

Some new results on modified diagonals

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O’Grady studied m^{th} modified diagonals for a smooth connected projective variety, generalizing the Gross–Schoen modified small diagonal. These cycles $\Gamma^m(X, a)$ depend on a choice of reference point $a \in X$ (or more generally a degree-1 zero-cycle). We prove that for any X, a , the cycle $\Gamma^m(X, a)$ vanishes for large m . We also prove the following conjecture of O’Grady: If X is a double cover of Y and $\Gamma^m(Y, a)$ vanishes (where a belongs to the branch locus), then $\Gamma^{2m-1}(X, a)$ vanishes, and we provide a generalization to higher-degree finite covers. We finally prove that $\Gamma^{n+1}(X, o_X) = 0$ when $X = S^{[m]}$, where S is a $K3$ surface, and $n = 2m$, which was conjectured by O’Grady and proved by him for $m = 2, 3$.

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1 Introduction

Let X be a connected smooth projective variety of dimension n . We will denote by $\text{CH}_i(X)$ the Chow groups of X with rational coefficients and $\text{CH}_i(X)/\text{alg}$ the groups of i -cycles of X with \mathbb{Q} -coefficients modulo algebraic equivalence. Let $a \in \text{CH}_0(X)$ be a 0-cycle of degree 1 on X . Following Gross and Schoen [9] and O’Grady [12], let us consider for $m \geq 2$ the following n -cycle $\Gamma^m(X, a)$ in X^m , which is a modification of the m^{th} small diagonal of X :

$$(1) \quad \Gamma^m(X, a) = \sum_{I \subset \{1, \dots, m\}, |I|=i < m} (-1)^i p_I^*(a^{*i}) \cdot p_J^* \Delta_{m-i} \in \text{CH}_n(X^m)_{\mathbb{Q}},$$

where

- $\{1, \dots, m\}$ is the disjoint union of I and J ,
- $p_I: X^m \rightarrow X^i$, resp. $p_J: X^m \rightarrow X^{m-i}$ are the projections onto the products of factors indexed by I , resp. J ,
- Δ_{m-i} is the small diagonal of X^{m-i} , $\Delta_1 = X$,
- $a^{*i} \in \text{CH}_0(X^i)$ is defined by

$$(2) \quad a^{*i} = p_1^* a \cdots p_i^* a.$$

For example, for $m = 2$, we have $\Gamma^2(X, a) = \Delta_X - a \times X - X \times a$ and $\Gamma^2(X, a) = 0$ if and only if $X = \mathbb{P}^1$ or a point. The modified small diagonal $\Gamma^3(X, a)$ appears in several recent works. Gross and Schoen prove that $\Gamma^3(X, a) = 0$ if X is a hyperelliptic curve and a is a Weierstrass point. This result was greatly extended in [5] by Colombo and van Geemen, who worked with 1-cycles modulo algebraic equivalence and proved that, for a d -gonal curve X , the cycle $\Gamma^{d+1}(X, a)$ is algebraically equivalent to 0. Although they do not state their result in this form, but as the vanishing modulo algebraic equivalence of the components Z_s , $s \geq d - 1$ of the Beauville decomposition [1] of X in its Jacobian, one can show that this is equivalent to the vanishing of $\Gamma^{d+1}(X, a)$ modulo algebraic equivalence. For completeness, we will prove this fact in Section 4.1.

Concerning higher-dimensional varieties, Beauville and Voisin proved in [2] the following theorem:

Theorem 1.1 *Let X be a K3 surface. Then there exists a canonical degree-1 zero-cycle o_X of X such that*

$$(3) \quad \Gamma^3(X, o_X) = 0 \quad \text{in } \text{CH}_2(X^3).$$

In fact, o_X can be defined as the class in $\text{CH}_0(X)$ of any point of X lying on a (singular) rational curve in X .

In the paper [12], O'Grady investigates $\Gamma^m(X, a)$ for higher m . He proves the following results (for X smooth projective connected):

Theorem 1.2 (O'Grady [12]) (i) *The cycle $\Gamma^{n+1}(X, a)$ is cohomologous to 0, for $n = \dim X$ and $q(X) = 0$. More generally $\Gamma^{m+1}(X, a)$ is cohomologous to 0 if and only if $m \geq \dim X + d$, where d is the dimension of the image of X in its Albanese variety.*

(ii) *If $\Gamma^m(X, a) = 0$ then $\Gamma^{m'}(X, a) = 0$ for $m' \geq m$.*

(iii) *If $p: X \rightarrow Y$ is a ramified double cover and a is a branch point such that $\Gamma^m(Y, a) = 0$, then for $m = 2$ or $m = 3$, $\Gamma^{2m-1}(X, b) = 0$, where $p(b) = a$.*

He conjectures that (iii) holds for any m (see [12, Conjecture 5.1]). One of our results is the proof of O'Grady's conjecture, see (i) below, and a generalization to any degree, see (ii) and (iii) below.

Theorem 1.3 *Let $p: X \rightarrow Y$ be a degree d finite morphism, where X, Y are smooth projective and connected.*

(i) *Assume $d = 2$, $a \in \text{CH}_0(Y)$ is a 0-cycle of degree 1 supported on the branch locus of p , and $b := \frac{1}{2}p^*a \in \text{CH}_0(X)$. If $\Gamma^m(Y, a) = 0$, then $\Gamma^{2m-1}(X, b) = 0$.*

- (ii) For any d , assume $a \in Y$ is a point such that the subscheme $p^{-1}(a)$ is supported on a point $b \in X$. If $\Gamma^m(Y, a) = 0$, then $\Gamma^{d(m-1)+1}(X, b) = 0$.
- (iii) For any d , let $b := \frac{1}{d} p^* a$ for some 0-cycle $a \in \text{CH}_0(Y)$ of degree 1. If

$$\Gamma^m(Y, a) = 0 \quad \text{in } \text{CH}_n(Y^m)/\text{alg},$$

then

$$\Gamma^{d(m-1)+1}(X, b) = 0 \quad \text{in } \text{CH}_n(X^{d(m-1)+1})/\text{alg}.$$

Statement (i) of Theorem 1.3 has been obtained independently by Moonen and Yin [11].

Remark 1.4 When $Y = \mathbb{P}^n$ and $d \leq n + 1$, there always exists a point $a \in \mathbb{P}^n$ as in (ii); see [8]. In this case, we have $\Gamma^m(Y, a) = 0$, with $m = n + 1$, hence we conclude that for d^{th} covers X of \mathbb{P}^n with $d \leq n + 1$, $\Gamma^{d(n+1)}(X, b) = 0$, with $b = \frac{1}{d} p^*(\text{pt})$. Note also that any curve X of genus g admits a morphism of degree $d \leq g + 1$ to \mathbb{P}^1 , which is totally ramified at one given point x . Hence we get $\Gamma^{g+2}(X, x) = 0$ for any $x \in X$. This last result is also proved by Moonen and Yin [11] using the Colombo–van Geemen vanishing result.

Remark 1.5 In the case where Y is \mathbb{P}^1 , so X is a d -gonal curve, Theorem 1.3(iii) gives the vanishing $\Gamma^{d+1}(X, b) = 0$ in $\text{CH}_1(X^{d+1})/\text{alg}$. As explained in Section 4.1, this is equivalent to the Colombo–van Geemen theorem [5] mentioned above.

Another application of Theorem 1.3 is the following result, which will be deduced from it in Section 2 using the smash nilpotence result of Voevodsky [14] for cycles algebraically equivalent to 0:

Corollary 1.6 *Let X be a smooth projective (connected) variety of dimension n . Then for any $a \in \text{CH}_0(X)$ of degree 1, there exists an integer m such that $\Gamma^m(X, a) = 0$ in $\text{CH}^{(m-1)n}(X^m)$.*

Our second result is the following more precise statement:

Theorem 1.7 *Let X be smooth projective connected of dimension n and let $a \in \text{CH}_0(X)$ be of degree 1. Then, if X is swept-out by irreducible curves of genus g supporting a zero-cycle rationally equivalent to a , and $m \geq (n + 1)(g + 1)$, one has $\Gamma^m(X, a) = 0$ in $\text{CH}^{(m-1)n}(X^m)$.*

Note that such a g always exists: Indeed, consider curves in X which are complete intersections of ample hypersurfaces containing the support of the cycle a . For sufficiently high degree such hypersurfaces, these curves will sweep out X , and thus

we can take for g the genus of the generic such curves. In the case where X is a $K3$ surface, we know that X is swept-out by elliptic curves supporting the canonical 0-cycle. Hence we get from Theorem 1.7 the vanishing $\Gamma^6(X, o_X) = 0$, which is not optimal in view of the relation (3) in Theorem 1.1.

We finally turn to the case of hyper-Kähler manifolds. For $K3$ surfaces, one can get as a consequence of (3) the following properties of o_X (note however that property (1) below is used to prove (3) so that we do not actually recover it from (3). Nevertheless, the consequences (1) and (2) indicate that surfaces satisfying (3) are quite special):

- (1) The intersection of two divisors D, D' on X is proportional to o_X in $\text{CH}_0(X)$.
- (2) The second Chern class $c_2(X)$ is equal to $24o_X$.

In the paper [12], O'Grady formulates the following generalization of (3):

Conjecture 1.8 (O'Grady, [12, Conjecture 0.1]) *Let X be a hyper-Kähler n -fold. Then there exists a canonical 0-cycle $o_X \in \text{CH}_0(X)$ of degree 1 such that*

$$\Gamma^{n+1}(X, o_X) = 0 \quad \text{in } \text{CH}_n(X^{n+1}).$$

Note that by Theorem 1.2, (i), we have $[\Gamma^{n+1}(X, o_X)] = 0$ in $H^*(X^{n+1}, \mathbb{Q})$ and that this is optimal. Conjecture 1.8 thus states that the cycles $\Gamma^k(X, o_X)$ vanish in $\text{CH}(X^k)$ once they vanish in $H^*(X^k, \mathbb{Q})$, which is very different from the situation encountered in the case of curves (except for the hyperelliptic ones).

O'Grady establishes this conjecture for the punctual Hilbert schemes $S^{[2]}$ and $S^{[3]}$ of a $K3$ surface. The canonical 0-cycle o_X , for $X = S^{[n]}$, is naturally defined as the class in $\text{CH}_0(X)$ of any point of X lying over $no_S \in S^{(n)}$, for some representative $o_S \in S$ of the canonical 0-cycle of S . We prove in Section 5 Conjecture 1.8 for punctual Hilbert schemes $X = S^{[n]}$ of $K3$ surfaces, and for any n , using methods from Voisin [17] and recent results of Yin [20]:

Theorem 1.9 *Let S be a $K3$ surface, and let $X = S^{[m]}$. Then*

$$(4) \quad \Gamma^{n+1}(X, o_X) = 0 \quad \text{in } \text{CH}_n(X^{n+1}),$$

where o_X is the canonical 0-cycle on X coming from the canonical 0-cycle of S , and $n = \dim X = 2m$.

Note that one can recover from (4) the following result, which had been in fact already proved in [17, Theorem 1.5].

Corollary 1.10 *The intersection of n divisors on X is proportional to o_X in $\text{CH}_0(X)$.*

For the proof of Theorem 1.9, we will need three tools. The first ingredient is similar to what we did in [17], namely we will use the de Cataldo–Migliorini theorem [4] and will prove Proposition 5.6 in order to reduce to computations in the Chow rings of the self-products S^k . The second ingredient is very new and it is provided by Yin’s recent result [20] saying that the cohomological relations between the big diagonals of a regular surface and the pull-back of the class of a point are generated (modulo trivial relations) by the pull-backs of the Kimura relation and the cohomological counterpart $[\Gamma^3(S, o_S)] = 0$ in $H^8(S^3, \mathbb{Q})$ of the relation (3) (see also [12, Proposition 1.3]). We then argue that the Kimura relation is not needed in our context, while the relation $\Gamma^3(S, o_S) = 0$ is satisfied in the Chow ring by Theorem 1.1.

To conclude, let us remark that the next conjecture in the same spirit as Conjecture 1.8 was stated first in [17] for $K3$ surfaces, and then by Shen and Vial in [13] for general hyper-Kähler manifolds:

Conjecture 1.11 *Let X be a projective hyper-Kähler manifold and $n > 0$ be an integer. Then there exists a canonical 0-cycle $o_X \in \text{CH}_0(X)$ such that any polynomial relation between the cohomology classes $\text{pr}_i^*[o_X]$, $i \leq n$, $\text{pr}_{ij}^*[\Delta_X]$, $i \neq j \leq n$, already holds in $\text{CH}(X^n)$.*

O’Grady’s Conjecture 1.8 is the particular case of Conjecture 1.11 which concerns the class $\Gamma^{n+1}(X, o_X)$, $n = \dim X$. As explained in [18] in the case of $K3$ surfaces, Conjecture 1.11 is extremely strong since it implies finite dimensionality in the Kimura sense, with very important consequences established by Kimura [10], in particular on the nilpotency of self-correspondences homologous to 0. O’Grady’s Conjecture 1.8 does not seem to have such implications, so it is possibly of a nature different from Conjecture 1.11.

The paper is organized as follows: In Section 2 we introduce variants $\Gamma^{1,m}(X, a)$ of the cycles $\Gamma^m(X, a)$ which lie in $\text{CH}_n(X^{m+1})$, $n = \dim X$, and relate them to $\Gamma^m(X, a)$. In Section 3, we will prove Theorem 1.7. Theorem 1.3 will be proved in Section 4 and Theorem 1.9 will be proved in Section 5. The last Section 5.2 is devoted to the sketch of the proof of a general theorem (Theorem 5.12) concerning universally defined cycles on quasiprojective surfaces, which is used in the proof of Theorem 1.9. This result is of independent interest and its complete proof will be given together with further applications in [15].

