities are mild cohomological singularities), we conclude that the central fiber Z_0 can be assumed to have symplectic singularities in the sense of Beauville [Bea00]. The rigidity results of Namikawa [Nam01, Nam06] then allow us to conclude that the degeneration can be modified to give a smooth family.

In the final section, Section 6, we make some remarks on the degenerations of hyper-Kähler manifolds with infinite monodromy. Namely, similarly to the case of degenerations of $K3$ surfaces, it is natural to define Type I, II, and III degenerations according to the index of nilpotency for monodromy N (where $N = \log T_u$, with T_u the unipotent part of the monodromy operator on H^2). For degenerations of $K3$ surfaces, a second theorem of Kulikov [Kul77] (see Theorem 6.1) gives a precise classification of the central fiber of the degeneration depending on Type (I, II, III). Our results (esp. Theorem 0.8) give a strong generalization of the Type I case of Kulikov. For the remaining Type II and III cases, we have weaker results, but which we believe to be of certain independent interest. Specifically, our focus is on the topology of the dual complex, a natural combinatorial gadget associated to semistable (or more generally *dlt degenerations*, by which we understand $(\mathcal{Z}, Z_t)$ is dlt for every $t \in \Delta$, where dlt (divisorial log terminal) is as in [Kol13, 2.8]).

Theorem 0.10. *Let $\mathcal{Z}/\Delta$ be a minimal dlt degeneration of $2n$-dimensional hyper-Kähler manifolds. Let Σ denote the dual complex of the central fiber (and $|\Sigma|$ its topological realization). Then*

(i) $\dim |\Sigma|$ *is* 0, n, *or* $2n$ *iff the Type of the degeneration is I, II, or III respectively (i.e.* $\dim |\Sigma| = (\nu - 1)n$, *where* $\nu \in \{1, 2, 3\}$ *is the nilpotency index of the log monodromy* N*).*

(ii) *If the degeneration is of Type III, then $|\Sigma|$ is a simply connected closed pseudo-manifold, which is a rational homology $\mathbb{CP}^n$.*

A few comments are in order here. First, this is clearly a (weak) generalization of Kulikov's theorem which states that for $K3$ surfaces, $|\Sigma|$ is either a point, an interval, or S^2 depending on the Type of the degeneration. Secondly, we note that under the assumption of minimal dlt degeneration, the dual complex is a well defined topological space (cf. [dFKX17], [MN15], and [NX13]). There is a significant interest in the study of the dual complex $|\Sigma|$ in connection with the SYZ conjecture in mirror symmetry, especially in the context of the work of Kontsevich–Soibelman [KS01, KS06]. In the strict Calabi-Yau case, it is expected that for maximal unipotent (MUM) degenerations $|\Sigma|$ is homeomorphic to the sphere S^n (in any case, it is always a simply connected rational homology S^n). The case $n = 2$ follows from Kulikov's Theorem, and the cases $n = 3$ and (conditionally) $n = 4$ were confirmed recently by Kollár–Xu [KX16]. Theorem 0.10 follows by arguments similar to the Calabi-Yau case (esp. [KX16] and [NX13]) and a result of Verbitsky [Ver96], which identifies the cohomology subalgebra generated by H^2 for a hyper-Kähler manifold. We also note that the occurrence of $\mathbb{CP}^n$ in Theorem 0.10 (see Theorem 6.15 for a more general statement) is in line with the predictions of mirror symmetry. Namely, in the case of hyper-Kähler manifolds, the base of the Lagrangian fibration occurring in SYZ can be identified (via a hyper-Kähler rotation) with the base of an algebraic Lagrangian fibration, and thus expected to be $\mathbb{CP}^n$ (see Hwang [Hwa08]).

We close by noting that in passing (in our study of dual complexes for HK degenerations) we essentially confirm a Conjecture of Nagai [Nag08] on the monodromy action on higher cohomology groups of hyper-Kähler manifolds (see Theorem 6.6 for a precise statement). Nagai has previously verified the conjecture for degenerations coming from Hilbert schemes of $K3$ surfaces or generalized Kummer varieties.

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1. Relative minimal models for degenerations of K-trivial varieties (aka Kulikov models)

The Kulikov-Persson-Pinkham Theorem ([Kul77, PP81]) states that any degeneration of $K3$ surfaces can be modified (after base change and birational transformations) to be semistable with trivial canonical bundle. In higher dimensions, the Minimal Model Program (MMP) guarantees for degenerations of K-trivial varieties the existence of a *minimal dlt model* $\mathcal{X}/\Delta$ (i.e. $K_{\mathcal{X}} \equiv 0$, and $(\mathcal{X}, X_t)$ is dlt for any $t \in \Delta$). The statement needed in this paper is due to Fujino [Fuj11, Theorem I]. The following is a version of Fujino's theorem with a focus on the relationship between the central fiber of the original degeneration and the central fiber in the resulting minimal dlt model (in particular we note that any non-uniruled component will survive in the resulting minimal dlt model):

Theorem 1.1. *Let $f : X \to C$ be a projective morphism to a smooth, projective curve C. Assume that*

(i) *the generic fiber X_g is irreducible and birational to a K-trivial variety with canonical singularities and*
(ii) *every fiber X_c has at least one irreducible component X_c^* that is not uniruled.*

Then there is a finite, ramified cover $\pi : B \to C$ such that $B \times_C X$ is birational to a projective morphism $f_B : Y \to B$ with the following properties.

(1) *The generic fiber Y_g is a K-trivial variety with terminal singularities,*
(2) *every fiber Y_b is a K-trivial variety with canonical singularities and*
(3) *if X_c^* has multiplicity 1 in X_c then Y_b is birational to X_c^* for $b \in \pi^{-1}(c)$.*

Remark 1.2. As a simple consequence of this theorem, we see that there can be at most one non-uniruled component for the central fiber of a degeneration of K-trivial manifolds.

Proof of Theorem 1.1. By the semistable reduction theorem, there is a finite ramified cover $\pi : B \to C$ such that $B \times_C X$ is birational to a projective morphism $q : Z \to B$ whose fibers are either smooth or reduced simple normal crossing divisors. Moreover, for $b \in \pi^{-1}(c)$ the fiber Z_b has at least one irreducible component Z_b^* that admits a generically finite, dominant morphism $\rho_b : Z_b^* \to X_c^*$. The degree of ρ_b divides the multiplicity of X_c^* in X_c. Thus if the multiplicity is 1 then ρ_b is birational.

By assumption the generic fiber is birational to a variety with canonical singularities and semiample canonical class. (This is called a *good minimal model*; in our case some multiple of the canonical class is actually trivial.) Thus by [Lai11, Thm.4.4] (see also [Fuj11]) the minimal model program for $q : Z \to B$ terminates with a model $f_B : Y \to B$ such that Y has terminal singularities and K_Y is f_B-nef.

A general fiber of f_B has terminal singularities and nef canonical class and it is also birational to a K-trivial variety. Thus general fibers of f_B are K-trivial varieties by (1.6). Therefore the canonical class $K_{Y/B}$ is numerically equivalent to a linear combination of irreducible components of fibers. A linear combination of irreducible components of fibers is nef iff it is numerically trivial (hence a linear combination of fibers). Thus $K_{Y/B}$ is numerically f_B-trivial.

The key point is to show that the fibers of $f_B : Y \to B$ are irreducible with canonical singularities. In order to do this, pick $b \in B$. By assumption (Z, Z_b) is a simple normal crossing (hence dlt) pair and Z_b is numerically q-trivial. Thus every step of the K_Z-minimal model program for $q : Z \to B$ is also a step of the $(K_Z + Z_b)$-minimal model program for $q : Z \to B$. Thus (Y, Y_b) is dlt (cf. [Kol13, 1.23]), in particular, every irreducible component of Y_b is normal (cf. [Fuj07, Sec.3.9] or [Kol13, 4.16]). The exceptional divisors contracted by a minimal model program are uniruled by [KMM87, 5-1-8]. Thus Z_b^* is not contracted and so it is birational to an irreducible component $Y_b^* \subset Y_b$ which is therefore not uniruled. Write $Y_b = Y_b^* + Y_b^\circ$ The adjunction formula (cf. [Kol13, Sec.4.1]) now gives that

$$K_{Y_b^*} \sim \big(K_{Y/B} + Y_b^*\big)|_{Y_b^*} \sim \big(K_{Y/B} - Y_b^\circ\big)|_{Y_b^*} \sim -Y_b^\circ|_{Y_b^*}.$$

(Note that in general we could have an extra term coming from singularities of Y along a divisor of Y_b^* but since (Y, Y_b) is dlt and Y_b is Cartier, this does not happen, cf. [Kol13, 4.5.5].) If $Y_b^\circ \neq 0$ then $-K_{Y_b^*}$ is effective and nonzero, hence Y_b^* is uniruled by [MM86]; a contradiction. Thus $Y_b = Y_b^*$ is irreducible. By the easy direction of the adjuction theorem (cf. [Kol13, 4.8]) it has only klt singularities and numerically trivial canonical class.

Let $\tau_b : Y_b^c \to Y_b$ be the canonical modification of Y_b (cf. [Kol13, 1.31]). If τ_b contracts at least 1 divisor then $K_{Y_b^c} \sim \tau_b^* K_{Y_b} - E$ where E is a positive linear combination of the τ_b-exceptional divisors. As before, we get that $-K_{Y_b^c}$ is effective and nonzero, hence Y_b^c is uniruled by [MM86]; a contradiction.

Thus τ_b is an isomorphism in codimension 1 and so $K_{Y_b^c} \sim \tau_b^* K_{Y_b}$. Since $K_{Y_b^c}$ is τ_b-ample, this implies that τ_b is an isomorphism. Hence Y_b has canonical singularities, as claimed. □

Remark 1.3. In general the above construction gives a model $Y \to B$ whose general fibers are only birational to the corresponding fibers of $X \to C$. We can, however, modify the construction in order to leave the general fibers unchanged. Assume first that general fibers of f are smooth K-trivial manifolds over an open subset $C^0 \subset C$. (This is the only case that we use in this note.) We can then choose $B \times_C X \to Z$ to be an isomorphism over $\pi^{-1}(C^0)$ and the minimal model program is then also an isomorphism over $\pi^{-1}(C^0)$. Thus we get $f_B : Y \to B$ that is isomorphic to $B \times_C X$ over $\pi^{-1}(C^0)$.

In general, assume that the generic fiber of f is a K-trivial variety with $\mathbb{Q}$-factorial terminal singularities. Let $C^0 \subset C$ be an open subset such that $f^{-1}(C^0)$ has $\mathbb{Q}$-factorial terminal singularities. Let $D_P := X \setminus f^{-1}(C^0)$, with reduced structure. First construct a dlt modification (cf. [Kol13, 1.34]) of (X, D_P) to get $(X', D'_P) \to C$ and then pick any $\pi : B \to C$ such that, for every $c \in P$, the multiplicities of all irreducible components of X'_c divide the ramification index of π over c. After base-change and normalization we get a model $q : Z \to B$ such that (Z, Z_b) is locally a quotient of a dlt pair for every $b \in B$. (See [dFKX17, Sec.5] for the precise definition of such qdlt pairs and their relevant properties.) The rest of the proof now works as before to yield $f_B : Y \to B$ that is isomorphic to $B \times_C X$ over $\pi^{-1}(C^0)$.

Remark 1.4. In the above proof it is essential that C be an algebraic curve. However, one can use [KNX15] to extend the theorem to the cases when C is either a smooth Riemann surface or a Noetherian, excellent, 1-dimensional, regular scheme over a field of characteristic 0. However, even when C a smooth Riemann surface, we still need to assume that $f : X \to C$ is at least locally projective, though this is unlikely to be necessary.

The next result follows from [Kaw08] but in fact it is part of the proof given there.

Lemma 1.5. *Let X_i be projective varieties with canonical singularities and nef canonical classes. Let $p_i : Y \to X_i$ be birational morphisms. Then $p_1^* K_{X_1} \sim_{\mathbb{Q}} p_2^* K_{X_2}$. (That is, the birational map $X_1 \dashrightarrow X_2$ is crepant in the terminology of* [Kol13, 2.23]*).*

Proof. We may assume that Y is normal and projective. Thus $K_Y \sim p_i^* K_{X_i} + E_i$ where E_i is p_i-exceptional and effective since X_i has canonical singularities. Thus $E_1 - E_2 \sim_{\mathbb{Q}} p_2^* K_{X_2} - p_1^* K_{X_1}$ is p_1-nef and $-(p_1)_*(E_1 - E_2) = (p_1)_*(E_2)$ is effective. Thus $-(E_1 - E_2)$ is effective by [KM98, 3.39]. Reversing the roles of p_1, p_2 gives that $-(E_2 - E_1)$ is effective, hence $E_1 = E_2$. □

Corollary 1.6. *Let X_i be birationally equivalent projective varieties with canonical singularities. Assume that $K_{X_1} \sim_{\mathbb{Q}} 0$ and K_{X_2} is nef. Then $K_{X_2} \sim_{\mathbb{Q}} 0$.* □

2. Cohomologically mild degenerations and Proof of Theorem 0.6

We first give an elementary proof of Theorem 0.6.

