Segre classes of tautological bundles on Hilbert schemes of surfaces

Claire Voisin*

Abstract

We first give an alternative proof, based on a simple geometric argument, of a result of Marian, Oprea and Pandharipande on top Segre classes of the tautological bundles on Hilbert schemes of $K3$ surfaces equipped with a line bundle. We then turn to the blow-up of $K3$ surface at one point and establish vanishing results for the corresponding top Segre classes in a certain range. This determines, at least theoretically, all top Segre classes of tautological bundles for any pair (Σ, H), $H \in \mathrm{Pic}\,\Sigma$.

1 Introduction

Let S be a smooth projective (or compact complex) surface. The Hilbert scheme $S^{[k]}$ is smooth projective (or compact complex) of dimension $2k$. For any line bundle H on S, we get an associated vector bundle $\mathcal{H}_{[k]}$ on $S^{[k]}$, whose fiber at a point $[Z] \in S^{[k]}$ is the vector space $H^0(H_{|Z})$. If S is a $K3$ surface and $c_1(H)^2 = 2g-2$, we denote

$$s_{k,g} := \int_{S^{[k]}} s_{2k}(\mathcal{H}_{[k]}).$$

This is indeed a number which depends only on k and g (see Theorem 2). The following result is proved in [4]:

Theorem 1. *One has* $s_{k,g} = 2^k \binom{g-2k+1}{k}$.

Here the binomial coefficient is defined for $k \geq 0$. It is always 1 for $k = 0$ and the formula for $\binom{n}{k}$ for any n is

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

In particular, we have $\binom{n}{k} = 0$ if $n \geq 0$ and $n < k$. The theorem above thus gives in particular the vanishing

$$s_{k,g} = 0 \text{ when } g-2k+1 \geq 0 \text{ and } k > g-2k+1. \qquad (1)$$

The proof of this vanishing statement in [4] is rather involved and we are going to give in Section 2 a direct geometric proof of (1), based on a small improvement of Lazarsfeld's arguments in [2].

We will then show how the vanishing (1), even only in the smaller range $g = 2k-1$, $g = 2k$, implies Theorem 1. We simply use for this the following result which is due to Tikhomirov [6] (see also Ellingsrud-Göttsche-Lehn [1] and Lehn [3] for related statements) :

Theorem 2. *The Segre numbers $\int_{S^{[k]}} s_{2k}(H_{[k]})$ for a projective surface S equipped with a line bundle H depend only on the four numbers*

$$\pi = H \cdot K_S,\ d = H^2,\ \kappa = K_S^2,\ e = c_2(S).$$

*Collège de France and ETH-ITS

We will denote these Segre numbers $s_{k,d,\pi,\kappa,e}$. It follows from Theorem 2 that the numbers $s_{k,g}$ can be computed as well by considering a surface Σ which is the disjoint union of a $K3$ surface S', equiped with a line bundle H' of self intersection $2(g-1)-2$, and an abelian surface A equiped with a line bundle θ with $\theta^2=2$. We will show in Section 3 that the formula obtained by this observation (this is a particular case of (3) below), combined with the vanishing result (1), uniquely determine the numbers $b_k := \int_{A^{[k]}} s_{2k}(\theta_{[2k]})$ and finally the numbers $s_{k,g}$ for all k, g, knowing that $s_{1,g}=2g-2$, $b_0=1$, $b_1=2$.

In Section 2, we will establish similar vanishing results for a $K3$ surface S blown-up at one point. Let $\widetilde{S}$ be such a surface and let $H=\tau^*L(-lE)$ with $2g-2=L^2$, where L generates $\operatorname{Pic} S$.

Theorem 3. *For $k\geq 2$, one has the following vanishing for the Segre numbers $\tilde{s}_{k,g,l} := \int_{\widetilde{S}^{[k]}} s_{2k}(\mathcal{H}_{[k]})$:*

$$\tilde{s}_{k,g,l}=0 \text{ for } k=l,\ l+1 \text{ and } g-\frac{l(l+1)}{2}=3k-2. \tag{2}$$

We will also prove that these vanishing statements together with Theorem 1 determine all Segre numbers $s_k(d,\pi,\kappa,e)$. We use for this the following complement to Theorem 2, (see [3], [1],) obtained by observing that the Hilbert scheme $S^{[k]}$ of a disjoint union $S_1\sqcup S_2$ is the disjoint union for $l=0,\ldots,k$, of $S_1^{[l]}\times S_2^{[k-l]}$, while all the data d, π, κ, c_2 for the pairs (Σ,H) are additive under disjoint unions $(S,L)=(S_1,L_1)\sqcup(S_2,L_2)$:

Lemma 4. *With the notation $s_{d,\pi,\kappa,e}(z)=\sum_k s_{k,d,\pi,\kappa,e}z^k$*

$$s_{d,\pi,\kappa,e}(z)=s_{d_1,\pi_1,\kappa_1,e_1}(z)s_{d_2,\pi_2,\kappa_2,e_2}(z) \tag{3}$$

with $d=d_1+d_2$, $\pi=\pi_1+\pi_2$ etc.