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2 Cycles $\Gamma^{1,m}(X, a)$

We first introduce the following notation: X being smooth projective, and $a \in \text{CH}_0(X)_{\mathbb{Q}}$ being a zero-cycle of degree 1, we define $\Gamma^{1,m}(X, a) \in \text{CH}_n(X^{m+1})_{\mathbb{Q}}$ by

$$(5) \quad \Gamma^{1,m}(X, a) := \prod_{1 \leq i \leq m} (p_{0i}^* \Delta_X - p_i^* a),$$

where:

- (1) $\Delta_X \subset X \times X$ is the diagonal of X .
- (2) $p_{0i}: X^{m+1} \rightarrow X \times X$ is the projection on the product of the first and $(i + 1)^{\text{st}}$ factors.
- (3) $p_i: X^{m+1} \rightarrow X$ is the projection on the $(i + 1)^{\text{st}}$ factor (our factors are indexed by $\{0, \dots, m\}$).

The cycles $\Gamma^m(X, a)$ and $\Gamma^{1,m}(X, a)$ are related as follows:

Lemma 2.1 *We have*

$$(6) \quad \Gamma^m(X, a) = p'_{1,\dots,m*} \Gamma^{1,m}(X, a),$$

where we index the factors of X^{m+1} by $\{0, \dots, m\}$ and $p'_{1,\dots,m}$ is the projection from X^{m+1} to the product X^m of its last m factors. We also have

$$(7) \quad \Gamma^{m+1}(X, a) = \Gamma^{1,m}(X, a) - p_0^* a \cdot p'_{1,\dots,m*} (\Gamma^m(X, a)).$$

Proof This is almost immediate. Developing the product in (5), we get

$$(8) \quad \Gamma^{1,m}(X, a) = \sum_{I \subset \{1, \dots, m\}, |I|=i} (-1)^i p_I^* (a^{*i}) \cdot p_{0,J}^* \Delta_{m+1-i},$$

where $I \sqcup J = \{1, \dots, m\}$, $p_{0,J}$ is the projection from X^{m+1} to the product X^{m+1-i} of factors indexed by $\{0\} \cup J$ and p_I^* is the projection from X^{m+1} to the product X^i of the factors indexed by $I \subset \{1, \dots, m\}$. Applying

$$p'_{1,\dots,m*}: \text{CH}_n(X^{m+1})_{\mathbb{Q}} \rightarrow \text{CH}_n(X^m)_{\mathbb{Q}},$$

we get by the projection formula, using the fact that $p'_I = p_I \circ p'_{1,\dots,m}$:

$$p'_{1,\dots,m*} \Gamma^{1,m}(X, a) = \sum_{I \subset \{1,\dots,m\}, |I|=i} (-1)^i p_I^*(a^{*i}) \cdot p'_{1,\dots,m*}(p_{0,J}^* \Delta_{m+1-i}).$$

Equation (6) then follows from the fact that $p_J^* \Delta_{m-i} = p'_{1,\dots,m*}(p_{0,J}^* \Delta_{m+1-i})$ in $\text{CH}_n(X^m)$.

As for (7), we first write formula (1) for X^{m+1} , where as above we index the factors of X^{m+1} by $\{0, \dots, m\}$. This gives us

$$(9) \quad \Gamma^{m+1}(X, a) = \sum_{I \subset \{0,\dots,m\}, i=|I| \leq m} (-1)^i p_I^*(a^{*i}) \cdot p'_J{}^* \Delta_{m+1-i} \in \text{CH}_n(X^{m+1})_{\mathbb{Q}}.$$

We now separate the terms where $0 \notin I$, which by (8) exactly give $\Gamma^{1,m}(X, a)$, and the terms where $0 \in I$, which exactly give $-p_0^* a \cdot p'_{1,\dots,m}{}^*(\Gamma^m(X, a))$. \square

We deduce the following:

Proposition 2.2 *The vanishing of $\Gamma^m(X, a)$ in $\text{CH}_n(X^m)$ is equivalent to the vanishing of $\Gamma^{1,m}(X, a)$ in $\text{CH}_n(X^{m+1})$.*

Proof If $\Gamma^{1,m}(X, a) = 0$ then $\Gamma^m(X, a) = 0$ by (6). Conversely, if $\Gamma^m(X, a) = 0$, then [12, Proposition 2.4] shows that also $\Gamma^{m+1}(X, a) = 0$. Equation (7) then implies that $\Gamma^{1,m}(X, a) = 0$. \square

A consequence of this result is the following statement comparing $\Gamma^m(X, a)$ and $\Gamma^m(X, b)$, for two 0–cycles $a, b \in \text{CH}_0(X)$ of degree 1.

Corollary 2.3 *If $\Gamma^m(X, a) = 0$ and the cycle $b - a$ satisfies $(b - a)^{*k} = 0$ in $\text{CH}_0(X^k)$, then $\Gamma^{m+k}(X, b) = 0$.*

Here we refer to (2) for the definition of the $*$ –product (or external product) of cycles.

Proof Indeed, by Proposition 2.2, the assumption is equivalent to the vanishing conditions

$$\Gamma^{1,m}(X, a) = \prod_{i=1}^{i=m} (p_{0i}^*(\Delta_X) - p_i^* a) = 0 \quad \text{in } \text{CH}_n(X^{m+1}),$$

$$\prod_{i=1}^{i=k} p_i^*(b - a) = 0 \quad \text{in } \text{CH}_0(X^k),$$

where $n = \dim X$. On the other hand, the conclusion is equivalent to the vanishing

$$\Gamma^{1,m+k}(X, a) = \prod_{i=1}^{m+k} (p_{0i}^*(\Delta_X) - p_i^*b) = 0 \quad \text{in } \text{CH}_n(X^{m+k+1}).$$

We now write $b = a + (b - a)$, getting

$$\Gamma^{1,m+k}(X, b) = \prod_{i=1}^{m+k} ((p_{0i}^*(\Delta_X) - p_i^*a) - p_i^*(b - a))$$

and develop the product. In the developed expression, the product of $\geq m$ terms of the form $p_{0i}^*(\Delta_X) - p_i^*a$ is 0 and the product of $\geq k$ terms of the form $p_i^*(b - a)$ is 0. Hence we conclude that each monomial in the development is 0. \square

Here is another corollary of Proposition 2.2. It shows how to deduce Corollary 1.6 from Theorem 1.3, and thus gives another proof of the nilpotency statement of Theorem 1.7, with no estimate on the nilpotency index.

Corollary 2.4 *Let X be a smooth projective connected variety and let a be a 0-cycle of degree 1 on X such that $\Gamma^m(X, a) = 0$ in $\text{CH}(X^m)/\text{alg}$. Then for any 0-cycle b of degree 1 on X , there is an integer M such that $\Gamma^M(X, b) = 0$ in $\text{CH}(X^M)$.*

Proof As a and b are algebraically equivalent, we also have $\Gamma^m(X, b) = 0$ in $\text{CH}(X^m)/\text{alg}$. By Proposition 2.2, which is true and proved in the same way for cycles modulo algebraic equivalence (observing that [12, Proposition 2.4] is true as well for cycles modulo algebraic equivalence), this is equivalent to the fact that $\Gamma^{1,m}(X, b)$ is algebraically equivalent to 0 in X^{m+1} . By the smash-nilpotence result of Voevodsky [14], there is an integer N such that the cycle $\Gamma^{1,m}(X, b)^{*N}$ vanishes identically in $\text{CH}(X^{N(m+1)})$. Thus its restriction to X^{Nm+1} embedded in $X^{N(m+1)}$ as the small diagonal on the factors of index $0, m + 1, 2m + 2, \dots, (N - 1)(m + 1)$ also vanishes in $\text{CH}(X^{Nm+1})$. But this restricted cycle is nothing but $\Gamma^{1,Nm}(X, b)$. \square

The following criterion for the vanishing of $\Gamma^m(X, a)$ will be used in Section 4. Here we consider more generally the vanishing of $\Gamma^m(X, a)$ modulo an adequate equivalence relation R which in applications will be rational or algebraic equivalence. We need an assumption on the 0-cycle a of degree 1, namely

$$(10) \quad p_1^*a \cdot p_2^*a = \Delta_*a \quad \text{in } \text{CH}_0(X \times X)/R,$$

where Δ is the diagonal inclusion map of X in $X \times X$. This assumption is satisfied for any R if a is a point, or for any 0-cycle if R is algebraic equivalence, and X is connected.

Proposition 2.5 Assume a satisfies (10). Then $\Gamma^m(X, a) = 0$ in $\text{CH}(X^m)/R$ if and only if

$$\Gamma^{1,m-1}(X, a) = p_0^* a \cdot \Gamma \quad \text{in } \text{CH}_n(X^m)/R, \quad n = \dim X,$$

for some cycle $\Gamma \in \text{CH}_{2n}(X^m)/R$.

The proof of Proposition 2.5 will use the following

Lemma 2.6 Assume the degree-1 zero-cycle a of X satisfies (10). Then for any Y and any cycle $\Gamma \in \text{CH}(X \times Y)/R$, we have

$$p_X^* a \cdot \Gamma = p_X^* a \cdot p_Y^* \Gamma_a \quad \text{in } \text{CH}(X \times Y)/R,$$

where

$$\Gamma_a := p_{Y*}(p_X^* a \cdot \Gamma) \quad \text{in } \text{CH}(Y)/R.$$

Proof Let $a = \sum_i n_i a_i$, where $a_i \in X$. Then

$$p_X^* a \cdot \Gamma = \sum_i n_i p_X^* a_i \cdot \Gamma = \sum_i n_i a_i \times \Gamma_{a_i} = \sum_i n_i p_X^* a_i \cdot p_Y^* \Gamma_{a_i},$$

where $\Gamma_{a_i} \in \text{CH}(Y)/R$ is the restriction of Γ to $a_i \times Y$. So we need to prove that, assuming (10),

$$(11) \quad \sum_i n_i p_X^* a_i \cdot p_Y^* \Gamma_{a_i} = p_X^* a \cdot p_Y^* \Gamma_a \quad \text{in } \text{CH}(X \times Y)/R,$$

where $\Gamma_a = \sum_i n_i \Gamma_{a_i} \in \text{CH}(Y)/R$. Note that (10) is exactly the case of (11) where $X = Y$ and Γ is the diagonal of X . The general case is then deduced from this one by introducing the following correspondence Γ' between $X \times X$ and $X \times Y$. Namely, let:

- $p_{X,Y}: X \times X \times X \times Y \rightarrow X \times Y$ be the projection onto the product of the second and last factors.
- $p_{X,X}: X \times X \times X \times Y \rightarrow X \times X$ be the projection onto the product of the first and third factors.

Let now $\Gamma' := p_{X,X}^* \Delta_X \cdot p_{X,Y}^* \Gamma \in \text{CH}(X \times X \times X \times Y)$. We clearly have

$$\Gamma'_*((x, y)) = p_X^* x \cdot p_Y^* \Gamma_y \quad \text{in } \text{CH}(X \times Y).$$

Formula (10) tells us that on $X \times X$, $p_1^* a \cdot \Delta_X = p_1^* a \cdot p_2^* a$ modulo R , so that by the above formula,

$$\begin{aligned} \Gamma'_*(p_1^* a \cdot \Delta_X) &= \sum_i n_i p_X^* a_i \cdot p_Y^* \Gamma_{a_i} \\ &= \Gamma'_*(p_1^* a \cdot p_2^* a) = \sum_{ij} n_i n_j p_X^* a_i \cdot p_Y^* \Gamma_{a_j} = p_X^* a \cdot p_Y^* \Gamma_a \end{aligned}$$

in $\text{CH}(X \times Y)/R$. This proves (11). □

Proof of Proposition 2.5 We have by (7)

$$\Gamma^m(X, a) = \Gamma^{1,m-1}(X, a) - p_0^*a \cdot p_{1,\dots,m-1}^* \Gamma^{m-1}(X, a)$$

so if $\Gamma^m(X, a) = 0$ in $\text{CH}(X^m)/R$, we get

$$\Gamma^{1,m-1}(X, a) = p_0^*a \cdot p_{1,\dots,m-1}^* \Gamma^{m-1}(X, a) \quad \text{in } \text{CH}(X^m)/R.$$

This proves one direction (for which we do not need (10)). In the other direction, we assume (10) and

$$(12) \quad \Gamma^{1,m-1}(X, a) = p_0^*a \cdot \Gamma \quad \text{in } \text{CH}_n(X^m)/R$$

for some cycle $\Gamma \in \text{CH}_{2n}(X^m)/R$. We now use Lemma 2.6 which gives

$$p_0^*a \cdot \Gamma = p_0^*a \cdot p_{1,\dots,m-1}^*(p_{1,\dots,m-1*}(p_0^*a \cdot \Gamma)).$$

By (12), this gives

$$p_0^*a \cdot \Gamma = p_0^*a \cdot p_{1,\dots,m-1}^*(p_{1,\dots,m-1*}(\Gamma^{1,m-1}(X, a))) \quad \text{in } \text{CH}_n(X^m)/R.$$

As $p_{1,\dots,m-1*}(\Gamma^{1,m-1}(X, a)) = \Gamma^{m-1}(X, a)$ by (6), we get

$$\Gamma^{1,m-1}(X, a) = p_0^*a \cdot p_{1,\dots,m-1}^*(\Gamma^{m-1}(X, a)) \quad \text{in } \text{CH}_n(X^m)/R.$$

Using (7), we conclude that

$$\Gamma^m(X, a) = \Gamma^{1,m-1}(X, a) - p_0^*a \cdot p_{1,\dots,m-1}^*(\Gamma^{m-1}(X, a)) = 0 \quad \text{in } \text{CH}_n(X^m)/R. \quad \square$$

3 Proof of Theorem 1.7

We prove in this section Theorem 1.7, that is the following statement:

Theorem 3.1 *Let X be a variety of dimension n and let $a \in \text{CH}_0(X)$ be of degree 1. If X is swept-out by irreducible curves of genus $\leq g$ supporting a 0-cycle rationally equivalent to a , and $m \geq (n + 1)(g + 1)$, then $\Gamma^m(X, a) = 0$.*

Note that for $g = 0$, we get the following corollary:

Corollary 3.2 *Let X be a rationally connected manifold of dimension n . Then $\Gamma^{n+1}(X, o) = 0$ for any point $o \in X$.*

This corollary will be improved at the end of this section in Theorem 3.6.