Proof of Theorem 0.6. Let $\mathcal{X} \to \Delta$ be as in Theorem 0.6, with hyper-Kähler fibers of dimension $2n$. According to Theorem 1.1, completed by Remark 1.3, we can find (after a finite base change) a model $\pi' : \mathcal{X}' \to \Delta$ isomorphic to $\mathcal{X} \to \Delta$ over Δ^* and such that the fiber X'_0 has canonical singularities. The morphism π' is projective and we can choose a relative embedding $\mathcal{X}' \subset \Delta \times \mathbb{P}^N$. Let $H \subset \mathbb{P}^N$ be a general linear subspace of codimension $2n - 2$. Then $S_0 := H \cap X_0$ is a surface with canonical singularities and for $t \neq 0$, $S_t := H \cap X_t$ is smooth (after shrinking Δ is necessary). The family $\mathcal{S} := \mathcal{X}' \cap (\Delta \times H)$ is thus a family of surfaces with smooth general fiber and central fiber with canonical singularities, hence the monodromy acting on $H^2(S_t, \mathbb{Z})$, $t \neq 0$ is finite. On the other hand, for $t \neq 0$, the restriction map

$$H^2(X_t, \mathbb{Z}) \to H^2(S_t, \mathbb{Z})$$

is injective by hard Lefschetz, and monodromy invariant. It follows that the monodromy acting on $H^2(X_t, \mathbb{Z})$, $t \neq 0$ is also finite. □

Remark 2.1. The same argument shows that in a projective degeneration $\mathcal{X} \to \Delta$ with smooth general fiber and special fiber X_0 satisfying $\operatorname{codim}(\operatorname{Sing} X_0) \geq k$, the monodromy acting on $H^l(X_t, \mathbb{Z})$ is trivial for $l < k$.

We are now going to discuss the result above from the viewpoint of Hodge theory and differential forms (similar arguments will be used again in Sections 4 and 6 below). The standard tool for studying 1-parameter degenerations $\mathcal{X}/\Delta$ of Kähler manifolds is the Clemens–Schmid exact sequence ([Cle77]). Specifically, this establishes a tight connection between the mixed Hodge structure (MHS) of the central fiber and the limit mixed Hodge Structure (LMHS), which depends only on the smooth family (and not the central fiber filling). As an application of this, under certain assumptions, one can determine the index of nilpotency for the monodromy $N = \log T$ for a degeneration purely in terms of the central fiber X_0 (e.g. as an application of Kulikov-Persson-Pinkham Theorem and Clemens–Schmid exact sequence, one obtains the properness of the period map for $K3$ surfaces). The big disadvantage of the Clemens–Schmid sequence is that it assumes that $\mathcal{X}/\Delta$ is a semistable family, which is difficult to achieve in practice. For surfaces, Mumford and Shah [Sha79] proved that one can allow X_0 to have mild singularities (called "insignificant limit singularities", which in modern terms is the same as Gorenstein semi-log-canonical (slc) singularities in dimension 2) and still get a tight connection between the MHS on X_0 and the LMHS. Shah's method was based on constructing explicit semistable models for this type of singularities and reducing to Clemens–Schmid. Steenbrink has noticed however that the true reason behind the close relationship between the MHS on the central fiber and the LMHS is the fact that Shah's insignificant singularities are du Bois. Specifically, we recall:

Definition 2.2 (Steenbrink [Ste81])**.** We say X_0 has *cohomologically insignificant singularities*, if for any 1-parameter smoothing $\mathcal{X}/\Delta$, the natural specialization map

$$sp_k : H^k(X_0) \to H^k_{\lim}$$

is an isomorphism on $I^{p,q}$-pieces (where $I^{p,q}$ denotes the Deligne's components of the MHS) with $p \cdot q = 0$.

Theorem 2.3 (Steenbrink [Ste81])**.** *If X_0 has du Bois singularities, then X_0 has cohomologically insignificant singularities.*

In other words, the original Shah [Sha79] theorem said that if X_0 has insignificant limit singularities (equivalently Gorenstein slc in dimension 2) then X_0 has cohomologically insignificant singularities. While Steenbrink noticed that the correct chain of implications is actually:

insignificant limit singularities ⇒ du Bois singularities ⇒ cohomologically insignificant singularities.

Three decades later, coming from a different motivation, Kollár–Kovács [KK10] (building on previous work by Kovács [Kov99] and others) have given a vast generalization of Shah's result:

Theorem 2.4 (Kollár–Kovács [KK10], [Kol13, 6.32])**.** *Let X_0 be a variety with slc singularities. Then X_0 has du Bois singularities.*

We immediately get the following consequence which will be improved later on:

Corollary 2.5. *Let $f : \mathcal{X} \to \Delta$ be a projective morphism. Assume that the generic fiber X_t is a smooth hyper-Kähler manifold, and that the special fiber X_0 has canonical singularities and $H^{2,0}(X_0) \neq 0$. Then the monodromy acting on $H^2(X_t)$, $t \neq 0$, is finite.*

Proof. By Theorem 2.4, the central fiber X_0, having canonical singularities (in fact, log canonical suffices), is du Bois. By Theorem 2.3, it follows that any degeneration is cohomologically insignificant, i.e. the natural specialization map $H^2(X_0) \to H^2_{\lim}$ is an isomorphism on the $I^{p,q}$ pieces with $p.q = 0$. The assumption is that $I^{2,0}(H^2(X_0)) \neq 0$. Since $\dim I^{2,0}_{\lim} + \dim I^{1,0}_{\lim} + \dim I^{0,0}_{\lim} = h^{2,0}(X_t) = 1$, the only possibility is that $H^2(X_0)$ and $H^2_{\lim}$ are both pure and agree on the $(2,0)$ and $(0,2)$ parts. In other words, $H^{2,0}_{lim}$ and its complex conjugate are contained in the monodromy invariant part H^2_{inv} of H^2_{lim}. The Hodge structure on H^2_{inv} is pure with $h^{2,0} = 1$ and the restriction to H^2_{inv} of the monodromy invariant pairing determined by the class l of a relatively ample line bundle is nondegenerate. By the Hodge index theorem, $M_{\mathbb{Z}} = (H^2_{inv})^{\perp} \cap H^2_{\lim,\mathbb{Z}} \subset H^2_{\lim,\mathbb{Q}}$ is a negative definite lattice. In particular, $O(M_{\mathbb{Z}})$ is a finite group, and as the monodromy action on $H^2_{\lim}$ factors up to a finite group through $O(M_{\mathbb{Z}})$, it is finite. □

To strengthen the previous corollary, we consider the situation coming from the KSBA theory of compactifications of moduli. Namely, we are interested in degenerations (flat and proper) $\mathcal{X}/\Delta$ which have the property that $K_{\mathcal{X}}$ is $\mathbb{Q}$-Gorenstein and the central fiber (is reduced and) has slc singularities, we call such a degeneration *KSBA degeneration*. If we assume additionally that $K_{\mathcal{X}}$ is relatively nef, we call it *minimal KSBA degeneration*. The total space of such a degeneration will have canonical singularities, and if needed, one can apply a terminalization, and obtain the so called *minimal dlt model*.

Theorem 2.6. *Let $\mathcal{X}/\Delta$ be a projective degeneration of hyper-Kähler manifolds. Assume that $\mathcal{X}/\Delta$ is a minimal KSBA degeneration (i.e. $K_{\mathcal{X}} \equiv 0$ and X_0 is slc). Then the following are equivalent:*

(1) *The monodromy action on $H^2(X_t)$ is finite.*
(2) *The special fiber X_0 has klt singularities (or equivalently, since Gorenstein degeneration, canonical singularities).*
(3) *The special fiber X_0 is irreducible and not uniruled (which in turn is equivalent to X_0 having a component that is not uniruled).*

Remark 2.7. The assumptions of K-triviality and minimality are clearly essential: a degeneration of curves to compact type has finite monodromy, but slc central fiber. Similarly, the blow-up of a family of elliptic curves gives a counterexample if we remove the minimality assumption.

Proof of Theorem 2.6. The equivalence between (2) and (3) is due to Fujino [Fuj11, Theorem II] (depending heavily on [MM86]); see also the proof of Theorem 1.1.

The implication (2) $\implies$ (1) is similar to Corollary 2.5. Namely, a variety X_0 with klt singularities has a pure Hodge structure on $H^1(X_0)$ and $H^2(X_0)$. This follows from the extension of holomorphic forms on such varieties (see [GKKP11]), and it is worked out in detail in Schwald [Sch16]. Since, klt varieties are du Bois (N.B. here, it suffices to apply an earlier result of Kovács [Kov99]: rational singularities are du Bois), it follows that the limit MHS in degree 2 is pure. As before, this is equivalent to the monodromy being finite.

To conclude the proof, it remains to see that if the monodromy is finite, the central fiber has to be klt. Assume not, then, possibly changing to a minimal dlt model, the components of X_0 are log Calabi-Yau varieties (V, D) with $D \not\equiv 0$. By adjunction, we see that D is a K-trivial variety. Either D has canonical singularities, in which case there is a holomorphic $(2n-1)$-form on D (where $2n = \dim X_t$, and $2n-1 = \dim D$). Also, as before, V is uniruled. Via a spectral sequence analysis as is [KX16, Claim 32.1] and [KX16, (32.2)] (N.B. while some of the arguments in [KX16] are only given for the normal crossing situation, one can adapt them to the dlt situation; see Appendix A, esp. Corollary A.7), we conclude $I^{2n-1,0}(H^{2n}(X_0))$, which in turn gives $I^{2n-1,0}(H^n_{\lim}) \neq 0$ (using again slc $\Longrightarrow$ du Bois $\Longrightarrow$ cohomologically insignificant). It follows (using Verbitsky's theorem [Ver96, Bog96] on the cohomology sub-algebra generated by H^2 and the arguments of Section 6, see esp. 6.18) that the Hodge structure on $H^2_{\lim}$ is not pure (i.e. if $H^2_{\lim}$ is pure, then $\dim I^{2n,0}(H^{2n}_{\lim}) = 1$, which contradicts $I^{2n-1,0}(H^{2n}_{\lim}) \neq 0$). The alternative (to D having canonical singularities) is that D is K-trivial, but with strict log canonical singularities. Then, the same argument applies inductively to D, leading to holomorphic forms that live deeper into the weight filtration of $H^{2n}_{\lim}$. The key observation here is that the deepest stratum, say of codimension k, (N.B. the dlt case behaves similarly to the snc case, see Section 6 and Appendix A) of X_0 has to be Calabi-Yau (with canonical singularities) leading to a holomorphic $(2n-k)$-form (which lives, via a Mayer-Vietoris spectral sequence, in $W_{2n-k}H^{2n}(X_0)$ and then $W_{2n-k}H^{2n}_{\lim}$). □

We conclude this section with another proof of Theorem 0.6.

Second proof of Theorem 0.6. By Theorem 1.1, after finite base change and birational transformations, we arrive at a minimal dlt model such that the central fiber consists of a unique non-uniruled component with canonical singularities. By Theorem 2.6, the monodromy acting on $H^2(X_t)$ is finite. □

3. Proof of Theorems 0.1, 0.3 and 0.8

Proof of Theorems 0.3 and 0.8. Let $\mathcal{X} \to \Delta$ be a projective morphism with smooth hyper-Kähler fibers over Δ^*, satisfying the hypothesis of Theorem 0.3. By assumption, one component of the central fiber is not uniruled. By Theorem 0.6, after performing a finite base change, we can assume that the monodromy acting on $H^2(X_t)$, $t \neq 0$ is trivial. We are now reduced to the situation of Theorem 0.8. Using a relatively ample line bundle on $\mathcal{X} \to \Delta$, the fibers X_t are projective with a given polarization $l := c_1(\mathcal{L}_{|X_t})$. Let q be the Beauville-Bogomolov form on $H^2(X_{t_0}, \mathbb{Q})$ for some given $t_0 \in \Delta^*$ and let

$$\mathcal{D}_l = \{\eta \in \mathbb{P}(H^2(X_{t_0}, \mathbb{C})),\, q(\eta) = 0,\, q(\eta, \overline{\eta}) > 0,\, q(\eta, l) = 0\}$$

be the polarized period domain for deformations of (X_{t_0}, l). The monodromy being trivial, the period map $\mathcal{P}^* : \Delta^* \to \mathcal{D}_l$ is well defined and by [Gri69], it extends to a holomorphic map $\mathcal{P} : \Delta \to \mathcal{D}_l$. (Note that this is one place where we seriously use the projectivity assumption. Griffiths' extension theorem holds for polarized period maps only.)