To conclude this introduction, we mention Lehn's conjecture [3, Conjecture 4.9]:

Conjecture 5. *One has*

$$s_{d,\pi,\kappa,e}(z)=\frac{(1-w)^a(1-2w)^b}{(1-6w+6w^2)^c}, \tag{4}$$

where $a=\pi-2\kappa$, $b=d-2\pi+\kappa+3\chi$, $c=\frac{d-\pi}{2}+\chi$, $\chi=\frac{\kappa+e}{12}$ and the variable w is related to z by

$$z=\frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3}.$$

This conjecture is proved in [4] for $K3$ and more generally K-trivial surfaces, that is for $\kappa=\pi=0$. Although we were not able to prove it in general, our results imply the following:

Corollary 6. *Lehn's conjecture is equivalent to the fact that the development in power series of z of the function $f_{d,\pi,\kappa,e}(z)$ defined as the right hand side in (4) has vanishing Taylor coefficient of order k for $e=25$, $\kappa=-1$ and $(d,\pi)=(7(k-1),k-1)$ or $(d,\pi)=(7(k-1)+1,k)$*

Thanks. I thank Rahul Pandharipande for discussions and in particular for suggesting, after I had given a geometric proof of the vanishings (1) on $K3$ surfaces, to look at surfaces other than $K3$'s. This work has been done during my stay at ETH-ITS. I acknowledge the support of Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation.

2 Geometric vanishing

Let S be a $K3$ surface with $\operatorname{Pic} S = \mathbb{Z}H$, where H is an ample line bundle of self-intersection $2g-2$. We give in this section a geometric proof of the vanishing result (1) proved in [4].

Proposition 7. *The Segre classes $s_{2k}(H_{[k]})$ vanish in the range*

$$3k-1 > g > 2k-2. \tag{5}$$

In particular, $s_{k,2k} = 0$ and $s_{k,2k-1} = 0$ when $k \geq 2$.

Proof. Sections of H provide sections of $\mathcal{H}_{[k]}$, or equivalently of the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{H}^*_{[k]})}(1)$. In fact, all sections of $\mathcal{H}_{[k]}$ come from $H^0(S,H)$. As we are on a $K3$ surface, $H^0(S,H)$ has dimension $g+1$. We thus have a rational map $\phi : \mathbf{P}(\mathcal{H}^*_{[k]}) \dashrightarrow \mathbf{P}^g$ such that $\phi^*\mathcal{O}_{\mathbf{P}^g}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{H}^*_{[k]})}(1)$. The top Segre class of $\mathcal{H}^*_{[k]}$ (or $\mathcal{H}_{[k]}$) is the top self-intersection of $c_1(\mathcal{O}_{\mathbf{P}(\mathcal{H}^*_{[k]})}(1))$ on $\mathbf{P}(\mathcal{H}^*_{[k]})$. We observe that the first inequality in (5) says that $\dim \mathbf{P}(\mathcal{H}^*_{[k]}) > \dim \mathbf{P}^g$, so the proposition is a consequence of the following lemma which is a mild generalization of Lazarsfeld's result in [2], saying that smooth curves in $\mid H \mid$ are brill-Noether generic:

Lemma 8. *If $g > 2k-2$, the vector bundle $\mathcal{H}_{[k]}$ is generated by the sections coming from $H^0(S,H)$.*

Indeed, this last statement says that the rational map ϕ is actually a morphism so that the top self-intersection of a line bundle pulled-back via ϕ is 0. □