Proof of Theorem 3.1 By Proposition 2.2, it suffices to prove the vanishing of $\Gamma^{1,m}(X, a)$. Let us see $\Gamma^{1,m}(X, a)$ as a correspondence between X and X^m . Then for any $x \in X$, we have

$$\Gamma^{1,m}(X, a)|_{x \times X^m} = (x - a)^{*m} \quad \text{in } \text{CH}_0(X^m)_{\mathbb{Q}}.$$

Recall now the following result proved in [14; 16]:

Lemma 3.3 *Let C be a smooth connected curve of genus g , and let $z \in \text{CH}_0(C)_{\mathbb{Q}}$ be a 0-cycle of degree 0 on C . Then for $k > g$, $z^{*k} = 0$ in $\text{CH}_0(C^k)_{\mathbb{Q}}$.*

Our assumption is now that X is swept out by irreducible curves of genus $\leq g$ supporting a 0-cycle rationally equivalent to a . This means that for any $x \in X$, there is a smooth connected curve C_x of genus $\leq g$ mapping to X via a morphism f_x , a point $x' \in C_x$ such that $f_x(x') = x$ and a 0-cycle $a' \in \text{CH}_0(C_x)_{\mathbb{Q}}$ of degree 1, such that $f_{x*}(a') = a$ in $\text{CH}_0(X)_{\mathbb{Q}}$. It is then clear that

$$f_x^k((x' - a')^{*k}) = (x - a)^{*k} \quad \text{in } \text{CH}_0(X^k)_{\mathbb{Q}}.$$

We thus conclude by Lemma 3.3 that for $k > g$, and for any $x \in X$

$$(13) \quad \Gamma^{1,k}(X, a)|_{x \times X^k} = (x - a)^{*k} = 0 \quad \text{in } \text{CH}_0(X^k)_{\mathbb{Q}}.$$

We use now the following general principle which is behind the Bloch–Srinivas decomposition of the diagonal [3]; see [18, Section 3.1]:

Theorem 3.4 *Let $\phi: W \rightarrow Y$ be a morphism, where W is smooth of dimension m . Let Z be a codimension- k cycle on W . Assume that, for general $y \in Y$, the restriction $Z|_{W_y}$ vanishes in $\text{CH}^k(W_y)$. Then there is a dense Zariski open set $U \subset Y$, such that $Z_U = 0$ in $\text{CH}^k(W_U)$. Equivalently, there exist a nowhere dense closed algebraic subset $D \subsetneq Y$ and a cycle $Z' \in \text{CH}_{m-k}(W_D)_{\mathbb{Q}}$ such that*

$$Z = Z' \quad \text{in } \text{CH}^k(W)_{\mathbb{Q}}.$$

(Here we use the notation $W_D := \phi^{-1}(D)$, $W_U := \phi^{-1}(U)$.) Applying this statement to $Y = X$, $W = X^{k+1}$, ϕ the projection to the first factor and $Z = \Gamma^{1,k}(X, a)$, we conclude from (13) that under the assumptions of Theorem 3.1, there exists for $k > g$ a proper closed algebraic subset $D \subsetneq X$, such that $\Gamma^{1,k}(X, a)$ is rationally equivalent to a cycle supported on $D \times X^k$.

Recall now the formula (5) defining $\Gamma^{1,k}$:

$$\Gamma^{1,k}(X, a) := \prod_{1 \leq i \leq k} (p_{0i}^* \Delta_X - p_i^* a).$$

It follows immediately that

$$(14) \quad \Gamma^{1,k+k'}(X, a) = p_{0,1 \leq i \leq k}^* \Gamma^{1,k}(X, a) \cdot p_{0,k+1 \leq i \leq k+k'}^* \Gamma^{1,k'}(X, a),$$

where

$$p_{0,1 \leq i \leq k}: X^{k+k'+1} \rightarrow X^{k+1}$$

is the projection on the product of the $k + 1$ first factors and

$$p_{0,k+1 \leq i \leq k+k'}: X^{k+k'+1} \rightarrow X^{k'+1}$$

is the projection on the product of the first factor (indexed by 0) and the last k' factors.

For $m \geq (n + 1)(g + 1)$, we write $m = (n + 1)(g + 1) + r$, for some $r \geq 0$ and we get from (14):

$$\begin{aligned} \Gamma^{1,m}(X, a) &= p_{0,1 \leq i \leq g+1}^* (\Gamma^{1,g+1}) \cdot p_{0,g+2 \leq i \leq 2(g+1)}^* (\Gamma^{1,g+1}) \\ &\cdots p_{0,n(g+1)+1 \leq i \leq (n+1)(g+1)}^* (\Gamma^{1,g+1}) \cdot p_{0,(n+1)(g+1)+1 \leq i \leq (n+1)(g+1)+r}^* (\Gamma^{1,r}). \end{aligned}$$

Now we proved that the cycle $\Gamma^{1,g+1}$ is supported (via the first projection $X^{g+2} \rightarrow X$) over a proper algebraic subset $D \subsetneq X$, and by the easy moving Lemma 3.5 below, we can choose closed algebraic subsets D_1, \dots, D_{n+1} such that $\bigcap_i D_i = \emptyset$ and $\Gamma^{1,g+1}$ is supported (via the first projection $X^{g+2} \rightarrow X$) over the proper algebraic subset $D_i \subsetneq X$ for each i .

Then we conclude that for $m \geq (n + 1)(g + 1)$, $\Gamma^{1,m}(X, a)$ is supported (via the first projection $X^{(n+1)(g+1)+r+1} \rightarrow X$) over the proper algebraic subset $\bigcap_i D_i = \emptyset$, and thus is equal to 0. □

Lemma 3.5 *Let Y be irreducible and let Z be a cycle on a product $Y \times W$. Assume there exists a proper closed algebraic subset $D \subsetneq Y$ such that Z is rationally equivalent to a cycle Z' supported on $D \times W$. Then for any finite set of points $y_1, \dots, y_l \in Y$, there is a $D' \subsetneq Y$ such that none of the y_j belongs to D' and Z is rationally equivalent to a cycle Z'' supported on $D' \times W$.*

Proof Let $\tau: \tilde{D} \rightarrow D$ be a desingularization of $D \xrightarrow{i} Y$. The cycle Z' of $D \times W$ with rational coefficients lifts to a cycle \tilde{Z}' of $\tilde{D} \times W$. Let $\tilde{i} = i \circ \tau: \tilde{D} \rightarrow Y$ be the natural map and let $\Gamma_{\tilde{i}} \subset \tilde{D} \times Y$ be its graph. Since $\Gamma_{\tilde{i}}$ has codimension $n = \dim Y$, and dimension $\leq n - 1$, there is a cycle $\Gamma' \subset \tilde{D} \times Y$ rationally equivalent to $\Gamma_{\tilde{i}}$ and not intersecting $\tilde{D} \times \{y_1, \dots, y_l\}$. In other words, $\text{pr}_2(\text{Supp } \Gamma')$ does not contain any of the points y_i . We have by assumption

$$Z = (i, \text{Id}_W)_* Z' = (\tilde{i}, \text{Id}_W)_* \tilde{Z}' = (\Gamma_{\tilde{i}}, \text{Id}_W)_* (\tilde{Z}') = (\Gamma', \text{Id}_W)_* (\tilde{Z}')$$

in $\text{CH}(Y \times W)$. Now, the cycle $(\Gamma', \text{Id}_W)_*(\tilde{Z}')$ is supported on $\text{pr}_2(\text{Supp } \Gamma') \times W$, so the result is proved with $D' = \text{pr}_2(\text{Supp } \Gamma')$, and $Z'' = (\Gamma', \text{Id}_W)_*(\tilde{Z}')$. \square

To conclude this section, let us observe that the same scheme of proof applies to give the following result, which is a generalization of Corollary 3.2:

Theorem 3.6 *Let X be a connected smooth projective variety with $\text{CH}_0(X) = \mathbb{Z}$. Then for the canonical degree-1 0-cycle o on X , $\Gamma^{n+1}(X, o) = 0$ in $\text{CH}_n(X^{n+1})$, where $n = \dim X$.*

Proof Indeed, the Bloch–Srinivas decomposition of the diagonal [3] gives an equality

$$\Delta_X - X \times o = Z \quad \text{in } \text{CH}_n(X \times X),$$

where Z is supported over $D \times X$, for some divisor $D \subset X$. By Lemma 3.5, we can write such a decomposition with $n + 1$ divisors D_1, \dots, D_{n+1} such that $\bigcap_i D_i = \emptyset$. We then conclude that $\Gamma^{1,n+1}(X, o) = \prod_{i=1}^{n+1} p_{0i}^*(\Delta_X - X \times o)$ is equal to 0 in $\text{CH}_n(X^{n+2})$, and it follows from Proposition 2.2 that

$$\Gamma^{n+1}(X, o) = 0 \quad \text{in } \text{CH}_n(X^{n+1}). \quad \square$$

4 Proof of Theorem 1.3

We will first give the proof of Theorem 1.3(i). Let us recall the statement:

Theorem 4.1 *Let Y be smooth projective, and let $\pi: X \rightarrow Y$ be a degree-2 finite morphism, where X is smooth projective. Let $a \in \text{CH}_0(Y)$ be a 0-cycle of degree 1 supported on the branch locus of π . Then if $\Gamma^m(Y, a) = 0$, we have $\Gamma^{2m-1}(X, b) = 0$, where $b = \frac{1}{2}\pi^*a \in \text{CH}_0(X)$.*

Remark 4.2 The assumption made on a and b is maybe not optimal, but in any case the condition $b = \frac{1}{2}\pi^*a$ is not sufficient. Indeed, consider the case where Y is connected with $\Gamma^m(Y, a) = 0$, and X consists of two copies of Y with $b = \frac{1}{2}\pi^*a \in \text{CH}_0(X)$. Then $\Gamma^k(X, b)$ is different from 0 for any k (in fact it is not even cohomologous to 0).

We will denote by $\pi_2 = (\pi, \pi): X \times X \rightarrow Y \times Y$. Let $i: X \rightarrow X$ be the involution of X over Y and $\Gamma_i \subset X \times X$ be its graph. We then have

$$\pi_2^*(\Delta_Y) = \Delta_X + \Gamma_i.$$

Let

$$\Delta_X^+ = \pi_2^*(\Delta_Y) = \Delta_X + \Gamma_i, \Delta_X^- = \Delta_X - \Gamma_i.$$

We thus have

$$(15) \quad 2\Delta_X = \Delta_X^+ + \Delta_X^-.$$

Lemma 4.3 *Under the assumptions of Theorem 4.1, we have the following equalities in $CH_n(X \times X \times X)$, where $n := \dim X$,*

$$(16) \quad p_{12}^* \Delta_X^- \cdot p_{13}^* \Delta_X^- = p_{12}^* \Delta_X^+ \cdot p_{23}^* \Delta_X^-,$$

$$(17) \quad p_2^* b \cdot p_{23}^* \Delta_X^- = 0,$$

hence

$$(18) \quad p_{12}^* \Delta_X^- \cdot p_{13}^* \Delta_X^- = p_{12}^*(\Delta_X^+ - 2p_2^* b) \cdot p_{23}^* \Delta_X^-.$$

Proof We compute the left-hand side of (16); we have

$$\begin{aligned} p_{12}^* \Delta_X^- \cdot p_{13}^* \Delta_X^- &= p_{12}^*(\Delta_X - \Gamma_i) \cdot p_{13}^*(\Delta_X - \Gamma_i) \\ &= p_{12}^* \Delta_X \cdot p_{13}^* \Delta_X - p_{12}^* \Delta_X \cdot p_{13}^* \Gamma_i - p_{12}^* \Gamma_i \cdot p_{13}^* \Delta_X + p_{12}^* \Gamma_i \cdot p_{13}^* \Gamma_i. \end{aligned}$$

We observe now that

$$\begin{aligned} p_{12}^* \Delta_X \cdot p_{13}^* \Delta_X &= p_{12}^* \Delta_X \cdot p_{23}^* \Delta_X, & p_{12}^* \Delta_X \cdot p_{13}^* \Gamma_i &= p_{12}^* \Delta_X \cdot p_{23}^* \Gamma_i, \\ p_{12}^* \Gamma_i \cdot p_{13}^* \Delta_X &= p_{12}^* \Gamma_i \cdot p_{23}^* \Gamma_i, & p_{12}^* \Gamma_i \cdot p_{13}^* \Gamma_i &= p_{12}^* \Gamma_i \cdot p_{23}^* \Delta_X. \end{aligned}$$

It thus follows that

$$(19) \quad \begin{aligned} p_{12}^* \Delta_X^- \cdot p_{13}^* \Delta_X^- \\ = p_{12}^* \Delta_X \cdot p_{23}^* \Delta_X - p_{12}^* \Delta_X \cdot p_{23}^* \Gamma_i - p_{12}^* \Gamma_i \cdot p_{23}^* \Gamma_i + p_{12}^* \Gamma_i \cdot p_{23}^* \Delta_X. \end{aligned}$$

The right-hand side of (19) is clearly equal to

$$(p_{12}^* \Delta_X + p_{12}^* \Gamma_i) \cdot (p_{23}^* \Delta_X - p_{23}^* \Gamma_i),$$

which is by definition $p_{12}^* \Delta_X^+ \cdot p_{23}^* \Delta_X^-$, thus proving formula (16).