By [Huy99], the unpolarized period map is surjective from any connected component of the marked deformation space of X_t to $\mathcal{D}_l$. Thus there is a hyper-Kähler manifold X'_0 which is deformation equivalent to X_t, with period point $\mathcal{P}(0) \in \mathcal{D}_l$. Finally, as $q(l) > 0$, X'_0 is projective by [Huy99]. The local period map $B_l \to \mathcal{D}_l$ is a local holomorphic diffeomorphism, where B_l is a ball in the universal deformation space of the pair (X'_0, l) consisting of a hyper-Kähler manifold and a degree 2 Hodge class on it, and thus the holomorphic disk $\mathcal{P} : \Delta \to \mathcal{D}_l$ can be seen as a curve $\Delta \to B_l$. Shrinking B_l and Δ if necessary and restricting to Δ the universal family $\mathcal{X}_{univ} \to B_l$ existing over B_l, there is thus a family $\mathcal{X}' \to \Delta$ of marked hyper-Kähler manifolds in the same deformation class as X_t with the property that the associated period map $\mathcal{P}'$ identifies to $\mathcal{P}$.

We now apply Verbitsky's Torelli theorem [Ver13], which allows us to conclude that for any $t \in \Delta^*$, X'_t and X_t are birational. Furthermore, as the family $\pi' : \mathcal{X}' \to \Delta$ is smooth proper with Kähler fibers, there exists a $\mathcal{C}^\infty$ family $(\omega_t)_{t \in \Delta_t}$ of Kähler classes in the fibers of π'. Furthermore, we also know that the morphism $\pi : \mathcal{X} \to \Delta$ is projective. It follows (see [Bis64]) that the relative Douady space over Δ (analytic version of the relative Hilbert scheme) of subschemes in fibers of $X'_t \times X_t$ is a countable union of analytic varieties which are proper over Δ. Furthermore, note that for each component S of this relative Douady space with corresponding family $\Gamma_S \to S \to \Delta$, $\Gamma_S \subset \mathcal{X}_S \times_S \mathcal{X}'_S$, with $\mathcal{X}_S = \mathcal{X} \times_\Delta S$, $\mathcal{X}'_S = \mathcal{X}' \times_\Delta S$, the property that $\Gamma_{S,t}$ is the graph of a birational map between X_s and X'_s is Zariski open in S. We thus conclude that

Δ^* is the union of the images of the maps $S^0 \to \Delta^*$, over all components S which admit a dense Zariski open subset S^0 over Δ^* such that the cycles $\Gamma_{S,s}$ parameterized by $s \in S^0$ are graphs of birational maps between the fibers of both families. By a Baire category argument, there exists such an S which dominates Δ. We may assume that S is smooth and, by properness, that the map $S \to \Delta$ is finite and surjective. The universal subvariety $\Gamma_S \subset \mathcal{X}_S \times_S \mathcal{X}'_S$ provides the desired fibered birational isomorphism. □

Remark 3.1. The arguments given here are very similar to those used in [BR75] and even much simpler since we have Verbitsky's theorem, while Burns and Rapoport use them to prove Torelli's theorem for $K3$ surfaces.

Let us note the following consequence of Theorem 0.3.

Corollary 3.2. *Under the assumptions of Theorem 0.3, the monodromy action on $H^k(X_t)$ is finite for any k.*

Proof. Huybrechts' theorem 0.2 tells us that X_t and X'_t are deformation equivalent. It also says a little more: for any $t \in \Delta^*$, there exists a cycle Γ_t in $X_t \times X'_t$ which is a limit of graphs of isomorphisms between deformations of X_t and X'_t and thus induces an isomorphism of cohomology rings

$$H^*(X_t, \mathbb{Z}) \cong H^*(X'_t, \mathbb{Z}). \tag{3.3}$$

As the two families are Kähler over Δ, we can use properness of the relative Douady spaces to conclude that possibly after base change, there exists a cycle $\Gamma \in \mathcal{X} \times_\Delta \mathcal{X}'$ whose restriction Γ_t induces the isomorphism (3.3). The monodromy action on $H^k(X_t)$ thus becomes trivial after base change for any k. □

Proof of Theorem 0.1. The proof follows closely the proof of Huybrechts [Huy99, Theorem 4.6]. Under the assumptions of Theorem 0.1, Theorem 0.3 gives us a birational map $\phi : \mathcal{X}'_S \dashrightarrow \mathcal{X}_S$ over a finite cover S of Δ, where $\mathcal{X}'_S$ is smooth over S with hyper-Kähler fibers. Let us blow-up $\mathcal{X}_S$ until it becomes smooth, say $\widetilde{\mathcal{X}}_S$, and then let us blow-up $\mathcal{X}'_S$ successively along smooth centers until the rational map ϕ induces a morphism $\tilde{\phi} : \widetilde{\mathcal{X}}'_S \to \widetilde{\mathcal{X}}_S$ over S. By assumption, the central fiber X_0 of our original family has a multiplicity 1 component V which is birational to the smooth hyper-Kähler manifold Z'_0 (which is projective, as it is Moïshezon and Kähler). The proper transform $\widetilde{V}$ of V is thus birational to V and also appears as a multiplicity 1 component of the central fiber of $\widetilde{\mathcal{X}}_S \to S$. (It is at this point that we use the fact that V is a multiplicity 1 component of X_0; otherwise the desingularization process needed to produce $\mathcal{X}'_S$ can involve a normalization which replaces V by a generically finite cover of it.) As $\tilde{\phi}$ is proper and birational, exactly one component V' of the central fiber of $\widetilde{\mathcal{X}}'_S \to S$ maps onto $\widetilde{V}$ and the morphism $V' \to \widetilde{V}$ is birational. Hence V' is birational to Z'_0. On the other hand, as $\mathcal{X}'_S$ is smooth, all the exceptional divisors of $\widetilde{\mathcal{X}}'_S \to \mathcal{X}'_S$ are uniruled, and thus the only component (called V' above) of the central fiber which can be birational to a hyper-Kähler manifold is the proper transform of X'_0 (via $\widetilde{\mathcal{X}}'_S \to \mathcal{X}'_S$). Thus we proved that X'_0 and Z'_0 are birational. This is now finished because by Huybrechts' theorem [Huy99] (Theorem 0.2), it follows that the two hyper-Kähler manifolds X'_0 and Z'_0 are deformation equivalent. On the other hand, X'_0 is by definition deformation equivalent to X'_t which is birational to X_t for $t \neq 0$, hence is deformation equivalent to X_t since X_t and X'_t are smooth. We conclude X_t is deformation equivalent to Z'_0 as claimed. □

4. Symplectic singularities; Alternative proof to Theorems 0.3 and 0.8

As previously discussed, the filling Theorems 0.3 and 0.8 are results specific to hyper-Kähler manifolds (see also Remark 0.9). In the previous section, we have proved these theorems by using the deep results (Torelli and surjectivity) for the period map for hyper-Kähler manifolds due to Huybrechts [Huy99] and Verbitsky [Ver13]. The MMP results are only tangentially used. In this section, we give an alternative proof to Theorems 0.3 and 0.8 relying essentially on MMP and on the results on Sections 1 and 2. The key point of this alternative proof is to notice that in the case of a minimal dlt degeneration $\mathcal{X}/\Delta$ of HK manifolds with finite monodromy, the central fiber X_0 has *symplectic singularities* in the sense of Beauville [Bea00]. The filling theorems now follow from the results of Namikawa [Nam01, Nam06], which roughly say that the symplectic singularities are rigid (and thus, if not smooth, there is no smoothing).

Another proof of Theorems 0.3 and 0.8. By Remark 1.4 we can apply Theorem 1.1 to the projective morphism $f : \mathcal{X} \to \Delta$. This gives, possibly after a base change $\pi : \Delta \to \Delta$, a projective morphism $\mathcal{Y} \to \Delta$ and a birational map $h : \mathcal{Y} \dashrightarrow \mathcal{X} \times_\Delta \Delta$ which induces a birational map from the central fiber Y_0, which is a Calabi–Yau with canonical singularities, to X_0^*. By Remark 1.3, we can also ensure that h induces an isomorphism between the fibers Y_t and $X_{\pi(t)}$, for $t \neq 0$.

We claim that Y_0 is *a symplectic variety* in the sense of Beauville [Bea00]. By definition, this means that Y_0 has canonical singularities and that the smooth locus of Y_0 carries a holomorphic symplectic form with the property that it extends to a holomorphic form on any resolution of Y_0. In our situation, as already noted Y_0 has canonical singularities. To check that the smooth locus of Y_0 carries a holomorphic 2-form that is symplectic and that it extends to resolutions, we use arguments similar to those of Section 2, but some extra care is needed to be able to interpret $H^{2,0}(Y_0)$ as holomorphic forms (N.B. Y_0 is singular). To start, since Y_0 has canonical singularities then Y_0 has rational singularities ([Kol13, (2.77) and (2.88)]) and then du Bois singularities ([Kov99]) (this also follows from [KK10], see Theorem 2.4). By [Ste81, Thms 1 and 2] this implies that $R^if_*\mathcal{O}_\mathcal{Y}$ is locally free (of rank one if i is even, zero otherwise) and satisfies base change, that the Hodge filtration on $H^i(Y_0)$ satisfies $Gr_F^0 H^i(Y_0) = H^i(Y_0, \mathcal{O}_{Y_0})$, and that the degeneration $\mathcal{Y} \to \Delta$ is cohomologically insignificant (cf. Section 2), i.e. that the specialization map $H^i(Y_0) \xrightarrow{sp} H^i_{\lim}$ induces an isomorphism on the (p,q)–pieces with $p \cdot q = 0$. Since Y_0 has rational singularities, $\pi^* : H^2(Y_0) \to H^2(\widetilde{Y}_0)$ is injective ([KM92, (12.1.3.2)]) and hence by [Del74, Cor. 8.2.5] the MHS on $H^2(Y_0)$ is pure of weight two. In particular, $Gr_F^0 H^2(Y_0) = H^{0,2}(Y_0)$ and hence $h^{2,0}(Y_0) = h^{0,2}(Y_0) = 1$. Let $\sigma_0 \in H^2(Y_0)$ be a generator of $H^{2,0}(Y_0) = F^2H^2(Y_0)$. We need to show that σ_0 defines a holomorphic symplectic form on the smooth locus of Y_0, which extends to a holomorphic 2–form on any resolution $\pi : \widetilde{Y}_0 \to Y_0$. We remarked that the pullback is injective on degree two cohomology, so $\pi^*(\sigma_0)$ defines a non–zero holomorphic 2–form $\widetilde{\sigma}_0$ on $\widetilde{Y}_0$. To show that it is generically symplectic, it is sufficient to show that $\widetilde{\sigma}_0^n \neq 0$. The cup–product is compatible with the specialization map and also with Deligne's MHS ([Del74, Cor. 8.2.11]), so $sp(\sigma_0^n) = sp(\sigma_0)^n$ lies in $F^{2n}H^{2n}_{\lim} \cap W_{2n} = H^{2n,0}_{\lim}$. Since $H^i_{\lim}$ is the i–th cohomology of a smooth hyper–Kähler manifold, by the cited result of Verbitsky (see [Bog96]) we know that $\mathrm{Sym}^i H^2_{\lim} \to H^{2i}_{\lim}$ is injective for $i \leq n$. Hence $sp(\sigma_0^n)$ generates $H^{2n,0}_{\lim}$ and, in particular, σ_0^n is non–zero. We are left with showing that the pullback $\widetilde{\sigma}_0^n = \pi^*(\sigma_0^n) \neq 0$. But this follows from the fact that the pullback morphism $\pi^* : H^i(Y_0) \to H^i(\widetilde{Y}_0)$ is injective one the weight i part of the MHS ([Del74, Cor. 8.2.5]).

Let $\pi : Y' \to Y_0$ be a $\mathbb{Q}$–factorial terminalization, i.e., Y' is $\mathbb{Q}$–factorial and terminal, and π is a crepant morphism. This always exists by [BCHM10, Cor. 1.4.3]. We use Namikawa's ([Nam06, Cor 2]) to show that Y' is smooth. For the readers sake we recall Namikawa's argument: The Main Theorem in [Nam06] shows that $\mathbb{Q}$–factorial symplectic varities with terminal singularities are locally rigid. Hence, to prove that Y' is smooth, it is enough to show that a smoothing of Y'_0 determines a smoothing of Y'. Indeed, since $R^1\pi_*\mathcal{O}_{Y'} = 0$, any deformation of Y' induces a deformation of Y_0 ([Wah76, Thm 1.4], [KM92, 11.4]). More specifically, let $\mathcal{Y}' \to \mathrm{Def}(Y')$ and $\mathcal{Y} \to \mathrm{Def}(Y_0)$ be versal deformation spaces for Y' and Y_0, respectively. By [Nam01, Thm 1] there is a finite morphism $\pi_* : \mathrm{Def}(Y') \to \mathrm{Def}(Y_0)$ which lifts to a morphism $\Pi_* : \mathcal{Y}' \to \mathcal{Y}$ inducing an isomorphism between the general deformation of Y' and the general deformation of Y_0. □

Corollary 4.1. *Let $\mathcal{X} \to \Delta$ be a projective degeneration with general fiber X_t a smooth hyper-Kähler manifold. Assume that one irreducible component V of the central fiber X_0 is not uniruled and appears with multiplicity one. Then any minimal model of V has a symplectic resolution (which is a smooth hyper-Kähler deformation equivalent to X_t) and the monodromy action on the cohomology of a smooth fiber of f is finite. Conversely, if the monodromy of $\mathcal{X} \to \Delta$ is finite, then there exists a smooth family $\mathcal{Y} \to \Delta$ of hyper-Kähler manifolds that is isomorphic over Δ^* to (a finite base change) of $\mathcal{X}^* \to \Delta^*$.*

Proof. The first part of the statement follows directly from the above proof. The second statement follows by using Remark 1.3, the equivalence (1) $\iff$ (3) of Theorem 2.6, and then again the arguments of this section. □

Remark 4.2. Notice that the second statement gives a stronger version of Theorem 0.8 (which leaves the general fibers unchanged).