Proof of Lemma 8. The proof is by contradiction. It is obtained by applying Lazarsfeld's arguments in [2]. For convenience of the reader and because Lazarsfeld considers only subschemes supported on smooth curves, we give the complete argument: If $z \in S^{[k]}$ is a point such that $H^0(S,H) \to \mathcal{H}_{[k],z}$ is not surjective, z corresponds to a length k subscheme $Z \subset S$ such that the restriction map $H^0(S,H) \to H^0(H_{|Z})$ is not surjective. By Serre duality, we have a nonzero class $e \in \operatorname{Ext}^1(\mathcal{I}_Z, H^{-1})$, which provides a torsion free rank 2 sheaf $\mathcal{E}$ fitting into an exact sequence

$$0 \to H^{-1} \to \mathcal{E} \to \mathcal{I}_Z \to 0. \tag{6}$$

Note that the original Lazarsfeld argument deals with all subschemes which are locally complete intersection, for which $\mathcal{E}$ is locally free (assuming k is minimal). We have $c_1(\mathcal{E}) = H^{-1}$ and $c_2(\mathcal{E}) = k$. It thus follows that

$$\chi(\mathcal{E},\,\mathcal{E}) := h^0(End(\mathcal{E})) - \dim Ext^1(\mathcal{E},\mathcal{E}) + \dim Ext^2(\mathcal{E},\mathcal{E}) = 4\chi(\mathcal{O}_S) + c_1(\mathcal{E})^2 - 4c_2(\mathcal{E})$$

$$= 8 + 2g - 2 - 4k.$$

The second inequality in (5) thus gives

$$\chi(\mathcal{E},\,\mathcal{E}) > 2.$$

We thus conclude (applying Serre duality showing that $\dim Ext^2(\mathcal{E},\mathcal{E}) = h^0(End(\mathcal{E}))$) that $\mathcal{E}$ has an endomorphism $f : \mathcal{E} \to \mathcal{E}$ which is not proportional to the identity, hence can be assumed to be of generic rank 1. Let B be the line bundle defined as $\mathcal{F}^{**}$ where $\mathcal{F}$ is the saturation of $\operatorname{Im} f$ in $\mathcal{E}$. The line bundle B must be a power of H. The non-split exact sequence (6) shows that $\operatorname{Hom}(\mathcal{E}, H^{-1}) = 0$ since the exact sequence (6) is not split, so B must be trivial or a positive power of H. It follows that $\mathcal{F}$ is equal to $H^{\otimes k} \otimes \mathcal{I}_W$ for some $k \geq 0$ and for some 0-dimensional subscheme $W \subset Z$ (which can appear only where $\mathcal{E}$ is not locally free). As $H^{\otimes k} \otimes \mathcal{I}_W$ is not contained in H^{-1}, it must map nontrivially to $\mathcal{I}_Z$ via $f : \mathcal{E} \to \mathcal{I}_Z$, so that finally $k = 0$ and $\mathcal{I}_W \subset \mathcal{I}_Z$. As $\mathcal{I}_Z \subset \mathcal{I}_W$ and $\operatorname{End}(\mathcal{I}_Z) = \mathbb{C}Id$, we conclude that in fact f induces an isomorphism $\mathcal{I}_W \cong \mathcal{I}_Z$ and the sequence (6) is split, which is a contradiction. □

We note for later reference the following simple fact on which the proof of Proposition 7 rests. We will say that H is k-ample if $\mathcal{H}_{[k]}$ is generated by its global sections. 1-ample means that H is generated by sections, and 2-ample means that H is very ample.

Lemma 9. *Let Σ be a surface, H a line bundle on Σ. Assume that H is k-ample and $h^0(\Sigma, H) < 3k$. Then $s_{2k}(\mathcal{H}_{[k]}) = 0$.*

3 Proof of Theorem 1

We are going to prove here Theorem 1 for $2g-2 \geq 0$, i.e. $g \geq 1$, by induction on g. The case where g is nonpositive works similarly, by induction on $-g$. Let S' be a $K3$ surface equiped with a line bundle H' such that $c_1(H')^2 = 2(g-1)-2$. Let A be an abelian surface with a principal polarization θ, so that $c_1(\theta)^2 = 2$. The surface $\Sigma = S' \sqcup A$ equiped with the line bundle H_Σ which is equal to H' on S' and θ on A, has the same characteristic numbers as our original pair (S, H) where S is a $K3$ surface, and H is a polarization with self-intersection $2g-2$. On the other hand, $\Sigma^{[k]}$ is the disjoint union

$$\Sigma^{[k]} = \sqcup_{l=0}^{l=k} {S'}^{k-l} \times A^{[l]},$$

and on each summand ${S'}^{k-l} \times A^{[l]}$, the vector bundle $H_{\Sigma,[k]}$ equals $pr_1^* H'_{[k-l]} \oplus pr_2^* \theta_{[l]}$. We thus conclude that we have the following formula, where $b_l := \int_{A^{[l]}} s_{2l}(\theta_{[2l]})$ (this is a particular case of (3)):

$$s_{k,g} = \sum_{l=0}^{l=k} b_l s_{k-l,g-1}. \tag{7}$$

Corollary 10. *The numbers $s_{k,g}$ for $g \geq 1$ are fully determined by the numbers b_l, $0 \leq l \leq k$ and the numbers $s_{l,1}$, $l \leq k$, $s_{1,g}$, $g \geq 1$ (or $s_{0,g}$).*