In order to prove formula (17), we use the fact that the 0-cycle b can be written as $\sum_j n_j x_j$, where the x_j are i -invariant. By linearity, it thus suffices to prove (17) when b is an i -invariant point of X . Now we have

$$p_2^* b \cdot p_{23}^* \Delta_X^- = p_{23}^*((b, b) - (b, ib)) = 0. \quad \square$$

Proof of Theorem 4.1 By (6), we have to prove that

$$p_{1,\dots,2m-1}^*(\Gamma^{1,2m-1}(X, b)) = 0 \quad \text{in } \text{CH}_n(X^{2m-1})_{\mathbb{Q}}.$$

Now, by (5) and (15), using

$$2b = \pi^*a, \quad \Delta_X^+ = \pi_2^* \Delta_Y,$$

we get

$$(20) \quad 2^{2m-1} \Gamma^{1,2m-1}(X, b) = p_{01}^*(\pi_2^* \Delta_Y^a + \Delta_X^-) \cdots p_{0,2m-1}^*(\pi_2^* \Delta_Y^a + \Delta_X^-).$$

Here we use the notation

$$\Delta_Y^a = \Delta_Y - p_2^*a \in \text{CH}_n(Y \times Y)_{\mathbb{Q}},$$

so that we have $\Delta_X^+ - 2p_2^*b = \pi_2^* \Delta_Y^a$ and (18) can be written as

$$(21) \quad p_{12}^* \Delta_X^- \cdot p_{13}^* \Delta_X^- = p_{12}^*(\pi_2^* \Delta_Y^a) \cdot p_{23}^* \Delta_X^-.$$

Our assumption $\Gamma^m(Y, a) = 0$ on Y can be written using Proposition 2.2 as

$$(22) \quad q_{01}^* \Delta_Y^a \cdot q_{02}^* \Delta_Y^a \cdots q_{0m}^* \Delta_Y^a = 0 \quad \text{in } \text{CH}_n(Y^{m+1})_{\mathbb{Q}},$$

where the $q_{0i}: Y^{m+1} \rightarrow Y \times Y$ are the projectors onto the product of the first and $(i + 1)^{\text{st}}$ factors.

Denote by $\pi_r: X^r \rightarrow Y^r$. We then clearly have for any r

$$(23) \quad \pi_{r+1}^*(q_{01}^* \Delta_Y^a \cdots q_{0r}^* \Delta_Y^a) = p_{01}^*(\pi_2^* \Delta_Y^a) \cdots p_{0r}^*(\pi_2^* \Delta_Y^a) \quad \text{in } \text{CH}(X^r),$$

and similarly for any choice of indices i_1, \dots, i_r in $\{1, \dots, 2m - 1\}$. Developing now the product in (20), we get a sum of monomials which up to reordering the factors take the form

$$(24) \quad p_{01}^*(\pi_2^* \Delta_Y^a) \cdots p_{0r}^*(\pi_2^* \Delta_Y^a) \cdot p_{0,r+1}^* \Delta_X^- \cdots p_{0,2m-1}^* \Delta_X^-$$

for some r . These terms vanish for $r \geq m$ by (23) and (22).

We now conclude the proof as follows: The terms $p_{0i}^* \Delta_X^-$ for $i \geq r + 1$ can be grouped by pairs, and there are at least $\lfloor (2m - 1 - r)/2 \rfloor$ such pairs. By (21), for each such pair, we have

$$p_{0i}^* \Delta_X^- \cdot p_{0,i+1}^* \Delta_X^- = p_{0i}^*(\pi_2^* \Delta_Y^a) \cdot p_{i,i+1}^* \Delta_X^-.$$

Hence each such pair produces a summand $p_{0i}^*(\pi_2^* \Delta_Y^a)$. In total we get in (24) at least $r + \lfloor (2m - 1 - r)/2 \rfloor$ factors of the form $p_{0j}^*(\pi_2^* \Delta_Y^a)$. We have $r + \lfloor (2m - 1 - r)/2 \rfloor \geq m$ unless $r = 0$, and it follows that (24) vanishes for $r \geq 1$. Hence we proved that the

only possibly nonzero monomial of the form (24) in the developed product (20) is $p_{01}^*(\Delta_X^-) \cdots p_{0,2m-1}^* \Delta_X^-$. Thus we proved that

$$(25) \quad 2^{2m-1} \Gamma^{1,2m-1}(X, b) = p_{01}^*(\Delta_X^-) \cdots p_{0,2m-1}^* \Delta_X^- \quad \text{in } \text{CH}(X^{2m}).$$

Let i' be the involution $(i, \text{Id}, \dots, \text{Id})$ acting on X^{2m} . Observe that each cycle $p_{0j}^* \Delta_X^-$ is skew-invariant under i'^* . It follows from (25) that $p_{01}^*(\Delta_X^-) \cdots p_{0,2m-1}^* \Delta_X^-$ is skew-invariant under i'^* , hence also under $i'_* = i'^*$. But as we have

$$p_{1,\dots,2m-1} \circ i' = p_{1,\dots,2m-1},$$

we get

$$\begin{aligned} \Gamma^{2m-1}(X, b) &= p_{1,\dots,2m-1*}(\Gamma^{1,2m-1}(X, b)) = p_{1,\dots,2m-1*} \circ i'_*(\Gamma^{1,2m-1}(X, b)) \\ &= -p_{1,\dots,2m-1*}(\Gamma^{1,2m-1}(X, b)) = -\Gamma^{2m-1}(X, b), \end{aligned}$$

so that $\Gamma^{2m-1}(X, b) = 0$ in $\text{CH}_n(X^{2m-1})$. □

We now turn to the proof of Theorem 1.3(ii), (iii): In fact, the result will take the following more precise form:

Theorem 4.4 *Let $\pi: X \rightarrow Y$ be a finite morphism of degree d . If $\Gamma^m(Y, a) = 0$ in $\text{CH}(Y^m)/R$ for some adequate equivalence relation R , and $b = \frac{1}{d} p^* a$ satisfies*

$$(26) \quad b * b = \Delta_*(b) \quad \text{in } \text{CH}_0(X \times X)/R,$$

where $\Delta: X \rightarrow X \times X$ is the diagonal inclusion map, then $\Gamma^{d(m-1)+1}(X, b) = 0$ in $\text{CH}(X^{dm})/R$.

Statement (ii) of Theorem 1.3 is the case where R is rational equivalence (that is $R = 0$) and b is the class of a point of X , as all points satisfy (26) modulo rational equivalence. Statement (iii) of Theorem 1.3 is the case where R is algebraic equivalence. Indeed, Theorem 4.4 applies since the equality $b * b = \Delta_*(b)$ in $\text{CH}_0(X \times X)$ modulo algebraic equivalence is satisfied by 0-cycles of degree 1 on a connected variety.

We first introduce some notation. Let as above $\Delta_Y^a := \Delta_Y - p_2^* a \in \text{CH}_n(Y \times Y)$ and similarly $\Delta_X^b := \Delta_X - p_2^* b \in \text{CH}_n(X \times X)$. In both expressions, p_2 is the projection from $Y \times Y$, resp. $X \times X$ onto its second factor. The proof of Theorem 4.4 will use the following result (which will replace formula (16) used previously when $d = 2$):

Proposition 4.5 *The morphism $\pi: X \rightarrow Y$ and the 0-cycle b being as in Theorem 4.4, there exist cycles $\Gamma_i \in \text{CH}^{(d-1)n}(X^{d+1})$ such that*

$$(27) \quad \prod_{i=1}^d p_{0i}^* \Delta_X^b = \sum_i p_{0i}^* (\pi_2^* \Delta_Y^a) \cdot p_{0, D \setminus \{i\}}^* \Gamma_i \quad \text{in } \text{CH}^{nd}(X^{d+1})/R,$$

where D is the set $\{1, \dots, d\}$ and as usual $p_{0, D \setminus \{i\}}$ is the projection onto the product of the factors indexed by the set $\{0\} \cup D \setminus \{i\}$.

Before giving the proof, we will first prove a similar statement of independent interest for Δ_X and Δ_Y , instead of Δ_X^b and Δ_Y^a , as the proof is much simpler to write and we will use similar but slightly more involved arguments to prove Proposition 4.5. Namely, we have the following result:

Proposition 4.6 *Let $\pi: X \rightarrow Y$ be a finite morphism of degree d . There exist cycles $\Gamma'_i \in \text{CH}^{n(d-1)}(X^{d+1})$ such that*

$$(28) \quad \prod_{i=1}^d p_{0i}^* \Delta_X = \sum_i p_{0i}^* (\pi_2^* \Delta_Y) \cdot p_{0, D \setminus \{i\}}^* \Gamma'_i \quad \text{in } \text{CH}^{nd}(X^{d+1}).$$

Proof Indeed, let us denote by $E_k \subset \text{CH}(X^{k+1})$ the ideal generated by the elements $p_{0i}^* (\pi_2^* \Delta_Y)$, $i = 1, \dots, k$. Next let

$$(29) \quad \Sigma_1 := \pi_2^{-1}(\Delta_Y) - \Delta_X \in \text{CH}(X \times X).$$

Note that, because π is finite of degree d , Σ_1 is the class of the Zariski closure in $X \times X$ of the subvariety $\{(x, x_1) \in X^0 \times X^0, \pi(x_1) = \pi(x), x_1 \neq x\}$ where $X^0 := \pi^{-1}(Y^0)$ and Y^0 is the open set of Y over which π is étale of degree d . The first projection $\text{pr}_1: \Sigma_1 \rightarrow X$ has degree $d - 1$. Let us denote more generally by $\Sigma_k \subset X^{k+1}$ the Zariski closure in X^{k+1} of the subvariety

$$(30) \quad \{(x, x_1, \dots, x_k) \in (X^0)^{k+1} \mid \pi(x_i) = \pi(x), x_i \neq x_j \text{ for } i \neq j, x_i \neq x \text{ for all } i\}.$$

The contents of formula (28) is that $\prod_{i=1}^d p_{0i}^* \Delta_X$ belongs to E_d . It is therefore a consequence of the following statement:

Claim 4.7 *For any integer $k \geq 1$, one has*

$$(31) \quad \alpha_k \prod_{i=1}^k p_{0i}^* \Delta_X = \Sigma_k \quad \text{in } \text{CH}(X^{k+1})/E_k,$$

with $\alpha_k = (-1)^k k!$. In particular, $\prod_{i=1}^d p_{0i}^* \Delta_X = 0$ in $\text{CH}(X^{d+1})/E_d$.

The second statement follows from the first since Σ_d is empty. The first statement is proved by induction on k . For $k = 1$, the result is (29). The induction step is immediate: we have the following equalities in $\text{CH}(X^{k+2})$:

$$\begin{aligned}
 (32) \quad \prod_{i=1}^{k+1} p_{0i}^* \Delta_X &= p_{0,\dots,k}^* \left(\prod_{i=1}^k p_{0i}^* \Delta_X \right) \cdot p_{0,k+1}^* \Delta_X \\
 &= - \left(\prod_{i=1}^k p_{0i}^* \Delta_X \right) \cdot p_{0,k+1}^* \Sigma_1 \pmod{E_{k+1}} \\
 &= - \frac{1}{\alpha_k} p_{0,\dots,k}^* (\Sigma_k) \cdot p_{0,k+1}^* \Sigma_1 \pmod{E_{k+1}}.
 \end{aligned}$$

On the other hand, we observe that Σ_{k+1} is obtained from $p_{0,\dots,k}^* (\Sigma_k) \cdot p_{0,k+1}^* \Sigma_1$ by removing in the fibered product the components where x_{k+1} equals one of the x_i for $i = 1, \dots, k$. This gives rise to the following identity:

$$(33) \quad p_{0,\dots,k}^* (\Sigma_k) \cdot p_{0,k+1}^* \Sigma_1 = \Sigma_{k+1} + \sum_{i=1}^k p_{0,\dots,k}^* (\Sigma_k) \cdot p_{i,k+1}^* \Delta_X.$$

In the right-hand side of (33), we can replace (using again the induction hypothesis) Σ_k by $\alpha_k \prod_{j=1}^k p_{0j}^* \Delta_X \pmod{E_k}$ and we also observe that

$$(34) \quad \prod_{j=1}^k p_{0j}^* \Delta_X \cdot p_{i,k+1}^* \Delta_X = \prod_{i=1}^{k+1} p_{0i}^* \Delta_X$$

for any $i = 1, \dots, k$. Hence we get, using (32), (33) and (34),

$$\prod_{i=1}^{k+1} p_{0i}^* \Delta_X = - \frac{1}{\alpha_k} \Sigma_{k+1} - k \prod_{i=1}^{k+1} p_{0i}^* \Delta_X.$$

This finally provides

$$\alpha_{k+1} \prod_{i=1}^{k+1} p_{0i}^* \Delta_X = \Sigma_{k+1}$$

with $\alpha_{k+1} = -(k + 1)\alpha_k$. □

Proof of Proposition 4.5 We follow the above argument with Δ_X, Δ_Y replaced by Δ_X^b and Δ_Y^a , in order to prove Lemma 4.8 below. We use the following notation: We will work with the n -cycle Σ_k^b of X^{k+1} obtained by replacing formally in the definition (30) of Σ_k each x_i by $x_i - b$ and developing multilinearly. More rigorously, Σ_k admits morphisms $p, p_i: \Sigma_k \rightarrow X$, obtained by restricting the projections $X^{k+1} \rightarrow X$ (where the factors are indexed by $\{0, \dots, k\}$ and $p = p_0$). Let $\Gamma_i \subset \Sigma_k \times X$

be the graphs of these projections. Then we can obviously define $\Sigma_k \subset X^{k+1}$ as $(p, \text{pr}_{X^k})_* (\prod_{i=1}^k \text{pr}_{\Sigma_k, i}^* \Gamma_i)$, where:

- $\text{pr}_{X^k}: \Sigma_k \times X^k \rightarrow X^k$ is the second projection and $(p, \text{pr}_{X^k}): \Sigma_k \times X^k \rightarrow X^{k+1}$ is the obvious morphism.
- $\text{pr}_{\Sigma_k, i}: \Sigma_k \times X^k \rightarrow \Sigma_k \times X$ is the projection on the product of the first factor and the i^{th} factor of X^k .