We also highlight the following useful remark of Greb–Lehn–Rollenske [GLR13, Prop. 6.4]:

Remark 4.3. If X_0 is symplectic variety having a smooth hyper-Kähler birational model, then X_0 admits a symplectic resolution.

5. Application: Deformation type of hyper-Kähler manifolds via degeneration methods

The main techniques for constructing hyper-Kähler manifolds is the Beauville-Mukai method, starting with a K3 or an abelian surface and considering moduli spaces of sheaves on them. This leads to $K3^{[n]}$ type and also after a delicate desingularization process, to the exceptional OG10 examples (and similarly to the generalized Kummer varieties and the exceptional OG6 manifolds when starting from an abelian surface). It turns out that there are other geometric constructions leading to hyper-Kähler manifolds, most notably starting with a cubic fourfold ([BD85], [LSV16], [LLSvS13]). In all these cases, a series of ad hoc geometric arguments were used to establish the deformation equivalence of these new constructions to the Beauville-Mukai examples. As an application of our results on degenerations of hyper-Kähler we give a somewhat unified and simplified method to obtain their deformation equivalence to the $K3^{[n]}$ or OG10 types. Namely, as investigated by Hassett [Has00], various codimension 1 loci in the moduli of cubic fourfolds are Hodge theoretically (and sometimes geometrically) related to $K3$ surfaces. Specializing to these loci often gives a clear link between the hyper-Kähler manifolds constructed from cubics and the ones constructed from $K3$ surfaces by Beauville-Mukai or O'Grady construction. In fact the easiest specializations of a cubic fourfold linking cubic fourfolds to $K3$ surfaces are specializations to nodal cubic fourfolds ($\mathcal{C}_6$ in Hassett's notation) or degenerations to the cubic secant to the Veronese surface in $\mathbb{P}^5$ (or $\mathcal{C}_2$; see also [Laz10]). In these cases, the associated $K3$ surface is obvious (e.g., in the nodal case it appears as the set of lines through the node of the cubic), and after specialization, a birational model of the associated hyper-Kähler manifold is easy to understand. The problem is that even though the cubic degeneration is as mild as possible, the associated hyper-Kähler manifolds (e.g. the Fano variety of lines) specialize to quite singular objects. Our main result Theorem 0.1 tells us that as long as the holomorphic 2-form survives for the degeneration, we can ignore the singularities. In this section, we are thus going to revisit [BD85], [DV10], [LSV16], [AL14] in the light of Theorem 0.1.

5.1. **Fano variety of lines of a cubic fourfold.** Let X be a smooth cubic fourfold. The variety of lines $F(X)$ is a smooth projective hyper-Kähler.fourfold by [BD85]. It is deformation equivalent to $S^{[2]}$, where S is $K3$ surface. More precisely, Beauville and Donagi prove the following:

Theorem 5.1. *Let X be a smooth Pfaffian cubic fourfold and S be the associated $K3$ surface. Then $F(X)$ is isomorphic to $S^{[2]}$.*

Here a Pfaffian cubic fourfold is defined as the intersection of the Pfaffian cubic in $\mathbb{P}^{14} = \mathbb{P}(\bigwedge^2 V_6)$ with a $\mathbb{P}^5 = \mathbb{P}(W_6) \subset \mathbb{P}^{14}$. The associated $K3$ surface S is defined in the Grassmannian $G(2, V_6^*)$ by the space W_6, seen as a set of Plücker linear forms on $G(2, V_6^*)$.

Note that Theorem 5.1 is used in [BD85] in order to prove that $F(X)$ is a smooth projective hyper-Kähler fourfold for general X. However, this last fact can be seen directly by saying that (1) $F(X)$ is smooth as all varieties of lines of smooth cubics are; (2) $F(X)$ has trivial canonical bundle as it is the zero set of a transverse section of $S^3\mathcal{E}$ on $G(2, V_6^*)$, where $\mathcal{E}$ is the dual of tautological rank 2 vector subbundle on $G(2, V_6^*)$, and (3) $F(X)$ has a holomorphic 2-form defined as $P^*\alpha_X$, where $P \subset F(X) \times X)$ is the incidence correspondence, and it can easily be proved to be generically nondegenerate.

Instead of considering the specialization to the Pfaffian case, let us consider the specialization to the nodal case, where X specializes to X_0 with one ordinary double point at $0 \in X_0$. Let thus be $\pi^X : \mathcal{X} \to \Delta$ be such a Lefschetz degeneration, and let $\pi^F : \mathcal{F} \to \Delta$ be the associated family of Fano varieties of lines. It is well-known (see [CG72]) that $F(X_0)$ is birational to $\Sigma^{[2]}$, where Σ is the surface of lines in X_0 passing through 0. Σ is the smooth intersection of a quadric and a cubic in $\mathbb{P}^4$, hence a $K3$ surface, and the birational map $\Sigma^{[2]} \dashrightarrow F(X_0)$ associates to a pair of lines l, l' through 0 the residual line of the intersection $P_{l,l'} \cap X_0$ where $P_{l,l'}$ is the plane generated by l and l'. Note also that the variety of lines of X_0 is smooth away from the surface Σ, hence $F(X_0)$ is a multiplicity 1 component of the central fiber of the family $\mathcal{F} \to \Delta$. Theorem 0.1 thus applies showing that $F(X_t)$ is deformation equivalent to $\Sigma^{[2]}$.

Remark 5.2. Note that in this example, we can check directly that the monodromy acting on $H^2(F(X_t))$ is finite (thus avoiding the use of Theorems 1.1 and 0.6). Indeed, the monodromy action on $H^4(X_t)$ is finite, being given by a Picard-Lefschetz reflection, and as the relative incidence correspondence $\mathcal{P} \subset \mathcal{F} \times_\Delta \mathcal{X}$ induces an isomorphism $\mathcal{P}^* : R^4\pi^X_*\mathbb{Q} \to R^2\pi^F_*\mathbb{Q}$ of local systems over Δ^*, the monodromy acting on $H^2(F(X_t))$ is also finite. The same remark applies in fact to all the cases described in section 5.

5.2. **Debarre-Voisin hyper-Kähler fourfolds.** The hyper-Kähler fourfolds constructed in [DV10] are defined as zero-sets Y_σ of general sections σ of the rank 20 vector bundle $\bigwedge^3 \mathcal{E}$ on the Grassmannian $G(6, V_{10})$, where $\mathcal{E}$ is the dual of the rank 6 tautological vector subbundle on $G(6, 10)$. Hence $\sigma \in \bigwedge^3 V_{10}^*$. It is proved in [DV10] that these varieties are deformation equivalent to $K3^{[2]}$. Let us now explain how the use of Theorem 0.1 greatly simplifies the proof of this statement. The choice of $\sigma \in \bigwedge^3 V_{10}^*$ also determines a hypersurface (a Plücker hyperplane section) $X_\sigma \subset G(3, V_{10})$. For general σ, X_σ is smooth of dimension 20 and there is an isomorphism

$$G^*_\sigma : H^{20}(X_\sigma, \mathbb{Q})_{prim} \to H^2(Y_\sigma, \mathbb{Q})_{prim} \tag{5.3}$$

induced by the incidence correspondence $G_\sigma \subset Y_\sigma \times X_\sigma$, where the fiber of G_σ over a point $[W_6] \in Y_\sigma$ is the Grassmannian $G(3, W_6)$ which is by definition contained in X_σ (see [DV10]). In the paper [DV10], the generic nodal degeneration $\pi^X : \mathcal{X} \to \Delta$ of X_σ is considered, with the associated family $\pi^Y : \mathcal{Y} \to \Delta$ and relative incidence correspondence $\mathcal{G} \subset \mathcal{Y} \times_\Delta \mathcal{X}$.

Finally we have the following result (see [DV10, Theorem 3.3]):

Theorem 5.4. *The variety Y_{σ_0} is reduced and birationally equivalent to $S^{[2]}$, where S is a $K3$ surface.*

We are thus in position to apply Theorem 0.1 and this shows that the smooth fibers $\mathcal{Y}_{\sigma_t}$ are deformation equivalent to $K3^{[2]}$. In the paper [DV10], the proof of this fact used a delicate analysis of the pull-back to $S^{[2]}$ of the Plücker line bundle, so as to apply a Proj argument in the spirit of Huybrechts. For the sake of completeness, let us recall how the $K3$ surface S is constructed in this case. Let X_σ be singular at $[W] \in G(3, V_{10})$. Then $\sigma_{|W} = 0$ in $\bigwedge^3 W^*$ and furthermore σ vanishes in $\bigwedge^2 W^* \otimes (V_{10}/W)^*$. Thus σ defines an element σ_2 of $W^* \otimes \bigwedge^2(V_{10}/W)^*$. Let $V_7 := V_{10}/W$. The surface S is defined as the set of 3-dimensional subspaces of V_7 whose inverse image in V_{10} belongs to Y_σ. This is a $K3$ surface: Indeed, σ_2 gives three sections of the bundle $\bigwedge^2 \mathcal{E}_3$ on the Grassmannian $G(3, V_7)$, where as usual $\mathcal{E}_3$ denotes the dual of the tautological subbundle on the Grassmannian $G(3, 7)$. On the vanishing locus of these three sections (that we can also see via the projection $V_{10} \to V_7$ as embedded in $G(6, V_{10})$), the section σ gives a section of $\bigwedge^3 \mathcal{E}_3$. Hence S is defined in $G(3, V_7)$ by three sections of $\bigwedge^2 \mathcal{E}_3$ and one section of $\bigwedge^3 \mathcal{E}_3$. Thus it has trivial canonical bundle and is in fact the general member of one of the families of $K3$ surfaces described by Mukai [Muk06].

5.3. **O'Grady 10-dimensional examples and intermediate jacobian fibrations.** This section is devoted to the hyper-Kähler manifolds $\mathcal{J}$ constructed in [LSV16] as a K-trivial compactification of the intermediate Jacobian fibration $\mathcal{J}_U \to U$ associated to the universal family $\mathcal{Y}_U \to U$ of smooth hyperplane sections of a general cubic fourfold $X \subset \mathbb{P}^5$. Here $U \subset (\mathbb{P}^5)^*$ is the Zariski open set parameterizing smooth hyperplane sections of X. Our aim is to give a new proof of [LSV16, Corollary 6.2]:

Theorem 5.5. *The varieties $\mathcal{J}$ are deformations of O'Grady's 10-dimensional hyper-Kähler manifolds.*

The original proof had been obtained by specializing X to a general Pfaffian cubic fourfold X_{Pf}. The proof that $\mathcal{J}_{X_{\mathrm{Pf}}}$ exists and is smooth does not necessitate much extra work but the proof that it is birational to the O'Grady moduli space $M_{4,2,0}(S)$ (where S is the associated $K3$ surface of degree 14 as in Section 5.1) is rather involved and uses work of Markushevich-Tikhomirov [MT01] and Kuznetsov [Kuz04] on Pfaffian geometry in the threefold case. We are going to use here a different degeneration which was introduced by Hassett [Has00], and plays an important role in [Laz10], [Loo09]. Let X_0 be the chordal cubic fourfold which is defined as the secant variety of the Veronese surface $V \subset \mathbb{P}^5$. Blowing up the parameter point $[X_0]$ in the space of all cubics, the general point of the exceptional divisor determines a cubic X_∞, (or rather its restriction to X_0). The restriction of X_∞ to V gives a sextic curve $C \subset \mathbb{P}^2 \cong V$, hence a $K3$ surface obtained as the double cover $r : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ ramified along C. It is proved in [Laz10], [Loo09] that the period map defined on the regular part of the pencil $\langle X_0, X_\infty \rangle$ extends over 0 (in particular the monodromy on degree 4 cohomology of the smooth fibers X_t is finite) and the limit Hodge structure is that of $H^2(S)$.