Remark 11. We have $b_0 = 1$, $b_1 = 2$, and similarly $s_{0,g} = 1$, $s_{1,g} = 2g-2$.

Lemma 12. *Suppose that the numbers b_l, $0 \leq l \leq k-1$ and the numbers $s_{l,1}$, $0 \leq l \leq k-1$ are given, with $b_0 = 1$, $b_1 = 2$. Then the numbers $s_{k,1}$ and b_k are determined by the condition $b_0 = 1$, $b_1 = 2$, equation (7), and the vanishing equations*

$$s_{k,2k} = 0,\ s_{k,2k-1} = 0 \tag{8}$$

for $k \geq 2$ proved in Proposition 7.

Proof. Indeed, by Corollary 10, all the numbers $s_{l,g'}$ for $g' \leq g-1$ and $l \leq k-1$ are determined by b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. We thus can write (7) as

$$s_{k,g} = s_{k,g-1} + (\ldots) + b_k,$$

$$s_{k,g-1} = s_{k,g-2} + (\ldots) + b_k,$$

$$\ldots$$

where the expressions $(\ldots)$ in the middle are determined by b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. Combining these equations, we get

$$\begin{aligned} s_{k,2k} &= s_{k,1} + (\ldots) + (2k-1)b_k \\ s_{k,2k-1} &= s_{k,1} + (\ldots) + (2k-2)b_k, \end{aligned} \tag{9}$$

hence we can see the equations $s_{k,2k} = 0$, $s_{k,2k-1} = 0$ as a system of two affine equations in the two variables $s_{k,1}$ and b_k, whose linear part is invertible and the constants are determined by b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. The numbers $s_{k,1}$ and b_k are thus uniquely determined by these equations and the numbers b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. □

Corollary 13. *There exist unique sequences of numbers $s_{k,g}$, $k \geq 0$, $g \geq 1$ and b_l, $l \geq 0$ satisfying:*

1. $b_0 = 1$, $b_1 = 2$,
2. $s_{0,g} = 1$, $s_{1,g} = 2g - 2$,
3. $s_{k,2k} = 0$, $s_{k,2k-1} = 0$ *for* $k \geq 2$.
4. $s_{k,g} = \sum_{l=0}^{l=k} b_l s_{k-l,g-1}$.

Proof of Theorem 1. The numbers $s'_{k,g} := 2^k \binom{g-2k+1}{k}$ satisfy the vanishings $s'_{k,2k} = 0$, $s'_{k,2k-1} = 0$ for $k \geq 2$, that is, condition 3 of Corollary 13. They also satisfy the condition $s'_{1,g} = 2g-2$, that is, condition 2 of Corollary 13. In order to show that $s_{k,g} = s'_{k,g}$, it suffices by Corollary 13 to show that they also satisfy condition 4 for adequate numbers b'_l, which is proved in the following Lemma 14. □

Lemma 14. *There exist numbers b'_l, $l \geq 0$ with $b'_0 = 1$, $b'_1 = 2$ such that for any $g \geq 1$*

$$s'_{k,g} = \sum_{l=0}^{k} b'_l s'_{k-l,g-1}. \tag{10}$$

Proof. We observe that $s'_{k,g}$ is, as a function of g, a polynomial of degree exactly k, with leading coefficient 2^k. Hence the $s'_{l,g}$ for $0 \leq l - 1$ form a basis of the space of polynomials of degree $k - 1$, and for k fixed, there exist uniquely defined numbers $b'_{l,k}$, $l = 0, \ldots, k$, with $b'_{0,k} = 1$, such that for any g:

$$s'_{k,g} = \sum_{l=0}^{k} b'_{l,k} s'_{k-l,g-1}. \tag{11}$$

Let us prove that $b'_{l,k} = b'_{l,k-1}$ for $l \leq k - 1$. We have

$$\binom{g-2k+1}{k} = \binom{g-2k}{k} + \binom{g-2k}{k-1},$$

that is,

$$2s'_{k-1,g-3} = s'_{k,g} - s'_{k,g-1}, \tag{12}$$

with the convention that $s'_{k,g} = 0$ for $k < 0$. It follows by definition of $b'_{l,k}$ that