On the other hand, we also have in $\Sigma_k \times X$ the graph $\Sigma_k \times \{b\}$ of the constant morphism mapping to b if b is a point, or more generally the n -cycle $\text{pr}_X^* b$ if b is any 0-cycle of degree 1. We then define analogously Σ_k^b as follows:

$$(35) \quad \Sigma_k^b = (p, \text{pr}_{X^k})_* \left(\prod_{i=1}^k \text{pr}_{\Sigma_k, i}^* (\Gamma_i - \text{pr}_X^* b) \right) \quad \text{in } \text{CH}(X^{k+1}).$$

Developing the product above, we see that the formula for Σ_k^b is of the form

$$(36) \quad \Sigma_k^b = \sum_{I \subset \{1, \dots, k\}} (-1)^{k-i} \lambda_{k, i, d} p_{0, I}^* \Sigma_i \cdot p_J^* b^{*j} \in \text{CH}_n(X^{k+1}),$$

where in the formula above, $I \sqcup J = \{1, \dots, k\}$, $i = |I|$, and the $\lambda_{k, j, d}$ are combinatorial coefficients given by the formula

$$(37) \quad \lambda_{k, i, d} = (d - i - 1)(d - i - 2) \cdots (d - k).$$

Indeed, the reason for (37) is the fact that the projection map

$$p_{0, I}: \Sigma_k \rightarrow \Sigma_i \subset X^{i+1}$$

has degree $(d - i - 1)(d - i - 2) \cdots (d - k)$. Note in particular, that $\Sigma_k^b = 0$ for $k \geq d$. Next we define $E_{k, a, R} \subset \text{CH}(X^{k+1})/R$ as the ideal generated by the $p_{0, i}^* \Delta_Y^a$ for $i = 1, \dots, k$. Recall that $\Gamma^{1, k}(X, b) = \prod_{i=1}^k p_{0, i}^* \Delta_X^b$.

Lemma 4.8 *The morphism $\pi: X \rightarrow Y$ and the 0-cycle b being as in Theorem 4.4, for any integer $k \geq 1$, one has*

$$(38) \quad \alpha_k \Gamma_{1, k}(X, b) = \Sigma_k^b \quad \text{in } \text{CH}(X^{k+1})/E_{k, a, R},$$

Proof We have by (36), (37)

$$\Delta_X^b = \pi_2^* \Delta_Y^a - \Sigma_1^b,$$

which can be written as $\Delta_X^b = -\Sigma_1^b \bmod E_{1,a,R}$, proving the case $k = 1$. Assume the formula is proved for k . Then we have

$$(39) \quad p_{0,1,\dots,k}^* \Sigma_k^b \cdot p_{0,k+1}^* \Sigma_1^b = -\alpha_k p_{0,1,\dots,k}^* \Gamma^{1,k}(X, b) \cdot p_{0,k+1}^* \Delta_X^b \\ = -\alpha_k \Gamma^{1,k+1}(X, b) \quad \text{in } \text{CH}(X^{k+2})/E_{k+1,b,R}.$$

Next we claim that we have the following relation in $\text{CH}(X^{k+2})/R$:

$$(40) \quad p_{0,1,\dots,k}^* \Sigma_k^b \cdot p_{0,k+1}^* \Sigma_1^b \\ = \Sigma_{k+1}^b + \sum_{i=1}^k p_{0,\dots,k}^* \Sigma_k^b \cdot p_{i,k+1}^* \Delta_X^b - \sum_{i=1}^k p_{0,\dots,\hat{i},k+1}^* \Sigma_k^b \cdot p_i^* b.$$

This relation uses in a crucial way the identity

$$(41) \quad \Delta_* b = p_1^* b \cdot p_2^* b \quad \text{in } \text{CH}_0(X \times X)/R.$$

The beginning

$$p_{0,1,\dots,k}^* \Sigma_k^b \cdot p_{0,k+1}^* \Sigma_1^b = \Sigma_{k+1}^b + \sum_{i=1}^k p_{0,\dots,k}^* \Sigma_k^b \cdot p_{i,k+1}^* \Delta_X^b + \dots$$

of the formula (40) is easily understood: it expresses the fact that in the left-hand side, we include all possible $x_{k+1} \neq x$, while in Σ_{k+1}^b , we have to take into account the restriction $x_{k+1} \neq x_i$ for $i = 1, \dots, k$. The last term in (40) is explained as follows. The intersection with $p_{i,k+1}^* \Delta_X^b = p_{i,k+1}^* \Delta_X - p_{k+1}^* b$ produces a term

$$\Delta_*(x_i - b) - (x_i - b, b) = (x_i, x_i) - \Delta_* b - (x_i, b) + p_i^* b p_{k+1}^* b = (x_i, x_i) - (x_i, b)$$

on the product of the i^{th} and $(k + 1)^{\text{st}}$ factors. On the other hand, we had on the left in (40) the term

$$(x_i - b) * (x_i - b) = (x_i, x_i) - (x_i, b) - (b, x_i) + (b, b)$$

in the product of the i^{th} and $(k + 1)^{\text{st}}$ factors, which is unwanted in the development of Σ_{k+1}^b . Hence we also have to add on the right the extra term $-(b, x_i - b)$ on the product of the i^{th} and $(k + 1)^{\text{st}}$ factors, which is exactly the meaning of the term $-p_{0,\dots,\hat{i},k+1}^* \Sigma_k^b \cdot p_i^* b$. Thus the claim is proved.

Combined with (39) and the inductive assumption, (40) gives

$$(42) \quad -\alpha_k \Gamma^{1,k+1}(X, b) = \Sigma_{k+1}^b + \alpha_k \left(\sum_{i=1}^k p_{0,\dots,k}^* \Gamma^{1,k}(X, b) \cdot p_{i,k+1}^* \Delta_X^b \right. \\ \left. - \sum_{i=1}^k p_{0,\dots,\hat{i},k+1}^* \Gamma^{1,k}(X, b) \cdot p_i^* b \right).$$

The equality above holds in $\text{CH}(X^{k+2})/E_{k+1,a,R}$. Let us now prove that for any i ,

$$(43) \quad p_{0,\dots,k}^* \Gamma^{1,k}(X, b) \cdot p_{i,k+1}^* \Delta_X^b - p_{0,\dots,\hat{i},k+1}^* \Gamma^{1,k}(X, b) \cdot p_i^* b = \Gamma^{1,k+1}(X, b)$$

in $\text{CH}(X^{k+2})/R$. As

$$\Gamma^{1,k}(X, b) = \prod_{i=1}^k p_{0i}^* \Delta_X^b, \quad \Gamma^{1,k+1}(X, b) = \prod_{i=1}^{k+1} p_{0i}^* \Delta_X^b,$$

it clearly suffices to show that the cycles

$$p_{01}^* \Delta_X^b \cdot p_{12}^* \Delta_X^b - p_1^* b \cdot p_{02}^* \Delta_X^b \quad \text{and} \quad p_{01}^* \Delta_X^b \cdot p_{02}^* \Delta_X^b$$

of X^3 are equal in $\text{CH}(X^3)/R$. We have

$$\begin{aligned} & p_{01}^* \Delta_X^b \cdot p_{12}^* \Delta_X^b - p_1^* b \cdot p_{02}^* \Delta_X^b \\ &= (p_{01}^* \Delta_X - p_1^* b) \cdot (p_{12}^* \Delta_X - p_2^* b) - p_1^* b \cdot p_{02}^* \Delta_X + p_1^* b \cdot p_2^* b \\ &= p_{01}^* \Delta_X \cdot p_{12}^* \Delta_X - p_{01}^* \Delta_X \cdot p_2^* b - p_1^* b \cdot p_{12}^* \Delta_X + p_1^* b \cdot p_2^* b - p_1^* b \cdot p_{02}^* \Delta_X + p_1^* b \cdot p_2^* b \\ &= p_{01}^* \Delta_X \cdot p_{02}^* \Delta_X - p_{01}^* \Delta_X \cdot p_2^* b - p_1^* b \cdot p_{02}^* \Delta_X + p_1^* b \cdot p_2^* b \end{aligned}$$

in $\text{CH}(X^3)/R$ because we assumed $p_1^* b \cdot p_{12}^* \Delta_X = p_1^* b \cdot p_2^* b$ in $\text{CH}(X^3)/R$; see (26). On the other hand,

$$\begin{aligned} p_{01}^* \Delta_X^b \cdot p_{02}^* \Delta_X^b &= (p_{01}^* \Delta_X - p_1^* b) \cdot (p_{02}^* \Delta_X - p_2^* b) \\ &= p_{01}^* \Delta_X \cdot p_{02}^* \Delta_X - p_{01}^* \Delta_X \cdot p_2^* b - p_1^* b \cdot p_{02}^* \Delta_X + p_1^* b \cdot p_2^* b. \end{aligned}$$

Hence we proved that both terms in (43) are equal; using (42), we then get

$$-\alpha_k \Gamma^{1,k+1}(X, b) = \Sigma_{k+1,b} + \alpha_k \left(\sum_{i=1}^k \Gamma^{1,k+1}(X, b) \right),$$

hence

$$-(k+1)\alpha_k \Gamma^{1,k+1}(X, b) = \Sigma_{k+1,b} \quad \text{in } \text{CH}(X^{k+2})/R$$

and Lemma 4.8 is proved. □

Finally, Lemma 4.8 for $k = d$ implies Proposition 4.5 since $\Sigma_d^b = 0$. □

Proof of Theorem 4.4 By Proposition 4.5 applied to each set of d indices $\{1, \dots, d\}$, $\{d+1, \dots, 2d\}$, $\{(m-2)d+1, \dots, (m-1)d\}$, we can write $\prod_{i=1}^{d(m-1)} p_{0i}^* \Delta_X^b$ as a sum of products of $m-1$ cycles, each of them being of the form $p_{0i}^* (\pi_2^* \Delta_Y^a) \cdot \Gamma''$ for an adequate index i (one in each of the sets above). We now apply Proposition 2.5 to

both Y and X . Thus the assumption $\Gamma^m(Y, a) = 0$ implies that for some cycle Γ_Y on Y^m ,

$$\prod_{i=1}^{m-1} p_{0i}^* \Delta_Y^a = p_0^* a \cdot \Gamma_Y \quad \text{in } \text{CH}(Y^m)/R.$$

Applying this relation to each product of $m-1$ factors $\prod_{k=1}^{m-1} p_{0i_k}^* (\pi_2^* \Delta_Y^a)$ for adequate indices i_k appearing above, we conclude that

$$\Gamma^{1,d(m-1)}(X, b) = \prod_{i=1}^{d(m-1)} p_{0i}^* \Delta_X^b = p_0^* b \cdot \Gamma_X \quad \text{in } \text{CH}(X^{d(m-1)+1})/R$$

for some cycle Γ_X on $X^{d(m-1)+1}$. By Proposition 2.5, and using the fact that b satisfies property (26), (that is, condition (10) in Proposition 2.5), we conclude that $\Gamma^{d(m-1)+1}(X, b) = 0$ in $\text{CH}(X^{d(m-1)+1})/R$. □

4.1 Case of curves

A special case of Theorem 1.3(iii) is the case where $Y = \mathbb{P}^1$, so X is a d -gonal curve. We then get the vanishing $\Gamma^{d+1}(X, b) = 0$ in $\text{CH}_1(X^{d+1})/\text{alg}$, where b is any point of X . Recall now the Beauville decomposition of cycles on an abelian variety A modulo rational or algebraic equivalence,

$$\text{CH}_i(A) = \bigoplus_s \text{CH}_i(A)_s,$$

with

$$\text{CH}_i(A)_s := \{z \in \text{CH}_i(A) \mid \mu_{k*} z = k^{2i+s} z \text{ for all } k \in \mathbb{Z}^*\}$$

and similarly for Chow groups modulo algebraic equivalence. Here $\mu_k: A \rightarrow A$ is the morphism $a \mapsto ka$. Let now X be a smooth genus g projective curve and $A := J(X)$. X has an embedding in $J(X)$ which is canonical up to translation, hence determines a 1-cycle Z in $J(X)$, well defined modulo algebraic equivalence. Thus we have a Beauville decomposition

$$Z = \sum_s Z_s \quad \text{in } \text{CH}_1(A)/\text{alg}.$$

For nonvanishing results concerning the cycles Z_s (when X is very general) and its decomposition, let us mention [7; 19] (in the later paper, it is proved that if $g \geq s^2/2$, then $Z_s \neq 0$ modulo algebraic equivalence for a very general curve X of genus g).