A hyperplane section $Y_0 = H \cap X_0$ of X_0 determines by restriction to V a conic $C = H \cap V$ in $\mathbb{P}^2$ whose inverse image $C' = r^{-1}(C)$ is a hyperelliptic curve of genus five. The degeneration of a smooth cubic threefold $Y_t = H \cap X_t$ to a pair (Y_0, Y_∞), consisting of the Segre cubic threefold (secant variety of a normal quartic curve $\mathbb{P}^1 \cong C_0 \subset \mathbb{P}^4$) and a cubic hypersurface section $Y_\infty = X_\infty \cap Y_0$ of it, is studied first in [Col82], see also [ACT11]. It is proved there that the intermediate Jacobian $J(Y_t)$ specializes to the Jacobian $J(C')$, where C' is the hyperelliptic curve defined as the double cover of C_0 ramified at the 12 points of $C_0 \cap Y_\infty$.

Remark 5.6. Note that under a general one-parameter degeneration of a cubic threefold to the Segre cubic threefold, the hyperelliptic Jacobian over 0 is a smooth (in particular reduced) fiber of the associated one-parameter family of intermediate Jacobians. This is clear since we are actually working with abelian varieties and not torsors (there is a 0-section).

It follows from this discussion that if $\mathcal{X} \to B$ is a general one-parameter family of cubic fourfolds with central fiber X_0 and fist order deformation determined by a generic X_∞, then the corresponding family $\mathcal{J}_\mathcal{X}$ (which is well defined over a Zariski open set of B and is a family of projective hyper-Kähler varieties) has a component of its central fiber which is birational to the Jacobian fibration $\mathcal{J}_{\mathcal{C}'}$ associated to the universal family $\mathcal{C}' \to (\mathbb{P}^5)^*$ of hyperelliptic curves on S.

The following fact already appears in [OR14]:

Proposition 5.7. *Let $r : S \to \mathbb{P}^2$ be a $K3$ surface which is a double cover of $\mathbb{P}^2$ ramified along a sextic curve. Assume $\operatorname{Pic} S = \mathbb{Z}$. Let $(\mathbb{P}^5)^*$ be the set of conics $C \subset \mathbb{P}^2$, that we see as parameterizing genus five hyperelliptic curves $C' = r^{-1}(C)$ in S. Then the Jacobian fibration $\mathcal{J}_{\mathcal{C}'} \to (\mathbb{P}^5)^*$ associated to the universal family of curves $\mathcal{C}' \to (\mathbb{P}^5)^*$ is birational to the O'Grady moduli space $M_{4,2,0}(S)$ of rank 2 vector bundles on S, with trivial determinant and $c_2 = 4$.*

Proof. Denoting $H = r^*\mathcal{O}(1) \in \operatorname{Pic} S$, the curves C' belong to $|2H[$. Let E be a general stable rank 2 vector bundle on S with $c_2 = 4$ and $c_1 = 0$. One has $\chi(S, E(H)) = 2$ and $H^1(S, E(H)) = 0 = H^2(S, E(H))$ as shows specialization to the case of the torsion free sheaf $I_z \oplus I_{z'}$ where z and z' are two general subschemes of length 2 on S. Thus E has two sections and is generically generated by them, again by the same specialization argument. So we have an injective evaluation map $W \otimes \mathcal{O}_S \to E(H)$, and its determinant vanishes along a curve $C' \in |2H|$. The cokernel of the evaluation map is then a line bundle L of degree 2 on C', as it has $H^0(C, L') = 0$, $H^1(C, L') \cong W$. Conversely, start with a general curve $C' \in |2H|$ and a general line bundle L' of degree 2 on C'. Then $H^0(C', K_{C'} - L')$ has dimension 2, and the Lazarsfeld-Mukai bundle associated to the pair (C', L') provides a rank 2 bundle with the desired Chern classes on S. Thus we constructed a birational map between $M_{4,2,0}(S)$ and the relative Picard variety $\mathcal{J}_{\mathcal{C}',2}$ of line bundles of degree 2 on the family $\mathcal{C}$ of curves C', which is in fact birational to $\mathcal{J}_{\mathcal{C}'}$ since the curves C' are hyperelliptic. Indeed, the hyperelliptic divisor gives a section of $\mathcal{J}_{\mathcal{C}',2}$ which provides the isomorphism above by translation. ☐

Theorem 5.5 now follows from Proposition 5.7 and Theorem 0.1. The only thing to check is the fact that under a general one-parameter degeneration of a cubic fourfold to the secant to Veronese X_0, the hyperelliptic Jacobian fibration $\mathcal{J}_{\mathcal{C}'}$ introduced above appears as a component in the central fiber of the associated family of intermediate Jacobian fibrations. As these varieties are fibered over a Zariski open set of $(\mathbb{P}^5)^*$, the fact that this component has multiplicity 1 follows from Remark 5.6. The proof is thus complete.

5.4. **LLSvS eightfolds.** The LLSvS eightfolds were constructed in [LLSvS13], and were proved in [AL14] (see also [Leh15]) to be deformation equivalent to $S^{[4]}$. These hyper-Kähler manifolds are constructed as follows: Start from a general cubic fourfold X and consider the Hilbert scheme $\mathcal{H}_3$ of degree 3 rational curves in X. Then $\mathcal{H}_3$ is birational to a $\mathbb{P}^2$-bundle over a hyper-Kähler manifold $Z(X)$. The following is proved in [AL14]:

Theorem 5.8. *If $X \subset \mathbb{P}(\bigwedge^2 V_6)$ is Pfaffian, then $Z(X)$ is birational to $S^{[4]}$, where $S \subset G(2, V_6)$ is the associated $K3$ surface as in Section 5.1.*

This result, combined with Huybrechts' Theorem 0.2, implies:

Corollary 5.9. *The varieties $Z(X)$ are deformation equivalent to $S^{[4]}$.*

Let us now give another proof of this last result, based on Theorem 0.1 and the degeneration to the chordal cubic. In [LSV16], it is noticed that the varieties $\mathcal{J}(X)$ and $Z(X)$ are related as follows: the intermediate Jacobian fibration $\mathcal{J}_U$ of X has a canonical Theta divisor which is birationally a $\mathbb{P}^1$-bundle over $Z(X)$. Indeed, we know by Clemens-Griffiths [CG72] that the Theta divisor in the intermediate Jacobian of a cubic threefold Y is parameterized via the Abel-Jacobi map of Y by degree 3 rational curves on Y, the fiber passing through a general curve $[C] \in \mathcal{H}_3(Y)$ being the $\mathbb{P}^2$ of deformations of C in the unique cubic surface $\langle C\rangle \cap X$ containing C. It follows from this result that the relative Theta divisor $\Theta \subset \mathcal{J}_U$ parameterizes the data of such a $\mathbb{P}^2_C \in \mathcal{H}_3(X)$ and of a hyperplane section Y of X containing the cubic surface $\langle C\rangle$. This is clearly birationally a $\mathbb{P}^1$-bundle over $Z(X)$. We now specialize X to the chordal cubic X_0, or more precisely to a point of the exceptional divisor of the blow-up of this point in the space of all cubics, which determines as in the previous section a degree 2 $K3$ surface $r : S \to \mathbb{P}^2 = V$. We use the fact already exploited in the previous section that the intermediate Jacobian fibration $\mathcal{J}_U$ then specializes birationally to the Jacobian fibration $\mathcal{J}_{\mathcal{C}'}$ associated to the family $\mathcal{C}'$ of hyperelliptic curves $C' = r^{-1}(C)$, C being a conic in $\mathbb{P}^2$. The Theta divisor $\Theta \subset \mathcal{J}_U$ specializes to the Theta divisor $\Theta_{\mathcal{C}'}$ which is indeed contained in $\mathcal{J}_{\mathcal{C}'}$ since the curves C' have a natural degree 4 divisor (the canonical Theta divisor is naturally contained in $\mathrm{Pic}^4(C')$ for a genus 5 curve C', so by translation using $H_{|C'}$, we get it contained in $\mathrm{Pic}^0(C')$). We now have:

Proposition 5.10. *The divisor $\Theta_{\mathcal{C}'} \subset \mathcal{J}_{\mathcal{C}'}$ is birational to a $\mathbb{P}^1$-bundle over $S^{[4]}$.*

Proof. Let us identify $\mathcal{J}_{\mathcal{C}'}$ to $\mathcal{J}^4_{\mathcal{C}'}$ via translation by the section $[C] \mapsto H_{|C'}$ of $\mathcal{J}^4_{\mathcal{C}'}$. Then $\Theta_{\mathcal{C}'} \subset \mathcal{J}^4_{\mathcal{C}'}$ is the family of effective divisors of degree 4 in curves $C' \subset S$. Such an effective divisor determines a subscheme of length 4 in S. This gives a rational map $\phi : \mathcal{J}^4_{\mathcal{C}'} \dashrightarrow S^{[4]}$. Given a generic subscheme $z \subset S$ of length four, z is contained in a pencil of curves $C' \subset S$ and determines an effective divisor of degree 4 in each of them, showing that the general fiber of ϕ in a $\mathbb{P}^1$. This shows that, via ϕ, $\Theta_{\mathcal{C}'}$ is birationally a $\mathbb{P}^1$-bundle over $S^{[4]}$. □

As a consequence of Proposition 5.10, we conclude that in the given degeneration, the central fiber of the family $\mathcal{Z}$ of LLSvS eightfolds has a component which is birational to $S^{[4]}$, so that (leaving to the reader to check the multiplicity 1 statement for the considered component of the central fiber), we can apply Theorem 0.1 and conclude that $\mathcal{Z}_s$ is deformation equivalent to $S^{[4]}$.

6. The dual complexes and monodromy actions for degenerations of hyper-Kähler manifolds

While most of the paper is concerned with the case of finite monodromy degenerations, we close here by making some remarks on the infinite monodromy case. We start by recalling the case of $K3$ surfaces. Namely, the Kulikov-Persson-Pinkham Theorem ([Kul77, PP81]) states that any projective 1-parameter degeneration $\mathcal{X}/\Delta$ of $K3$ surfaces can be arranged (after base change and birational transformations) to be semistable with trivial canonical bundle; such a degeneration is called a *Kulikov degeneration* of $K3$s. Since a Kulikov degeneration is semistable, the monodromy operator T on $H^2(X_t, \mathbb{Q})$ is unipotent. Let $N = \log T$, then $N^3 = 0$, and one distinguishes three cases, called Type I, II, III respectively, corresponding to the three possibilities for the index of nilpotency ν of N (i.e. $N^\nu = 0$, $N^{\nu-1} \neq 0$). Depending on the Type of the degeneration, Kulikov [Kul77] (see also [FM83]) has classified the possible central fibers X_0:

Theorem 6.1 (Kulikov [Kul77, Theorem II])**.** *Let $\mathcal{X}/\Delta$ be a Kulikov degeneration of $K3$ surfaces. Then, depending on the Type of the degeneration (or equivalently the nilpotency index of N) the central fiber X_0 of the degeneration is as follows:*

i) *Type I: X_0 is a smooth $K3$ surface.*
ii) *Type II: X_0 is a chain of surfaces, glued along smooth elliptic curves. The end surfaces are rational surfaces, and the corresponding double curves are anticanonical divisors. The intermediary surfaces (possibly none) are (birationally) elliptically ruled; the double curves for such surfaces are two distinct sections which sum up to an anticanonical divisor.*
iii) *Type III: X_0 is a normal crossing union of rational surfaces such that the associated dual complex is a triangulation of S^2. On each irreducible component V of X_0, the double curves form a cycle of rational curves giving an anticanonical divisor of V.*

Remark 6.2. As usual, we can let Σ be the dual complex associated to the normal crossing variety X_0, the central fiber of the Kulikov degeneration. Then, the topological realization $|\Sigma|$ is either a point, an interval, or S^2 according to the three Types. In particular, $\dim|\Sigma| = \nu - 1$ (where ν is the nilpotency index of N).

The purpose of this section is to give partial generalizations of Kulikov classification of the central fiber in a degeneration of hyper-Kähler manifolds (and make some remarks on the general K-trivial case). To start, we note that for a degeneration of HK manifolds, it is natural to define a Type (I, II, III) based on the nilpotency index of the monodromy on the second cohomology group. Then, Theorem 0.8 is nothing but a strong generalization of Kulikov's Theorems in the Type I case. Informally, *a finite monodromy degeneration of HK manifolds admits a smooth filling.* The focus here is some discussion of the Type II and III cases. Namely, we will discuss some generalization of Remark 6.2 to the higher dimensional case and a partial resolution of a conjecture of Nagai [Nag08] concerning the monodromy action on higher cohomology groups.

Definition 6.3. Let $\mathcal{X}^*/\Delta^*$ be a degeneration of HK manifolds. Let $\nu \in \{1,2,3\}$ be the nilpotency index for the associated monodromy operator N on $H^2(X_t)$ (i.e. $N = \log T_u$, where T_u is the unipotent part of the monodromy $T = T_sT_u$). We say that the degeneration is of Type I, II, or III respectively if $\nu = 1,2,3$ respectively.