$$2s'_{k-1,g-3} = \sum_{l=0}^{k} b'_{l,k} s'_{k-l,g-1} - \sum_{l=0}^{k} b'_{l,k} s'_{k-l,g-2},$$

which gives, by applying (12) again:

$$2s'_{k-1,g-3} = 2\sum_{l=0}^{k} b'_{l,k} s'_{k-l-1,g-4} = 2\sum_{l=0}^{k-1} b'_{l,k} s'_{k-l-1,g-4}.$$

By definition of $b'_{l,k-1}$, this provides $b'_{l,k} = b'_{l,k-1}$. □

4 Further geometric vanishing

We discuss in this section similar geometric vanishing results for the Segre classes on the blow-up of a $K3$ surface at one point. The setting is thus the following: S is a $K3$ surface with $\operatorname{Pic} S = \mathbb{Z}L$, $L^2 = 2g-2$, and $x \in S$ is a point. The surface $\tau : \widetilde{S} \to S$ is the blow-up of S at x with exceptional curve E, and $H := \tau^*L(-lE) \in \operatorname{Pic} \widetilde{S}$ for some positive integer l. Our main goal is to discuss the analogue of Lemma 8 in this context. Note that, when H is very ample, the curve E has degree l in the embedding given by $|H|$, so that the vector bundle $\mathcal{H}_{[k]}$ can be generated by sections only when $k \le l+1$.

To start with, we have:

Proposition 15. *Let S be a $K3$ surface with Picard group generated by L, $L^2 = 2g-2$. Let $\tau : \widetilde{S} \to S$ be the blow-up at a point $x \in S$. Then, denoting $H = \tau^*L(-lE)$, if*

$$4 + 2g > (l+1)^2, \tag{13}$$

one has $H^1(\widetilde{S}, H) = 0$. It follows that $h^0(\widetilde{S}, H) = g + 1 - \frac{l(l+1)}{2}$.

Proof. We argue by contradiction. The proof follows Reider's [5] and Lazarsfeld's [2] methods. Assume $H^1(\widetilde{S}, H) \neq 0$. Then $\operatorname{Ext}^1(H, \mathcal{O}_{\widetilde{S}}(E)) \neq 0$, which provides a rank 2 vector bundle $\mathcal{E}$ on $\widetilde{S}$ which fits in an exact sequence

$$0 \to \tau^*L^{-1}((l+1)E) \to \mathcal{E} \to \mathcal{O}_{\widetilde{S}} \to 0. \tag{14}$$

The fact that the extension class of (14) is not trivial translates into $h^0(\widetilde{S}, \mathcal{E}) = 0$. We have $c_2(\mathcal{E}) = 0$ and $c_1(\mathcal{E})^2 = 2g - 2 - (l+1)^2$, so that (13) gives the inequality

$$\chi(End\,\mathcal{E}) = 8 + c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) > 2.$$

It follows that $h^0(\widetilde{S}, End\,\mathcal{E}) + h^0(\widetilde{S}, End\,\mathcal{E}(E)) > 2$, hence $h^0(\widetilde{S}, End\,\mathcal{E}(E)) > 1$. Thus there exists a $\phi \in \operatorname{Hom}(\mathcal{E}, \mathcal{E}(E))$ which is which is not proportional to the identity. Looking at its characteristic polynomial and using the fact that all sections of $\mathcal{O}_{\widetilde{S}}(\mu E)$ are constant, we can even assume that ϕ is generically of rank 1. Let $A = \operatorname{Ker} \phi \subset \mathcal{E}$. We have $A = \tau^*L^{\alpha}(\beta E)$ and $\mathcal{E}$ fits in an exact sequence

$$0 \to A \to \mathcal{E} \to B \otimes \mathcal{I}_W \to 0, \tag{15}$$

where B is the line bundle $\tau^*L^{-1-\alpha}((l+1-\beta)E)$. As $B = \operatorname{Im} \phi$, we have $B \hookrightarrow \mathcal{E}(E)$. From the exact sequence (14), we immediately conclude that $\alpha \le 0$ and $(-1-\alpha) \le 0$, so that $\alpha = 0$ or $\alpha = -1$.