Let us show the following:

Proposition 4.9 *The vanishing of $\Gamma^{d+1}(X, b)$ in $\text{CH}_1(X^{d+1})/\text{alg}$ is equivalent to the vanishing of Z_s , for all $s \geq d - 1$, in $\text{CH}_1(J(X))/\text{alg}$.*

Proof It suffices to prove the result for $d \leq g - 1$, because we know by Theorem 1.3(iii) (see Remark 1.5) that $\Gamma^{d+1}(X, b) = 0$ in $\text{CH}_1(X^{d+1})/\text{alg}$ for some $d \leq g - 1$. Assuming the proposition proved for $d \leq g - 1$, this implies that $Z_s = 0$ in $\text{CH}_1(J(X))/\text{alg}$ for all $s \geq g - 1$, and thus for $d \geq g$, both vanishing statements are true.

We thus assume $d \leq g - 1$; note that the cycle $\Gamma^{d+1}(X, b)$ is a 1-cycle of X^{d+1} which is invariant under the action of the symmetric group \mathfrak{S}_{d+1} , so that its vanishing in $\text{CH}_1(X^{d+1})/\text{alg}$ is equivalent to the vanishing of its image $\bar{\Gamma}^{d+1}(X, b)$ in $\text{CH}_1(X^{(d+1)})/\text{alg}$. We now consider the inclusion

$$b_{g-d-1}: X^{(d+1)} \rightarrow X^{(g)}, \quad z \mapsto z + (g - d - 1)b$$

and claim that

$$\bar{\Gamma}^{d+1}(X, b) = 0 \quad \text{in } \text{CH}_1(X^{(d+1)})/\text{alg}$$

if and only if

$$b_{g-d-1*}(\bar{\Gamma}^{d+1}(X, b)) = 0 \quad \text{in } \text{CH}_1(X^{(g)})/\text{alg}.$$

Indeed, there is an incidence correspondence

$$\Sigma \subset X^{(d+1)} \times X^{(g)}, \quad \Sigma = \{(z, z') \mid z' = z + z'' \text{ for some } z'' \in X^{(g-d-1)}\}.$$

It is not hard to see that, due to its special form, the cycle $\bar{\Gamma}^{d+1}(X, b)$ satisfies

$$\Sigma^*(b_{g-d-1*}(\bar{\Gamma}^{d+1}(X, b))) = \bar{\Gamma}^{d+1}(X, b),$$

which proves the claim.

The next step is to observe that the Griffiths group of 1-cycles homologous to 0 modulo algebraic equivalence is a birational invariant. This is elementary to show using resolution of indeterminacies of birational maps, as it is invariant under blow-up and is functorial under pushforward and pullbacks under generically finite morphisms. As $X^{(g)}$ is birational to $J(X)$ via the Abel map, we conclude that $\bar{\Gamma}^{d+1}(X, b) = 0$ in $\text{CH}_1(X^{(d+1)})/\text{alg}$ if and only if its image $W := a_{X*}(\bar{\Gamma}^{d+1}(X, b))$ in $J(X)$ under the Abel map $a_X: C^{(d+1)} \rightarrow J(X)$ vanishes in $\text{CH}_1(J(X))/\text{alg}$.

Finally, we observe that a cycle appearing in the formula (1) for $\Gamma^{d+1}(X, b)$, which is up to permutation of the form

$$\{(x, \dots, x, b, \dots, b) \mid x \in X\},$$

where x appears k times and b appears $d + 1 - k$ times, maps under the Abel map a_X to a 1-cycle of $J(X)$ algebraically equivalent to $\mu_{k*}(Z)$. The vanishing of W in

$\text{CH}_1(J(X))/\text{alg}$ thus gives

$$(44) \quad \sum_{k=1}^{d+1} (-1)^{d+1-k} \binom{d+1}{k} \mu_{k*} Z = 0 \quad \text{in } \text{CH}_1(J(X))/\text{alg}.$$

Writing the Beauville decomposition

$$Z = \sum_s Z_s,$$

the vanishing of W in $\text{CH}_1(J(X))/\text{alg}$ is equivalent to

$$(45) \quad \sum_{k=1}^{d+1} (-1)^{d+1-k} \binom{d+1}{k} k^{2+s} Z_s = 0 \quad \text{in } \text{CH}_1(J(X))/\text{alg}$$

for any s .

We now have the following easy lemma:

Lemma 4.10 *We have*

$$\begin{aligned} \sum_{k=1}^{d+1} (-1)^{d+1-k} \binom{d+1}{k} k^{2+s} &= 0 \quad \text{for } s \leq d-2, \\ \sum_{k=1}^{d+1} (-1)^{d+1-k} \binom{d+1}{k} k^{2+s} &\neq 0 \quad \text{for } s \geq d-1. \end{aligned}$$

This shows that the vanishing (45) is equivalent to the vanishing of Z_s for $s \geq d-1$. \square

Remark 4.11 Proposition 4.9 is also proved in [11], where it is used to deduce the vanishing $\Gamma^{g+2}(X, a) = 0$ of Remark 1.4, for any point $a \in X$, from the main result of Colombo and van Geemen [5].

5 Hyper-Kähler manifolds

5.1 Proof of Theorem 1.9

We prove in this section the following theorem (see Theorem 1.9 of the introduction):

Theorem 5.1 *Let S be a K3 surface, and let $X = S^{[n]}$. Then*

$$(46) \quad \Gamma^{2n+1}(X, o_X) = 0 \quad \text{in } \text{CH}_{2n}(X^{2n+1}),$$

where o_X is the canonical 0-cycle on X constructed from the canonical 0-cycle of S .

Here the cycle o_S appears in the following theorem from [2] providing a list of relations which hold in the Chow ring of a self-product of a $K3$ surface.

Theorem 5.2 *Let S be a smooth projective $K3$ surface. Then there is a degree-1 zero-cycle $o_S \in \text{CH}_0(S)$ satisfying the following equalities (which are all polynomial relations in $\text{CH}(S^k)$ for adequate k , between the cycles $p_i^*o_S, p_j^*L, p_{st}^*\Delta_S$):*

- (1) $L^2 - \text{deg}(L^2)o_S = 0$ in $\text{CH}_0(S)$, for any $L \in \text{Pic } S$.
- (2) $\Delta_S \cdot p_1^*L - L \times o_S - o_S \times L = 0$ in $\text{CH}_1(S \times S)$ for any $L \in \text{Pic } S$, where p_1 is the first projection from $S \times S$ to S , and $L \times o_S = p_1^*L \cdot p_2^*o_S$.
- (3) $\Gamma^3(S, o_S) = 0$ in $\text{CH}_2(S \times S \times S)$. (Using formula (1) and the identity $\Delta_3 = p_{12}^*\Delta_S \cdot p_{13}^*\Delta_S$, we can also view (3) as a polynomial relation in $\text{CH}(S^3)$ involving the classes $p_{ij}^*\Delta_S$ and $p_k^*(o_S)$.)
- (4) $\Delta_S^2 = 24p_1^*o_S \cdot p_2^*o_S$ in $\text{CH}_0(S \times S)$.
- (5) $\Delta_S \cdot p_1^*o_S - p_1^*o_S \cdot p_2^*o_S = 0$ in $\text{CH}_0(S \times S)$.

Note that property (5) is (26) and is easily satisfied because o_S is the class of a point in S . Property (4) is a consequence of Property (3) which implies $c_2(S) = 24o_S$ in $\text{CH}_0(S)$, and Property (5).

Remark 5.3 The above relations are the nontrivial relations involving $p_i^*(o_S), p_j^*L, L \in \text{CH}^1(S)$ and the $p_{kl}^*\Delta_S$ and with the property that in at least one monomial, an index is repeated. To make a complete list of such relations, one should add the “trivial relations”, which hold on any surface, namely:

- (1) $o_X \cdot L = 0$ in $\text{CH}(S)$, $L \in \text{CH}^1(S)$.
- (2) $o_X \cdot o_X = 0$ in $\text{CH}(S)$.
- (3) $p_{12}^*\Delta_S \cdot p_{23}^*\Delta_S = p_{13}^*\Delta_S \cdot p_{23}^*\Delta_S$ in $\text{CH}(S \times S \times S)$.

As in [17], the ingredients of the proof of Theorem 5.1 are (1) the results of de Cataldo–Migliorini [4], which will allow, thanks to Proposition 5.6, to translate the problem into computations in ordinary self-products $S^N, N \leq (2n + 1)n$, of a $K3$ surface; (2) the relations listed in Theorem 5.2; (3) the recent result of Yin [20]. The latter says basically that for a regular surface S , the *cohomological* polynomial relations on S^N between the diagonal classes and the pull-back under the various projections of the class of a point are generated by the relations listed above (or rather, their cohomological counterpart) and the Kimura relation (see [10] and [18, Section 3.2.3]) which holds on the Chow level when the motive of S is finite-dimensional. A key point of the proof

will be thus the fact that the Kimura relation is not needed to express the pull-back to S^N of the vanishing relation $[\Gamma^{2n+1}(X, o_X)] = 0$.

We first recall some notation related to S^n and $S^{[n]}$, for any smooth surface S . Let $\mu = \{A_1, \dots, A_l\}$, $l = l(\mu)$ be a partition of $\{1, \dots, n\}$, where all the A_i are nonempty. Let $S^\mu \cong S^{l(\mu)} \subset S^n$ be the set

$$\{(s_1, \dots, s_n) \mid s_i = s_j \text{ if } i, j \in A_k \text{ for some } k\}.$$

The image $\overline{S^{(\mu)}}$ of S^μ in $S^{(n)}$ is a stratum of $S^{(n)}$. It is not normal in general, but its normalization $S^{(\mu)}$ is the quotient of S^μ by the subgroup \mathfrak{S}_μ of \mathfrak{S}_n preserving S^μ , that is acting on $\{1, \dots, n\}$ by permuting the A_i with the same cardinality. Let $c: S^{[n]} \rightarrow S^{(n)}$ be the Hilbert–Chow morphism and let $E_\mu := S^\mu \times_{S^{(n)}} S^{[n]} \subset S^\mu \times S^{[n]}$. It is known that E_μ is irreducible of dimension $n + l(\mu)$. We see E_μ as a correspondence between S^μ and $S^{[n]}$.

Theorem 5.4 (de Cataldo–Migliorini [4]) *The collection $(E_\mu)_\mu$ of correspondences identifies the motive of $S^{[n]}$ with a submotive of the disjoint union $\bigsqcup_\mu S^\mu$. More precisely, for some combinatorial coefficients λ_μ ,*

$$\Delta_X = \sum_\mu \lambda_\mu (E_\mu, E_\mu)_*(\Delta_{S^\mu}) \quad \text{in } \text{CH}_{2n}(X \times X).$$

The result above implies in particular:

Corollary 5.5 *Let $X := S^{[n]}$. For any integer k , the map*

$$\bigoplus_{(\mu_1, \dots, \mu_k)} (E_{\mu_1}, \dots, E_{\mu_k})^*: \text{CH}^*(X^k) \rightarrow \bigoplus_{(\mu_1, \dots, \mu_k)} \text{CH}^*(S^{\mu_1} \times \dots \times S^{\mu_k})$$

is injective.

We now have the following result: Let n and k be fixed. Let us denote by $\Delta_k \subset X^k$ the small diagonal of X^k , where $X := S^{[n]}$, for a smooth projective surface S .

Proposition 5.6 *For any k -tuple (μ_1, \dots, μ_k) of partitions of $\{1, \dots, n\}$, there exists a universal (ie independent of S) polynomial P_μ . (in many variables) with the following property: For any smooth quasiprojective surface S ,*

$$(E_{\mu_1}, \dots, E_{\mu_k})^*(\Delta_k) = P_\mu. (\text{pr}_i^* c_2(S), \text{pr}_j^*(K_S), \text{pr}_{st}^*(\Delta_S))$$

in $\text{CH}(S^{\mu_1} \times \dots \times S^{\mu_k})$, where the pr_i are the projections from $\prod_i S^{\mu_i} \cong S^N$ to its factors (isomorphic to S), and the pr_{st} are the projections from $\prod_i S^{\mu_i}$ to the products of two of its factors (isomorphic to $S \times S$).

Proof Proposition 5.6 is a particular case of Theorem 5.12 whose proof will be sketched in Section 5.2 and will be completed in [15], because the cycles

$$(E_{\mu_1}, \dots, E_{\mu_k})^*(\Delta_k) \in \text{CH}(S^N)$$

are clearly universally defined cycles in the sense of Definition 5.11. Indeed, for any family $S \rightarrow B$ of smooth quasiprojective surfaces, we can construct the smooth family of relative Hilbert schemes $\mathcal{X} := \mathcal{S}^{[n/B]}$ and its relative small diagonals

$$\Delta_{k/B}(\mathcal{X}) \subset \mathcal{X}^{k/B}.$$

Then we have the relative correspondences $E_{\mu_i} \subset S^{\mu_i/B} \times_B \mathcal{X}$, which are proper over the first summand, and we have thus the relative cycle

$$E_{\mu.}^*(\Delta_{k/B}(\mathcal{X})) \in \text{CH}(S^{[N/B]}), N = l(\mu_1) + \dots + l(\mu_k),$$

satisfying the functoriality properties stated in Definition 5.11, because the morphisms $E_{\mu_i} \rightarrow B$ are flat. □

Remark 5.7 One may have the feeling that the canonical class is not necessary in Proposition 5.6, as set theoretically one wants the set of $(s_1, \dots, s_k) \in S^{\mu_1} \times \dots \times S^{\mu_k}$ such that there is a subscheme $x \in S^{[n]}$ whose associated cycle is s_i (or rather its image in $S^{(n)}$) and this does not seem to involve the intrinsic geometry of S , except for the self-intersection of the diagonal, thus only c_2 . In fact, due to excess formulas, the canonical class actually appears, as the simplest example shows: Let X be $S^{[2]}$, $k = 3$, and $\mu_1 = \mu_2 = \mu_3$ be the partition of $\{1, 2\}$ consisting of a single set with 2 elements. Then $E_{\mu_1} = E_{\mu_2} = E_{\mu_3} = E$ is the exceptional divisor of $S^{[2]}$ and we have

$$(E_{\mu_1}, E_{\mu_2}, E_{\mu_3})^*(\Delta_3) = \Delta_*(p_*(E^2|_E)),$$

where $\Delta: S \rightarrow S^3$ is the diagonal inclusion, and $p: E \rightarrow S$ is the natural map. But $p_*(E^2|_E) \in \text{CH}^1(S)$ is a nonzero multiple of the canonical class of S .