In contrast to the case of $K3$ surfaces, for higher dimensional HK manifolds, the higher (than 2) degree cohomology is non-trivial, and thus a natural first question is to what extent the nilpotency index for the monodromy on this higher cohomology is determined by the Type (or equivalently the nilpotency index on H^2). This question was investigated by Nagai [Nag08] who obtained specific results in the case of degenerations of Hilbert schemes of $K3$s and Kummer case, and made the following natural conjecture:

Conjecture 6.4 (Nagai [Nag08, Conjecture 5.1])**.** *For a degeneration of HK*

$$\mathrm{nilp}(N_{2k}) = k(\mathrm{nilp}(N_2) - 1) + 1.$$

(i.e. the nilpotency order on H^{2k} is determined by that on H^2).

Remark 6.5. There is difference of 1 between our nilpotency index, and that used by Nagai: for us N has index ν if ν is minimal such that $N^\nu = 0$, while in [Nag08], N has index ν if $N^{\nu+1} = 0$ (and $N^\nu \neq 0$).

The main result of Nagai ([Nag08, Thm. 2.7, Thm. 3.6]) is that the conjecture is true for degenerations arising from Hilbert schemes of $K3$s or generalized Kummers associated to families of abelian surfaces. Below, we check the conjecture in the Type I and III cases (see Corollary 6.19), and furthermore, we get some results on the topological type of the dual complex of the degeneration (see Theorem 0.10 and Theorem 6.15).

Theorem 6.6. *Nagai's Conjecture holds in Type I and III cases. For Type II, it holds* $\mathrm{nilp}(N_{2k}) \in \{k+1,\dots,2k-1\}$ *for* $k \in \{2,\dots,n-1\}$*.*

Remark 6.7. The case of type I is Corollary 3.2, a consequence of Theorem 0.8.

6.1. **Essential Skeleton of a K-trivial degeneration.** Let $\mathcal{X}/\Delta$ be a semistable degeneration of algebraic varieties. An important gadget associated to the degeneration is the dual complex Σ of the normal crossing variety X_0 (the central fiber of the degeneration). The dual complex encodes the combinatorial part of the degeneration and can be used to compute the 0-weight piece (which reflects the combinatorial part) of the MHS on X_0 and of the LMHS. Specifically, an easy consequence of the Clemens-Schmid exact sequence (see [Mor84], [ABWo13]) gives:

$$H^k(|\Sigma|) \cong W_0H^k(X_0) \cong W_0H^k_{\mathrm{lim}}. \tag{6.8}$$

The first identification is almost tautological; it follows from the Mayer–Vietoris spectral sequence computing the cohomology of X_0. While the second follows from a weight analysis of the Clemens–Schmid sequence, which (in particular) shows that the natural specialization map $H^k(X_0) \to H^k_{\mathrm{lim}}$ has to be an isomorphism in weight 0. We note that there is a much more general version of the second identity. Namely, as explained in Section 2, as a consequence of [KK10] and [Ste81], as long as X_0 is semi log canonical (e.g. normal crossing), the specialization map $H^k(X_0) \to H^k_{\mathrm{lim}}$ is an isomorphism on the $I^{p,q}$ pieces with $p \cdot q = 0$. In particular, we get an isomorphism for the weight 0 pieces (corresponding to $p = q = 0$) of the MHS on X_0 and the LMHS.

The semistable models are not unique, and thus the topological space $|\Sigma|$ depends on the model (e.g. $|\Sigma|$ might be a point, but after a blow-up might become an interval). In order to obtain a more canonical topological space one needs to require some "minimality" for the semistable model. While many ideas towards an intrinsic definition for $|\Sigma|$ occur in the literature (e.g. Kulikov's results can be regarded as a the starting point), the right definitions were only recently identified by de Fernex–Kollár–Xu [dFKX17]. Namely, the minimality corresponds to a relative minimal model in the sense of MMP. This, however leads to singularities of $\mathcal{X}/\Delta$. It turns out that the right singularities that still allow the definition of a meaningful dual complex is that $(\mathcal{X}, X_0)$ is dlt. In other words, the correct context for defining an intrinsic dual complex associated to a degeneration is that of minimal dlt degeneration (see Appendix A). The minimal dlt model $\mathcal{X}/\Delta$ is not unique, but changing the model has no effect on $|\Sigma|$ (the associated topological spaces will be related by a PL homeomorphism, see [dFKX17, Prop. 11]). On the other hand, resolving $\mathcal{X}/\Delta$ to a semistable model will lead to a retraction of the topological space associated to the dual complex of the semistable model to this canonical $|\Sigma|$ (and thus homotopically equivalent).

Remark 6.9. Let us note that the semi-log-canonical (slc) singularities are too degenerate to lead to a good notion of dual complex. For instance, it is easy to produce KSBA degenerations of $K3$ surfaces of Type III such that the central fiber X_0 is a normal surface with a single cusp singularity (e.g. such examples occur in the GIT analysis for quartic surfaces, see [Sha81]). The (naive) dual complex in this situation would be just a point, while from Kulikov's Theorem, the intrinsic dual complex is in fact S^2.

In the case of K-trivial degenerations, there is an alternative approach (yet producing the same outcome) coming from mirror symmetry in the Kontsevich–Soibelman interpretation. This in carefully worked out in Mustata–Nicaise [MN15] (via Berkovich analytification). Relevant for us is the fact that associated to a K-trivial degeneration $\mathcal{X}/\Delta$ there is an intrinsic (depending only on $\mathcal{X}^*/\Delta^*$) topological space, that we call (following [MN15]) *the essential skeleton*, $\mathrm{Sk}(\mathcal{X})$ associated to the degeneration. For a minimal dlt degeneration of K-trivial varieties, the essential skeleton $\mathrm{Sk}(\mathcal{X})$ can be identified with the topological realization $|\Sigma|$ of the dual complex (cf. Nicaise-Xu [NX13, Thm. 3.3.3]). (As discussed in Section 1 and [Fuj11, Theorem 1.1], a minimal dlt model always exists. Two such models are birationally crepant, leading to $\mathrm{Sk}(\mathcal{X})$ being well defined.) Finally, Nicaise-Xu [NX13, Thm. 3.3.3] show that $\mathrm{Sk}(\mathcal{X})$ is a pseudo-manifold with boundary.

The purpose of this section is to make some remarks on the structure of the essential skeleton $\mathrm{Sk}(\mathcal{X})$ for a degeneration of hyper-Kähler manifolds (depending on the Type of the degeneration). We note that there is an extensive literature on the related case of Calabi-Yau varieties (esp. relevant here is Kollár–Xu [KX16]), and that several papers (esp. [MN15], [NX13], [KX16]) treat the general K-trivial case. However, to our knowledge, none of the existing literature discusses the skeleton $\mathrm{Sk}(\mathcal{X})$ in terms of the Type (I, II, III) of the HK degeneration.

Remark 6.10. Recently, Gulbrandsen–Halle–Hulek [GHH16] (see also [GHHZ17]) have studied explicit models for certain types of degenerations of Hilbert schemes of surfaces. In particular, starting with a Type II degeneration of $K3$ surfaces $\mathcal{S}/\Delta$, it is constructed in [GHHZ17] an explicit minimal dlt degeneration for the associated Type II family of Hilbert schemes $\mathcal{X}/\Delta$ of n-points on $K3$ surfaces (with $X_t = (S_t)^{[n]}$). From our perspective here, most relevant is the fact that the $\mathrm{Sk}(\mathcal{X})$ is the n-simplex. For comparison, our results (see Theorem 0.10) will only say $\dim \mathrm{Sk}(\mathcal{X}) = n$ and that $\mathrm{Sk}(\mathcal{X})$ has trivial rational cohomology.

6.2. **Type III is equivalent to the MUM case.** In the above situation $f : \mathcal{X} \to \Delta$, we now assume f is projective. It is then well-known that the monodromy γ_k acting on $H^k(X_t, \mathbb{Q})$, $t \in \Delta^*$ is quasi-unipotent, that is $(\gamma_k^N - Id)^m = 0$ for some integers N, m. Furthermore one can take $m \leq k+1$.

Definition 6.11. We will say that the monodromy on H^k is maximally unipotent if the minimal order m is $k+1$. Let $\mathcal{X}/\Delta$ be a degeneration of K-trivial varieties of dimension n. We say that the degeneration is maximally unipotent (or MUM) if the nilpotency index for the monodromy action on $H^n(X_t, \mathbb{Q})$ is $n+1$ (the maximal possible index).

It is immediate to see that in a MUM degeneration, the Skeleton has dimension at least n. For K-trivial varieties, a strong converse also holds:

Theorem 6.12 (Nicaise-Xu [NX13])**.** *Let $\mathcal{X}/\Delta$ be a degeneration of K-trivial varieties of dimension n.*

i) *If the degeneration is MUM, then* $\mathrm{Sk}(\mathcal{X})$ *is a pseudo-manifold of dimension* n.
ii) *Conversely, if* $\mathrm{Sk}(\mathcal{X})$ *is of dimension* n*, the degeneration is MUM.*

Remark 6.13. We note here that both the minimality and K-triviality are essential conditions. Dropping the K-triviality, we can consider a family of genus $g \geq 2$ curves degenerating to a compact type curve. Then the monodromy is finite, but the dual graph of the central fiber is an interval. Similarly, one can start with a family of elliptic curves and blow-up a point. This will give a non-minimal family, with trivial monodromy, and dual graph of the central fiber an interval.

We note that one additional topological constraint on the skeleton $\mathrm{Sk}(\mathcal{X})$ is that it is simply connected.

Proposition 6.14. *Let* $\mathcal{X}/\Delta$ *be a degeneration such that* $\pi_1(X_t) = 1$*. Then* $\pi_1(\mathrm{Sk}(\mathcal{X})) = 1$*.*

Proof. [KX16, §34 on p. 541]. □

Mirror symmetry makes some predictions on the structure of essential skeleton $\mathrm{Sk}(\mathcal{X})$ for MUM degenerations. Briefly, the situation is as follows:

6.2.1. The SYZ conjecture ([SYZ96]) predicts the existence of a special Lagrangian fibration X/B for K-trivial varieties near the large complex limit point (the cusp of the moduli corresponding to the MUM degeneration). Furthermore, SYZ predicts that the mirror variety is obtained by dualizing this Lagrangian fibration.

6.2.2. Kontsevich–Soibelman [KS01, KS06] predict that the basis B of the Lagrangian fibration is homeomorphic to the essential skeleton $\mathrm{Sk}(\mathcal{X})$. In fact, B is predicted to be the Gromov–Hausdorff limit associated to (X_t, g_t) where g_t is an appropriately scaled Ricci-flat Yau metric on the (polarized) smooth fibers X_t. This gives a much richer structure to B (*Monge-Ampere manifold*, [KS06, Def. 6]). The underlying topological space is expected to be $\mathrm{Sk}(\mathcal{X})$ (e.g. [KS06, §6.6]). As already mentioned, the Kontsevich–Soibelman predictions led to the Mustata-Nicaise [MN15] definition of $\mathrm{Sk}(\mathcal{X})$.

6.2.3. The case of $K3$ surfaces is quite well understood (see [KS06], [GW00]). In higher dimensions, there is a vast literature on the case of (strict) Calabi-Yau's, most notably the Gross–Siebert program (e.g. [GS11]). From our perspective, we note that the $\mathrm{Sk}(\mathcal{X})$ for a MUM degeneration of Calabi-Yau n-folds is predicted to be the sphere S^n. This is true in dimension 2 by Kulikov's Theorem, and in dimensions 3 and 4 (conditional) by Kollár–Xu [KX16].

6.2.4. The SYZ conjecture and the Kontsevich–Soibelman picture for hyper-Kählers is similar to the $K3$ case (see especially Gross–Wilson [GW00]). Specifically, the special Lagrangian fibration X/B near the large complex limit point can be constructed via the hyper-Kähler rotation. Briefly, let $[\Omega] \in H^2(X, \mathbb{C})$ and $[\omega] \in H^2(X, \mathbb{R})$ be the classes of the holomorphic form and of the polarization (a Kähler class) on X. The MUM condition implies the existence of a vanishing cycle $\gamma \in H^2(X, \mathbb{Q})$ with $q(\gamma) = 0$ (where q is the Beauville-Bogomolov form on H^2). The problem is that γ is not an algebraic class. However, using the fact that on a hyper-Kähler manifold there is S^2 space of complex structures, one can modify the complex structure on X (call the resulting complex manifold X') such that γ is an algebraic class with $q(\gamma) = 0$ (essentially, after an appropriate $\mathbb{C}^*$-scaling of Ω, we can arrange $\Omega' = \mathrm{Im}(\Omega) + i\omega$ and $\omega' = \mathrm{Re}(\Omega)$ to be the holomorphic and respectively Kähler classes on X', and γ to be orthogonal to Ω'). The so-called hyper-Kähler SYZ conjecture (which is known in various cases) then says that (a multiple of) γ is class of a (holomorphic) Lagrangian fibration X'/B. Of course, in the C^∞ category, X'/B is the same as the desired special Lagrangian X/B. (From a slightly different perspective, a study of the mirror symmetry for hyper-Kähler manifolds was done by Verbitsky [Ver99].)