Assume first $\alpha = 0$. Then as $h^0(\widetilde{S}, \mathcal{E}) = 0$, we conclude that $\beta < 0$, hence $l + 1 - \beta > 0$. Then (15) gives

$$c_2(\mathcal{E}) = A \cdot B + \deg W = -\beta(l+1-\beta) > 0,$$

which is a contradiction.

In the remaining case $\alpha = -1$, we conclude that $B = \mathcal{O}_{\widetilde{S}}((l+1-\beta)E)$, so that we have a nonzero morphism $\mathcal{O}_{\widetilde{S}}((l-\beta)E) \to \mathcal{E}$, to which we can apply the previous argument, getting a contradiction. ☐

Pushing forward the arguments above, we now prove the following result:

Theorem 16. *Let S be a general $K3$ surface with Picard group generated by L, and $x \in S$ a general point. Then for $k \ge 2$, $H = \tau^*L(-lE)$ is k-ample for $k = l$ or $k = l+1$, and $g - \frac{l(l+1)}{2} = 3k - 2$.*

Remark 17. *When $g - \frac{l(l+1)}{2} = 3k - 2$, with $k = l$ or $k = l+1$, one has for $l > 0$*

$$4 + 2g = l(l+1) + 6k \ge (l+7)l > (l+1)^2$$

so that Proposition 15 applies, which gives $H^1(\widetilde{S}, H) = 0$ and $h^0(\widetilde{S}, H) = g + 1 - \frac{l(l+1)}{2} = 3k - 1$.

Proof of Theorem 16. With the assumptions of Theorem 16, assume H is not k-ample. Therefore there exists a 0-dimensional subscheme $Z \subset \widetilde{S}$ of length k such that $H^1(\widetilde{S}, H \otimes \mathcal{I}_Z) \neq 0$. Using the duality $H^1(\widetilde{S}, H \otimes \mathcal{I}_Z)^* = \operatorname{Ext}^1(\mathcal{I}_Z, -H + E)$, this provides us with a rank 2 torsion free sheaf $\mathcal{E}$ on $\widetilde{S}$ fitting in an exact sequence

$$0 \to \tau^* L^{-1}((l+1)E) \to \mathcal{E} \to \mathcal{I}_Z \to 0. \tag{16}$$

The numerical invariants of $\mathcal{E}$ are given by

$$c_2(\mathcal{E}) = k,\ c_1(\mathcal{E})^2 = 2g - 2 - (l+1)^2,$$

from which we conclude that

$$\chi(\mathcal{E}, \mathcal{E}) = 8 + 2g - 2 - (l+1)^2 - 4k,$$

hence

$$h^0(End\,\mathcal{E}) + h^0(End\,\mathcal{E}(E)) \geq 8 + 2g - 2 - (l+1)^2 - 4k. \tag{17}$$

By assumption, $g - \frac{l(l+1)}{2} = 3k - 2$, so $2g - 2 - (l+1)^2 = 6k - 6 - (l+1)$ and (17) gives

$$2h^0(End\,\mathcal{E}(E)) \geq 2 + 2k - (l+1),$$

hence $2h^0(End\,\mathcal{E}(E)) > 2$ because $k \geq 2$ and $k = l$ or $k = l+1$. Thus there exists a morphism

$$\phi : \mathcal{E} \to \mathcal{E}(E)$$

which is not proportional to the identity. As before, we can even assume that ϕ is generically of rank 1. One difference with the previous situation is the fact that $\mathcal{E}$ is not necessarily locally free, and furthermore $c_2(\mathcal{E}) \neq 0$. The kernel of ϕ and its image are torsion free of rank 1, hence are of the form $A \otimes \mathcal{I}_W$, $B \otimes \mathcal{I}_{W'}$ for some line bundles A, B on $\widetilde{S}$ which are of the form

$$A = \tau^* L^{\alpha}(\beta E),\ B = \tau^* L^{-1-\alpha}((l+1-\beta)E).$$

As before, we must have $\alpha \leq 0$ and $-1 - \alpha \leq 0$ because B injects into $\mathcal{E}(E)$. Hence we conclude that $\alpha = 0$ or $\alpha = -1$.