Remark 5.8 We proved in [17] a similar statement where instead of the small diagonal, arbitrary polynomials in the Chern classes of the tautological sheaf on $X^{[n]}$ and the Chern classes of the ideal sheaf of the incidence correspondence in $S^{[n-1]} \times S$ are considered; the same kind of arguments used there, which are in fact borrowed from [6], can be applied to prove Proposition 5.6, but the proofs are very intricate and lengthy and in fact all these results can also be obtained as Proposition 5.6, as a consequence of Theorem 5.12.

We now show how Theorem 5.1 follows from Proposition 5.6.

Proof of Theorem 5.1 We have to prove the vanishing of $\Gamma^{2n+1}(X, o_X)$, where S is a smooth projective $K3$ surface and $X = S^{[n]}$. By Corollary 5.5, it suffices to show that for any $2n + 1$ -tuple $(\mu_1, \dots, \mu_{2n+1})$ of partitions of $\{1, \dots, n\}$, we have

$$(47) \quad (E_{\mu_1, \dots, \mu_{2n+1}})^*(\Gamma^{2n+1}(X, o_X)) = 0$$

in $\text{CH}(S^{\mu_1} \times \dots \times S^{\mu_{2n+1}})$. As $\Gamma^{2n+1}(X, o_X)$ is a combination of cycles which up to permutation of factors are of the form $\Delta_k \times o_X^{2n+1-k}$ and $E_{\mu}^* o_X = 0$ if $\mu \neq \{\{1\}, \dots, \{n\}\}$, and is equal to $n!(o_S, \dots, o_S)$ if $\mu = \{\{1\}, \dots, \{n\}\}$, it follows from Proposition 5.6 that there exists a polynomial Q_{μ} . (in many variables) with the following property: For any smooth projective surface S , and any point $o_S \in S$,

$$(48) \quad (E_{\mu_1, \dots, \mu_{2n+1}})^*(\Gamma^{2n+1}(X, o_X)) = Q_{\mu} \cdot (\text{pr}_i^* c_2(S), \text{pr}_j^*(K_S), \text{pr}_l^* o_S, \text{pr}_{st}^*(\Delta_S))$$

in $\text{CH}(S^{\mu_1} \times \dots \times S^{\mu_{2n+1}})$. We know by [12, Proposition 1.3] (see also Theorem 1.2(i)) that for any regular surface S , and any point $o_S \in S$, $\Gamma^{2n+1}(X, o_X)$ is cohomologous to 0, where o_X is any point of $X = S^{[n]}$ over $no_S \in S^{(n)}$. It follows that for each $2n + 1$ -tuple $(\mu_1, \dots, \mu_{2n+1})$ as above, the cycle

$$(E_{\mu_1, \dots, \mu_{2n+1}})^*(\Gamma^{2n+1}(X, o_X))$$

is cohomologous to 0 in $S^{\mu_1} \times \dots \times S^{\mu_{2n+1}}$. Hence the polynomial Q_{μ} . has the property that for a regular surface S ,

$$(49) \quad Q_{\mu} \cdot (\text{pr}_i^*[c_2(S)], \text{pr}_j^*([K_S]), \text{pr}_l^*[o_S], \text{pr}_{st}^*([\Delta_S])) = 0$$

in $H^*(S^{\mu_1} \times \dots \times S^{\mu_{2n+1}}, \mathbb{Q})$. Here the brackets denote the cohomology class of the corresponding cycles. In this equation, we can of course replace $[c_2(S)]$ by $\chi_{\text{top}}(S)[o_S]$, with $\chi_{\text{top}}(S)$ determined by the polynomial relation (this is relation (4) in Theorem 5.2) $[\Delta_S]^2 = \chi_{\text{top}}(S) \text{pr}_1^*[o_S] \cup \text{pr}_2^*[o_S]$ in $H^4(S \times S, \mathbb{Q})$. We now follow [17] (see also [20]): The cohomological version of the equations given in Theorem 5.2 with $L = K_S$ holds on any smooth projective surface with $b_1 = 0$, and if the canonical class satisfies $[K_S] = 0$ or $[K_S]^2 \neq 0$, one can reduce modulo these relations any polynomial expression in the variables

$$\text{pr}_i^*[\text{pt}], \quad \text{pr}_j^*[K_S], \quad \text{pr}_{st}^*[\Delta_S]$$

to a linear combination of monomials in the variables $\text{pr}_i^*[\text{pt}]$, $\text{pr}_j^*[K_S]$, $\text{pr}_{st}^*[\Delta_S]^0$, with the property that no index appears twice in the monomial. Here the class $[\Delta_S]^0$ is the class

$$[\Delta_S] - \text{pr}_1^*[\text{pt}] - \text{pr}_2^*[\text{pt}] - \lambda \text{pr}_1^*[K_S] \cup \text{pr}_2^*[K_S],$$

where the coefficient λ , when $K_S \neq 0$, is determined by the relation $\lambda[K_S]^2 = 1$ (the class $[\Delta_S]^0 \in H^4(S \times S, \mathbb{Q})$) is the projector onto $H^2(S, \mathbb{Q})^{\perp[K_S]}$). Now, it is clear by Künneth decomposition that if a linear combination of such monomials vanishes in $H^*(S^N, \mathbb{Q}) = H^*(S, \mathbb{Q})^{\otimes N}$, then for fixed distinct indices $i_1, \dots, i_m, j_1, \dots, j_p, k_1, \dots, k_q$, the sum of such monomials of the form

$$\text{pr}_{i_1}^*[\text{pt}] \cdots \text{pr}_{i_m}^*[\text{pt}] \cdot \text{pr}_{j_1}^*[K_S] \cdots \text{pr}_{j_p}^*[K_S] \cdot \text{pr}_{k_1}^* 1_S \cdots \text{pr}_{k_q}^* 1_S \cdot \prod_{s_1, t_1, \dots, s_l, t_l} \text{pr}_{s_i, t_i}^*[\Delta_S]^0,$$

where the indices s_i, t_j exhaust the remaining indices, are all distinct and are different from the i_s, j_s, k_s , has to be 0. This way, we reduced the problem to linear combinations of monomials of the form

$$(50) \quad \text{pr}_{s_1 t_1}^*[\Delta_S]^0 \cdots \text{pr}_{s_l t_l}^*[\Delta_S]^0$$

on S^{2l} , where no index is repeated. We now have the following result due to Yin [20]: The “Kimura relation” is a relation between monomials of the above type. It says that, for $M = \dim H^2(S, \mathbb{Q})^{\perp[K_S]}$, the cohomology class of the projector onto $\bigwedge^{M+1} H^2(S, \mathbb{Q})^{\perp[K_S]} \subset H^{2M+2}(S^{M+1}, \mathbb{Q})$ is 0, which is obvious since $\bigwedge^{M+1} H^2(S, \mathbb{Q})^{\perp[K_S]} = 0$. The class of this projector is the class

$$(51) \quad \sum_{\sigma \in \mathfrak{S}_{M+1}} \epsilon(\sigma) \prod_{i=1}^{M+1} \text{pr}_{i, M+1+\sigma(i)}^*[\Delta_S]^0 \in H^{4M+4}(S^{2M+2}, \mathbb{Q})$$

and the Kimura relation is thus the vanishing of (51).

Theorem 5.9 (Yin [20]) *For any integer m , the relations in $H^*(S^m, \mathbb{Q})$ between the monomials (50) with no repeated indices are generated by the pull-back to S^m of the Kimura relation via a projection (and a permutation) $S^m \rightarrow S^{2N+2}$.*

We deduce the following:

Corollary 5.10 *The polynomial Q_μ belongs to the ideal generated by the trivial relations (see Remark 5.3), the relation $c_2(S) = \chi_{\text{top}}(S)o_S$ (where we recover $\chi_{\text{top}}(S)$ as the self-intersection of Δ_S) and the relations listed in Theorem 5.2 with $L = K_S$.*

Proof Indeed, choose for S a smooth projective surface with $b_1(S) = 0$ and $b_2(S) > n(2n+1)/2$. Then by Theorem 5.9, there are no linear relations between the monomials (50) with no repeated index if $s \leq (2n+1)n$. On the other hand, we have the vanishing of the cohomology class

$$Q_\mu. (\text{pr}_i^*[c_2(S)], \text{pr}_j^*[K_S], \text{pr}_l^*[o_S], \text{pr}_{s_t}^*[\Delta_S]) \in H^*(S^N, \mathbb{Q}),$$

where $N = \sum_i l(\mu_i) \leq (2n + 1)n$. It then follows from the above reduction that the polynomial $Q_{\mu.}$, where one substitutes $\chi_{\text{top}}(S)[o_S]$ to $[c_2(S)]$, belongs to the ideal generated by the cohomological version of the relations given in Theorem 5.2, with $L = K_S$. □

The proof of Theorem 5.1 is now finished. Indeed, S being now a $K3$ surface, we know by Theorem 5.2 that the relation $\chi_{\text{top}}(S)o_S = c_2(S)$ holds in $\text{CH}_0(S)$ and that the relations listed in Theorem 5.2 hold in $\text{CH}(S^k)$ for adequate k . As the polynomial $Q_{\mu.}$, where one substitutes $\chi_{\text{top}}(S)o_S$ to $c_2(S)$, belongs to the ideal generated by the relations given in Theorem 5.2 and the trivial relations, we conclude that $Q_{\mu.} = 0$ in $\text{CH}(S^N)$. By (48), we proved the vanishing (47)

$$(E_{\mu_1}, \dots, E_{\mu_{2n+1}})^*(\Gamma^{2n+1}(X, o_X)) = 0 \quad \text{in } \text{CH}(S^N),$$

which concludes the proof. □

5.2 Universally defined cycles

This subsection is devoted to introducing the notion of “universally defined cycles” and to sketching the proof of a quite general statement which will be fully proved in [15]. It concerns “universally defined” cycles on self-products of surfaces. We first explain the meaning of this expression. In the following, we work with Chow groups with integral coefficients, and we will write $\text{CH}(X)_{\mathbb{Q}}$ for cycles with \mathbb{Q} -coefficients.

Definition 5.11 Let n, N be integers. A universally defined cycle on the N^{th} power of smooth algebraic varieties X of a given dimension n consists in the following data: For each smooth family of n -dimensional algebraic varieties $\mathcal{X} \rightarrow B$ defined over a field k , where B is smooth quasiprojective, a cycle $z_{\mathcal{X}} \in \text{CH}(\mathcal{X}^{N/B})$ is given, satisfying the following conditions:

- (i) If $r: B' \rightarrow B$ is a morphism, with induced morphism

$$R_N: (\mathcal{X}')^{N/B'} \rightarrow \mathcal{X}^{N/B}, \quad \mathcal{X}' := \mathcal{X} \times_B B',$$

then

$$z_{\mathcal{X}'} = R_N^* z_{\mathcal{X}} \quad \text{in } \text{CH}((\mathcal{X}')^{N/B'}).$$

- (ii) If $\mathcal{X} \rightarrow B$ is a family as above and $\mathcal{Y} \subset \mathcal{X}$ is a Zariski open set, then

$$z_{\mathcal{Y}} = z_{\mathcal{X}|_{\mathcal{Y}^{N/B}}} \quad \text{in } \text{CH}(\mathcal{Y}^{N/B}).$$

Theorem 5.12 For any universally defined cycle z on N^{th} powers of surfaces, there exists a polynomial P with rational coefficients, depending only on z , such that for any smooth algebraic surface S defined over \mathbb{C} ,

$$z_S = P(\text{pr}_i^* c_1(S), \text{pr}_j^* c_2(S), \text{pr}_{rs}^* \Delta_S) \text{ in } \text{CH}(S^N)_{\mathbb{Q}}.$$

Remark 5.13 One could introduce as well universally defined cycles with \mathbb{Q} -coefficients, by replacing everywhere in the definition above CH by $\text{CH}_{\mathbb{Q}}$. It is possible that the conclusion holds as well for universally defined cycles with \mathbb{Q} -coefficients, but our present proof uses the integral structure.