6.2.5. Finally, the basis of an (algebraic) Lagrangian fibration X'/B is expected to be $\mathbb{CP}^n$ (for $2n$-dimensional HK manifolds). For instance, if B is smooth, then $B \cong \mathbb{CP}^n$ by a theorem of Hwang [Hwa08].

To conclude, mirror symmetry (via SYZ conjecture and Kontsevich-Soibelman) predicts that *the essential skeleton* $\mathrm{Sk}(\mathcal{X})$ *for a MUM degeneration is* S^n *and respectively* $\mathbb{CP}^n$ *for Calabi-Yau's and respectively hyper-Kähler's.* The following result is a weaker version of this statement, saying that it holds in a cohomological sense.

If X is a simply connected compact Kähler manifold with trivial canonical bundle, the Beauville-Bogomolov decomposition theorem [Bea83] says that $X \cong \prod_i X_i$ where the X_i are either Calabi-Yau of dimension k_i (that is with $\mathrm{SU}(k_i)$ holonomy group), or irreducible hyper-Kähler of dimension $2l_j$ (that is with $\mathrm{Sp}(2l_j)$ holonomy group). The type of the decomposition will be the collection of the dimensions k_i, $2l_j$ (with their multiplicities).

Theorem 6.15. *Let $\mathcal{X}/\Delta$ be a minimal dlt degeneration of K-trivial varieties. Assume that the general fiber X_t is a simply connected K-trivial variety.*

Assume that the degeneration is maximal unipotent. Then

i) $H^*(\mathrm{Sk}(\mathcal{X}), \mathbb{Q}) \cong \prod_i H^*(S^{k_i}, \mathbb{Q}) \times \prod_j H^*(\mathbb{CP}^{l_j}, \mathbb{Q}))$*, where k_i represent the dimensions of the Calabi-Yau factors and $2l_j$ the dimensions of the hyper-Kähler factors in the Beauville-Bogomolov decomposition of the general fiber X_t.*

ii) *Conversely, the cohomology algebra of the skeleton* $\mathrm{Sk}(\mathcal{X})$ *determines the type of the Beauville-Bogomolov decomposition of X_t.*

Proof of Theorem 6.15. Let $\mathcal{X}/\Delta$ be a minimal dlt degeneration. By the du Bois arguments of Section 2, the weight 0 pieces of the limit mixed Hodge structure on $H^*_{\lim}$ is identified with the weight 0 pieces of the mixed Hodge structure on $H^*(X_0)$. Next, a Mayer–Vietoris argument identifies $W_0H^k(X_0)$ with $H^k(\mathrm{Sk}(\mathcal{X}))$ (recall $\mathrm{Sk}(\mathcal{X})$ is nothing but the topological realization of the dual complex in this situation). In other words, we see that (6.8) holds in the situation of minimal dlt degenerations. Summing over all degrees k gives an algebra structure, and then an identification of the algebra associated to the weight 0 piece of the LMHS with the cohomology algebra of $\mathrm{Sk}(\mathcal{X})$. Here it is important to note the multiplication structure on both sides is given by cup product, and by Lemma 6.16, this is compatible with taking limit mixed Hodge structures.

It remains to understand the algebra structure for the weight 0 piece of the LMHS (under the MUM assumption). It is immediate to see that on $H^k_{\lim}$ the weight 0 piece is non-zero if and only if the monodromy action on $H^k(X_t)$ is maximally unipotent. When this is satisfied, we have $N^k : Gr^W_{2k}H^k_{\lim} \cong W_0H^k_{\lim}$, and then $Gr^W_{2k}H^k_{\lim} \subset F^kH^k_{\lim} \cong H^{k,0}(X_t)$ (as vector spaces). Thus, the weight 0 piece can be identified with a subspace in the space of degree k holomorphic forms on X_t. Under the K-triviality assumption and assuming X_t has a single factor in the Beauville-Bogomolov decomposition, the assumption $W_0H^k_{\lim} \neq 0$ implies in fact $W_0H^k_{\lim} \cong H^{k,0}(X_t)$ (since it is 1-dimensional; it is here where we use essentially the assumption of the theorem that X_t is simply connected, so that we can exclude the abelian variety factors in the Beauville-Bogomolov decomposition). These identifications are compatible with the cup product by the following lemma: Let $f : \mathcal{X} \to \Delta$ be a proper holomorphic map, smooth over Δ^*, with a central fiber X_0 which is a (global) normal crossing divisor. We assume for simplicity that if X_i, $i \in I$ are the components of X_0, for each $J \subset I$, $X_J := \cap_{j\in J}X_j$ is either empty or connected. Let Σ be the dual graph of X_0. It has vertices I and one simplex $J \subset I$ for each non-empty X_J.

Lemma 6.16. *The two natural maps*

$$a : H^*(|\Sigma|, \mathbb{Z}) \to H^*(X_0, \mathbb{Z}),$$

$$b : H^*(X_0, \mathbb{Z}) \to H^*(X_t, \mathbb{Z})$$

are compatible with the cup-product.

Proof. The map b is the specialization map of the Clemens–Griffiths exact sequence, and is obtained by observing that X_0 is a deformation retract of $\mathcal{X}$, hence has the same homotopy type as $\mathcal{X}$. The map b is then the restriction map $H^*(\mathcal{X}, \mathbb{Z}) \to H^*(X_t, \mathbb{Z})$ composed with the inverse of the restriction isomorphism $H^*(\mathcal{X}, \mathbb{Z}) \to H^*(X_0, \mathbb{Z})$. Thus it is clearly compatible with cup-product.

The map a (which can be defined using Corollary A.7 as the composite map $H^p(|\Sigma|, \mathbb{Z}) = E_2^{p,0} = E_\infty^{p,0} \to H^q(D, \mathbb{Z})$) can also be constructed as follows: The realization $|\Sigma|$ of Σ is the union over all the faces J of Σ of the simplices Δ_J, with identifications given by faces: for $J' \subset J$ the simplex $\Delta_{J'}$ is naturally a face of Δ_J. Next we have a simplicial topological space $X_0^\bullet$ associated to X_0, given by the X_J and the natural inclusions $X_{J'} \subset X_J$ for each $J \subset J'$. Let $r(X_0^\bullet)$ be the topological constructed as the union over all $J \in \Sigma$ of the $X_J \times \Delta_J$ with gluings given by the natural maps $X_J \times \Delta_{J'} \to X_{J'} \times \Delta_J$ for each inclusion $J' \subset J$.

There are two obvious continuous maps

$$g : r(X_0^\bullet) \to X_0,$$
$$f : r(X_0^\bullet) \to r(\Sigma).$$

The first map is just the projection to X_J on each $X_J \times \Delta_J$, followed by the inclusion in X_0. This map is clearly a homotopy equivalence. The second map is the projection to Δ_J on each $X_J \times \Delta_J$. The map b can be defined as the composition of $f^* : H^*(r(\Sigma), \mathbb{Z}) \to H^*(r(X_0^\bullet), \mathbb{Z})$ composed with the inverse of the isomorphism $g^* : H^*(X_0, \mathbb{Z}) \cong H^*(r(X_0^\bullet), \mathbb{Z})$. It follows that b is also compatible with cup-product. □

Together with the previous analysis, we now conclude that in the MUM case, the cohomology algebra $H^*(|\Sigma|, \mathbb{C})$ is isomorphic to the algebra of holomorphic forms $\oplus_i H^0(X_0, \Omega^i_{X_0})$ and also to the algebra of holomorphic forms $\oplus_i H^0(X_t, \Omega^i_{X_t})$. We next have the following lemma.

Lemma 6.17. *Let X be a simply connected compact Kähler manifold with trivial canonical bundle. Then the type of the Beauville-Bogomolov decomposition of X is determined by the algebra $\oplus_i H^0(X, \Omega^i_X)$.*

Proof. For a Calabi-Yau manifold of dimension k_i, there is exactly one holomorphic form ω_i of degree k_i and it satisfies $\omega_i^2 = 0$, while for a hyper-Kähler factor X_j, the algebra $H^0 X_j, \Omega^\cdot_{X_j})$ is generated in degree 2 with one generator σ_j satisfying the equation $\sigma_j^{l_j+1} = 0$. The algebra $A^\cdot_X := H^0(X, \Omega^\cdot_X)$ is the tensor product of algebras of these types. Consider for any integer k the set $(A^2_X)_k =: \{u \in A^2_X,\ u^{k+1} = 0\}$. Let k_0 be the smallest k such that $(A^2_X)_k \neq 0$. Then it is immediate that the hyper-Kähler summands are all of dimension $\geq 2k_0$ and that there are exactly $a_k := \dim (A^2_X)_k$ summands of dimension $2k_0$. The quotient of $A^\cdot_X$ by the ideal generated by $(A^2_X)_k$ is the algebra $A^\cdot_{X'}$ of holomorphic forms on the variety X' which is the product of all Calabi-Yau factors of X and hyper-Kähler factors which are of dimension $> 2k_0$. Continuing with X', we see that the multiplicities of the dimensions of the hyper-Kähler factors are determined by $A^\cdot_X$, and that $A^\cdot_X$ determines the algebra $A^\cdot_{X''}$ of holomorphic forms on the variety X'' which is the product of all Calabi-Yau summands of X of dimension > 2. It is clear that the latter determines the dimensions (with multiplicities) of the Calabi-Yau summands of X, as they correspond to the degrees (with multiplicities) of generators of the algebra $A^\cdot_{X''}$. □

Together with Proposition 6.18, this concludes the proof of the theorem. □

Proposition 6.18. *Let $\mathcal{X}/\Delta$ be a projective degeneration of K-trivial varieties.*

i) *Assume the fibers X_t are simply connected Calabi-Yau manifolds (so $h^i(X_t, \mathcal{O}_{X_t}) = 0$ for $0 < i < n = \dim X_t$). Then the only degree in which the monodromy can be maximally unipotent is n.*
ii) *Assume the fibers X_t are hyper-Kähler manifolds (so $h^i(X_t, \mathcal{O}_{X_t}) = 0$ for i odd and $\mathbb{C}$ for $i = 2j$, $0 < i < 2n = \dim X_t$). Then the only degrees where the monodromy can be maximally unipotent are the even degrees $2i$ and the monodromy is maximally unipotent in some degree $k = 2i$ if and only if it is maximally unipotent in all degrees $2i \leq 2n$. In particular, MUM degeneration is equivalent to Type III degeneration (for hyper-Kähler manifolds).*

Proof. (i) As we have $H^{i,0}(X_t) = 0$ for $0 < i < n$, the Hodge structure on $H^i(X_t, \mathbb{Q})$ has coniveau ≥ 1. The variation of Hodge structure on $R^i f_*\mathbb{Q}$ is thus the Tate twist of an effective polarized variation of Hodge structure of weight $i - 2$. Hence its quasiunipotency index is $\leq i - 1$.

(ii) The same argument applies to show that monodromy is not maximally unipotent on cohomology of odd degree if X_t is hyper-Kähler, since $H^{2i+1,0}(X_t) = 0$. We know by Verbitsky [Bog96] that in degree $2i \leq 2n$, we have an injective map given by cup-product

$$\mu_{i,t} : \mathrm{Sym}^i H^2(X_t, \mathbb{Q}) \hookrightarrow H^{2i}(X_t, \mathbb{Q}),$$

which more generally induces an injection of local systems on Δ^*

$$\mu_i : \mathrm{Sym}^i(R^2 f_*\mathbb{Q}) \hookrightarrow R^{2i} f_*\mathbb{Q}.$$

Note that μ_i is an morphism of variations of Hodge structures. Next, using a relatively ample line bundle on f, we have an orthogonal decomposition

$$R^{2i} f_*\mathbb{Q} = \mathrm{Im}\, \mu_i \oplus B^{2i}$$

where the local system B^{2i} carries a polarized variation of Hodge structures of weight $2i$ with trivial $(2i,0)$-part, as the map $\mu_{i,t}$ induces a surjection on $(2i,0)$-forms. Applying the same argument as before, we conclude that the monodromy action on B^{2i} is of quasiunipotency index $\leq 2(i-1)+1(<2i+1)$, so the monodromy acting on H^{2i} is maximally unipotent if and only if it is maximally unipotent on $\mathrm{Sym}^i H^2(X_t,\mathbb{Q})$. It is then an easy exercise to show that this is the case if and only if it is maximally unipotent on $H^2(X_t,\mathbb{Q})$. □

Corollary 6.19. *Nagai's Conjecture 6.4 holds for Type III degenerations of hyper-Kähler manifolds.*

6.3. **The Type II case.** We now focus on the intermediary Type II case. The aim of the subsection is to prove the following result (which together with the results in the Type I and III cases completes the proof of Theorems 0.10 and 6.6).