(i) If $\alpha = 0$, then we have a nonzero morphism $\mathcal{O}(\beta E) \otimes \mathcal{I}_W \to \mathcal{I}_Z$. It follows that $\beta \leq 0$. If $\beta = 0$, this says that $\mathcal{I}_W \subset \mathcal{I}_Z$ and that the extension class of (16) vanishes in $\operatorname{Ext}^1(\mathcal{I}_W, \tau^* L^{-1}((l+1)E))$. But the restriction map

$$\operatorname{Ext}^1(\mathcal{I}_Z, \tau^* L^{-1}((l+1)E)) \to \operatorname{Ext}^1(\mathcal{I}_W, \tau^* L^{-1}((l+1)E))$$

is injective as it is dual to the map $H^1(\widetilde{S}, \mathcal{I}_W(H)) \to H^1(\widetilde{S}, \mathcal{I}_Z(H))$ which is surjective. Indeed, the spaces are respective quotients of $H^0(H_{|W})$, $H^0(H_{|Z})$ by Proposition 15 which applies in our case as noted in Remark 17. So we conclude that $\beta < 0$. We now compute $c_2(\mathcal{E})$ using the exact sequence

$$0 \to A \otimes \mathcal{I}_W \to \mathcal{E} \to B \otimes \mathcal{I}_{W'} \to 0,$$

with $A = \mathcal{O}(\beta E)$, $B = \tau^* L^{-1}((l+1-\beta)E)$. This gives

$$c_2(\mathcal{E}) = \deg W + \deg W' - \beta(l+1-\beta) \geq -\beta(l+1-\beta) \geq l+2.$$

This contradicts $c_2(\mathcal{E}) = k \leq l+1$.

(ii) If $\alpha = -1$, then we use instead the inclusion $B \otimes \mathcal{I}_{W'} \subset \mathcal{E}(E)$, with $B = \mathcal{O}((l+1-\beta)E)$ and argue exactly as before. $\square$

We deduce the following Corollary 18 concerning the numbers $s_k(d, \pi, \kappa, e)$ (we adopt here Lehn's notation [3]) defined as the top Segre class of $\mathcal{H}_{[k]}$ for a pair (Σ, H) where Σ is a smooth compact surface, and

$$d = H^2,\ \pi = H \cdot c_1(K_\Sigma),\ \kappa = c_1(\Sigma)^2,\ e = c_2(\Sigma).$$

Corollary 18. *(Cf. Theorem 3.) One has the following vanishing for $s_k(d, \pi, -1, 25)$*

$$s_k(7(k-1), k-1, -1, 25) = 0,\ s_k(7(k-1)+1, k, -1, 25) = 0 \tag{18}$$

for $k \geq 2$.

Proof. Take for Σ the blow-up of a $K3$ surface at a point so

$$\kappa = -1,\ e = 25.$$

Furthermore, assuming $\mathrm{Pic}\, S = \mathbb{Z}L$ with $L^2 = 2g-2$, and letting $H = \tau^* L(-lE)$ as above, we have

$$d = H^2 = 2g - 2 - l^2,\ \pi = H \cdot c_1(K_\Sigma) = l. \tag{19}$$

We consider the cases where

$$g - \frac{l(l+1)}{2} = 3k - 2 \tag{20}$$

with (i) $k = l+1$ or (ii) $k = l$.

Using (19), (20) gives in case (i), $d = 7(k-1)$, $\pi = k-1$ and in case (ii), $d = 7(k-1)+1, \pi = k$, so we are exactly computing $s_k(7(k-1), k-1, -1, 25) = 0$ in case (i) and $s_k(7(k-1)+1, k, -1, 25)$ in case (ii). Remark 17 says that assuming (20),

$$H^1(\widetilde{S}, H) = 0,\ h^0(\widetilde{S}, H) = 3k - 1$$

in cases (i) and (ii). Theorem 16 says that under the same assumption, H is k-ample on $\widetilde{S}$. Lemma 9 thus applies and gives $s_{2k}(\mathcal{H}_{[k]}) = 0$ in both cases, which is exactly (18). $\square$

Remark 19. Lehn gives in [3, Section 4] the explicit polynomial formulas for $2!s_2, \ldots, 5!s_5$ as polynomial functions of d, π, κ, e with huge integral coefficients. For example

$$\begin{aligned} 5!s_5 = d^5 - 100d^4 + d^3(3740 + 10e - 50\pi - 10\kappa) \\ -d^2(62000 - 3420\pi + 700e - 860\kappa) + d(384384 + 15e^2 \\ +15960e - 30e\kappa - 150\pi e + 15\kappa^2 + 150\kappa\pi - 75610\pi - 24340\kappa + 375\pi^2) \\ -400e^2 - 117120e + 3920\pi e + 960\kappa e + 226560\kappa - 4720\kappa\pi \\ -560\kappa^2 + 530880\pi - 9600\pi^2 \end{aligned} \tag{21}$$

It is pleasant to check the vanishing statements (18) for $k = 2, \ldots, 5$ using these formulas. For $k = 5$, one just has to plug-in the values $e = 25$, $\kappa = -1$, $d = 28$ and $\pi = 4$, or $e = 25$, $\kappa = -1$, $d = 29$ and $\pi = 5$ in (21).