We will give some hints on the proof, with a complete proof only in the case $N = 1$ (Proposition 5.18) and the construction of the desired polynomials (Corollary 5.15 and Proposition 5.17). We refer to [15] for a full treatment. Let us first show how to produce such polynomials. Let $G := G(2, 5)$ be the Grassmannian of 2-dimensional vector subspaces in k^5 . Any smooth complex projective surface can be embedded in G , for example by choosing 5 general sections of a very ample vector bundle E on S . Let $\mathcal{O}_G(1)$ be the Plücker line bundle on G , and let $c \in \text{CH}^2(G)$ be the second Chern class of the tautological rank-2 vector bundle on G . We choose an integer d , and consider the universal family $S_d \rightarrow B$ of smooth surfaces in G which are complete intersections of 4 members of $|\mathcal{O}_G(d)|$. The smooth variety B is thus the vector space $H^0(G, \mathcal{O}_G(d))^4$ and

$$S_d \subset S_{d,\text{univ}}$$

is the Zariski open set consisting of points where $S_{d,\text{univ}} \rightarrow B$ is smooth. Here

$$S_{d,\text{univ}} := \{(b, x) \in B \times G \mid b = (f_{1,b}, \dots, f_{4,b}), f_{i,b}(x) = 0 \text{ for all } i\}.$$

There is an obvious morphism

$$f: S_d \rightarrow G$$

given by the restriction to S_d of the second projection $S_{d,\text{univ}} \rightarrow G$, which induces for any $N \geq 1$ the morphism

$$f_N: S_d^{N/U} \rightarrow G^N$$

with induced pull-back morphism $f_N^*: \text{CH}(G^N) \rightarrow \text{CH}(S_d^{N/U})$. We now use the following result, which is one of the main ingredients in the proof of Theorem 5.12:

Proposition 5.14 For any integer $N > 0$ and sufficiently large d , $\text{CH}(S_d^{N/U})$ is generated as a $\text{CH}(G^N)$ -module by the relative partial diagonals $\Delta_{I/U}(S_d)$.

Here I denotes as usual a partition of $\{1, \dots, N\}$, determining a partial diagonal.

Proof of Proposition 5.14 By the localization exact sequence, it suffices to prove the result with S_d replaced by $S_{d,\text{univ}}$. Next consider the natural morphism

$$f_N: S_{d,\text{univ}}^{N/B} \rightarrow G^d.$$

The fiber of f_N over a N -tuple (x_1, \dots, x_N) consists of the set of 4-tuples $(\sigma_1, \dots, \sigma_4)$ in $H^0(G, \mathcal{O}_G(d))^4$ having the property that the σ_i vanish at all points x_i . As d is large compared to N , any k distinct points of G with $k \leq N$ impose independent conditions to $H^0(G, \mathcal{O}_G(d))$, and thus, denoting by G_k^N the locally closed subvariety of G^N consisting of N -tuples with exactly k -distinct points, which is the disjoint union of the diagonals $\Delta_I(G)$ with $l(I) = k$ (or rather of the $\Delta_I^0(G) := \Delta_I(G) \setminus \bigcup_{J, l(J) < k} \Delta_J(G)$), we find that $f_N^{-1}(G_k^N)$ is a Zariski open set in a vector bundle over G_k^N . It follows from the localization exact sequence and A^1 -invariance that

$$\text{CH}(G_k^N) \xrightarrow{f_N^*} \text{CH}(f_N^{-1}(G_k^N))$$

is surjective. Writing G^N as the disjoint union of the $\Delta_I^0(G)$, we conclude from the above and the localization exact sequence that

$$\bigoplus_I \text{CH}(\Delta_I(G)) \xrightarrow{(j_{I*} \circ f_I^*)} \text{CH}(S_{d,\text{univ}}^{N/B})$$

is surjective, where f_I is the restriction of f_N to $f_N^{-1}(\Delta_I(G)) \subset G^N$ and j_I is the inclusion of $f_N^{-1}(\Delta_I(G))$ in $S_d^{N/U}$. Note that $f_I^{-1}(\Delta_I(G)) = \Delta_{I/B}(S_{d,\text{univ}})$. Finally, we observe that the restriction map

$$\text{CH}(G^N) \rightarrow \text{CH}(\Delta_I(G))$$

is surjective, and that for any $\alpha \in \text{CH}(G^N)$,

$$j_{I*} \circ f_I^*(\alpha|_{\Delta_I(G)}) = f_N^* \alpha \cdot (j_{I*} \circ f_I^*)(1) = f_N^* \alpha \cdot \Delta_{I/B}(S_{d,\text{univ}}),$$

and this finishes the proof. □

Corollary 5.15 *For any universally defined cycle z on N^{th} powers of surfaces and for sufficiently large d , there exists a polynomial P_d with rational coefficients, depending only on z and d such that for any smooth complete intersection surface $S_d \subset G$ as above,*

$$(52) \quad z_{S_d} = P_d(\text{pr}_i^* c_1(S_d), \text{pr}_j^* c_2(S_d), \text{pr}_{rs}^* \Delta_{S_d}) \quad \text{in } \text{CH}(S_d^N)_{\mathbb{Q}}.$$

Furthermore, $(4d - 5)^{2N} P_d$ has integral coefficients.

Proof As z is universal, there exists a cycle $z_{S_d} \in \text{CH}(S_d^{N/B})$ such that for any surface S_d as above,

$$z_{S_d} = (z_{S_d})|_{S_d^N},$$

where we see S_d as a fiber of the universal family $S_d \rightarrow B$. We next use Proposition 5.14 to write, for $d \gg 0$, Z_d as a combination $\sum_I f_N^* \alpha_I \cdot \Delta_{I/B}(S_d)$, where $\alpha_I \in \text{CH}(G(2, 5))$. Furthermore, it is immediate to prove that $\text{CH}(G^N) = \text{CH}(G)^{\otimes N}$, so that we can write each α_I as a polynomial with integral coefficients in $\text{pr}_i^* m$, $m := c_1(\mathcal{O}_G(1)) = c_1(E)$ and $\text{pr}_j^* c$, $c := c_2(E)$, where E is the dual of the tautological subbundle on G . Of course, under restriction to S_d^N , only polynomials of weighted degree ≤ 2 in each set of variables $\text{pr}_i^* m$, $\text{pr}_i^* c$ will survive. We now observe that the restriction of m to S_d is a rational multiple of $c_1(S)$ (more precisely, $K_{S_d} = \mathcal{O}_G(4d - 5)|_S$ by the adjunction formula), and the restriction of c to S_d is an adequate linear combination of $\frac{1}{(4d-5)^2} c_1(S_d)^2, c_2(S_d)$. Putting everything together and using the fact that the relative diagonals $\Delta_{I/B}(S_d)$ restrict to $\Delta_I(S_d)$, we get the result. \square

Remark 5.16 Note that Corollary 5.15 is true more generally for the regular and complete intersection locus S_{reg} of any set of 4 degree d equations on G . The proof uses Proposition 5.14 (which works for the family $S_d \rightarrow B$ of smooth complete intersection quasiprojective surfaces), and both conditions (i) and (ii) in Definition 5.11.

The corollary above proves Theorem 5.12 for smooth complete intersection surfaces of degree d , and more generally for the regular and complete intersection locus of any set of 4 degree d equations on G . What remains to be done is to prove that the polynomial above works for all surfaces. Note that the polynomial P_d is in fact not uniquely defined as only its value on the set of variables $\text{pr}_i^* c_1(S_d), \text{pr}_j^* c_2(S_d), \text{pr}_{st}^* \Delta_{S_d}$ is well defined in $\text{CH}(S_d^N)_{\mathbb{Q}}$. Hence *a priori* P_d is only defined modulo the relations in $\text{CH}(S_d^N)_{\mathbb{Q}}$ between these variables. However, the following result shows that a part of P_d is in fact independent of d for large d .

Proposition 5.17 *For any universally defined cycle z on N^{th} powers of surfaces, there exists a polynomial Q in the variables $\text{pr}_{st}^* \Delta_S$, depending only on z , with the following property: For any smooth surface S , there is a Zariski dense open set $V \subset S$ such that $z_V = Q(\text{pr}_{st}^* \Delta_S)$ in $\text{CH}(V^N)_{\mathbb{Q}}$.*

Proof Let Q_d be the part of the polynomial P_d which involves only the diagonals. Then let $U_d \subset S_d$ be the complement of a hyperplane section defined by the choice of

a codimension-2 vector subspace $W \subset \mathbb{C}^5$ in general position. As $c_1(\mathcal{O}_{S_d}(1))$ and $c_2(E)$ vanish in $\text{CH}(U_d)$, we deduce from (52) that

$$(53) \quad z_{U_d} = Q_d(\text{pr}_{st}^* \Delta_{U_d}) \quad \text{in } \text{CH}(U_d^N).$$

We observe now that for $d' \leq d$, a surface $S_{d'}$ which is the complete intersection in G of hypersurfaces of degree d' is an irreducible component of a (singular) surface $\Sigma_d = S_{d'} \cup T$ defined as the complete intersection in G of four degree d hypersurfaces containing $S_{d'}$ and that, denoting $C := S_{d'} \cap T$, the open set $U'_{d'} := S_{d'} \setminus C$ is contained in the smooth locus of Σ_d . From Remark 5.16, we thus get that

$$(54) \quad z_{U'_{d'}} = P_d(\text{pr}_i^* c_1(U'_{d'}), \text{pr}_j^* c_2(U'_{d'}), \text{pr}_{st}^* \Delta_{U'_{d'}}) \quad \text{in } \text{CH}(U'_{d'}{}^N)_{\mathbb{Q}},$$

and after restriction to $V_{d'} := U_d \cap U'_{d'}$, this becomes

$$(55) \quad z_{V_{d'}} = Q_d(\text{pr}_{st}^* \Delta_{V_{d'}}) \quad \text{in } \text{CH}(V_{d'}^N)_{\mathbb{Q}},$$

On the other hand, we also have (53) for d' , which provides after restriction to $V_{d'}$

$$(56) \quad z_{V_{d'}} = Q_{d'}(\text{pr}_{st}^* \Delta_{V_{d'}}) \quad \text{in } \text{CH}(V_{d'}^N)_{\mathbb{Q}}.$$

Hence $Q_d - Q_{d'}$ belongs to the kernel of the map

$$\text{ev}_{d'}: \mathbb{Q}[X_{rs}]_{1 \leq r \neq s \leq N} \rightarrow \text{CH}(V^N), \quad f \mapsto f(\text{pr}_{rs}^* \Delta_V),$$

where V is a sufficiently small Zariski open set of a general complete intersection of four hypersurfaces of degree d' in G . On the other hand, it follows from the above construction that $\text{Ker } \text{ev}_d \subset \text{Ker } \text{ev}_{d'}$ for $d' \leq d$. As the polynomials we consider are homogeneous of given degree (equal to half the codimension of z), they live in a finite-dimensional vector space and we conclude that these kernels are in fact stationary, equal to K for $d \geq d_0$. So we finally conclude that there exists a d_0 such that for $d \geq d' \geq d_0$, $Q_d - Q_{d'}$ belongs to K . It follows that for any d , for any reduced complete intersection of four degree d hypersurfaces in G , and for a dense Zariski open set $V \subset S$

$$z_V = Q_{d_0}(\text{pr}_{st}^* \Delta_V) \quad \text{in } \text{CH}(V^N)_{\mathbb{Q}}.$$

As any smooth quasiprojective surface has a dense Zariski open set which is contained in the smooth locus of such a complete intersection for d large enough, the proposition is proved, with $Q = Q_{d_0}$. □

We finish this section with the proof of Theorem 5.12 in the case $N = 1$ and for codimension one cycles.

Proposition 5.18 *Let z be a universally defined codimension one cycle on surfaces. Then there is an integer m independent of S such that for any smooth quasiprojective surface S ,*

$$z_S = mc_1(S) \quad \text{in } \text{CH}^1(S).$$

Proof For complete intersections S_d of four hypersurfaces of degree d in G , we must have by Corollary 5.15

$$z = \alpha_d K_{S_d},$$

for some rational number α_d such that $(4d-5)\alpha_d \in \mathbb{Z}$, and for any surface S , choosing a very ample vector bundle E of rank 2 on S to embed S in G , and choosing d large enough, we get

$$(57) \quad z_{S|U} = \alpha_d K_U \quad \text{in } \text{CH}^1(U)_{\mathbb{Q}},$$

where $U = S \setminus C$, the surface $S \cup_C T = \Sigma_d$ being the complete intersection of four degree- d hypersurfaces containing S in G . The curve C belongs to the linear system $|(4d-5)L - K_S|$, where $L = \det E = \mathcal{O}_G(1)|_S$. For a general choice of equations and d large enough, the curve C will be irreducible, so by the localization exact sequence, (57) rewrites as

$$(58) \quad z_S = \alpha_d K_S + \beta_d C \quad \text{in } \text{CH}^1(S)_{\mathbb{Q}},$$

If K_S and L are linearly independent in $\text{CH}^1(S)_{\mathbb{Q}}$, this implies, because the left-hand side is independent of L , that $\beta_d = 0$ and thus $z_S = \alpha_d K_S$, with $\alpha_d =: \alpha$ necessarily independent of d . If not, we simply blow up S at one point and choose L on \tilde{S} linearly independent of $K_{\tilde{S}}$ in $\text{CH}^1(\tilde{S})_{\mathbb{Q}}$. Then the above conclusion applies to \tilde{S} , hence we get

$$(59) \quad z_{\tilde{S}} = \alpha K_{\tilde{S}} \quad \text{in } \text{CH}^1(\tilde{S}).$$

As

$$S \setminus \{p\} \cong \tilde{S} \setminus E_p, \quad \text{CH}^1(S)_{\mathbb{Q}} \cong \text{CH}^1(S \setminus \{p\})_{\mathbb{Q}},$$

(59) is also true for S by condition (ii) in Definition 5.11. Finally α has to be an integer since $(4d-5)\alpha \in \mathbb{Z}$ for any d . This proves Proposition 5.18. \square

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