Theorem 6.20. *Let $\mathcal{X}/\Delta$ be a projective degeneration of hyper-Kähler manifolds, with X_t smooth of dimension $2n$. Assume that the degeneration has Type II (i.e. $N^2=0$ and $N\neq 0$ on $H^2(X_t)$). Then the following hold:*

i) $\dim \mathrm{Sk}(\mathcal{X})=n$;
ii) *For $k\in\{2,n\}$, the index of nilpotency for the monodromy action on H^{2k} is at least $k+1$ and at most $2k-1$.*

Proof. The same arguments as before (based on Verbitsky's Theorem [Ver96, Bog96] describing the subalgebra of $H^*(X_t)$ generated by H^2), we obtain that in the Type II case, it holds

$$I^{k,0}\neq 0$$

on $H^{2k}_{\lim}$ (and of course, $I^{l,0}=0$ for $l=0,\dots,2k$, $l\neq k$). This implies that the nilpotency index on H^{2k} (for $k\in\{1,\dots,n\}$) is at least k. From Proposition 6.18, it also follows that the monodromy can not be maximally unipotent on any H^{2k} (if this it were case, from the proposition, then the monodromy would be unipotent on H^2, giving Type III case). In fact, since $H^{2k}/\operatorname{Sym}^2 H^2$ is a Hodge structure of level $2k-2$, it follows that the monodromy action has nilpotency index at most $2k-1$.

To conclude, we note that the arguments of [KX16, Claim 32.1] show that the dimension of $\mathrm{Sk}(\mathcal{X})$ is precisely n. This is equivalent to saying that the codimension of the deepest stratum in a dlt Type II degeneration is n. For a minimal dlt degeneration, we know X_0 has trivial canonical bundle. This means that its components are log Calabi-Yau (V,D) with $K_V+D=0$. Inductively, each component of the strata is log Calabi-Yau (e.g. in the $K3$ situation the codimension 1 components are either elliptic curves or $\mathbb{P}^1$ with 2 marked points) and is K–trivial if and only if it is contained in every component of X_0 that intersects it. Hence, a stratum $W\subset X_0^{[p]}$ is minimal with respect to inclusion if and only if it has a top holomorphic form (and is thus a K–trivial variety with at worst canonical singularities). Moreover, all minimal strata are birational [Kol13, 4.29]. It follows that to show that the dual complex has dimension n, we only need to produce a top holomorphic form on an n–dimensional stratum. To show this look at the spectral sequence (A.8). We first notice that there is a non zero class in $H^1(\mathcal{O}_{X_0^{[1]}})$ which generates $H^2(\mathcal{O}_{X_0})$. To see this we only need to show that there is no contribution from $H^2(\mathcal{O}_{X_0^{[0]}})$ and from $H^0(\mathcal{O}_{X_0^{[2]}})$. By weights considerations, both statements are clearly true for the spectral sequence of a snc filling. However, since the strata of a dlt filling have rational singularities (Prop. A.2) the statement for a snc filling implies that for a dlt filling. Hence, the only possibility is that a generator for $\bar{\eta}\in H^2(\mathcal{O}_{X_0})$ has to come from a class $\eta\in H^1(\mathcal{O}_{X_0^{[1]}})$. By Lemma 6.21, we may consider the product $\eta^n\in H^n(\mathcal{O}_{X_0^{[n]}})$ which is non zero since $\bar{\eta}^n$ has to be non zero and we may conclude that the deepest stratum has dimension n. □

Lemma 6.21. *The spectral sequence of Corollary A.7 for $\mathcal{O}_{X_0}$ is endowed with an algebra structure that is compatible with the cup product on $H^*(\mathcal{O}_{X_0})$.*

Proof. By Proposition A.6, it is enough to produce a morphism of complexes

$$\varphi:\mathcal{O}_{X_0^\bullet}\otimes\mathcal{O}_{X_0^\bullet}\to\mathcal{O}_{X_0^\bullet} \tag{6.22}$$

which induces the regular cup product on $\mathcal{O}_{X_0}$. For $\alpha=\{\alpha_J\}$ a section of $\mathcal{O}_{X_0^{[s]}}$ and $\beta=\{\beta_K\}$ a section of $\mathcal{O}_{X_0^{[t]}}$ we set $\varphi(\alpha\otimes\beta)_{j_0j_1\cdots j_{s+t+1}}=\alpha_{j_0j_1\cdots i_s}|_{X_{0j_0j_1\cdots j_{s+t+1}}}\cdot\beta_{j_sj_{s+1}\cdots i_{s+t+1}}|_{X_{0j_0j_1\cdots j_{s+t+1}}}$. The verification that φ is a morphism of complexes is formally the same as that for the cup product in Cech cohomology. We can lift φ to a morphism of the Cech resolutions of each of the two complexes, getting a product structure on the corresponding spectral sequence and hence a product $H^q(\mathcal{O}_{X_0^{[p]}}) \otimes H^{q'}(\mathcal{O}_{X_0^{[p']}}) \to H^{q+q'}(\mathcal{O}_{X_0^{[p+p']}})$ which is compatible with the cup product on $H^*(\mathcal{O}_{X_0})$. □

Remark 6.23. J. Nicaise pointed out that the $\dim \mathrm{Sk}(\mathcal{X}) = n$ follows also from Halle-Nicaise [HN17] (see Theorem 3.3.3 and esp. (3.3.4) of loc. cit.).

Appendix A. Reduced dlt pairs

The purpose of this section is to show that for many aspects reduced dlt pairs behave like snc. Most of the results are well known to the experts (cf. [dFKX17]).

A lc pair (X, D) is called dlt if for every divisor E over (X, D) with discrepancy -1, the pair (X, D) is snc at the generic point of $center_X(E)$. Given a reduced dlt pair (X, D) (i.e. the divisors appearing in $D = \sum_I D_i$ have coefficient 1) a stratum of (X, D) is an irreducible component of $D_J := \cap_J D_i$, for some $J \subset I$. By [Kol13, 4.16], [Fuj07] the strata of (X, D) have the expected codimension (i.e. the strata of codimension k in X are the irreducible components of the intersection of k components of D) and are precisely the log–canonical centers of (X, D). In particular, (X, D) is snc at the generic point of every strata and every stratum of codimension k is contained in exactly $k+1$ strata of codimension $k-1$. As noticed in [dFKX17, (8)], this is observation is enough to specify the glueing maps needed to define a dual complex. Hence, the dual complex of a dlt pair can be defined just as in the snc case and it satisfies

$$\Sigma((X, D)) = \Sigma((X, D)^{snc}),$$

where $(X, D)^{snc}$ is the largest open subset of X where the pair (X, D) is snc. In Proposition A.6 we show another instance of the fact that "dlt is almost snc", namely that given a reduced dlt pair (X, D) we can use the Mayer–Vietoris sequence [GS75] to compute the cohomology of D. We will apply this in Section 6.3 to a minimal dlt degeneration $\mathcal{X}/D$ of K–trivial varieties, since the pair $(\mathcal{X}, X_0)$ is dlt.

Definition A.1. [Kol13, (2.78)] Let X be a normal variety and $D \subset X$ be a reduced divisor such that $(Y, D_Y := f_*^{-1}(D))$ is a snc pair. Then $f : (Y, D_Y) \to (X, D)$ is called rational if

(1) $f_*\mathcal{O}_Y(-D_Y) = \mathcal{O}_X(-D)$;
(2) $R^i f_*\mathcal{O}_Y(-D_Y) = 0$ for $i > 0$;
(3) $R^i f_*\omega_Y(D_Y) = 0$ for $i > 0$.

Proposition A.2. *Let (X, D) be a reduced dlt pair with $D = \sum_I D_i$ and let $f : (Y, D_Y) \to (X, D)$ be a rational resolution. For every reduced divisor $D' \leq D$, setting $D'_Y := f_*^{-1} D'$ we have*

$$f_*\mathcal{O}_Y(-D'_Y) = \mathcal{O}_X(-D') \quad \text{and} \quad R^i f_*\mathcal{O}_Y(-D'_Y) = 0, \quad \text{for } i > 0. \tag{A.3}$$

and for every $J \subset I$

$$f_*\mathcal{O}_{(D_Y)_J} = \mathcal{O}_{D_J} \quad \text{and} \quad R^i f_*\mathcal{O}_{(D_Y)_J} = 0 \quad \text{for } i > 0. \tag{A.4}$$

In particular, the induced resolution $f_{|(D_Y)_J} : (D_Y)_J \to D_J$ has connected fibers and every connected component of D_J is irreducible, normal, and has rational singularities.

Proof. Item (1) follows from [Kol13, (2.87),(2.88)]. Then (A.4) by induction on $|J|$. □

Remark A.5. From this corollary it follows that in the definition of dual complex of a reduced dlt pair we could consider the connected components of the intersections, rather than the irreducible components (cf. [dFKX17, (8)]).

Let (X, D) be a reduced dlt pair (X, D), with $D = \sum_I D_i$, and fix an ordering of I. Denote by $D^{[k]}$ the disjoint union of the codimension $k+1$ strata. For a sheaf $\mathcal{F}$ on D, the Mayer–Vietoris complex of $\mathcal{F}$ is

$$\mathcal{F}_{D^\bullet} : \mathcal{F}_{D^{[0]}} \to \mathcal{F}_{D^{[1]}} \to \cdots \to \mathcal{F}_{D^{[d]}},$$

where $d = \dim |\Sigma(D)|$, where $\mathcal{F}_{D^{[k]}}$ denotes the pullback of $\mathcal{F}$ to $D^{[k]}$ via the natural morphism $i_k : D^{[k]} \to D$ and where the differential of the complex is induced by the natural restriction maps $\mathcal{F}_{D_J} \to \mathcal{F}_{D_{J \cup j}}$, with a plus or a minus sign according to the parity of the position of j in $J \cup j$.

Proposition A.6. *Let (X, D) be a reduced dlt pair with $D = \sum D_i$. If $\mathcal{F} = \mathcal{O}_D$ (or is locally free) or $\mathcal{F} = \mathbb{Q}$ (or is a constant sheaf), then $\mathcal{F}_{D^\bullet}$ is a resolution of $\mathcal{F}$.*

Proof. We start with $\mathcal{F} = \mathcal{O}_D$. Since (Y, D_Y) is a snc pair, $\mathcal{O}_{D_Y^\bullet}$ is a resolution of $\mathcal{O}_{D_Y}$ (see for example [FM83]). From Corollary A.2 it follows both that the complex $f_*\mathcal{O}_{D_Y^\bullet}$ is exact and that $f_*\mathcal{O}_{D_Y^\bullet} = \mathcal{O}_{D^\bullet}$. Now the case $\mathcal{F} = \mathbb{Q}$. Let $U \subset X$ be any open set such that on $U \cap X^{snc}$ the divisor D is given by the vanishing of a product of local coordinates. The complex $\Gamma(i_\bullet^{-1}(U \cap X^{snc}), \mathbb{Q}_{D^{[\bullet]}})$ is exact except in degree zero, where it has cohomology equal to $\Gamma(U \cap X^{snc} \cap D, \mathbb{Q})$ (e.g. [Mor84]). As a consequence, the complex is exact on the snc locus. By the dlt assumption, every connected (hence irreducible) component of $D^{[k]}$ intersects the snc locus of (X, D) so $\Gamma(i_k^{-1}(U), \mathbb{Q}_{D^{[k]}}) = \Gamma(i_k^{-1}(U \cap X^{snc}), \mathbb{Q}_{D^{[k]}})$. Hence, for any $x \in D$ there is a sufficiently small open neighborhood U such that the complex $\Gamma(i_\bullet^{-1}(U), \mathbb{Q}_{D^{[\bullet]}}) \cong \Gamma(i_\bullet^{-1}(U \cap X^{snc}), \mathbb{Q}_{D^{[\bullet]}})$ is exact except in degree zero where it has cohomology equal to $\Gamma(U \cap D, \mathbb{Q})$ and the proposition follows. □

Corollary A.7. *For (X, D) and $\mathcal{F}$ as above there is a spectral sequence with E_1 term*

$$E_1^{p,q} = H^q(\mathcal{F}_{D^{[p]}}) \tag{A.8}$$

abutting to $H^*(\mathcal{F})$.

Proof. Resolving every term of the complex with its Čech complex we get a double complex which yields a spectral sequence with E_1 term equal to (A.8). □

Remark A.9. We notice that Corollary A.7 for $\mathcal{F} = \mathbb{C}$ implies (A.8) for $\mathcal{F} = \mathcal{O}_D$. Indeed, since the connected components of $D^{[q]}$ are rational, by [Kov99] they are Du Bois and hence there is a surjection $H^p(D^{[q]}, \mathbb{Q}) \to Gr_F^0 H^p(D^{[q]}, \mathbb{Q}) = H^p(D^{[q]}, \mathcal{O}_{D^{[q]}})$. By [Del71, Thm 2.3.5] Gr_F^k is an exact functor and hence $Gr_F^0 H^p(D^{[q]})$ abuts to $Gr_F^0 H^{p+q}(D, \mathbb{C}) = H^{p+q}(D, \mathcal{O}_D)$.

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