We conclude this note by showing that all the Segre numbers are formally determined by the above results and formula (3).

Proposition 20. *The vanishings (18) together with the data of the numbers $s_k(d, 0, 0, 24)$ and $s_k(d, 0, 0, 0)$ determine all numbers $s_k(d, \pi, \kappa, e)$.*

Note that $s_k(d, 0, 0, 24)$ is for $d = 2g-2$ the number $s_{k,g}$ of the introduction, and these numbers are given by Marian-Oprea-Pandharipande's Theorem 1. The numbers $s_k(d, 0, 0, 0)$ correspond for d even to the Segre classes of tautological sheaves on Hilbert schemes of abelian surfaces equipped with a line bundle of self-intersection d. They are fully determined, by multiplicativity, by the case of self-intersection 2, where one gets the numbers b'_k appearing in our proof of Theorem 1.

Proof of Proposition 20. According to [3], [1], and as follows from (3), the generating series

$$s(z) = \sum_k s_k(d, \pi, \kappa, e) z^k$$

is of the form

$$s(z) = A(z)^d B(z)^e C(z)^\pi D(z)^\kappa, \tag{22}$$

for power series A, B, C, D with 0-th order coefficient equal to 1. Theorem 1 determines the series $A(z)$ and $B(z)$. We thus only have to determine $C(z)$ and $D(z)$. The degree 1 coefficients of the power series $C(z)$, $D(z)$ are immediate to compute as $s_1 = d$. We now assume that the coefficients of the power series $C(z)$ and $D(z)$ are computed up to degree $k-1$. The degree k coefficient of $s(z) = A(z)^d B(z)^e C(z)^\pi D(z)^\kappa$ is of the form $\pi C_k + \kappa D_k + \nu$ where ν is determined by d, e, π, κ, the coefficients of A and B, and the coefficients of order $\leq k-1$ of C and D. The vanishings (18) thus give the equations

$$0 = (k-1)C_k - D_k + \nu, \ 0 = kC_k - D_k + \nu',$$

which obviously determines C_k and D_k as functions of ν and ν'. □

We finally prove Corollary 6 of the introduction.

Proof of Corollary 6. Let $f_{d,\pi,\kappa,e}(z)$ be the Lehn function introduced in Conjecture 5. As Lehn's conjecture is proved by [4] for $\pi = \kappa = 0$ (the K-trivial case), the coefficients $f_{k,d,\pi,\kappa,e}$ of the Taylor expansion of $f_{d,\pi,\kappa,e}$ in z (not w) are the Segre numbers $s_{k,d,0,0,e}$ when $\pi = 0$, $\kappa = 0$. If furthermore they satisfy the vanishings $f_{k,d,\pi,\kappa,e} = 0$ for $e = 25$, $\kappa = -1$ and $d = 7(k-1)$, $\pi = k-1$ or $d = 7(k-1)+1$, $\pi = k$, the proof of Proposition 20 shows that $f_{k,d,\pi,\kappa,e} = s_{k,d,\pi,\kappa,e}$ for all k, d, π, κ, e as, by definition, f has the same multiplicative form (22) as s. □

References

[1] G. Ellingsrud, L. Göttsche, M. Lehn. On the cobordism class of the Hilbert scheme of a surface. J. Algebraic Geom. 10 (2001), no. 1, 81-100.

[2] R. Lazarsfeld. Brill-Noether-Petri without degenerations. J. Differential Geom. 23 (1986), no. 3, 299-307.

[3] M. Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. Invent. Math. 136 (1999), no. 1, 157-207.

[4] A. Marian, D. Oprea, R. Pandharipande. Segre classes and Hilbert schemes of points. Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 1, 239-267.

[5] I. Reider. Vector bundles of rank 2 and linear systems on algebraic surfaces. Ann. of Math. (2) 127 (1988), no. 2, 309-316.

[6] A. S. Tikhomirov, Standard bundles on a Hilbert scheme of points on a surface. In: *Algebraic geometry and its applications*, Yaroslavl,1992. Aspects of Mathematics, Vol. E25. Vieweg Verlag, 1994.

Collège de France, 3 rue d'Ulm 75005 Paris
claire.voisin@imj-prg.fr